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Research Article

A Novel Modeling Technique for the Forecasting of Multiple-Asset Trading Volumes: Innovative Initial-Value-Problem Differential Equation Algorithms for Reinforcement Machine Learning

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Liquidity risk arises from the inability to unwind or hedge trading positions at the prevailing market prices. The risk of liquidity is a wide and complex topic as it depends on several factors and causes. While much has been written on the subject, there exists no clear-cut mathematical description of the phenomena and typical market risk modeling methods fail to identify the effect of illiquidity risk. In this paper, we do not propose a definitive one either, but we attempt to derive novel mathematical algorithms for the dynamic modeling of trading volumes during the closeout period from the perspective of multiple-asset portfolio(s), as well as for financial entities with different subsidiary firms and multiple agents. The robust modeling techniques are based on the application of initial-value-problem differential equations technique for portfolio selection and risk management purposes. This paper provides some crucial parameters for the assessment of the trading volumes of multiple-asset portfolio(s) during the closeout period, where the mathematical proofs for each theorem and corollary are provided. Based on the new developed econophysics theory, this paper presents for the first time a closed-form solution for key parameters for the estimation of trading volumes and liquidity risk, such as the unwinding constant, half-life, and mean lifetime and discusses how these novel parameters can be estimated and incorporated into the proposed techniques. The developed modeling algorithms are appealing in terms of theory and are promising for practical econophysics applications, particularly in developing dynamic and robust portfolio management algorithms in light of the 2007–2009 global financial crunch. In addition, they can be applied to artificial intelligence and machine learning for the policymaking process, reinforcement machine learning techniques for the Internet of Things (IoT) data analytics, expert systems in finance, FinTech, and within big data ecosystems.

1. Introduction

Liquidity risk measures and the illiquidity of assets and trading volumes, in both financial and commodity markets, have been a subject of much debate, interest, and controversy in the last few decades. There is a longstanding Wall Street saying that "*it takes trading volume to make prices move*." Thus, there are plentiful empirical conclusions to reinforce the impact of trading volume on absolute value of price changes, conditional volatility, and the illiquidity of multiple trading assets during the closeout period [1–9].

In an earlier strand of research papers, Jain and Joh [1] investigate the dependence between hourly prices and trading volume and provide evidence on the joint features of hourly common stock trading volume and returns on the New York Stock Exchange. Their study shows that the average volume traded displays substantial changes across trading hours of the day (i.e., average trading volumes across six trading hours of the day) and across days of the week, while the average returns deviate across hours of the day and, to a certain extent, across days of the week. In addition, the relation between trading volume absolute returns is significantly more different for positive returns than for nonpositive returns. In a similar vein, the dynamics between stock returns, trading volume, and volatility in nine key national stock markets (i.e., index returns and trading volume from USA, Japan, UK, France, Canada, Italy, Switzerland, Netherlands, and Hong Kong) are examined by Chen et al. [2]. Their empirical outcomes indicate a positive correlation between trading volume and the absolute value of the stock price change, as well as persistence in volatility, and indicate that trading volume provides certain information to the returns process.

The 2007–2009 global financial crisis (GFC) has stressed the necessity for more efficient liquidity measures and management tools, in both normal and stressed market conditions, and placed the market microstructure of liquidity risk measures and trading volumes [1, 2] at the forefront agenda for research and development, particularly in emerging and illiquid markets [3–6]. One of the raising concerns in the wake of the GFC is that typical Value-at-Risk (VaR) market risk measures omit a key component, the risk associated with illiquidity of trading assets. As a result, financial markets and institutions would like to implement robust models and optimization algorithms that can place a cost not only for market risk but also for liquidity risk [4, 7]. In addition, the 2007–2009 financial meltdown has underlined the deficiencies of the Value-at-Risk (VaR) risk measures for the computation of market risk because such metrics do not integrate liquidity risk into the total risk process [3]. Furthermore, the GFC stressed the fact that illiquidity risk is challenging to quantify as it relies on multiple market microstructure parameters and that scenario testing is critical to any contemporary liquidity risk modeling process. Thus, robust liquidity modeling techniques and appropriate scenario identification risk processes have become major issues for the financial community to address. In fact, very little is written about why modeling the dynamics of the trading volume is important in measuring liquidity risk. To that end, the connection of the trading volume modeling under consideration to illiquidity risk is the key motivation of this paper.

The notion of illiquidity refers to the aptitude to convert in timely manner trading assets into cash at the prevailing market prices with little or no cost, risk, or disruption and without affecting its asset price [4]. Thus, liquidity risk is synonymous with both the necessary time to liquidate trading assets and the cost of liquidation, a trade-off that is a key feature in modeling market liquidity risk. The market liquidity risk depends on quite a few parameters and grounds. For instance, certain trading assets, such as highly traded equites, are integrally more liquid than other assets. In addition, the position size is a significant parameter and plays a key role in liquidity risk since it refers to ability of the financial trading entity to unwind into the financial markets, in one trading day, the aggregate number of shares compared with that day's total trading volume. Nevertheless, under stressed market conditions, liquidity can drop drastically, as financial markets' participants generally tend to be more risk-averse. In this case, the financial trading entity may be involuntary obliged to retain its assets positions for a much longer period, thereby intensifying liquidity risk [6].

Market liquidity risk has currently gained a great deal of consideration in light of the aftermath of the 2007–2009 GFC. During the past few years, quite a few research papers were written on the consideration of illiquidity risk in the VaR methodology. While a great deal has been written on the topic, the notions of liquidity risk, position size and traded volume, and the unwinding of assets during the closeout periods lack clear definitions, let alone the risk processes themselves.

Yet, despite this widespread acknowledgment of the phenomena, there exists no specific mathematical description of illiquidity risk and conventional VaR methods fail to identify the effect of illiquidity risk. In this paper, we do not propose a complete one either, but we propose novel modeling algorithms of some classes of illiquidity risk which are helpful for complementing the description of market risk and for forecasting dynamic trading volumes declines (or decays) during the closeout horizon under illiquid and stressed market circumstances and within a multivariate context. Undeniably, the reinforcement machine learning processes of the liquidity modeling algorithms proposed in this paper do not integrate all the microstructure features of the illiquidity risk measures(machine learning, which has been identified as a technology with significant impacts for portfolio construction and risk management, can enable the development of far more accurate risk-return forecasting techniques by identifying sophisticated, multifaceted, and stochastic trends in big data among all types of investments [10–13]. The categorization of machine learning approaches encompasses a wide range of methodologies and technologies that convey a shared goal from many perspectives. In this way, machine learning approaches may be classified using quite diverse metrics and conventions depending on the type of learning desired. As a result, the authors direct readers to [10–13] for some of the most recent research on machine learning, big data, and expert systems in finance for modern portfolio optimization and management. Furthermore, some literature addressed the genuine concerns of big data and associated green challenges and applications, while others examined the notion of information and communication technologies (ICT) for sustainable development goals (SDG). For further insights on these two important topics, the authors refer the readers to [14, 15].). However, it is useful as a tool for estimating dynamic trading volumes and liquidity risk when the influence of liquidity of certain financial assets is substantial.

A range of liquidity risk modeling techniques have been suggested in the academic literature. For convenience, and to be faithful to the literature, we focus on some contemporary challenges to tackle the issue of illiquidity risk and liquidity-adjusted VaR (LVaR) with special emphasis on merely the recent attempts and literature.

In effect, previous researchers have endeavored to study the notion of illiquidity risk but not fundamentally within the perception of multiple-asset portfolios (for other relevant literature on liquidity risk, internal risk modeling techniques, asset pricing, and portfolio choice and diversification, one can refer as well to Al Janabi [12]; Ruozi and Ferrari [16]; Grillini et al. (2019); Roch and Soner [17]; Al

Janabi et al. [6]; Weiß and Supper [8]; Al Janabi, Ferrer, and Shahzad [7]; Madoroba and Kruger [18]; Madhavan et al. [19]; Takahashi and Alexander [20]; Cochrane (2005); and Meucci [21]; among others). Their main motivation, in fact, was on examining transaction costs (i.e., the expanding of the bid-ask spreads); however, the special impacts of undesirable market prices with distinct dependence measures have not been researched systematically, yet Hung et al. [3], Weiß and Supper [8], Al Janabi et al. [6], and Al Janabi et al. [7] are exemptions. In their research study, Weiß and Supper [8] propose a multivariate model for evaluating liquidity-adjusted intraday VaR based on vine copulas and the attained research outcomes validate that the recommended method functions adequately in forecasting possible intraday liquidity-adjusted portfolio shortfalls. Al Janabi et al. [7] and Al Janabi et al. [6] used a modified version of Al Janabi model for liquidity risk management [18] combined with different copula function and presented an endogenous liquidity adjusted VaR method, which measured the liquidity risk triggered by illiquid t holdings and difficult supply-demand imbalances. Al Janabi et al. [6] present a portfolio optimization model based on the integration of dynamic conditional correlation t-copula and LVaR algorithms. In a similar vein, Al Janabi, Ferrer, and Shahzad [7] examine a robust portfolio optimization approach based on vine-copula and LVaR modeling algorithm. While Al Janabi et al. [6] study a portfolio that consists of nine assets (i.e., G-7 stock market indexes, crude oil, and gold commodities), [7] a research paper substitutes oil by a general commodity index and it incorporates Bitcoin as an additional asset. Among the conclusions therein is the fact that the presence of liquidity dimension in market risk is feasible to improve the robustness of market risk computations.

On another front, Hung et al. [3] study the influence of illiquidity on calculating VaR predictions using multivariate GARCH-t and GJR-GARCH-t models for portfolios with unlike liquidity strengths. The obtained conclusions show that calculating portfolio VaR predictions with multivariate techniques outperform the univariate techniques for full and subsample cycles in terms of precision and estimations of effectiveness. Allaj [5] presented a general framework of a one period risk measurement technique that incorporates illiquidity risk into standard risk measures. These risk techniques are decomposed into two terms, one determining the risk of the future value of a given position in an asset or a portfolio of assets and the other the initial cost of that position. Finally, Careta and Jaimungal [9] examined an optimal execution policy for an investor seeking to complete a large order using limit and market orders. They demonstrated that the different considered strategies outperform the Almgren and Chriss [22] modeling technique because those strategies benefited from the optimal mix of limit orders.

In this backdrop, though many attempts in liquidity risk modeling focused on transaction costs and VaR measures, it is well known that traditional risk measures such as VaR models are incoherent [23] and can lead to regulatory arbitrage to reduce capital requirements. As such, conventional VaR methods cannot satisfy the subadditivity requirement to be classified as coherent measures like other popular measures such as Expected Shortfall (ES). The lack of subadditivity in a risk measure can be manipulated to form regulatory arbitrage as financial institutions can establish different subsidiary firms to save regulatory capital. This is where this research study comes in as we attempt to make clear the fundamental nature of illiquidity risk measures and propose novel modeling techniques and robust algorithms to tackle the problem of illiquidity risk and trading volumes during the closeout (or unwinding) period for multiple-asset portfolios within the same financial entity, as well as for financial holding enterprises with different subsidiary firms and multiple agents.

Despite the large amount of research on liquidity risk, our special interest in introducing the trading volume, as a key factor for a multiagent dynamic model of trading activities and then in modeling the dynamics of the trading volume during the closeout period (i.e., the liquidation or unwinding horizon), rests on the lack of research studies that apply trading volume in an attempt to boost the forecasting of risk measures. Our study contributes to the understanding of liquidity risk and the dynamic modeling of trading volume during the closeout period in several ways. First, this paper is the first econophysics attempt, to the best of my knowledge, to develop a novel modeling technique and robust techniques for the estimation of trading volumes and illiquidity risk during the closeout period and within the context of multiple-asset portfolios, as well as for financial holding entities with different subsidiary firms and multiple agents. Second, in contrast to all known models for liquidity risk, this paper implements an innovative dynamic modeling technique in deriving the liquidity risk process and trading volume using initial-value-problem differential equations algorithms, as other authors have done heretofore. The proposed robust modeling techniques can resolve some of the main drawbacks of the traditional VaR method of being incoherent because of its lack of subadditivity and its tendency to lead to a form of regulatory arbitrage [23]. Third, this paper provides some new important parameters, which are the first of their kind to the best of the author's knowledge, for the assessment of the trading volumes of multiple-asset portfolios during the closeout period, where the mathematical proofs for each theorem and corollary are provided. Based on the new developed econophysics theory, this paper presents for the first time a closed-form solution for key parameters for the estimation of trading volumes and liquidity risk, such as the unwinding constant, half-life, and mean lifetime, and discusses how these novel parameters can be estimated and incorporated into the recommended techniques. Fourth, we examine potential reinforcement machine learning algorithms for the implementation of the proposed novel econophysics modeling techniques to risk forecasting and multiple-asset portfolio selection practices. To that end, in line with Al Janabi et al. [7] and Al Janabi [12], we define a modified dynamic process of Al Janabi model [18] for multiple-asset portfolio selection and risk forecasting and combine it with the innovative initial-value-problem differential equations algorithms. As a result, the alternative reinforcement machine learning algorithms can address some of the drawbacks of the traditional mean-variance VaR technique, presenting robust generalizations and meaningful improvements on Markowitz's [24] mean-variance solution. Fifth, the developed modeling techniques are appealing in terms of theory and are promising for likely real-world applications, particularly in developing dynamic and robust portfolio management algorithms that financial markets and institutions could put into effect in the wake of the 2007–2009 global financial meltdown. In addition, it can be applied to artificial intelligence and machine learning for the policymaking process, reinforcement machine learning techniques for the Internet of Things (IoT) data analytics, expert systems in finance, FinTech, and within big data ecosystems.

The remainder of the paper is organized as follows. In Section 2, we provide details of mathematical definition and derivation of the modeling algorithms and liquidity risk process using initial-value-problem differential equations techniques. Section 3 provides expansion of the modeling algorithms to multiple-asset portfolios. Section 4 examines portfolio selection and risk management practices and highlights certain reinforcement machine learning issues. In addition, in this section, we present one potential reinforcement machine learning algorithm for the possible implementation of the proposed modeling techniques to multiple-asset portfolio selection and risk management practices, and we discuss the overall reinforcement machine learning process with the aid of an operational flowchart. Section 5 concludes the paper and provides future directions for research on the topic and the possible applications of the proposed novel modeling techniques and robust algorithms.

2. The Model

We define the dynamic process of continuous changes in the trading volume of any portfolio of multiple assets as an initial-value-problem differential equation of the form

$$\frac{dV(H)}{dH} = -\mu V(H) + V_N(H), \tag{1}$$

where $V(H)$ is the current trading volume of a multiple-asset portfolio, whose domain consists of all nonnegative real numbers ($\mathbb{R}^+$); that is, $V(H) \geq 0$; $V_N(H)$ is the rate of incorporation of "*new*" trading volume of multiple assets to the current trading portfolio, whose domain consists of all nonnegative real numbers ($\mathbb{R}^+$); that is, $V_N(H) \geq 0$; H is the closeout period (i.e., unwinding horizon or holding period), which can take nonnegative real numbers that are above or equal to 1.0 ($\mathbb{R}^+_{\geq 1.0}$) only during the unwinding process to convert trading assets into cash at the prevailing market prices; that is, $H \geq 1.0$; and $\mu =$ is a constant of proportionality that we can label as the "*unwinding constant*." This unwinding constant can be defined as the probability per unit time that the current trading volume of any portfolio of multiple assets will undergo a decline (or a decay) in its holding assets, given that multiples of trading assets will be sold (i.e., unwound) during the closeout period H.

In fact, the first term on the right-hand side of (1) denotes the rate of decline (or the rate of the decay process) of the current trading volume of the existing multiple-asset portfolio, whereas the second term indicates the rate of incorporating (i.e., the rate of the "*production*" or "*creation*" process) new multiple-asset trading volume to the existing portfolio. Certainly, the above statements are rather ambiguous and, hence, we need to explain the economic foundations in some mathematically and financially meaningful means. To begin, we need some means to explain the rationality and usefulness of the proposed dynamic trading volume model and to link its assumptions to market dynamics.

There are multiple rationalities behind the mathematical foundation and the financial dentition of the proposed differential equation model. To that end, and in line with other liquidity risk research papers discussed earlier, which usually make many ad hoc assumptions, we attempt to typically link in some way the proper economic foundation and assumptions to market dynamics and provide full justification, detailed as follows:

(1) In our modeling technique, we are attempting to explain both terms of (1) as the rate of decline (or decay) process of trading volume and the rate of the incorporation process (i.e., the rate of the "production" or "creation" process) of new trading volume. This is because there are some resemblances in our econophysics modeling technique and other physical science and social science processes, such as the decay and production of radionuclides and the process of the decline in the population of nations and the simultaneous process of the incorporation of new immigrants to those nations.

(2) Portfolio managers and their respective markets' traders (i.e., multiple agents) can unwind certain assets and add new multiple assets to the current trading portfolio on a daily basis. This is because the decline in the trading volume of certain multiple assets is accompanied by the incorporation (i.e., "*production*" or "*creation*") of new trading volume by other multiple agents inside the same financial entity or within multiple agents of different subsidiaries of the principal financial holding firm. In addition, in our model, multiple agents are provided with various trading volumes that are for a limited time only associated with their particular circumstances and the market conditions and, as such, we propose a dynamic multiagent model with agent-dependent and time-dependent trading volumes.

(3) The level and depth of public and private information available to the different portfolio managers and traders are imbalanced (i.e., asymmetric levels of different news, statistical datasets, figures and facts, communications, and lines of evidence are accessible and available to the contrasting market participants). This assumption is quite relevant as it allows the expansion of the proposed dynamic trading volume

model from the perspective of a single multiple-asset portfolio to multiple-asset portfolios. It also permits financial holding entities, with different subsidiary firms and multiple agents, to consider different trading volumes and closeout horizons for all multiple-asset portfolios.

(4) Within a large pool of markets' traders, the activities, behaviors, and actions of the individual traders are uncorrelated in many instances, even though these traders are engaging and executing buying-selling market orders of multiple-asset portfolios inside the same financial entity or within multiple agents of different subsidiaries of the primary financial holding company.

(5) One key assumption that we make in our proposed model is that the trading positions of multiple-asset portfolio(s) are unwound only at the quoted or prevailing market prices. As such, the quantity (or position size) to liquidate each day into the markets is limited to a preset fraction of that day's trading volume. Thus, in our suggested modeling technique, the multiple-asset trading positions will be unwound into the markets at that fraction of trading volume over each time period until the overall trading positions are completely liquidated, and the income from the unwinding process is settled into cash.

(6) The rooted effects of the "*call option-like*" embedded incentives given to traders to induce them to undertake additional risk as the potential upside rewards are quite appealing, whereas the downside impacts and consequences are very limited for their irrational and/or exorbitant risk-takings.

(7) In fact, some of these rational assumptions embedded in our econophysics differential equation model can contradict some traditional market theories and perhaps disappoint fully devoted believers in efficient markets hypothesis. However, the actual realities on the ground of how financial markets work, as evidenced by the latest severe financial crises and meltdowns besides the scandalous events of several rogue traders and trading entities, have placed efficient market hypothesis on the edge of rationality and judgment to questioning its assumptions and validity. Based on our particular working experiences in diverse financial markets and institutions, the authors of this paper fundamentally believe that some financial markets (probably in the western hemisphere) are more efficient than other markets. As such, the authors are strong believers that the large bulks of emerging markets are considerably less efficient than their western counterparts.

In this backdrop, rearranging (1) yields

$$\frac{dV(H)}{dH} + \mu V(H) = V_N(H). \tag{2}$$

Indeed, (2) is a part of a general differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + r(x)y = q(x). \tag{3}$$

In fact, the algebraic step to solve (3) does not separate variables. However, it does remove the y variable from the right side of the equation and at the same time sets up the left side for multiplication by an "*integrating factor*" trailed by a vital use of the "*product rule*" for differentiation.

Theorem 1. *Assume that $r(x)$ and $q(x)$ are continuous functions and let $R(x)$ be any antiderivative of $r(x)$. The general solution of the differential equation $dy/dx + r(x)y = q(x)$ is then*

$$y(x) = e^{-R(x)} \int e^{R(x)} q(x)\mathrm{d}x + Ce^{-R(x)}, \tag{4}$$

where C is an arbitrary constant.

Proof of Theorem 1. The existence of both $\mathrm{d}y/\mathrm{d}x$ and $y(x)$ in the sum on the left side of (3) instructs us to contemplate the "*product rule*" for differentiation. If u is a function of x that is never 0, then we have the following:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(uy) &= u\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)y \\ &= u\left(\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{u}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)y\right).\end{aligned} \tag{5}$$

The notion is to obtain a function u so that the factor of y on the right side of the former equation matches the factor of y in (3). Namely, we seek to find a function u such that

$$\frac{1}{u(x)}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) = r(x). \tag{6}$$

This separable differential equation can be solved by rewriting it as

$$\int \frac{1}{u}\mathrm{d}u = \int r(x)\mathrm{d}x. \tag{7}$$

Given that $R(x)$ is an antiderivative of $r(x)$, the universal solution of the prior equation is $\ln(|u|) = R(x) + C_0$, where C_0 is a constant of integration. Further, since any specific solution u will help in achieving our purpose, we can streamline the algebra by selecting a solution u with $u(x) > 0$ and $C_0 = 0$. With these selections, we can obtain the following $\ln(u) = R(x)$, or $u(x) = e^{R(x)}$ which is our integrating factor. Now, on multiplying both sides of (3) by $e^{R(x)}$, we can get

$$r(x)e^{R(x)} \cdot y + e^{R(x)}\frac{\mathrm{d}y}{\mathrm{d}x} = e^{R(x)}q(x). \tag{8}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{R(x)}.y\right) = \left(\frac{d}{dx}\,e^{R(x)}\right).y + e^{R(x)}\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$= R'(x)e^{R(x)} \cdot y + e^{R(x)}\frac{dy}{dx} \tag{9}$$
$$= r(x)e^{R(x)} \cdot y + e^{R(x)}\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$= e^{R(x)}q(x).$$

In other words, $e^{R(x)} \cdot y$ is an antiderivative of $e^{R(x)}q(x)$, such that

$$e^{R(x)} \cdot y = \int e^{R(x)}q(x)\mathrm{d}x + C. \tag{10}$$

To that end, formula (4) can be obtained on dividing each side of (10) by $e^{R(x)}$. Thus, the Proof of Theorem 1 is completed.

Set against this background, linear equations [such as in the case of (1) and/or (2)] with constant coefficients arise frequently in practical applications, such as in the case of the trading volume of multiple-asset portfolio posed in this paper. In general terms, these differential equations have the form $\mathrm{d}y(t)/\mathrm{d}t + \theta y(t) = \alpha$. We can realize that by rewriting this equation in the form $\mathrm{d}y(t)/\mathrm{d}t = -\theta y(t) + \alpha$, which is a separable linear differential equation. Though this equation may be solved using other mathematical techniques, we can simplify the calculations by implementing Theorem 1. □

Theorem 2. *Assume that θ and α are constants with $\theta \neq 0$; then the linear differential equation is*

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \theta y(t) = \alpha. \tag{11}$$

It has a general solution:

$$y(t) = \frac{\alpha}{\theta} + Ce^{-\theta t}. \tag{12}$$

As a result, the initial-value-problem

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \theta y(t) = \alpha, \quad y(0) = y_0, \tag{13}$$

has the following unique single solution:

$$y(t) = \frac{\alpha}{\theta} + \left(y_0 - \frac{\alpha}{\theta}\right)e^{-\theta t}. \tag{14}$$

Proof of Theorem 2. In fact, (11) is the special case of (3) that arises from placing $q(t) = \alpha$ and $r(t) = \theta$. Given that $R(t) = \theta t$ is an antiderivative of $r(t)$, (4) states that (11) has the following solution:

$$y(t) = e^{-\theta t}\int e^{\theta t}\alpha\mathrm{d}t + Ce^{-\theta t} = \frac{\alpha}{\theta} + Ce^{-\theta t}. \tag{15}$$

This proves (12). However, if $y(0) = y_0$, then $\alpha/\theta + Ce^{-\theta.0} = y_0$, or $C = y_0 - \alpha/\theta$.

After substituting the above obtained value of C into (12), one can obtain formula (14); that is, $y(t) = \alpha/\theta + (y_0 - \alpha/\theta)e^{-\theta t}$ and, hence, it finalizes the proof of Theorem 2.

In this backdrop, we can now apply the above theorems to the case of portfolio management with structural asset allocations, specifically for the assessment of trading volume of multiple-asset portfolios at different closeout periods. □

Corollary 1. *As denoted earlier, the closeout period (H) can take nonnegative real numbers that are above or equal to 1.0 ($\mathbb{R}^{+}_{\geq 1.0}$) only during the unwinding process to convert trading assets into cash at the prevailing market prices; that is, $H \geq 1.0$. However, to simplify the solution of the initial-value-problem differential equation at the beginning of the trading process and before the initiation of the actual liquidation process of any asset, we are assuming here that the notion of $H \geq 1.0$ is still valid before the acquisition of any multiple assets at the initial conditions of the trading process (i.e., when $t = 0$).*

In a similar vein and following the differential equations process of the above two theorems and their respective proofs, for the singular case of an initial-value-problem of a multiple-asset trading portfolio at which $V(0) = V_0$ (when $t = 0$, at the start of the trading process) and $V_N(H) = V_N$ (for the special case of a constant rate for the incorporation of new volumes of the further multiple assets to the existing trading portfolio), the solution to (2) is easily obtained as

$$V(H) = V_0 e^{-\mu H} + \left(\frac{V_N}{\mu}\right)\left(1 - e^{-\mu H}\right). \tag{16}$$

In addition, for the exceptional case when $V_0 = 0$ at $t = 0$, (16) is confined to

$$V(H) = \left(\frac{V_N}{\mu}\right)\left(1 - e^{-\mu H}\right). \tag{17}$$

Furthermore, for the special case in which there is not any incorporation of new volumes of multiple assets to the existing trading portfolio during the closeout horizon (i.e., $V_N = 0$), the solution of (2) via (16) can be reduced to

$$V(H) = V_0 e^{-\mu H}. \tag{18}$$

Proof of Corollary 1. Equation (2), that is, $\mathrm{d}V(H)/\mathrm{d}H + \mu V(H) = V_N(H)$, has the following general solution, where C is an arbitrary constant:

$$V(H) = \frac{V_N(H)}{\mu} + Ce^{-\mu H}. \tag{19}$$

The initial-value-problem can be structured when $t = 0$ at the start of the trading process, such that

$$\frac{\mathrm{d}V(H)}{\mathrm{d}H} + \mu V(H) = V_N(H),$$
$$V(0) = V_0, \quad \text{for } t = 0, \tag{20}$$

and indeed, when $V_N(H) = V_N$ for the easiest case when the incorporation or "*generation rate*" is constant in time (i.e., the special case of a constant rate for the incorporation of new volumes of multiple assets to the existing trading portfolio during the closeout horizon), (20) can be evaluated analytically to give a unique solution:

$$V(H) = \left(\frac{V_N}{\mu}\right) + \left[V_0 - \left(\frac{V_N}{\mu}\right)\right] e^{-\mu H}. \tag{21}$$

Further, (21) can be written as

$$V(H) = V_0 e^{-\mu H} + \left(\frac{V_N}{\mu}\right)\left(1 - e^{-\mu H}\right). \tag{22}$$

For the special case when $V_0 = 0$ at $t = 0$, (22) is reduced to

$$V(H) = \left(\frac{V_N}{\mu}\right)\left(1 - e^{-\mu H}\right). \tag{23}$$

As a result, (23) is the same as (17) and, for the exceptional case in which there is not any addition of new trading volume to the trading portfolio (i.e., $V_N = 0$), the solution of (2) via (22) can be reduced to

$$V(H) = V_0 e^{-\mu H}. \tag{24}$$

This establishes (18) and ends the proof of Corollary 1.

Set against this background, we can now proceed to determine the required parameters for solving (16) and/or its special case (18), detailed as follows. □

Corollary 2. *Using the special case formula $V(H) = V_0 e^{-\mu H}$ as discussed earlier in (24) and Corollary 1 along with its proof, it is not difficult to show that the half-life $(H_{1/2})$ and the mean lifetime $(\overline{H})$ for a multiple-asset portfolio to unwind its assets throughout the closeout period (H) can be expressed as*

$$H_{1/2} = \frac{\ln 2}{\mu} \cong \frac{0.693}{\mu}. \tag{25}$$

$$\overline{H} = \frac{1}{\mu} = 1.44\, H_{1/2}. \tag{26}$$

Proof of Corollary 2. The unwinding of a multiple-asset portfolio holding is a statistical random or stochastic process.

It is not possible to foresee whether or not a single asset can be sold out (closed out) in a given time period along the unwinding horizon H. Nevertheless, we can forecast the probable or typical closing-out performance of sizeable multiple-asset portfolios. We can contemplate a sample portfolio comprising sizeable volume V of assets. In a very small holding period interval ΔH, ΔV of the assets can be sold into the market. The probability that the holding assets are being sold into the market in ΔH is thus $\Delta V/V$. Clearly as ΔH becomes smaller, so will the probably of unwinding of particular assets within the trading portfolio $\Delta V/V$. However, as ΔH gets smaller, the statistical variation in the unwinding rate of trading volume would turn out to be apparent and the determined unwinding probability per unit time would have higher statistical variations. As a result, the statistically averaged unwinding probability per unit time, in the limit of infinitely small ΔH, comes close to a constant μ; that is, we can express

$$\mu \equiv \lim_{\Delta H \longrightarrow 0} \left(\frac{\Delta V/V}{\Delta H}\right). \tag{27}$$

Indeed, each trading portfolio has its distinguishing unwinding constant μ, which, for the above characterization, is the probability a particular trading volume unwinds in a unit time for an infinitesimal time period (i.e., holding horizon). The same concept can be applied to individual assets within the trading portfolio, or in other words we can generalize the case of trading volume of a multiple-asset portfolio to its constituent's assets. In this sense, each trading asset within the large portfolio can have its own unwinding constant, namely, μ_i. In fact, the smaller μ is, the more slowly the trading volume can be sold to financial markets. Certainly, for stable multiple-asset portfolios, which is indeed a rare case in practical asset management norms, $\mu = 0$.

We can now consider a sample multiple-asset portfolio constituted of a considerable number of assets with unwinding constant μ. With a sizable volume portfolio $(V >> 1)$, we can apply continuous mathematics to define an intrinsically discrete process. Thus, $V(H)$ can be understood as the average or expected trading volume (a continuous quantity) of all assets at time H. Therefore, the probability that holding assets are being sold (or unwound) in an interval dH is $\mu\, dH$, and the expected number of declines (diminishes) in the trading volume that happen in dH at time H is $\mu\, dH\, V(H)$. In essence, this must be equivalent to the decline, dH, in the number of multiple assets in the sample portfolio (i.e., which is also equal to the decrease in the number of unsold assets in time dH); that is,

$$-dV = \mu V(H) dH \tag{28}$$

or

$$\frac{\mathrm{d}V(H)}{\mathrm{d}H} = -\mu V(H). \tag{29}$$

This differential equation can be integrated to obtain the exponential variation with time formula, which governs the behavior of a process characterized by a constant rate of change, as follows:

Dividing by $V(H)$, we may integrate (29) from time zero to time H to obtain

$$\int_{V(0)}^{V(H)} \frac{\mathrm{d}V(H)}{V(H)} = -\mu \int_0^H \mathrm{d}H, \tag{30}$$

where $V(0) = V_0$ is the initial trading volume of multiple assets at the launching of the trading process at $t = 0$. Noting that $\mathrm{d}V(H)/V(H) = d \ln(V(H))$, (30) becomes

$$\ln[V(H)/V(0)] = -\mu H. \tag{31}$$

The characteristic exponential rate of declining of a multiple-asset portfolio trading volume is yielded.

$$V(H) = V_0 e^{-\mu H}. \quad (32)$$

This dynamic process, which is governed by exponential decline of portfolio holding assets, has a notable characteristic. The necessary time it takes for the multiple-asset portfolio to diminish to one-half the original volume, $H_{1/2}$, is a constant termed half-life. In other words, the half-life is a more intuitive measure of the time during which the trading activity of unwinding of assets falls by a factor of two. From (32), we can get

$$\begin{aligned} V(H_{1/2}) &\equiv \frac{V_0}{2} \\ &= V_0 e^{-\mu H_{(1/2)}}. \end{aligned} \quad (33)$$

Solving for $H_{1/2}$ yields

$$H_{1/2} = \frac{\ln 2}{\mu} \cong \frac{0.693}{\mu}. \quad (34)$$

It is essential to note that the half-life is independent from the holding-horizon time H. Therefore, after n half-lives, the original trading volume has reduced by a multiplicative parameter of $1/2^n$; that is,

$$V(nH_{1/2}) = \frac{1}{2^n} V_0. \quad (35)$$

The number of half-lives n required for any given multiple-asset portfolio to diminish to a fraction ω of its original volume is obtained as

$$\omega \equiv \frac{V(nH_{1/2})}{V_0} = \frac{1}{2^n}. \quad (36)$$

Upon solving for n, it gives

$$n = -\frac{\ln \omega}{\ln 2} \cong -1.44 \ln \omega. \quad (37)$$

On the other hand, the exponential decline of portfolio holding assets of (32) could be stated by means of half-life in this way:

$$V(H) = V_0 \left(\frac{1}{2}\right)^{H/H_{1/2}}. \quad (38)$$

From the exponential decline of portfolio holding assets, we can define some suitable probabilities and averages. If we start by having a portfolio trading volume of V_0 at $t = 0$, we can expect to have $V_0 e^{-\mu H}$ trading assets after a period of time H. Accordingly, the probability $\widehat{P}$ that the holding assets do not diminish in value (i.e., are not being sold as yet) in a time horizon H can be deduced as

$$\begin{aligned} \widehat{P}(H) &= \frac{V(H)}{V(0)} \\ &= e^{-\mu H}. \end{aligned} \quad (39)$$

The probability P that trading assets do diminish (i.e., are being unwound or sold) in a time interval H is

$$\begin{aligned} P(H) &= 1 - \widehat{P}(H) \\ &= 1 - e^{-\mu H}. \end{aligned} \quad (40)$$

As the time interval turns out to be small, that is, $H \longrightarrow \Delta H \ll 1$, we can observe that

$$\begin{aligned} P(\Delta H) &= 1 - e^{-\mu H} \\ &= 1 - \left[1 - \mu\Delta H + \left(\frac{1}{2}\right)(\mu\Delta H)^2 - \left(\frac{1}{6}\right)(\mu\Delta H)^3 + \ldots\right], \\ &\cong \mu\Delta H. \end{aligned} \quad (41)$$

This approximate solution is in line t with our former clarification of the unwinding constant μ as being the decline probability per infinitesimal unwinding time horizon.

Given the above outcomes, we are now able to get the probability distribution function for when a trading volume declines in value. Explicitly, let $p(H)dH$ be the probability that a trading volume that exists at $t = 0$ declines in value in the time period between H and $H + dH$. Evidently,

$$\begin{aligned} p(H)dH &= \{\text{Prob} \cdot \text{it does not decline in value}\,(0, H)\} \\ &\quad \times \{\text{Prob} \cdot \text{it declines in value in thenext}\; dH \;\text{time interval}\} \\ &= \left\{\widehat{P}(H)\right\}\{P(H)\} \\ &= \left\{e^{-\mu H}\right\}\{\mu\; dH\} \\ &= \mu e^{-\mu H} dH\,. \end{aligned} \quad (42)$$

The same results of (42) can be obtained if we consider a sample multiple-asset portfolio with an initial volume of V_0 at time $t = 0$. In view of (32), there will be $V_0 e^{-\mu H}$ trading volume remaining after H time interval. The fraction of the original volume which has not declined in value is therefore $e^{-\mu H}$. This fraction can also be viewed as the probability that the portfolio trading volume will not decline in value in the time interval from $t = 0$ to $t = H$. Now let $p(H)dH$ be the probability that the trading volume declines in value in the time dH between H and $H + dH$. This is evidently equal to the probability that the trading volume has not declined in value up to time H times the probability that it does in fact decline in value in the additional time dH. It follows therefore that $p(H)dH = e^{-\mu H} \times \mu\; dH = \mu e^{-\mu H} dH$, which is the same as (42).

If (42) is integrated over all H, it is obtained that

$$\begin{aligned} \int_0^\infty p(H)\mathrm{d}H &= \mu \int_0^\infty e^{-\mu H}\mathrm{d}H \\ &= 1. \end{aligned} \quad (43)$$

This shows that the probability that a particular trading volume eventually declines in value (i.e., unwinds throughout the holding horizon H) is equal to unity, as would be expected.

In a large sample multiple-asset portfolio, trading assets can be sold (i.e., always unwound). From (32), we see that a large time period (probably an infinite time) is needed to unwind all multiple assets within the portfolio. However, as time escalates, less trading assets can be sold into the financial markets. In fact, we can calculate the mean lifetime

($\overline{H}$) for a multiple-asset portfolio to unwind its assets throughout the closeout period (H) by using the decline probability distribution $p(H)dH$ of (42). The average or mean lifetime can now be determined by finding the average value of H over the probability distribution $p(H)$. Denoting the mean lifetime by $\overline{H}$,

$$\begin{aligned}\overline{H} &= \int_0^\infty H p(H)\mathrm{d}H = \int_0^\infty H\mu e^{-\mu H}\mathrm{d}H \\ &= \frac{1}{\mu}.\end{aligned} \tag{44}$$

Moreover, the mean lifetime of (44) can be obtained by defining $\overline{H}$ as the average time that a particular trading volume is likely to survive before it is being sold into the market. The trading volume that survives to time H is just $V(H)$, and the volume that declines in value between H and $H + dH$ is $|\mathrm{d}V/\mathrm{d}H|\mathrm{d}H$. The mean lifetime is then

$$\overline{H} = \frac{\int_0^\infty H\,|dV/dH|\mathrm{d}H}{\int_0^\infty |\mathrm{d}V/\mathrm{d}H|\mathrm{d}H}, \tag{45}$$

where the denominator gives the total number of declines in trading volume. Evaluating the integrals of (45) gives $\overline{H} = 1/\mu$. Thus, the mean lifetime is the inverse of the unwinding constant.

In view of (34), the mean lifetime can also be written as

$$\begin{aligned}\overline{H} &= \frac{H_{1/2}}{0.693} \\ &= 1.44H_{1/2}.\end{aligned} \tag{46}$$

This confirms (26) and finalizes the proof of Corollary 2. □

3. Expansion to Multiple-Asset Portfolios

While in the previous section we looked at the dynamic model of trading volume from the perspective of a single multiple-asset portfolio, we can now expand and generalize differential (2) to multiple-asset portfolios, as well as, for financial holding entities, with different subsidiary firms and multiple agents, by considering different trading volumes and closeout horizons for all multiple-asset portfolios, as follows:

$$\sum_{i=1}^{k} \frac{\mathrm{d}V_i(H_i)}{\mathrm{d}H_i} = \sum_{i=1}^{k} -\mu_i V_i(H_i) + V_{N_i}(H_i); \quad \forall i = 1, 2, \ldots, k. \tag{47}$$

The solution of the differential equation for each trading asset within the multiple-asset portfolio of (47) can be obtained via (16) to yield

$$V_i(H_i) = V_{0_i} e^{-\mu_i H_i} + \left(\frac{V_{N_i}}{\mu_i}\right)\left(1 - e^{-\mu_i H_i}\right). \tag{48}$$

Furthermore, for the exceptional case when $V_{0_i} = 0$ at $t_i = 0$ (i.e., at the initiation of the trading process), (48) is restricted to

$$V_i(H_i) = \left(\frac{V_{N_i}}{\mu_i}\right)\left(1 - e^{-\mu_i H_i}\right). \tag{49}$$

Likewise, for the special case in which there is not any buildup of new volume of some assets (i.e., $V_{N_i} = 0$), the solution of the above differential equation can be reduced to yield

$$V_i(H_i) = V_{0_i} e^{-\mu_i H_i}. \tag{50}$$

Therefore, we can now define the variation in the total portfolio trading volume (V) throughout the closeout period (H) as

$$\begin{aligned}V(H) &= \sum_{i=1}^{k} V_i(H_i) \\ &= \sum_{i=1}^{k} \left[V_{0_i} e^{-\mu_i H_i} + \left(\frac{V_{N_i}}{\mu_i}\right)\left(1 - e^{-\mu_i H_i}\right)\right]; \\ &\forall i = 1, 2, \ldots, k.\end{aligned} \tag{51}$$

The above equation can be reduced for the special case in which there is not any addition of new volumes for all assets during the unwinding horizon (i.e., $\sum_{i=1}^{k} V_{N_i} = 0$; $\forall i = 1, 2, \ldots, k$.) to yield

$$\begin{aligned}V(H) &= \sum_{i=1}^{k} V_i(H_i) \\ &= \sum_{i=1}^{k} V_{0_i} e^{-\mu_i H_i}; \\ &\forall i = 1, 2, \ldots, k.\end{aligned} \tag{52}$$

Finally, in order to solve for the unwinding constant for multiple-asset portfolios, we can determine μ_i for each trading asset via a trial-and-error process along the individual closeout horizons of all assets. For instance, if traders can determine H_i (i.e., the necessary number of trading days to completely unwind any particular asset), $V_i(H_i), V_{N_i}, V_{0_i}$ then one can solve for the unwinding constants (μ_i; $\forall i = 1, 2, \ldots, k$) for the different multiple assets under consideration in a given portfolio as follows.

In real-world practices, the overnight trading volume of multiple assets is appraised as the average volume over certain time horizon, usually a month of trading operations. In typical operations, the overnight trading volume of multiple assets can be viewed as the average overnight volume that can be unwound during unfavorable market conditions. One the contrary, the trading volume during a crisis period can be roughly approximated as the average daily trading volume minus a few standard deviations. While this alternate tactic is relatively unpretentious, it is still fairly unbiased approach. Furthermore, it is relatively straightforward to assemble the needed datasets to implement the required unwinding events.

In fact, the liquidation days H_i required to completely unwind any particular asset are associated with the selection of the unwinding threshold. Nevertheless, the magnitude of this threshold is expected to alter under stressed ecosystems. In reality, the selection of the unwinding period (i.e., the closeout horizon) can be projected from the overall trading volume and the overnight volume that can be unwound into the financial markets without substantially interrupting multiple-asset values. To that end, in real-world applications, the absolute values of the different closeout periods are generally estimated as

$$H_i = \left|\frac{\text{Total Trading Position Size of Asset}_i}{\text{Daily Trading Volume of Asset}_i}\right|, \quad s.t. \quad H_i \geq 1.0. \tag{53}$$

Equation (53) can also be written as

$$H_i = \left|\frac{\text{Total Number of Asset}_i \text{ in Trading Portfolio } \times \text{Price of Asset}_i}{\text{Daily Number of Asset}_i \text{ to Unwind from Trading Portfolio } \times \text{Price of Asset}_i}\right|, \quad s.t.\ H_i \geq 1.0. \tag{54}$$

Accordingly, if for instance we determine that it is necessary to have $H_i = 4.0$ days to unwind $V_i(4.0)$ of volume of any particular asset i and at the same time V_{N_i} and V_{0_i} are known data at that specific time horizon, then one can solve for μ_i by a trial-and-error process or by using a numerical procedure such as the iterative process of the Newton-Raphson method, yielding (The Newton-Raphson (NR) procedure is an iteration process. This NR method involves the process of iterating (or repeating) technique in which repetition of a sequence of operations yields outcomes uninterruptedly closer to a preferred outcome. To that end, Newton-Raphson method is intended to resolve an equation of the kind $f(x) = 0$. It begins with an estimation of the solution: $x = x_0$. It then yields sequentially improved guesses of the solution: $x = x_1, x = x_2, x = x_3, \ldots\ldots$ using the formula $x_{i+1} = x_i - f(x_i)/f'(x_i)$. Typically, x_2 is very close to the factual answer.)

$$V_i(4.0) = V_{0_i} e^{-\mu_i 4.0} + \left(\frac{V_{N_i}}{\mu_i}\right)\left(1 - e^{-\mu_i 4.0}\right). \tag{55}$$

Similarly, for the special case in which there is not any new addition of volume of this particular asset (i.e., $V_{N_i} = 0$), (55) can be shortened to

$$V_i(4.0) = V_{0_i} e^{-\mu_i 4.0}. \tag{56}$$

Thus, once the various unwinding (or decay) constants (i.e., $\mu_i; \forall i = 1, 2, \ldots, k$) for the different multiple assets under consideration are computed, we can now apply the statistics of the decay constants in (51) to solve for the various characteristics of the total trading volumes (i.e., $V(H) = \sum_{i=1}^{k} V_i(H_i); \forall i = 1, 2, \ldots, k$) by using different closeout periods (i.e., $H_i; \forall i = 1, 2, \ldots, k$).

4. Portfolio Selection and Risk Management Practices with a Reinforcement Machine Learning Process

In this backdrop, we present one potential reinforcement machine learning algorithm for the implementation of the proposed econophysics techniques to risk assessment and multiple-asset portfolio selection practices, detailed as follows.

In line with Al Janabi et al. [7] and Al Janabi [12], we can define the following multiple-asset portfolio selection and risk assessment process:

Stage 1: liquidity-adjusted value-at-risk (LVaR) model and multiple-asset portfolio algorithm:

(1) Formally, the Value-at-Risk (VaR) for any individual trading asset i can be computed in this way:

$$VaR_i = \left|\left(E(R_i) - \Phi * \sigma_i\right)\left(\text{Asset}_i * Fx_i\right)\right|, \tag{57}$$

where $E(R_i)$ is the expected return, Φ is the confidence level, σ_i is the conditional risk factor (volatility), Asset_i denotes the mark-to-market value, and Fx_i is the foreign exchange unit for any trading asset.

(2) To get the global VaR of a multiple-asset portfolio, the correlations parameters $[\rho_{i,j}]$ among the diverse assets are considered and the computational process can be presented in terms of matrices; thus,

$$VaR_P = \sqrt{\sum_{i=1}^{k}\sum_{j=1}^{k} VaR_i VaR_j \rho_{i,j}} = \sqrt{[VaR]^T [\rho] [VaR]}. \tag{58}$$

It is feasible that the risk engine and objective function can include the special impacts of nonlinearity and nonnormality of assets returns in the optimization process. This can be achieved by employing Kendall's tau algorithm (or any other copula-based modeling methods) as a yardstick to evaluate the degree of nonlinear dependence and to be used instead of the linear Pearson's correlation factors. Similarly, Cornish-Fisher expansion as a yardstick of nonnormality can simply be tailored to solve the problem of nonnormality in multiple-asset returns.

(3) As in the work of Al Janabi et al. [6], the LVaR algorithm for the computation of multiple-asset portfolios for any closeout period H_i can be expressed as

$$LVaR_{i_{\mathrm{adj}}} = VaR_i\left(\sqrt{\frac{(2H_i+1)(H_i+1)}{6H_i}}\right), \tag{59}$$

where the closeout period can be estimated using (53) or (54) above. In order to compute the $LVaR$ for the entire multiple-asset portfolio (i.e., $LVaR_{P_{\mathrm{adj}}}$), the next model, which is an extension of (58), can be used:

$$\begin{aligned} LVaR_{P_{\mathrm{adj}}} &= \sqrt{\sum_{i=1}^{k}\sum_{j=1}^{k} LVaR_{i_{\mathrm{adj}}} LVaR_{j_{\mathrm{adj}}} \rho_{i,j}} \\ &= \sqrt{\left[LVaR_{\mathrm{adj}}\right]^{T}[\rho]\left[LVaR_{\mathrm{adj}}\right]}. \end{aligned} \tag{60}$$

(4) The simulation experiments for this first stage can be conducted as follows:

(i) From the obtained multiple-asset datasets, the distribution parameters are estimated (i.e., the vector of expected returns and variance/covariance matrices or the association matrices of any other dependence measures)
(ii) Next, these parameters are used to produce samples of independent identically distributed random vectors from multivariate elliptical distributions or any other selected distributions
(iii) The forecasting returns distribution is obtained using time series modeling techniques
(iv) Each trading asset's expected return and conditional risk parameters are computed and sample of k observations is generated

Stage 2: portfolio optimization algorithm and financial and operational constraints:

(1) The portfolio optimization problem and the procedures of solving the proposed modeling algorithm are formulated as follows:

$$\begin{aligned} \text{Min: } LVaR_{P_{\mathrm{adj}}} &= \sqrt{\sum_{i=1}^{k}\sum_{j=1}^{k} LVaR_{i_{\mathrm{adj}}} LVaR_{j_{\mathrm{adj}}} \rho_{i,j}} \\ &= \sqrt{\left[LVaR_{\mathrm{adj}}\right]^{T}[\rho]\left[LVaR_{\mathrm{adj}}\right]}. \end{aligned} \tag{61}$$

(2) The risk objective function of (61) could be minimized dependent on complying with the following operational and financial constraints:

$$\sum_{i=1}^{k} E(R_i)x_i = E(R_P); \quad l_i \le x_i \le u_i\, i = 1, 2, \ldots, k, \tag{62}$$

$$\sum_{i=1}^{k} x_i = 1.0; \quad l_i \le x_i \le u_i\, i = 1, 2, \ldots, k, \tag{63}$$

$$\sum_{i=1}^{k} V_i(H_i) = V(H) \quad i = 1, 2, \ldots, k, \tag{64}$$

$$[\mathrm{LRF}] \ge 1.0; \quad \forall i = 1, 2, \ldots, k, \tag{65}$$

where LRF is the liquidation risk factor that is expressed for any particular asset i as

$$\mathrm{LRF}_i = \left[\sqrt{\frac{(2H_i+1)(H_i+1)}{6H_i}}\right] \ge 1.0; \quad i = 1, 2, \ldots, k. \tag{66}$$

In (62)–(66) above, $E(R_P)$ and $V(H)$ symbolize the target portfolio expected return and the total volume of the multiple-asset portfolio, respectively, and x_i is the fraction (i.e., the weights) for every trading asset. The values l_i and u_i, for $i = 1, 2, \ldots, k$ in (62)–(63), stand for the lower and upper limits for the portfolio weights x_i. Moreover, [LRF] denotes a $(k \times 1)$ vector of the closeout periods for each trading asset of the multiple-asset portfolio. Therefore, as indicated earlier, the various unwinding constants (i.e., $\mu_i; \forall i = 1, 2, \ldots, k$) for the different multiple assets under consideration can be computed using the procedure discussed in (51)–(56). Next, we can use the statistics of the decay constants in (51) to solve for the various characteristics of the total trading volumes (i.e., $V(H) = \sum_{i=1}^{k} V_i(H_i)$; $\forall i = 1, 2, \ldots, k$) by applying different closeout periods (i.e., $H_i; \forall i = 1, 2, \ldots, k$). Thereafter, the obtained individual trading volumes for each of the multiple assets (i.e., $V_i(H_i); \forall i = 1, 2, \ldots, k$) can be used in an iteration process to recompute the specific closeout periods, H_i, using (53) or (54). Finally, the obtained total trading volume (i.e., $V(H)$) can be integrated into the optimization process as a constraint in line with (64).

(3) The simulation experiments for this second stage can be implemented as follows:

(i) For each modeling technique, the related optimization problems are solved to structure the different efficient frontiers of the multiple-asset portfolios and the related asset allocation is determined

(ii) Out-of-sample features of the attained portfolios are computed applying the factual factors of the underlying distribution

(iii) The above two steps are reiterated several times to calculate more steady assessment of out-of-sample traits s of the chosen portfolios and structure the corresponding efficient frontiers

Stage 3: construction of efficient frontiers, comparison, and validation with the mean-variance method:

(1) Solve the modeling algorithm in (61)–(66) by using a quadratic programming (QP) technique to obtain the best allocation weights of each trading asset and then construct the equivalent efficient frontiers for multiple-asset portfolios.

(2) In this final stage, construct different efficient frontiers of the LVaR proposed algorithm versus the traditional mean-variance method [24], using (53)–(66).

(3) Validate and compare the output results of the multiple-asset portfolio(s) obtained in Stage (2) with the optimum portfolio(s) defined in Stage (1).

(4) Repeat the optimization process until another convergence to meaningful portfolio(s) is accomplished. These portfolios should comply with the optimization parameters and budget restrictions defined above.

(5) At this final phase of the operational method validation process, new meaningful portfolio(s) with coherent asset-allocations, which conform to the constraints' settings stated in Stage (2), are satisfied consistently.

(6) Finally, repeat the process to compute more steady valuation of the characteristics of the out-of-sample designated portfolios.

Indeed, this extension to multiple-asset portfolios, as well as for financial holding entities with different subsidiary firms and multiple agents, can resolve some of the main drawbacks of the incoherence of traditional VaR models due to its lack of subadditivity [23]. In fact, the conventional VaR measure is not coherent because it does not comply with the subadditivity restriction, which is a clear requirement for any coherent risk measure; otherwise, there would be no risk advantage in combining uncorrelated multiple assets into trading portfolios. Artzner et al. [23] described the following set of rational criteria that a measure of risk, $\Omega(X)$, where X is a set of outcomes, should comply with the following requirements: Subadditivity can be defined to satisfy this condition: $\Omega(X+Y) \leq \Omega(X) + \Omega(Y)$. Thus, by adding two multiple-asset portfolios together, the overall risk cannot get any worse than combining the two risks independently because of diversification effects that reduce the total risk substantially. As such, the conventional VaR measure is not coherent, since it does not satisfy the subadditivity specification. For instance, if we have two multiple-asset portfolios X and Y, then this diversification benefit can be defined as $\Omega(X) + \Omega(Y) - \Omega(X+Y)$, which according to the subadditivity condition can only take nonnegative values. In fact, the nonexistence of subadditivity in a risk measure can be exploited to structure a regulatory arbitrage since a financial entity can create different subsidiary firms, in an opposite procedure of the above example two multiple-asset portfolios, to avert and/or save regulatory capital cushion. Thus, with a coherent measure of risk (e.g., Expected Shortfall (ES) risk measure), explicitly due to its subadditivity, one can easily incorporate risks of separate multiple-asset portfolios to obtain moderate assessments of the aggregate risk.

In this backdrop and to capitalize on its usefulness as a risk management and portfolio selection tool, we have structured the portfolio management modeling algorithms such that the suggested risk engine and robust optimization modeling techniques can be used for computer programming and machine learning objectives, machine learning for the policymaking process, and reinforcement machine learning techniques for the IoT data analytics. To that end, the graphical flowchart in Figure 1 demonstrates a succinct framework of the different computational steps of the overall market and liquidity risk modeling algorithms and their association for computer programming and reinforcement machine learning objectives.

The above robust modeling techniques and algorithms can be applied to reinforcement machine learning processes for portfolio selection and risk management to determine the most suitable risk-return profiles and assets allocation. Given the iterative nature of the above novel modeling algorithms, it can be applied to reinforcement machine learning processes for portfolio optimization and risk management conditional on using credible operational and financial constraints. Moreover, it can be of interest to machine learning for the policymaking process, reinforcement machine learning techniques for the Internet of Things (IoT) data analytics, financial engineering, FinTech, and within big data ecosystems.

As an extension to this work, we aspire in another research study to use the proposed modeling algorithms to the case of multiple asset portfolios and to examine the impact of adverse prices on the global market and illiquidity risk profiles. To that end, the following natural step in this research is to choose certain emerging markets or a particular region and strive to apply the proposed modeling techniques and algorithms to specific multiple-asset portfolios and then to compute the effect of market and illiquidity risks on the global potential risk exposure. In addition, it is quite feasible to apply the novel econophysics modeling techniques and algorithms to the case of selected developed, emerging, and commodity markets. In this case, the proposed robust reinforcement machine learning processes and modeling algorithms can be applied to multiple securities trading and/or asset management portfolios to examine the effect of unfavorable price impact on the total market and liquidity risk potential exposures. Our aim would be to demonstrate different experiments and to test empirically our proposed approach by simulations using real market datasets and to

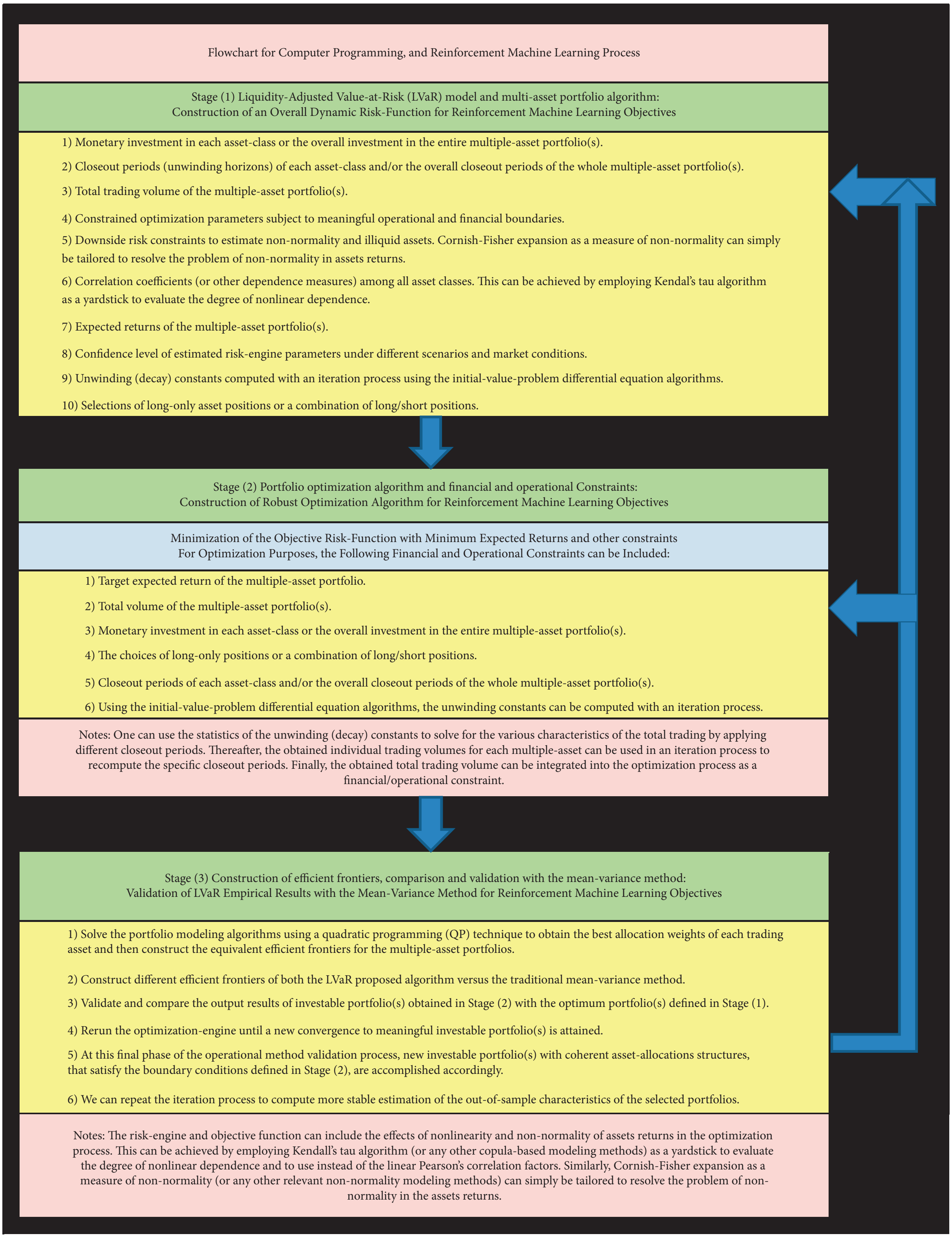

FIGURE 1: A graphical flowchart of the operational stages of the proposed modeling algorithm.

examine the impact of adverse prices on the global market/ and illiquidity risk profiles by implementing realistic operational and financial constraints.

The flowchart in Figure 1 represents a summarized view of the three stages of the proposed modeling algorithms and the connections among the different phases. This flowchart can be very useful for computer programming, comprehending the indispensable input factors for the risk-engine, constrained optimization procedure, and reinforcement machine learning. The figure is designed by the author.

5. Conclusion and Future Directions

Liquidity as the ease of trading of assets has recently acquired a great deal of attention in the academic literature and in market practices. Illiquidity risk grows with the size of the holding positions and refers to the inability to unwind trading assets at the prevailing market conditions without incurring additional costs. Illiquidity happens over some short term but disappears over a longer horizon. However, unlike other risk factors, liquidity risk cannot be diversified or hedged. Furthermore, the liquidity risk depends on several factors and causes and, thus, there are not any standard methods or techniques for its estimation and control.

In this paper, after a concise review of certain contemporary literature on liquidity risk, we propose a novel econophysics mathematical technique and robust algorithms for the modeling of trading volumes and illiquidity risk during the closeout period for multiple-asset portfolios, as well as for financial holding entities with different subsidiary firms and multiple agents.

Liquidity risk should be quantified in a dynamic setting, accounting directly or indirectly for the influence of the multiple market risk factors, including the term structure of the time-varying volatility. In this paper and in contrast to most of the previous works on market illiquidity, we put in perspective the analytical components to modeling the impact of liquidity risk with the use of daily trading volumes of multiple-asset portfolios.

The contributions of this paper to the academic literature, in this specific field of quantitative methods for financial markets applications, are severalfold. As such, this paper is the first attempt, to the best of my knowledge, to develop a novel econophysics modeling technique and robust algorithms for the estimation of trading volumes and illiquidity risk during the closeout period and within the context of multiple-asset portfolios, as well as for financial holding entities with different subsidiary firms and multiple agents, using initial-value-problem differential equations algorithms. Furthermore, this paper provides some new important parameters, which are the first of their kind to the best of my knowledge, for the assessment of the trading volumes of multiple-asset portfolios during the closeout period, where the mathematical proofs for each theorem and corollary are provided. Based on the new developed econophysics theory, this paper presents for the first time a closed-form solution for key parameters for the estimation of trading volumes and liquidity risk, such as the unwinding constant, half-life, and mean lifetime, and discusses how these novel parameters can be estimated and incorporated into the recommended modeling techniques. Finally, in this backdrop, the robust reinforcement machine learning modeling techniques and algorithms are promising and interesting in terms of theory as well as for possible real-world uses for multiple-asset portfolios and can have a variety of applications in financial markets and institutions, predominantly in light of the 2007–2009 global financial crunch. In addition, the proposed novel techniques and risk computation algorithms can contribute to improving risk management and portfolio optimization and selection processes in emerging, developed, and commodity markets, especially in the wake of the 2007–2009 financial crunch. Furthermore, the proposed modeling processes could have fundamental uses and applications for expert systems in finance, financial technology (FinTech), machine learning for the policymaking process, Internet of Things (IoT), and within big data environments.

Several venues are still open for future research in this specific econophysics field. To that end, for further directions on trading volume and liquidity risk, we recommend the following:

(1) The constant of proportionality, μ, that we have denoted earlier as the "*unwinding constant*" and defined as being the decline (or the decay) probability per infinitesimal unwinding time horizon can be possibly expanded to multiple unwinding constants using decline or decay chains by competing processes. The trading volumes of some multiple assets will decline by more than one operational process (or mode) and the market impact of trading on multiple-asset prices may well not be one time but possibly will cause second-phase impacts. As a result, every decay form (or mode) is differentiated by its specific unwinding constant $\mu_{i.}$ To obtain the relevant unwinding constant when the decline process has n participating decay modes, we need to rewrite the differential equation in slightly different form by expressing the unwinding constant of the ith mode by μ_i and solving for the overall unwinding constant, namely, $\mu = \sum_{i=1}^{n} \mu_i$. Nevertheless, for general decay chain processes, the structure of the differential equations can become rather complex and would require the use of the so-called Bateman equation(s) with the set of initial conditions to find a general solution for the coefficients using the Laplace transform. In a similar fashion, we can now determine the half-life and mean lifetime during each decay mode and then find the effective or overall half-life and mean lifetime when the decline process has n participating decay modes.

(2) A general solution to the above posed issue can be given by the Bateman equation(s) in which the activity of the ith mode of the chain is given in terms of the unwinding constants of all preceding modes. The Bateman equations can be derived using the Laplace transform. Thus, the system of differential equations for all decline or decay modes may be transformed to a system of linear equations by taking the Laplace transform. Next, using the notion that the Laplace transform of a derivative is achieved by integration by parts, the Laplace transform of the first derivatives can be obtained. Now, these equations may be solved

successively, and a solution can be obtained for the specific coefficients using the inverse transform and that will yield the Bateman equation(s). These simple properties of the Laplace transform make it a very convenient tool for solving systems of first-order linear differential equations, such as the differential equations posed in this paper. They allow these differential equations to be treated as if they were systems of simple transformed linear equations without derivatives.

(3) Although our proposed robust modeling techniques clearly reflect liquidity risk with the use of trading volumes of multiple-asset portfolios, we did not explore the time-varying characteristics of market liquidity and the dynamic relationships between trading operations and the movements in assets prices with respect to market's spreads. A potential next step would be to examine the dynamics of market impact by employing time-series techniques. Likewise, the market impact of trading on multiple-asset prices might not be a unique event; rather it could initiate second phase outcomes. Moreover, since the motivating strengths on the back of the tick-by-tick markets price movements are related to not only trading actions but also the influx of fresh information and statistics, recognition of these factors would be crucial to comprehending intraday price changes in a dynamic framework.

(4) One of the key issues for liquidity risk measures is the handling of delay risk since the dynamics of delay and its relation to the price dynamics are yet indistinct in times of stressed market conditions and crises. The incorporation of delay risk and how to measure and forecast delay are still unresolved research topics and, thus, further intuitions into when and under which conditions delay arises can aid to evolve this line of reasoning.

(5) As discussed earlier, the risk of illiquidity should be quantified in a dynamic framework. One alternate measure of liquidation risk and cost can be accomplished with the application of a piecewise linear function that models trading volume discounting, as the larger the trading position size, the lower the liquidation price. For this purpose, Laplace transform, which is a more generalized transform, as well as Fourier transform, can be used to solve piecewise linear functions. In fact, the Fourier transform, which is a subset of Laplace transform, is the extension of the Fourier series to nonperiodic signals (e.g., the Fourier transform of triangle and rectangular pulse functions). To that end, the Fourier transform is used largely for steady-state signal analysis, while Laplace transform is used for transient signal analysis. Therefore, the Laplace transform is useful at looking for the response to pulses, step-functions, and delta-functions, while the Fourier transform is beneficial for continuous signals. However, it is important to emphasize that one of the main drawbacks of the Fourier transform is that it can be clearly defined merely for stable systems, whereas Laplace transform can be defined for both stable and unstable systems. Nevertheless, it is possible to convert Laplace transform to Fourier transform and vice versa.

(6) In a similar fashion to the piecewise linear function discussed above, it is also possible to model liquidation risk and cost with the application of pulse function (i.e., the pulse-transfer function), which are commonly used in control systems and signal processes via the Laplace transform, where the Z-transform (i.e., the discrete Laplace transform) is the most suitable for the analytical study of the linear pulse control systems. As a result, the pulse control system in which the control is accomplished by pulses (i.e., signals of short-durations) produced at preset times can be explained as a system of finite differences equations.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study and the article describes entirely theoretical research.

Additional Points

A novel modeling technique for forecasting trading volumes is proposed. The modeling techniques are based on initial-value-problem differential equations. We develop algorithms for optimizing multiple-asset portfolios with liquidity constraints. We propose operational stages for computer programming and reinforcement machine learning.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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