Research Article

The Analysis of Fractional-Order Nonlinear Systems of Third Order KdV and Burgers Equations via a Novel Transform

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In this article, we solve nonlinear systems of third order KdV Equations and the systems of coupled Burgers equations in one and two dimensions with the help of two different methods. The suggested techniques in addition with Laplace transform and Atangana–Baleanu fractional derivative operator are implemented to solve four systems. The obtained results by implementing the proposed methods are compared with exact solution. The convergence of the method is successfully presented and mathematically proved. The results we get are compared with exact solution through graphs and tables which confirms the effectiveness of the suggested techniques. In addition, the results obtained by employing the proposed approaches at different fractional orders are compared, confirming that as the value goes from fractional order to integer order, the result gets closer to the exact solution. Moreover, suggested techniques are interesting, easy, and highly accurate which confirm that these methods are suitable methods for solving any partial differential equations or systems of partial differential equations as well.

1. Introduction

In recent years, fractional calculus has surpassed ordinary calculus in popularity. The standard calculus has reached its pinnacle in terms of discovery. Mathematicians and engineers require fractional calculus as a solution. This permits a more accurate description of real-world phenomena than the traditional “integer” order. Numerous mathematicians, including Fourier, Laplace, Riesz, and others, were engaged and made substantial contribution to the subject. Modern definitions of fractional order derivatives and integrals, such as the Atangana–Baleanu fractional integral [1], the Caputo fractional derivative [2], and the Caputo–Fabrizio fractional derivative [3], have ushered in a new era in the literature of fractional derivatives. Models based on fractional calculus can accurately depict numerous engineering, physics, and chemistry processes, among others [4]. In addition, fractional calculus is used to simulate the frequency-dependent damping behavior of a variety of viscoelastic materials [5], the dynamics of interfaces between nanoparticles and substrates [6], economics [7], and numerous other applications [8–11].

Finding the actual or approximate solutions of FDEs is crucial in all of these areas of study, but because we lack a method for obtaining the precise solution of these sorts of FDEs, we must focus on approximating the exact solution. Determining the exact answer to such FDEs and other scientific applications is a difficult task in mathematics. Unlike the approximate answer [12], the exact solution enables us to comprehend the problem’s mechanism and
complexity. Obtaining exact analytical expressions to FDEs is exceedingly difficult, if not impossible, due to the complexity of computation involved in these equations. As a result, it is necessary to seek out some useful approximations and numerical techniques, such as the homotopy perturbation method [13], variation iteration method [14], residual power series method [15], approximate-analytical method [16], Elzaki transform decomposition method [17], Iterative Laplace transform method [18], Adomian decomposition method [19], reduced differential transform method, and others [20–23].

In this paper, we provided two analytic approaches in conjunction with the Laplace transformation and fractional derivatives in Antagana–Baleanu solution to satisfy fractional-order problems [24]. The first method is the mixing of Laplace transform (LT) and variational iteration method known as variational iteration transform method (VITM) which was first developed by He [25] and is an effective solution for a broad variety of problems in scientific fields [26, 27]. The second significant methodology for solving nonlinear functional equations is the combination of the Adomian decomposition method and Laplace transform, which was first developed by George Adomian (1923–1996) in the 1980s. This technique depends on the decomposition of a nonlinear equation result into a series of functions. A polynomial produced by a power series expansion of an analytic function returns each series term. This method for solving several nonlinear fractional-order differential equations is interesting, straightforward, and accurate.

Harry Bateman introduced the Burgers equation in 1915 [28], and it was subsequently dubbed by the Burgers equation. The Burgers equation has many applications in science and engineering, especially when dealing with nonlinear problems. Burgers equation applications have grown in prominence and attention among mathematical scientists and researchers. This equation is acknowledged to represent a range of phenomena, such as dynamic modeling, heat conduction, acoustic waves, and turbulence [29–31]. In 1895, Korteweg and Vries initially derived the Korteweg–De Vries (KDV) equation. The KDV equation is used to predict long waves, tides, solitary waves, and wave propagation in a shallow canal. The KDV equation is utilized in several disciplines, including fluid mechanics, signal processing, hydrology, viscoelasticity, and fractional kinetics.

2. Preliminaries

In this section, we presented some basic definitions of fractional calculus related to our present work.

Definition 1. The derivative by Caputo having order fraction is defined as

$$^{c}D_{\sigma}^{\alpha}[h(\xi)] = \frac{1}{(n-\sigma)} \int_{0}^{\sigma} (\xi-k)^{n-\sigma-1} h^{(n)}(k)dk, \text{ where } n<\sigma<n+1. \quad (1)$$

Definition 2. The derivative by Caputo having order fraction with the aid of Laplace transform \( {^L}D_{\sigma}^{\alpha}[h(\xi)] \) is defined as

$$\mathcal{L}\left\{{^L}D_{\sigma}^{\alpha}[h(\xi)]\right\}(\omega) = \frac{1}{\omega^{\alpha-\sigma}} \left[ \omega^n \mathcal{L}[h(\xi)](\omega) - \omega^{n-1} h(\xi,0) - \cdots - h^{n-1}(\xi,0) \right]. \quad (2)$$

Definition 3. The fractional Atangana–Baleanu derivative in terms of Caputo manner is defined as

$$^{ABC}D_{\sigma}^{\alpha}[h(\xi)] = \frac{B(\sigma)}{1-\sigma} \int_{a}^{\xi} h'(k)E_{\sigma}\left[\frac{\sigma}{1-\sigma}(1-k)^{\sigma}\right]dk. \quad (3)$$

where normalization function is denoted by \( B(\sigma) \) with \( B(0) = B(1) = 1, h \in H^{1}(a,b), b > a, \sigma \in [0,1] \) and \( E_{\sigma} \) is the Mittag–Leffler function.

Definition 4. In terms of Riemann–Liouville, the Atanga- na–Baleanu fractional derivative is defined as

$$^{ABC}D_{\sigma}^{\alpha}[h(\xi)] = \frac{B(\sigma)}{1-\sigma} \frac{d}{d\xi} \int_{a}^{\xi} h(k)E_{\sigma}\left[\frac{\sigma}{1-\sigma}(1-k)^{\sigma}\right]dk. \quad (4)$$

Definition 5. The Atangana–Baleanu operator in connection with Laplace transform is given by
\[
\begin{align*}
\text{Definition 6.} \text{ Consider } 0 < \sigma < 1, \text{ and } h \text{ is a function of order } \\
\text{then the fractional integral operator for } \sigma \text{ is defined as } \\
\int_0^\tau h(k)(\mathfrak{G} - k)^{-\sigma - 1}dk.
\end{align*}
\]

3. Idea of LTDM

Here, we discuss the methodology of LTDM for solving fractional-order partial differential equations.

\[
L[\varphi(\xi, \mathfrak{G})] = \Theta(\xi, \omega) - \frac{\omega^\sigma + \sigma(1 - \sigma)}{\omega^\sigma}L[\mathcal{F}(\xi, \mathfrak{G})].
\]

where \( \Theta(\xi, \omega) \) shows the term that come from the source term. LTDM generates the result of the infinite series of \( \varphi(\xi, \mathfrak{G}) \)

\[
\varphi(\xi, \mathfrak{G}) = \sum_{m=0}^{\infty} \varphi_m(\xi, \mathfrak{G}),
\]

and decomposing the nonlinear operator \( \mathcal{N}_1 \) as

\[
\sum_{m=0}^{\infty} \varphi_m(\xi, \mathfrak{G}) = \Theta(\xi, \omega) - L^{-1}\left\{ \frac{\omega^\sigma + \sigma(1 - \sigma)}{\omega^\sigma}L[\mathcal{F}(\xi, \mathfrak{G})] \right\}.
\]

The terms listed below are defined as

\[
\begin{align*}
\varphi_0(\xi, \mathfrak{G}) &= \Theta(\xi, \omega), \\
\varphi_1(\xi, \mathfrak{G}) &= L^{-1}\left\{ \frac{\omega^\sigma + \sigma(1 - \sigma)}{\omega^\sigma}L[\mathcal{F}(\xi, \mathfrak{G})] + \mathcal{A}_0 \right\}.
\end{align*}
\]

As a result, all of the components for \( m \geq 1 \) are determined as

\[
\begin{align*}
\mathcal{N}_1(\xi, \mathfrak{G}) &= \sum_{m=0}^{\infty} \mathcal{A}_m, \\
\mathcal{A}_m &= \frac{1}{m!} \left[ \frac{\gamma_0^m}{\partial \xi} \left\{ \sum_{k=0}^{\infty} \xi^k \sum_{\ell=0}^{\infty} \xi^\ell \mathcal{A}_1 \right\} \right]_{\xi=0}.
\end{align*}
\]

4. VITM Formulation

Here, we discuss the methodology of VITM for solving fractional-order partial differential equations.
having initial term

\[ \varphi (\xi, 0) = g_1 (\xi). \]  

(19)

Here, the fractional-order AB operator is indicated from \( AB^\sigma \) and the inverse Laplace transform. \( \mathcal{L} \) is the Laplace transform operator. Using the differentiation property of the inverse Laplace transform, we obtain

\[ \varphi_m (\xi, 3) = \varphi (0) + \mathcal{L}^{-1} [\sigma (s) \mathcal{L}[-\mathcal{F} (\xi, 3)]], \]

\[ \varphi_1 (\xi, 3) = \mathcal{L}^{-1} [\sigma (s) \mathcal{L}[\mathcal{M}(\xi, 3) + \mathcal{N}(\xi, 3)]]], \]

\[ \vdots \]

\[ \varphi_{m+1} (\xi, 3) = \mathcal{L}^{-1} [\sigma (s) \mathcal{L}[\mathcal{M}(\varphi_0 (\xi, 3) + \varphi_1 (\xi, 3) + \cdots + \varphi_n (\xi, 3))]. \]

(24)

5. Applications

To show the validity and capability of the suggested techniques, we implemented proposed methods for solving four nonlinear systems.

5.1. Problem 1. Consider system of homogeneous KdV equation having order three

\[ \frac{\partial^\sigma \varphi}{\partial \xi^\sigma} = \varphi \frac{\partial^3 \varphi}{\partial \xi^3} + \varphi \frac{\partial \varphi}{\partial \xi} + \varphi \frac{\partial \varphi}{\partial \xi} \]

\[ \quad - 2 \frac{\partial^3 \varphi}{\partial \xi^3} + \varphi \frac{\partial \varphi}{\partial \xi} < \sigma \leq 1, \]

(25)

On taking the Laplace transform of (25), we get

\[ \frac{\omega^\sigma \mathcal{L}[\varphi (\xi, 3)] - \omega^{-1} \varphi (\xi, 0)}{\omega^\sigma + \sigma (1 - \omega^\sigma)} = \mathcal{L} \left[ \frac{\partial^3 \varphi}{\partial \xi^3} + \varphi \frac{\partial \varphi}{\partial \xi} + \varphi \frac{\partial \varphi}{\partial \xi} \right]. \]

(27)

We obtain when we use the Laplace inverse transform...
\[
\phi(\xi, \mathfrak{F}) = \left(3 - 6\tanh^2 \frac{\xi}{2}\right) + L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left[ \frac{\partial^3 \phi}{\partial \xi^3} + \frac{\partial \phi}{\partial \xi} + \gamma \frac{\partial \psi}{\partial \xi} \right] \right].
\]

\[
\nu(\xi, \mathfrak{F}) = \left(3l\sqrt{2} \tanh^2 \frac{\xi}{2}\right) + L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left[ -2 \frac{\partial^3 \psi}{\partial \xi^3} + \frac{\partial \psi}{\partial \xi} \right] \right].
\]

Assume that the solution, \(\phi(\xi, \mathfrak{F})\) and \(\nu(\xi, \mathfrak{F})\) in series form as

\[
\phi(\xi, \mathfrak{F}) = \sum_{m=0}^{\infty} \phi_m(\xi, \mathfrak{F}), \quad \nu(\xi, \mathfrak{F}) = \sum_{m=0}^{\infty} \nu_m(\xi, \mathfrak{F}),
\]

where \(\phi_0 = \sum_{m=0}^{\infty} \mathfrak{A}_m, \nu_0 = \sum_{m=0}^{\infty} \mathfrak{B}_m\), and \(\phi_1 = \sum_{m=0}^{\infty} \mathfrak{C}_m\) are Adomian polynomials that characterize the nonlinear terms, and so equation (28) is rewritten as

\[
\sum_{m=0}^{\infty} \phi_m(\xi, \mathfrak{F}) = \left(3 - 6\tanh^2 \frac{\xi}{2}\right) + L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left[ \frac{\partial^3 \phi}{\partial \xi^3} + \sum_{m=0}^{\infty} \mathfrak{A}_m + \sum_{m=0}^{\infty} \mathfrak{B}_m \right] \right].
\]

\[
\sum_{m=0}^{\infty} \nu_m(\xi, \mathfrak{F}) = \left(3l\sqrt{2} \tanh^2 \frac{\xi}{2}\right) + L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left[ -2 \frac{\partial^3 \psi}{\partial \xi^3} + \sum_{m=0}^{\infty} \mathfrak{C}_m \right] \right].
\]

The decomposition of nonlinear terms by Adomian polynomials is defined as in equation (14),

\[
\mathfrak{A}_0 = \phi_0 \phi_0 \xi, \quad \mathfrak{A}_1 = \phi_1 \phi_0 \xi + \phi_0 \phi_1 \xi, \quad \mathfrak{A}_2 = \phi_2 \phi_0 \xi + \phi_1 \phi_1 \xi + \phi_0 \phi_2 \xi,
\]

\[
\mathfrak{B}_0 = \nu_0 \nu_0 \xi, \quad \mathfrak{B}_1 = \nu_1 \nu_0 \xi + \nu_0 \nu_1 \xi, \quad \mathfrak{B}_2 = \nu_2 \nu_0 \xi + \nu_1 \nu_1 \xi + \nu_0 \nu_2 \xi,
\]

\[
\mathfrak{C}_0 = \nu_0 \nu_0 \xi, \quad \mathfrak{C}_1 = \nu_1 \nu_0 \xi + \nu_0 \nu_1 \xi, \quad \mathfrak{C}_2 = \nu_2 \nu_0 \xi + \nu_1 \nu_1 \xi + \nu_0 \nu_2 \xi.
\]

As a result, when comparing the two sides of equation (30)

\[
\phi_0(\xi, \mathfrak{F}) = \left(3 - 6\tanh^2 \frac{\xi}{2}\right), \quad \nu_0(\xi, \mathfrak{F}) = \left(3l\sqrt{2} \tanh^2 \frac{\xi}{2}\right),
\]

For \(m = 0\),

\[
\phi_1(\xi, \mathfrak{F}) = -6\text{sech}^2 \frac{\xi}{2} \tan \frac{\xi}{2} \left[ \frac{\sigma \mathfrak{F}^\sigma}{\mathfrak{F}^2 + 1} + (1 - \sigma) \right],
\]

\[
\nu_1(\xi, \mathfrak{F}) = 3l\sqrt{2} \text{sech}^2 \frac{\xi}{2} \tan \frac{\xi}{2} \left[ \frac{\sigma \mathfrak{F}^\sigma}{\mathfrak{F}^2 + 1} + (1 - \sigma) \right].
\]

For \(m = 1\),

\[
\phi_2(\xi, \mathfrak{F}) = \frac{3}{2} \left[ 2\text{sech}^2 \frac{\xi}{2} + 7\text{sech}^4 \frac{\xi}{2} - 15\text{sech}^6 \frac{\xi}{2} \right] \left[ \frac{\sigma^2 \mathfrak{F}^{2\sigma}}{\mathfrak{F}^2 + 1} + 2\sigma (1 - \sigma) \frac{\mathfrak{F}^\sigma}{\mathfrak{F}^2 + 1} + (1 - \sigma)^2 \right],
\]

\[
\nu_2(\xi, \mathfrak{F}) = \frac{3l\sqrt{2}}{4} \left[ 2\text{sech}^2 \frac{\xi}{2} + 21\text{sech}^4 \frac{\xi}{2} - 24\text{sech}^6 \frac{\xi}{2} \right] \left[ \frac{\sigma^2 \mathfrak{F}^{2\sigma}}{\mathfrak{F}^2 + 1} + 2\sigma (1 - \sigma) \frac{\mathfrak{F}^\sigma}{\mathfrak{F}^2 + 1} + (1 - \sigma)^2 \right],
\]
The approximate solution to the series is written as

$$
\varphi(\xi, \mathcal{S}) = \sum_{m=0}^{\infty} \varphi_m(\xi, \mathcal{S}) = \varphi_0(\xi, \mathcal{S}) + \varphi_1(\xi, \mathcal{S}) + \varphi_2(\xi, \mathcal{S}) + \cdots \\
\nu(\xi, \mathcal{S}) = \sum_{m=0}^{\infty} \nu_m(\xi, \mathcal{S}) = \nu_0(\xi, \mathcal{S}) + \nu_1(\xi, \mathcal{S}) + \nu_2(\xi, \mathcal{S}) + \cdots
$$

We achieve the exact solution by putting $\sigma = 1$

$$
\varphi(\xi, \mathcal{S}) = 3 - 6\tanh^2\left(\frac{\mathcal{S} + \xi}{2}\right),
$$

$$
\nu(\xi, \mathcal{S}) = -3\sqrt{2} \tanh^2\left(\frac{\mathcal{S} + \xi}{2}\right).
$$

For Equation (25), we have the iteration formula:

$$
\varphi_{m+1}(\xi, \mathcal{S}) = \varphi_m(\xi, \mathcal{S}) - L^{-1}\left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L\left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma(1 - \omega^\sigma)} \frac{\partial^3 \varphi_m}{\partial \xi^3} + \frac{\partial \varphi_m}{\partial \xi} \frac{\partial \nu_m}{\partial \xi} \right\} \right],
$$

$$
\nu_{m+1}(\xi, \mathcal{S}) = \nu_m(\xi, \mathcal{S}) - L^{-1}\left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L\left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma(1 - \omega^\sigma)} - 2 \frac{\partial^3 \nu_m}{\partial \xi^3} \frac{\partial \varphi_m}{\partial \xi} \right\} \right],
$$

where

$$
\varphi_0(\xi, \mathcal{S}) = \left(3 - 6\tanh^2\frac{\xi}{2}\right),
$$

$$
\nu_0(\xi, \mathcal{S}) = \left(-3\sqrt{2} \tanh^2\frac{\xi}{2}\right).
$$

For $m = 0, 1, 2, \cdots$,
The analytical solution and exact solution is shown in Figures 1(a) and 1(b) at \( \sigma = 1 \) and \(-5 \leq \psi \leq 5\). Figure 1(c) shows the absolute error, and Figure 1(d) gives the solution at various fractional-order graph for \( \phi (\xi, 3) \). The behavior of the exact solution and analytical solution for \( \nu (\xi, 3) \) is seen in Figures 2(a) and 2(b). Tables 1 and 2 show the comparison of the exact and our methods solution in addition with the absolute error at different fractional-order. From the figure and tables, it is clear that our methods solution is in good agreement with the exact solution.

We achieve the exact solution by putting \( \sigma = 1 \)

\[
\phi (\xi, 3) = 3 - 6 \tanh^2 \left( \frac{3 + \xi}{2} \right),
\]

\[
\nu (\xi, 3) = -3 \sqrt{2} \tanh^2 \left( \frac{3 + \xi}{2} \right).
\]

The analytical solution and exact solution is shown in Figures 1(a) and 1(b) at \( \sigma = 1 \) and \(-5 \leq \psi \leq 5\). Figure 1(c) shows the absolute error, and Figure 1(d) gives the solution at various fractional-order graph for \( \phi (\xi, 3) \). The behavior of the exact solution and analytical solution for \( \nu (\xi, 3) \) is seen in Figures 2(a) and 2(b). Tables 1 and 2 show the comparison of the exact and our methods solution in addition with the absolute error at different fractional-order. From the figure and tables, it is clear that our methods solution is in good agreement with the exact solution.

\[
\phi (\xi, 3) = 3 - 6 \tanh^2 \left( \frac{3 + \xi}{2} \right),
\]

\[
\nu (\xi, 3) = -3 \sqrt{2} \tanh^2 \left( \frac{3 + \xi}{2} \right).
\]
Figure 1: Exact solution, analytical solution, absolute error, and various fractional order solution for $\varphi(\xi, \mathfrak{I})$ of problem 1.

Figure 2: Exact solution and analytical solution for $v(\xi, \mathfrak{I})$ problem 1.
Table 1: Comparison of the proposed method and exact results with absolute error (AE) at various fractional order for \( \varphi(\xi, \Omega) \) of problem 1.

<table>
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<th>( \Omega )</th>
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<th>Proposed techniques solution</th>
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<th>AE of proposed techniques</th>
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Table 2: Comparison of the proposed method and exact results with Absolute Error (AE) at various fractional order for \( \nu(\xi, \Omega) \) of problem 1.

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</table>

\[
\frac{\omega^\alpha L[\varphi(\xi, \Omega)] - \omega^{-1}\varphi(\xi, 0)}{\omega^\alpha + \sigma(1 - \omega^\alpha)} = L \left[ \frac{\partial^3 \varphi}{\partial \xi^3} - 3\frac{\partial \varphi}{\partial \xi} + \varphi + \frac{\partial}{\partial \xi} (\nu \varphi) \right],
\]

\[
\frac{\omega^\alpha L[\nu(\xi, \Omega)] - \omega^{-1}\nu(\xi, 0)}{\omega^\alpha + \sigma(1 - \omega^\alpha)} = L \left[ 3\frac{\partial \nu}{\partial \xi} - \frac{\partial^3 \varphi}{\partial \xi^3} \right],
\]

\[
\frac{\omega^\alpha L[\ell(\xi, \Omega)] - \omega^{-1}\ell(\xi, 0)}{\omega^\alpha + \sigma(1 - \omega^\alpha)} = L \left[ 3\frac{\partial \ell}{\partial \xi} - \frac{\partial^3 \varphi}{\partial \xi^3} \right].
\]

We obtain when we use the Laplace inverse transform

\[
\varphi(\xi, \Omega) = \frac{1}{3} + 2\tanh^2 \xi + L^{-1} \left[ \frac{\omega^\alpha + \sigma(1 - \omega^\alpha)}{\omega^\alpha} L \left[ \frac{\partial^3 \varphi}{\partial \xi^3} - 3\frac{\partial \varphi}{\partial \xi} + \varphi + \frac{\partial}{\partial \xi} (\nu \varphi) \right] \right].
\]

\[
\nu(\xi, \Omega) = \tanh \xi + L^{-1} \left[ \frac{\omega^\alpha + \sigma(1 - \omega^\alpha)}{\omega^\alpha} L \left[ 3\frac{\partial \nu}{\partial \xi} - \frac{\partial^3 \varphi}{\partial \xi^3} \right] \right],
\]

\[
\ell(\xi, \Omega) = \frac{8}{3} \tanh \xi + L^{-1} \left[ \frac{\omega^\alpha + \sigma(1 - \omega^\alpha)}{\omega^\alpha} L \left[ 3\frac{\partial \ell}{\partial \xi} - \frac{\partial^3 \varphi}{\partial \xi^3} \right] \right].
\]
Assume that the solution, $\varphi(\xi, \mathfrak{F})$, $\nu(\xi, \mathfrak{F})$, and $\ell(\xi, \mathfrak{F})$ in series form as

$$
\varphi(\xi, \mathfrak{F}) = \sum_{m=0}^{\infty} \varphi_m(\xi, \mathfrak{F}), \quad \nu(\xi, \mathfrak{F}) = \sum_{m=0}^{\infty} \nu_m(\xi, \mathfrak{F}), \quad \ell(\xi, \mathfrak{F}) = \sum_{m=0}^{\infty} \ell_m(\xi, \mathfrak{F}),
$$

(45)

where $\varphi_\ell = \sum_{m=0}^{\infty} \mathcal{A}_m$, $(\forall)\xi = \sum_{m=0}^{\infty} \mathcal{B}_m$, $\varphi_\nu = \sum_{m=0}^{\infty} \mathcal{C}_m$, and $\varphi_\ell = \sum_{m=0}^{\infty} \mathcal{D}_m$ are Adomian polynomials that characterize the nonlinear terms, and so equation (30) is rewritten as

$$
\sum_{m=0}^{\infty} \varphi_m(\xi, \mathfrak{F}) = \frac{1}{3} + \frac{2\tanh^{3}\xi}{L} + \left[ \frac{\omega^2 + \sigma(1 - \omega^2)}{\omega^2} L \left\{ \frac{\partial^2 \varphi}{\partial \xi^2} - 3 \sum_{m=0}^{\infty} \mathcal{A}_m + 3 \sum_{m=0}^{\infty} \mathcal{B}_m \right\} \right].
$$

(46)

The decomposition of nonlinear terms by Adomian polynomials is defined as in equation (7),

$$
\mathcal{A}_0 = \varphi_0 \varphi_0, \quad \mathcal{A}_1 = \varphi_1 \varphi_0 + \varphi_0 \varphi_1, \quad \mathcal{A}_2 = \varphi_2 \varphi_0 + \varphi_1 \varphi_1 + \varphi_0 \varphi_2,
$$

$$
\mathcal{B}_0 = \nu_0 \ell_0 + \ell_0 \nu_0, \quad \mathcal{B}_1 = (\nu_0 \ell_0 + \nu_1 \ell_0) + (\ell_0 \nu_0 + \ell_1 \nu_0), \quad \mathcal{B}_2 = (\nu_2 \ell_0 + \nu_1 \ell_1 + \nu_0 \ell_2) + (\ell_2 \nu_0 + \ell_1 \nu_1 + \ell_0 \nu_2),
$$

$$
\mathcal{C}_0 = \varphi_0 \nu_0, \quad \mathcal{C}_1 = \varphi_1 \nu_0 + \varphi_0 \nu_1, \quad \mathcal{C}_2 = \varphi_2 \nu_0 + \varphi_1 \nu_1 + \varphi_0 \nu_2,
$$

$$
\mathcal{D}_0 = \nu_0 \ell_0, \quad \mathcal{D}_1 = \nu_1 \ell_0 + \nu_0 \ell_1, \quad \mathcal{D}_2 = \nu_2 \ell_0 + \nu_1 \ell_1 + \nu_0 \ell_2.
$$

(47)

As a result, when comparing the two sides of (46),

$$
\varphi_0(\xi, \mathfrak{F}) = \frac{1}{3} + 2\tanh^{2}\xi,
$$

$$
\nu_0(\xi, \mathfrak{F}) = \tanh^{2}\xi,
$$

$$
\ell_0(\xi, \mathfrak{F}) = \frac{8}{3} \tanh^{2}\xi.
$$

(48)

For $m = 0$,

$$
\varphi_2(\xi, \mathfrak{F}) = 4\text{sech}^2\xi(1 - 3\tanh^{2}\xi) \left[ \frac{\sigma \mathfrak{F}^2}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\mathfrak{F}^2}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right],
$$

$$
\nu_2(\xi, \mathfrak{F}) = -\text{sech}^2\xi \tanh^{2}\xi \left[ \frac{\sigma \mathfrak{F}^2}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\mathfrak{F}^2}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right],
$$

$$
\ell_2(\xi, \mathfrak{F}) = -\frac{8}{3}\text{sech}^2\xi \tanh^{2}\xi \left[ \frac{\sigma \mathfrak{F}^2}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\mathfrak{F}^2}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right].
$$

(50)
The approximate solution to the series is written as

\[ \varphi(\xi, \mathfrak{F}) = \sum_{m=0}^{\infty} \varphi_m(\xi, \mathfrak{F}) = \varphi_0(\xi, \mathfrak{F}) + \varphi_1(\xi, \mathfrak{F}) + \varphi_2(\xi, \mathfrak{F}) + \cdots \]

\[ \nu(\xi, \mathfrak{F}) = \sum_{m=0}^{\infty} \nu_m(\xi, \mathfrak{F}) = \nu_0(\xi, \mathfrak{F}) + \nu_1(\xi, \mathfrak{F}) + \nu_2(\xi, \mathfrak{F}) + \cdots \]

\[ \ell(\xi, \mathfrak{F}) = \sum_{m=0}^{\infty} \ell_m(\xi, \mathfrak{F}) = \ell_0(\xi, \mathfrak{F}) + \ell_1(\xi, \mathfrak{F}) + \ell_2(\xi, \mathfrak{F}) + \cdots \]

\[ \varphi(\xi, \mathfrak{F}) = \frac{1}{3} + 2 \tanh^2 \xi + 4 \text{sech}^2 \xi \text{tanh} \xi \left[ \frac{\sigma \mathfrak{F}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma) \right] + 4 \text{sech}^2 \xi \left( 1 - 3 \tanh^2 \xi \right) \left[ \frac{\sigma^2 \mathfrak{F}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\mathfrak{F}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right] + \cdots \]

\[ \nu(\xi, \mathfrak{F}) = \tanh \xi + \text{sech}^2 \xi \left[ \frac{\sigma \mathfrak{F}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma) \right] - \text{sech}^2 \xi \text{tanh} \xi \left[ \frac{\sigma^2 \mathfrak{F}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\mathfrak{F}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right] + \cdots \]

\[ \ell(\xi, \mathfrak{F}) = \frac{8}{3} \tanh \xi + \frac{8}{3} \text{sech}^2 \xi \left[ \frac{\sigma \mathfrak{F}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma) \right] - \frac{8}{3} \text{sech}^2 \xi \text{tanh} \xi \left[ \frac{\sigma^2 \mathfrak{F}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\mathfrak{F}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right] + \cdots \]

We achieve the exact solution by putting \( \sigma = 1 \),

\[ \varphi(\xi, \mathfrak{F}) = \frac{1}{3} + 2 \tanh^2 (\mathfrak{F} + \xi), \]

\[ \nu(\xi, \mathfrak{F}) = \tanh (\mathfrak{F} + \xi), \]

\[ \ell(\xi, \mathfrak{F}) = \frac{8}{3} \tanh (\mathfrak{F} + \xi). \]

\[ \varphi_{m+1}(\xi, \mathfrak{F}) = \varphi_m(\xi, \mathfrak{F}) - L^{-1} \left[ \frac{\omega^\sigma + \sigma (1 - \omega^\sigma)}{\omega^\sigma} L \left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma (1 - \omega^\sigma)} \left\{ \frac{1}{2} \frac{\partial^3 \varphi_m}{\partial \xi^3} - 3 \frac{\varphi_m}{\omega^\sigma} \frac{\partial^2 \varphi_m}{\partial \xi^2} + 3 \frac{\partial \varphi_m}{\partial \xi} \nu_m \right\} \right\} \right], \]

\[ \nu_{m+1}(\xi, \mathfrak{F}) = \nu_m(\xi, \mathfrak{F}) - L^{-1} \left[ \frac{\omega^\sigma + \sigma (1 - \omega^\sigma)}{\omega^\sigma} L \left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma (1 - \omega^\sigma)} \left\{ 3 \frac{\varphi_m}{\omega^\sigma} \frac{\partial \nu_m}{\partial \xi} - \frac{\partial^3 \nu_m}{\partial \xi^3} \right\} \right\} \right], \]

\[ \ell_{m+1}(\xi, \mathfrak{F}) = \ell_m(\xi, \mathfrak{F}) - L^{-1} \left[ \frac{\omega^\sigma + \sigma (1 - \omega^\sigma)}{\omega^\sigma} L \left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma (1 - \omega^\sigma)} \left\{ 3 \frac{\varphi_m}{\omega^\sigma} \frac{\partial \ell_m}{\partial \xi} - \frac{\partial^3 \ell_m}{\partial \xi^3} \right\} \right\} \right], \]
where

\[ \varphi_0(\xi, \mathfrak{g}) = \frac{1}{3} + 2\tanh^2\xi, \]

\[ \gamma_0(\xi, \mathfrak{g}) = \tanh\xi, \]

\[ \ell_0(\xi, \mathfrak{g}) = \frac{8}{3} \tanh\xi. \]

For \( m = 0, 1, 2, \ldots \),

\[ \varphi_1(\xi, \mathfrak{g}) = \varphi_0(\xi, \mathfrak{g}) - L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma(1 - \omega^\sigma)} \frac{1}{2} \frac{\partial^3 \varphi_0}{\partial \xi^3} - 3 \frac{\partial \varphi_0}{\partial \xi} \frac{\partial \varphi_0}{\partial \xi} + 3 \frac{\partial \varphi_0}{\partial \xi} (\gamma_0') \right\} \right], \]

\[ \varphi_1(\xi, \mathfrak{g}) = 4 \text{sech}^2\xi \tanh\xi \left[ \frac{\sigma \mathfrak{g}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma) \right], \]

\[ \gamma_1(\xi, \mathfrak{g}) = \gamma_0(\xi, \mathfrak{g}) - L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma(1 - \omega^\sigma)} \frac{3 \varphi_0}{\partial \xi} \frac{\partial \varphi_0}{\partial \xi} \right\} \right], \]

\[ \gamma_1(\xi, \mathfrak{g}) = \text{sech}^2\xi \left[ \frac{\sigma \mathfrak{g}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma) \right], \]

\[ \ell_1(\xi, \mathfrak{g}) = \ell_0(\xi, \mathfrak{g}) - L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma(1 - \omega^\sigma)} \frac{3 \varphi_0}{\partial \xi} \frac{\partial \varphi_0}{\partial \xi} \right\} \right], \]

\[ \ell_1(\xi, \mathfrak{g}) = \frac{8}{3} \text{sech}^2\xi \left[ \frac{\sigma \mathfrak{g}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma) \right], \]

\[ \varphi_2(\xi, \mathfrak{g}) = \varphi_1(\xi, \mathfrak{g}) - L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma(1 - \omega^\sigma)} \frac{1}{2} \frac{\partial^3 \varphi_1}{\partial \xi^3} - 3 \frac{\partial \varphi_1}{\partial \xi} \frac{\partial \varphi_1}{\partial \xi} + 3 \frac{\partial \varphi_1}{\partial \xi} (\gamma_1') \right\} \right], \]

\[ \varphi_2(\xi, \mathfrak{g}) = 4 \text{sech}^2\xi (1 - 3 \tanh^2\xi) \left[ \frac{\sigma^2 \mathfrak{g}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2 \sigma (1 - \sigma) \frac{\mathfrak{g}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right], \]

\[ \gamma_2(\xi, \mathfrak{g}) = \gamma_1(\xi, \mathfrak{g}) - L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma(1 - \omega^\sigma)} \frac{3 \varphi_1}{\partial \xi} \frac{\partial \varphi_1}{\partial \xi} \right\} \right], \]

\[ \gamma_2(\xi, \mathfrak{g}) = \text{sech}^2\xi \tanh\xi \left[ \frac{\sigma^2 \mathfrak{g}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2 \sigma (1 - \sigma) \frac{\mathfrak{g}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right], \]

\[ \ell_2(\xi, \mathfrak{g}) = \ell_1(\xi, \mathfrak{g}) - L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma(1 - \omega^\sigma)} \frac{3 \varphi_1}{\partial \xi} \frac{\partial \varphi_1}{\partial \xi} \right\} \right], \]

\[ \ell_2(\xi, \mathfrak{g}) = \frac{8}{3} \text{sech}^2\xi \tanh\xi \left[ \frac{\sigma^2 \mathfrak{g}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2 \sigma (1 - \sigma) \frac{\mathfrak{g}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right], \]

\[ \varphi(\xi, \mathfrak{g}) = \frac{1}{3} + 2 \tanh^2\xi + 4 \text{sech}^2\xi \tanh\xi \left[ \frac{\sigma \mathfrak{g}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma) \right] + 4 \text{sech}^2\xi (1 - 3 \tanh^2\xi) \left[ \frac{\sigma^2 \mathfrak{g}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2 \sigma (1 - \sigma) \frac{\mathfrak{g}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right] + \cdots \]

\[ \gamma(\xi, \mathfrak{g}) = \tanh\xi + \text{sech}^2\xi \left[ \frac{\sigma \mathfrak{g}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma) \right] - \text{sech}^2\xi \tanh\xi \left[ \frac{\sigma^2 \mathfrak{g}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2 \sigma (1 - \sigma) \frac{\mathfrak{g}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right] + \cdots \]

\[ \ell(\xi, \mathfrak{g}) = \frac{8}{3} \tanh\xi + \frac{8}{3} \text{sech}^2\xi \left[ \frac{\sigma \mathfrak{g}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma) \right] - \frac{8}{3} \text{sech}^2\xi \tanh\xi \left[ \frac{\sigma^2 \mathfrak{g}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2 \sigma (1 - \sigma) \frac{\mathfrak{g}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right] + \cdots \]
We achieve the exact solution by putting \(\sigma = 1\)
\[
\varphi(\xi, 3) = \frac{1}{3} + 2\tanh^2(3 + \xi),
\]
\[
\nu(\xi, 3) = \tanh(3 + \xi)
\]
\[
\ell(\xi, 3) = \frac{8}{3}\tanh(3 + \xi)
\] (56)

The analytical solution and exact solution for \(\varphi(\xi, 3)\) of example 2 at \(\sigma = 1\) and \(-5 \leq \varphi \leq 5\) are shown in Figures 3(a) and 3(b), whereas Figures 3(c) and 3(d) show the absolute error and the solution at various fractional-order. The graphical behavior of exact solution and analytical solution for \(\nu(\xi, 3)\) are shown in Figures 4(a) and 4(b), while Figures 4(c) and 4(d) show the absolute error and the solution at different fractional-order. Figures 5(a) and 5(b) give the solution graph for \(\ell(\xi, 3)\). Tables 3–5 show the comparison of the exact and suggested methods solution in addition with the absolute error at various fractional order. From the results of the figures and Tables, it is confirmed that our method solution converges quickly towards exact solution.

\[
\varphi(\xi, 3) = \cos \xi + L^{-1}\left[\frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L\left[\frac{\partial^2 \varphi}{\partial \xi^2} + 2 \frac{\partial \varphi}{\partial \xi} - \frac{\partial}{\partial \xi} (\varphi \nu)\right]\right].
\]
\[
\nu(\xi, 3) = \cos \xi + L^{-1}\left[\frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L\left[\frac{\partial^2 \nu}{\partial \xi^2} + 2 \frac{\partial \nu}{\partial \xi} - \frac{\partial}{\partial \xi} (\varphi \nu)\right]\right].
\] (60)

Assume that the solution, \(\varphi(\xi, 3)\) and \(\nu(\xi, 3)\), in series form as
\[
\varphi(\xi, 3) = \sum_{m=0}^{\infty} \varphi_m(\xi, 3), \quad \nu(\xi, 3) = \sum_{m=0}^{\infty} \nu_m(\xi, 3),
\] (61)

The decomposition of nonlinear terms by Adomian polynomials is defined as in equation (14),
\[
\frac{\partial^\sigma \varphi}{\partial \xi^\sigma} = \frac{\partial^2 \varphi}{\partial \xi^2} + 2 \frac{\partial \varphi}{\partial \xi} - \frac{\partial}{\partial \xi} (\varphi \nu),
\]
\[
\frac{\partial^\sigma \nu}{\partial \xi^\sigma} = \frac{\partial^2 \nu}{\partial \xi^2} + 2 \frac{\partial \nu}{\partial \xi} - \frac{\partial}{\partial \xi} (\varphi \nu) 0 < \sigma \leq 1.
\] (63)

As a result, when comparing the two sides of (62),

5.3. Problem 3. Consider the coupled Burgers equation in one dimension
\[
\frac{\partial^\sigma \varphi}{\partial \xi^\sigma} = \frac{\partial^2 \varphi}{\partial \xi^2} + 2 \frac{\partial \varphi}{\partial \xi} - \frac{\partial}{\partial \xi} (\varphi \nu),
\]
\[
\frac{\partial^\sigma \nu}{\partial \xi^\sigma} = \frac{\partial^2 \nu}{\partial \xi^2} + 2 \frac{\partial \nu}{\partial \xi} - \frac{\partial}{\partial \xi} (\varphi \nu) 0 < \sigma \leq 1,
\] (57)

with initial source
\[
\varphi(\xi, 0) = \cos \xi, \quad \nu(\xi, 0) = \cos \xi.
\] (58)

On taking the Laplace transform of (57), we get
\[
\frac{\omega^\sigma L[\varphi(\xi, 3)] - \omega^{-1} \varphi(\xi, 0)}{\omega^\sigma + \sigma(1 - \omega^\sigma)} = L\left[\frac{\partial^2 \varphi}{\partial \xi^2} + 2 \frac{\partial \varphi}{\partial \xi} - \frac{\partial}{\partial \xi} (\varphi \nu)\right].
\]
\[
\frac{\omega^\sigma L[\nu(\xi, 3)] - \omega^{-1} \nu(\xi, 0)}{\omega^\sigma + \sigma(1 - \omega^\sigma)} = L\left[\frac{\partial^2 \nu}{\partial \xi^2} + 2 \frac{\partial \nu}{\partial \xi} - \frac{\partial}{\partial \xi} (\varphi \nu)\right].
\] (59)

We obtain when we use the Laplace inverse transform

\[
\varphi(\xi, 3) = \sum_{m=0}^{\infty} \varphi_m(\xi, 3), \quad \nu(\xi, 3) = \sum_{m=0}^{\infty} \nu_m(\xi, 3),
\]

where \(\varphi_{\nu} = \sum_{m=0}^{\infty} \varphi_m(\xi, 3)\), \((\varphi \nu)_{\nu} = \sum_{m=0}^{\infty} \nu_m(\xi, 3)\) are Adomian polynomials that characterize the nonlinear terms, and so (60) is rewritten as

\[
\sum_{m=0}^{\infty} \varphi_m(\xi, 3) = \cos \xi + L^{-1}\left[\frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L\left[\frac{\partial^2 \varphi}{\partial \xi^2} + 2 \sum_{m=0}^{\infty} \varphi_m(\xi, 3) - \sum_{m=0}^{\infty} \nu_m(\xi, 3)\right]\right].
\]
\[
\sum_{m=0}^{\infty} \nu_m(\xi, 3) = \cos \xi + L^{-1}\left[\frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L\left[\frac{\partial^2 \nu}{\partial \xi^2} + \sum_{m=0}^{\infty} \nu_m(\xi, 3) - \sum_{m=0}^{\infty} \varphi_m(\xi, 3)\right]\right].
\] (62)

\[
\varphi(\xi, 0) = \cos \xi, \quad \nu(\xi, 0) = \cos \xi.
\] (64)

For \(m = 0\),
\[
\frac{\partial^\sigma \varphi}{\partial \xi^\sigma} = \frac{\partial^2 \varphi}{\partial \xi^2} + 2 \frac{\partial \varphi}{\partial \xi} - \frac{\partial}{\partial \xi} (\varphi \nu),
\]
\[
\frac{\partial^\sigma \nu}{\partial \xi^\sigma} = \frac{\partial^2 \nu}{\partial \xi^2} + 2 \frac{\partial \nu}{\partial \xi} - \frac{\partial}{\partial \xi} (\varphi \nu) 0 < \sigma \leq 1.
\] (65)

For \(m = 1\),
Figure 3: Exact solution, analytical solution, absolute error, and various fractional order solutions for \( \phi(\xi, \mathbf{z}) \) of problem 2.

Figure 4: Continued.
Figure 4: Exact solution, analytical solution, absolute error, and various fractional order solutions for $\gamma(\xi, \zeta)$ of problem 2.

Figure 5: Exact solution and analytical solution for $\ell(\xi, \zeta)$ problem 2.

Table 3: Comparison of the proposed method and exact results with absolute error (AE) at various fractional order for $\varphi(\xi, \zeta)$ of problem 2.

<table>
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<th>Proposed techniques solution</th>
<th>AE of proposed techniques $\sigma = 1$</th>
<th>AE of proposed techniques $\sigma = 0.9$</th>
<th>AE of proposed techniques $\sigma = 0.8$</th>
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</tr>
</tbody>
</table>
Table 4: Comparison of the proposed method and exact results with absolute error (AE) at various fractional order for \( \nu(\xi, \mathfrak{S}) \) of problem 2.

<table>
<thead>
<tr>
<th>( \mathfrak{S} = 0.0001 )</th>
<th>Exact solution</th>
<th>Proposed techniques solution</th>
<th>AE of proposed techniques</th>
<th>AE of proposed techniques</th>
<th>AE of proposed techniques</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma )</td>
<td>( \sigma = 1 )</td>
<td>( \sigma = 1 )</td>
<td>( \sigma = 1 )</td>
<td>( \sigma = 0.9 )</td>
<td>( \sigma = 0.8 )</td>
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<tr>
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<td>0.000100000000000</td>
<td>3.3000000000E-09</td>
<td>1.6117443060E-04</td>
<td>1.6442509900E-03</td>
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<tr>
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<td>1.5957453000E-04</td>
<td>1.6279175300E-03</td>
</tr>
<tr>
<td>0.2</td>
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<td>3.3000000000E-09</td>
<td>1.3791040000E-04</td>
<td>1.4068874000E-03</td>
</tr>
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<td>0.462195802100000</td>
<td>3.7000000000E-09</td>
<td>1.2675890000E-04</td>
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<td>3.2000000000E-09</td>
<td>6.7692400000E-05</td>
<td>6.9054610000E-04</td>
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</tbody>
</table>

Table 5: Comparison of the proposed method and exact results with absolute error (AE) at various fractional order for \( \ell(\xi, \mathfrak{S}) \) of problem 2.

<table>
<thead>
<tr>
<th>( \mathfrak{S} = 0.0001 )</th>
<th>Exact solution</th>
<th>Proposed techniques solution</th>
<th>AE of proposed techniques</th>
<th>AE of proposed techniques</th>
<th>AE of proposed techniques</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma )</td>
<td>( \sigma = 1 )</td>
<td>( \sigma = 1 )</td>
<td>( \sigma = 1 )</td>
<td>( \sigma = 0.9 )</td>
<td>( \sigma = 0.8 )</td>
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<td>9.0000000000E-09</td>
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<td>2.6000000000E-09</td>
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<td>1.013425403000000</td>
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<td>1.432321933000000</td>
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<td>1.618116670000000</td>
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<tr>
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<td>1.910257499000000</td>
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<td>1.8051200000E-04</td>
<td>6.4670200000E-04</td>
</tr>
</tbody>
</table>

\[
\varphi_2(\xi, \mathfrak{S}) = \cos \xi \left[ \frac{\sigma^2 \mathfrak{S}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma (1 - \sigma) \frac{\mathfrak{S}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right], \tag{66}
\]

\[
\nu_2(\xi, \mathfrak{S}) = \cos \xi \left[ \frac{\sigma^2 \mathfrak{S}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma (1 - \sigma) \frac{\mathfrak{S}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right].
\]

The approximate solution to the series is written as

\[
\varphi(\xi, \mathfrak{S}) = \sum_{m=0}^{\infty} \varphi_m(\xi, \mathfrak{S}) = \varphi_0(\xi, \mathfrak{S}) + \varphi_1(\xi, \mathfrak{S}) + \varphi_2(\xi, \mathfrak{S}) + \cdots
\]

\[

\nu(\xi, \mathfrak{S}) = \sum_{m=0}^{\infty} \nu_m(\xi, \mathfrak{S}) = \nu_0(\xi, \mathfrak{S}) + \nu_1(\xi, \mathfrak{S}) + \nu_2(\xi, \mathfrak{S}) + \cdots \tag{67}
\]
We achieve the exact solution by putting $\sigma = 1$,
\[ \varphi(\xi, \zeta) = \cos \xi \left( 1 - \zeta + \frac{\zeta^2}{2} - \cdots \right), \]
\[ \nu(\xi, \zeta) = \cos \xi \left( 1 - \zeta + \frac{\zeta^2}{2} - \cdots \right). \]  
(68)

In closed form, $\varphi(\xi, \zeta) = \cos \xi \exp^{-\zeta}$ and $\nu(\xi, \zeta) = \cos \xi \exp^{-\zeta}$.

5.3.1. VITM Analytical Results. For (57), we have the iteration formula:

\[ \varphi_{m+1}(\xi, \zeta) = \varphi_m(\xi, \zeta) - L^{-1} \left[ \frac{\omega^\prime + \sigma (1 - \omega^\prime)}{\omega^\prime} \right] \left[ \frac{\omega^\prime}{\omega^\prime + \sigma (1 - \omega^\prime)} \right] \frac{\partial^2 \varphi_m}{\partial \xi^2} + \frac{2 \omega^\prime}{\omega^\prime + \sigma (1 - \omega^\prime)} \frac{\partial \varphi_m}{\partial \xi} \left( \varphi_m \right)_m \right], \]
\[ \nu_{m+1}(\xi, \zeta) = \nu_m(\xi, \zeta) - L^{-1} \left[ \frac{\omega^\prime + \sigma (1 - \omega^\prime)}{\omega^\prime} \right] \left[ \frac{\omega^\prime}{\omega^\prime + \sigma (1 - \omega^\prime)} \right] \frac{\partial^2 \nu_m}{\partial \xi^2} + \frac{2 \omega^\prime}{\omega^\prime + \sigma (1 - \omega^\prime)} \frac{\partial \nu_m}{\partial \xi} \left( \nu_m \right)_m \right], \]
(69)

where
\[ \varphi(\xi, 0) = \cos \xi, \quad \nu(\xi, 0) = \cos \xi. \]  
(70)

\[ \varphi_1(\xi, \zeta) = \varphi_0(\xi, \zeta) - L^{-1} \left[ \frac{\omega^\prime + \sigma (1 - \omega^\prime)}{\omega^\prime} \right] \left[ \frac{\omega^\prime}{\omega^\prime + \sigma (1 - \omega^\prime)} \right] \frac{\partial^2 \varphi_0}{\partial \xi^2} + \frac{2 \omega^\prime}{\omega^\prime + \sigma (1 - \omega^\prime)} \frac{\partial \varphi_0}{\partial \xi} \left( \varphi_0 \right)_0 \right], \]
\[ \varphi_1(\xi, \zeta) = -\cos \xi \left[ \frac{\sigma \zeta^\sigma}{\Gamma(1 - \sigma)} + (1 - \sigma) \right], \]
\[ \nu_1(\xi, \zeta) = \nu_0(\xi, \zeta) - L^{-1} \left[ \frac{\omega^\prime + \sigma (1 - \omega^\prime)}{\omega^\prime} \right] \left[ \frac{\omega^\prime}{\omega^\prime + \sigma (1 - \omega^\prime)} \right] \frac{\partial^2 \nu_0}{\partial \xi^2} + \frac{2 \omega^\prime}{\omega^\prime + \sigma (1 - \omega^\prime)} \frac{\partial \nu_0}{\partial \xi} \left( \nu_0 \right)_0 \right], \]
\[ \nu_1(\xi, \zeta) = -\cos \xi \left[ \frac{\sigma \zeta^\sigma}{\Gamma(1 - \sigma)} + (1 - \sigma) \right], \]
\[ \varphi_2(\xi, \zeta) = \varphi_1(\xi, \zeta) - L^{-1} \left[ \frac{\omega^\prime + \sigma (1 - \omega^\prime)}{\omega^\prime} \right] \left[ \frac{\omega^\prime}{\omega^\prime + \sigma (1 - \omega^\prime)} \right] \frac{\partial^2 \varphi_1}{\partial \xi^2} + \frac{2 \omega^\prime}{\omega^\prime + \sigma (1 - \omega^\prime)} \frac{\partial \varphi_1}{\partial \xi} \left( \varphi_1 \right)_1 \right], \]
\[ \varphi_2(\xi, \zeta) = \cos \xi \left[ \frac{\sigma^2 \zeta^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\zeta^\sigma}{\Gamma(1 - \sigma)} + (1 - \sigma)^2 \right], \]
\[ \nu_2(\xi, \zeta) = \nu_1(\xi, \zeta) - L^{-1} \left[ \frac{\omega^\prime + \sigma (1 - \omega^\prime)}{\omega^\prime} \right] \left[ \frac{\omega^\prime}{\omega^\prime + \sigma (1 - \omega^\prime)} \right] \frac{\partial^2 \nu_1}{\partial \xi^2} + \frac{2 \omega^\prime}{\omega^\prime + \sigma (1 - \omega^\prime)} \frac{\partial \nu_1}{\partial \xi} \left( \nu_1 \right)_1 \right], \]
\[ \nu_2(\xi, \zeta) = \cos \xi \left[ \frac{\sigma^2 \zeta^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\zeta^\sigma}{\Gamma(1 - \sigma)} + (1 - \sigma)^2 \right], \]
\[ \varphi(\xi, \zeta) = \cos \xi - \cos \xi \left[ \frac{\sigma \zeta^\sigma}{\Gamma(1 - \sigma)} + (1 - \sigma) \right] + \cos \xi \left[ \frac{\sigma^2 \zeta^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\zeta^\sigma}{\Gamma(1 - \sigma)} + (1 - \sigma)^2 \right] + \cdots \]
\[ \nu(\xi, \zeta) = \cos \xi - \cos \xi \left[ \frac{\sigma \zeta^\sigma}{\Gamma(1 - \sigma)} + (1 - \sigma) \right] + \cos \xi \left[ \frac{\sigma^2 \zeta^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\zeta^\sigma}{\Gamma(1 - \sigma)} + (1 - \sigma)^2 \right] + \cdots \]
We achieve the exact solution by putting $\sigma = 1$,

$$
\varphi(\xi, \mathfrak{I}) = \cos \xi \left(1 - \mathfrak{I}^2 + \frac{\mathfrak{I}^4}{2} - \cdots\right),
$$

(72)

$$
\nu(\xi, \mathfrak{I}) = \cos \xi \left(1 - \mathfrak{I}^2 + \frac{\mathfrak{I}^4}{2} - \cdots\right).
$$

In closed form, $\varphi(\xi, \mathfrak{I}) = \cos(\xi)\exp^{-\mathfrak{I}^2}$ and $\nu(\xi, \mathfrak{I}) = \cos(\xi)\exp^{-\mathfrak{I}^2}$.

In Figures 6(a) and 6(b), we display the solution graph at $\sigma = 1$ for $\varphi(\xi, \mathfrak{I}), \varphi(\xi, \mathfrak{I})$ in the domain $-5 \leq \psi, \phi \leq 5$, and Figures 6(c) and 6(d) show solution graph of absolute error and various fractional order solution. Also Table 6 demonstrates the error comparison of the exact and suggested methods solution at various orders. It is verified from the figures and table that our solution is closely related with the exact solution.

5.4. Problem 4. Consider the coupled Burgers equation two dimension

$$
\frac{\partial \varphi}{\partial \mathfrak{I}^2} + 2 \frac{\partial \varphi}{\partial \mathfrak{I}} - \frac{\partial}{\partial \mathfrak{I}} (\varphi \nu),
$$

(73)

$$
\frac{\partial \nu}{\partial \mathfrak{I}^2} + 2 \frac{\partial \nu}{\partial \mathfrak{I}} - \frac{\partial}{\partial \mathfrak{I}} (\varphi \nu) 0 < \sigma \leq 1,
$$

with initial source

$$
\varphi(\xi, 0) = \cos \xi, \quad \nu(\xi, 0) = \cos \xi.
$$

(74)

On taking the Laplace transform of (73), we get

$$
\omega^\sigma L[\varphi(\xi, \omega, \mathfrak{I})] - \omega^{-1} \varphi(\xi, 0) = L \left[ \frac{\partial^2 \varphi}{\partial \mathfrak{I}^2} + \frac{\partial^2 \varphi}{\partial \mathfrak{I} \partial \omega^2} + 2 \frac{\partial \varphi}{\partial \mathfrak{I}} \frac{\partial \varphi}{\partial \omega} - \frac{\partial}{\partial \mathfrak{I}} (\varphi \nu) \right].
$$

(75)

$$
\omega^\sigma L[\nu(\xi, \omega, \mathfrak{I})] - \omega^{-1} \nu(\xi, 0) = L \left[ \frac{\partial^2 \nu}{\partial \mathfrak{I}^2} + \frac{\partial^2 \nu}{\partial \mathfrak{I} \partial \omega^2} + 2 \frac{\partial \nu}{\partial \mathfrak{I}} \frac{\partial \nu}{\partial \omega} - \frac{\partial}{\partial \mathfrak{I}} (\varphi \nu) \right].
$$

(76)

We obtain when we use the Laplace inverse transform

$$
\varphi(\xi, \omega, \mathfrak{I}) = \cos(\xi + \omega) + L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega \sigma} L \left[ \frac{\partial^2 \varphi}{\partial \mathfrak{I}^2} + \frac{\partial^2 \varphi}{\partial \mathfrak{I} \partial \omega^2} + 2 \frac{\partial \varphi}{\partial \mathfrak{I}} \frac{\partial \varphi}{\partial \omega} - \frac{\partial}{\partial \mathfrak{I}} (\varphi \nu) \right] \right].
$$

(77)

$$
\nu(\xi, \omega, \mathfrak{I}) = \cos(\xi + \omega) + L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega \sigma} L \left[ \frac{\partial^2 \nu}{\partial \mathfrak{I}^2} + \frac{\partial^2 \nu}{\partial \mathfrak{I} \partial \omega^2} + 2 \frac{\partial \nu}{\partial \mathfrak{I}} \frac{\partial \nu}{\partial \omega} - \frac{\partial}{\partial \mathfrak{I}} (\varphi \nu) \right] \right].
$$

Assume that the solution, $\varphi(\xi, \omega, \mathfrak{I})$ and $\nu(\xi, \omega, \mathfrak{I})$ in series form as

$$
\varphi(\xi, \omega, \mathfrak{I}) = \sum_{m=0}^{\infty} \varphi_m(\xi, \omega, \mathfrak{I}), \quad \nu(\xi, \omega, \mathfrak{I}) = \sum_{m=0}^{\infty} \nu_m(\xi, \omega, \mathfrak{I}),
$$

(78)

where $\varphi_m = \sum_{m=0}^{\infty} \mathfrak{A}_m$, $(\varphi \nu)_m = \sum_{m=0}^{\infty} \mathfrak{B}_m$, and $\nu(\xi, \omega, \mathfrak{I}) = \sum_{m=0}^{\infty} \mathfrak{C}_m$ are Adomian polynomials that characterize the non-linear terms, and so (76) is rewritten as

$$
\sum_{m=0}^{\infty} \varphi_m(\xi, \omega, \mathfrak{I}) = \cos(\xi + \omega) + L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega \sigma} L \left[ \frac{\partial^2 \varphi}{\partial \mathfrak{I}^2} + \frac{\partial^2 \varphi}{\partial \mathfrak{I} \partial \omega^2} + 2 \sum_{m=0}^{\infty} \mathfrak{A}_m - \sum_{m=0}^{\infty} \mathfrak{B}_m \right] \right].
$$

(79)

$$
\sum_{m=0}^{\infty} \nu_m(\xi, \omega, \mathfrak{I}) = \cos(\xi + \omega) + L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega \sigma} L \left[ \frac{\partial^2 \nu}{\partial \mathfrak{I}^2} + \frac{\partial^2 \nu}{\partial \mathfrak{I} \partial \omega^2} + \sum_{m=0}^{\infty} \mathfrak{C}_m - \sum_{m=0}^{\infty} \mathfrak{B}_m \right] \right].
$$

(80)
Figure 6: Exact solution, analytical solution, absolute error, and various fractional order solution for $\varphi(\xi, \Omega)$ and $v(\xi, \Omega)$ of problem 3.

Table 6: Comparison of the proposed method and exact results with absolute error (AE) at various fractional order for $\varphi(\xi, \Omega)$ and $v(\xi, \Omega)$ of problem 3.

<table>
<thead>
<tr>
<th>$\Omega$ = 0.0001</th>
<th>Exact solution</th>
<th>Proposed techniques solution</th>
<th>AE of proposed techniques</th>
<th>AE of proposed techniques</th>
<th>AE of proposed techniques</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$ = 1</td>
<td>$\sigma$ = 1</td>
<td>$\sigma$ = 1</td>
<td>$\sigma$ = 0.9</td>
<td>$\sigma$ = 0.8</td>
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<td>5.0000000000E-09</td>
<td>9.5960000000E-05</td>
<td>5.7744580000E-04</td>
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<tr>
<td>0.1</td>
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<td>5.0000000000E-09</td>
<td>9.5480600000E-05</td>
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</tr>
<tr>
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<td>4.9000000000E-09</td>
<td>9.4071000000E-05</td>
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<tr>
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<td>0.920968879900000000</td>
<td>4.6000000000E-09</td>
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<td>4.4000000000E-09</td>
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</tr>
<tr>
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<td>4.2000000000E-09</td>
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<tr>
<td>0.8</td>
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<td>3.8000000000E-09</td>
<td>6.8656000000E-05</td>
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<td>0.621547810400000000</td>
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<td>3.6000000000E-09</td>
<td>5.9649700000E-05</td>
<td>3.5894610000E-04</td>
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<tr>
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<td>0.540248275700000000</td>
<td>2.7000000000E-09</td>
<td>5.1847400000E-05</td>
<td>3.1199530000E-04</td>
</tr>
</tbody>
</table>
The decomposition of nonlinear terms by Adomian polynomials is defined as in equation (14),

$$
A_0 = \varphi_0 \varphi_{0\xi}, \quad A_1 = \varphi_1 \varphi_{0\xi} + \varphi_0 \varphi_{1\xi}, \quad A_2 = \varphi_2 \varphi_{0\xi} + \varphi_1 \varphi_{1\xi} + \varphi_0 \varphi_{2\xi}, \\
B_0 = \varphi_0 \varphi_{0\xi} + \varphi_0 \varphi_{0\xi}, \quad B_1 = \left( \varphi_0 \varphi_{0\xi} + \varphi_0 \varphi_{1\xi} \right) + \left( \varphi_1 \varphi_{0\xi} + \varphi_0 \varphi_{1\xi} \right), \\
B_2 = \left( \varphi_2 \varphi_{0\xi} + \varphi_1 \varphi_{1\xi} + \varphi_0 \varphi_{2\xi} \right) + \left( \varphi_2 \varphi_{0\xi} + \varphi_1 \varphi_{1\xi} + \varphi_0 \varphi_{2\xi} \right), \\
C_0 = \varphi_0 \varphi_{0\xi}, \quad C_1 = \varphi_1 \varphi_{0\xi} + \varphi_0 \varphi_{1\xi}, \quad C_2 = \varphi_2 \varphi_{0\xi} + \varphi_1 \varphi_{1\xi} + \varphi_0 \varphi_{2\xi}. 
$$

As a result, when comparing the two sides of (78),

$$
\varphi(\xi, 0) = \cos \xi, \quad \nu(\xi, 0) = \cos \xi. \quad \text{(80)}
$$

For $m = 0$,

$$
\varphi(\xi, \mathfrak{J}) = \cos \xi \left( 1 - \mathfrak{J} + \frac{\mathfrak{J}^2}{2} - \cdots \right), \\
\nu(\xi, \mathfrak{J}) = \cos \xi \left( 1 - \mathfrak{J} + \frac{\mathfrak{J}^2}{2} - \cdots \right). \quad \text{(81)}
$$

For $m = 1$,

$$
\varphi_2(\xi, \omega, \mathfrak{J}) = 2 \cos(\xi + \omega) \left[ \frac{\sigma^2 \mathfrak{J}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\mathfrak{J}^{\sigma}}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right], \\
\nu_2(\xi, \omega, \mathfrak{J}) = 2 \cos(\xi + \omega) \left[ \frac{\sigma^2 \mathfrak{J}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\mathfrak{J}^{\sigma}}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right]. \quad \text{(82)}
$$

The approximate solution to the series is written as

$$
\varphi(\xi, \omega, \mathfrak{J}) = \sum_{m=0}^{\infty} \varphi_m(\xi, \mathfrak{J}) \\
\nu(\xi, \omega, \mathfrak{J}) = \sum_{m=0}^{\infty} \nu_m(\xi, \mathfrak{J})
$$

$$
\varphi(\xi, \omega, \mathfrak{J}) = \cos(\xi + \omega) - 2 \cos(\xi + \omega) \left[ \frac{\sigma \mathfrak{J}^{\sigma}}{\Gamma(\sigma + 1)} + (1 - \sigma) \right] \\
+ 2 \cos(\xi + \omega) \left[ \frac{\sigma^2 \mathfrak{J}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\mathfrak{J}^{\sigma}}{\Gamma(\sigma - 1)} + (1 - \sigma)^2 \right] + \cdots \quad \text{(83)}
$$

$$
\nu(\xi, \omega, \mathfrak{J}) = \cos(\xi + \omega) - 2 \cos(\xi + \omega) \left[ \frac{\sigma \mathfrak{J}^{\sigma}}{\Gamma(\sigma + 1)} + (\sigma + 1) \right] + \\
\cdot 2 \cos(\xi + \omega) \left[ \frac{\sigma^2 \mathfrak{J}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\mathfrak{J}^{\sigma}}{\Gamma(\sigma + 1)} + (\sigma + 1)^2 \right] + \cdots
$$
We achieve the exact solution by putting $\sigma = 1$, \[ \varphi(\xi, \omega, \mathfrak{F}) = \cos(\xi + \omega) \left(1 - 2\mathfrak{F} + \frac{4\mathfrak{F}^2}{2!} - \cdots\right), \]
\[ \gamma(\xi, \omega, \mathfrak{F}) = \cos(\xi + \omega) \left(1 - 2\mathfrak{F} + \frac{4\mathfrak{F}^2}{2!} - \cdots\right). \] (84)

5.4.1. VITM Analytical Results. For (73), we have the iteration formula:

\[ \varphi_{m+1}(\xi, \omega, \mathfrak{F}) = \varphi_m(\xi, \mathfrak{F}) - L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma(1 - \omega^\sigma)} \frac{\partial^2 \varphi_m}{\partial \xi^2} + \frac{\partial^2 \varphi_m}{\partial \omega^2} + 2\varphi_m \frac{\partial \varphi_m}{\partial \xi} - \frac{\partial}{\partial \xi} (\varphi_m^2) \right\} \right], \]
\[ \gamma_{m+1}(\xi, \omega, \mathfrak{F}) = \gamma_m(\xi, \mathfrak{F}) - L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma(1 - \omega^\sigma)} \frac{\partial^2 \gamma_m}{\partial \xi^2} + \frac{\partial^2 \gamma_m}{\partial \omega^2} + 2\gamma_m \frac{\partial \gamma_m}{\partial \xi} - \frac{\partial}{\partial \xi} (\gamma_m^2) \right\} \right], \] (85)

where

\[ \varphi_0(\xi, \omega, \mathfrak{F}) = \cos(\xi + \omega), \]
\[ \gamma_0(\xi, \omega, \mathfrak{F}) = \cos(\xi + \omega). \] (86)

\[ \begin{align*}
\varphi_1(\xi, \omega, \mathfrak{F}) &= \varphi_0(\xi, \mathfrak{F}) - L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma(1 - \omega^\sigma)} \frac{\partial^2 \varphi_0}{\partial \xi^2} + \frac{\partial^2 \varphi_0}{\partial \omega^2} + 2\varphi_0 \frac{\partial \varphi_0}{\partial \xi} - \frac{\partial}{\partial \xi} (\varphi_0^2) \right\} \right], \\
\varphi_1(\xi, \omega, \mathfrak{F}) &= -2 \cos(\xi + \omega) \left[ \frac{\sigma \mathfrak{F}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma) \right], \\
\gamma_1(\xi, \omega, \mathfrak{F}) &= \gamma_0(\xi, \mathfrak{F}) - L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma(1 - \omega^\sigma)} \frac{\partial^2 \gamma_0}{\partial \xi^2} + \frac{\partial^2 \gamma_0}{\partial \omega^2} + 2\gamma_0 \frac{\partial \gamma_0}{\partial \xi} - \frac{\partial}{\partial \xi} (\gamma_0^2) \right\} \right], \\
\gamma_1(\xi, \omega, \mathfrak{F}) &= -2 \cos(\xi, \mathfrak{F}) \left[ \frac{\sigma \mathfrak{F}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma) \right], \\
\varphi_2(\xi, \omega, \mathfrak{F}) &= \varphi_1(\xi, \mathfrak{F}) - L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma(1 - \omega^\sigma)} \frac{\partial^2 \varphi_1}{\partial \xi^2} + \frac{\partial^2 \varphi_1}{\partial \omega^2} + 2\varphi_1 \frac{\partial \varphi_1}{\partial \xi} - \frac{\partial}{\partial \xi} (\varphi_1^2) \right\} \right], \\
\varphi_2(\xi, \omega, \mathfrak{F}) &= 2 \cos(\xi, \mathfrak{F}) \left[ \frac{\sigma^2 \mathfrak{F}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\mathfrak{F}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right], \\
\gamma_2(\xi, \omega, \mathfrak{F}) &= \gamma_1(\xi, \mathfrak{F}) - L^{-1} \left[ \frac{\omega^\sigma + \sigma(1 - \omega^\sigma)}{\omega^\sigma} L \left\{ \frac{\omega^\sigma}{\omega^\sigma + \sigma(1 - \omega^\sigma)} \frac{\partial^2 \gamma_1}{\partial \xi^2} + \frac{\partial^2 \gamma_1}{\partial \omega^2} + 2\gamma_1 \frac{\partial \gamma_1}{\partial \xi} - \frac{\partial}{\partial \xi} (\gamma_1^2) \right\} \right], \\
\gamma_2(\xi, \omega, \mathfrak{F}) &= 2 \cos(\xi, \mathfrak{F}) \left[ \frac{\sigma^2 \mathfrak{F}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\mathfrak{F}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right], \\
\varphi(\xi, \omega, \mathfrak{F}) &= \cos(\xi, \mathfrak{F}) - 2 \cos(\xi + \omega) \left[ \frac{\sigma \mathfrak{F}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma) \right] + 2 \cos(\xi + \omega) \left[ \frac{\sigma^2 \mathfrak{F}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\mathfrak{F}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right], \\
\gamma(\xi, \omega, \mathfrak{F}) &= \cos(\xi, \mathfrak{F}) - 2 \cos(\xi + \omega) \left[ \frac{\sigma \mathfrak{F}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma) \right] + 2 \cos(\xi + \omega) \left[ \frac{\sigma^2 \mathfrak{F}^{2\sigma}}{\Gamma(2\sigma + 1)} + 2\sigma(1 - \sigma) \frac{\mathfrak{F}^\sigma}{\Gamma(\sigma + 1)} + (1 - \sigma)^2 \right].
\end{align*} \]
Figure 7: Exact solution, analytical solution, absolute error, and various fractional order solution for $\phi(\xi, \mathcal{I})$ and $\psi(\xi, \mathcal{I})$ of problem 4.

Table 7: Comparison of the proposed method and exact results with absolute error (AE) at various fractional order for $\phi(\xi, \mathcal{I})$ and $\psi(\xi, \mathcal{I})$ of problem 4.

<table>
<thead>
<tr>
<th>$\mathcal{I}$</th>
<th>Exact solution</th>
<th>Proposed techniques solution</th>
<th>AE of proposed techniques</th>
<th>AE of proposed techniques</th>
<th>AE of proposed techniques</th>
</tr>
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<tr>
<td>$\sigma = 1$</td>
<td>$\sigma = 1$</td>
<td>$\sigma = 1$</td>
<td>$\sigma = 0.9$</td>
<td>$\sigma = 0.8$</td>
<td></td>
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<td>0</td>
<td>0.877407062900000</td>
<td>0.877407045400000</td>
<td>1.7500000000E-08</td>
<td>2.8290530000E-04</td>
<td>1.0135215000E-03</td>
</tr>
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<td>0.1</td>
<td>0.825170564300000</td>
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<td>1.6500000000E-08</td>
<td>2.6606260000E-04</td>
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<td>0.2</td>
<td>0.764689234200000</td>
<td>0.764689218900000</td>
<td>1.5300000000E-08</td>
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<td>1.0800000000E-08</td>
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<td>0.453505402200000</td>
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<td>5.2385880000E-04</td>
</tr>
<tr>
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<td>0.362285282900000</td>
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<td>1.1681290000E-04</td>
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</tr>
<tr>
<td>0.8</td>
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<td>0.267445328800000</td>
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<td>8.6233300000E-05</td>
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<tr>
<td>0.9</td>
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<td>1.4100000000E-09</td>
<td>2.2803470000E-05</td>
<td>8.1694500000E-05</td>
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</tbody>
</table>
We achieve the exact solution by putting $\sigma = 1$,

$$
\varphi(\xi, \omega, \mathfrak{F}) = \cos(\xi + \omega) \left( 1 - 2\mathfrak{F} + \frac{4\mathfrak{F}^2}{2!} - \cdots \right),
$$

$$
\upsilon(\xi, \omega, \mathfrak{F}) = \cos(\xi + \omega) \left( 1 - 2\mathfrak{F} + \frac{4\mathfrak{F}^2}{2!} - \cdots \right).
$$

(88)

In closed form, $\varphi(\xi, \omega, \mathfrak{F}) = \cos(\xi + \omega)\exp^{-2\mathfrak{F}}$, and $\upsilon(\xi, \omega, \mathfrak{F}) = \cos(\xi + \omega)\exp^{-2\mathfrak{F}}$.

Figures 7(a) and 7(b) show the behavior of the exact and analytical solutions, respectively, whereas Figures 7(c) and 7(d) show the graphical perspective for various fractional orders. Table 7 also shows the behaviour of the exact and suggested method solutions in addition with absolute error at various orders of $\sigma$. Finally, the figures and tables show that the suggested techniques have higher degree of accuracy and rapid convergence towards the exact results.

6. Conclusion

The LTDM and VITM were used for solving of coupled nonlinear partial differential equations. On comparing the results of these methods with the exact solution, it is observed that proposed methods are extremely simple and easy to handle the nonlinear terms. The obtained results converge quickly in the form of series towards the exact solution. Four nonlinear systems are solved which shows that the suggested techniques solution are in strong agreement with the exact solution. It is confirmed that the proposed methods need much less computational work which shows fast convergence. Furthermore, LTDM and VITM are very effective and efficient for finding out the approximate analytic solutions for a wide range of real world problems arising in engineering and science.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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