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Research Article

Multivalued Impulsive SDEs Driven by G-Brownian Noise: Periodic Averaging Result

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This paper aims to study two approximation theorems in view of the periodic averaging results for non-Lipschitz multivalued stochastic differential equations with impulses and G-Brownian motion (MISDEGs). By adopting G-Itô's formula and non-Lipschitz condition, the solutions to the simplified MSDEGs without impulses may replace those of the initial MISDEGs in view of approximation in L^2-sense and capacity. Finally, we bring a couple of two examples to enhance our theoretical results.

1. Introduction

The theory of averaging principles provides a useful account of how to simplify the complex systems to be more amenable in numerical calculations and analysis. Recently, a considerable amount of literature has emerged around the theme of approximation theorems for stochastic differential equations (SDEs) [1–8]. As one of the most significant models, recently, multivalued stochastic differential equations (MSDEs) received considerable critical attention. In [9, 10], averaging principles are established for MSDEs with Gaussian noise. Guo and Pei in [11] extended the technique proposed by the authors of [10] to MSDEs fluctuating with the Poisson point process. Meanwhile, by Bihari's inequality, Mao et al. [12] proved that the solutions of the initial non-Lipschitz MSDEs perturbed by Poisson jumps can be replaced by those of simplified MSDEs both in the probability and mean square.

Impulses are of pressing need for the theory of SDEs due to their contributions in modelling processes with rapid changes at certain moments of time. Nowadays, there has been a surge of interest concerning existence, uniqueness, and stability for SDEs with impulses (ISDEs) [13–16]. Although more research has been carried out on averaging principles for MSDEs, no controlled studies have been reported for MISDEs. Basically, the idea behind the periodic averaging method for ISDEs is to allow a simplified autonomous SDE without impulses to replace original nonautonomous ISDEs [17–19].

The theory of *G*-expectation is at the heart of our understanding of uncertainty problems, risk measures, and optimization problems [20, 21]. Peng [20] constructed the cornerstone of *G*-expectation theory with its related random calculus, and SDEs perturbed with G-Brownian motion (SDEGs) became the subject of much systematic investigation [22–33]. In 2017, Ren et al. [34] studied a new model of multivalued SDEGs and satisfied the existence and uniqueness problem by means of the penalized method as well as its related stochastic control problem. However, the periodic averaging method for SDEGs and MSDEGs is rarely considered, and the only paper dealing with this issue is the one mentioned in [35].

Based on the above, with the aid of the G-Itô formula and G-stochastic calculus, we, in this work, present the periodic averaging principle for MSDEGs with impulses (MISDEGs) under the Taniguchi non-Lipschitz condition [36]. This article's contributions are highlighted as follows:

(i) The model uncertainty described by G-Brownian motion fluctuation and jumps presented by impulses show our MISDEG model's generality.

(ii) It is observed that the proofs of Theorems 2.1, 3.1, and 3.9 in [10–12] [35], respectively, depend on the second moment boundedness property of the solution. However, in our case, Theorem 2 does not depend on the boundedness property of the second moment for MISDEG solutions. Moreover, the multivalued term in our model is of a subdifferential term, which is different from the multivalued term in [9–12].

(iii) Our non-Lipschitz condition is more general than the one used in [9–12, 35] and considers them as special cases. Therefore, the results in [9–12, 35] are generalized and extended.

Section 2 is concerned with some preliminary notions, definitions, lemmas, and interpretation of the MISDEG model. In Section 3, we give the approximation in capacity and L^2-sense between the initial MISDEG and simplified MSDEGs without impulses as well as the approximation order to (1) in a bounded interval of time. Finally, we bring a couple of examples to enhance our theoretical results in Section 4.

2. Preliminaries

Here, we mention some notions and facts on random calculus with respect to G-Brownian motion and prepare our model.

2.1. Notations. In this section, we first give the notion of sublinear expectation space $(\Omega, \mathscr{H}, \mathbb{E})$, where Ω is a given state set and $\mathscr{H}$ is a linear space of real valued functions defined on Ω. The space $\mathscr{H}$ can be considered the space of random variables.

Assume Ω be the space of all continuous $\mathbb{R}^n$-valued functions $\{\omega_t\}_{t\in\mathbb{R}_+}$, with $\omega_0 = 0$, equipped with the distance

$$\rho\left(\omega^1, \omega^2\right) = \sum_{i=1}^{\infty} 2^{-i}\left[\max_{t\in[0,i]}\left|\omega_t^1 - \omega_t^2\right|(\cdot)\wedge 1\right]. \tag{1}$$

Then, (Ω, ρ) is a metric space.

For all $\omega \in \Omega$, we define the canonical process $W_t(\omega) = \omega_t$, $t \in [0,\infty)$. The filtration generated by the canonical process $(W_t)_{t\geq 0}$ is defined as $\mathscr{F}_t = \sigma(W_s, 0\leq s\leq t)$. Let

$$L^{\text{lip}}(\mathscr{F}_T) \coloneqq \left\{\psi\left(W_{t_1}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}}\right): 0\leq t_1 < \cdots < t_n \leq T, \psi \in \mathbb{C}^{b.\text{lip}}\left(\mathbb{R}^{n\times n}\right)\right\}, \tag{2}$$

where $n\geq 1$ and $\mathbb{C}^{b.\text{lip}}(\mathbb{R}^n\times\mathbb{R}^n)$ refers to the space of Lipschitz-bounded functions on $\mathbb{R}^n\times\mathbb{R}^n$ and $L^{\text{lip}}(\mathscr{F}) = \cup_{i=1}^{\infty} L^{\text{lip}}(\mathscr{F}_i)$.

For any $\vartheta(\omega) = \psi(W_{t_1}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}}) \in L^{\text{lip}}(\mathscr{F})$ with $0 < t_1 < \cdots < t_n < \infty$, we define

$$\mathbb{E} = \left[\psi\left(W_{t_1}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}}\right)\right] \coloneqq \psi_n, \tag{3}$$

where ψ_n is defined iteratively by

$$\begin{aligned}
\psi_1(y_1, y_2, \dots, y_{n-1}) &= \mathbb{E}\left[\psi\left(y_1, y_2, \dots, y_{m-1}, W_{t_n} - W_{t_{n-1}}\right)\right],\\
\psi_2(y_1, y_2, \dots, y_{n-2}) &= \mathbb{E}\left[\psi_1\left(y_1, y_2, \dots, y_{m-2}, W_{t_{n-1}} - W_{t_{n-2}}\right)\right],\\
&\vdots\\
\psi_{n-1}(y_1) &= \mathbb{E}\left[\psi_{n-2}\left(y_1, W_{t_2} - W_{t_1}\right)\right],\\
\psi_n(y_1) &= \mathbb{E}\left[\psi_{n-1}\left(W_{t_1}\right)\right].
\end{aligned} \tag{4}$$

The conditional expectation of $\vartheta(\omega) = \psi(W_{t_1}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}})$ is defined as

$$\mathbb{E}\left[\frac{\vartheta}{\mathscr{F}_{t_i}}\right] \coloneqq \psi_{n-i}\left(W_{t_1}, W_{t_2} - W_{t_1}, \dots, W_{t_i} - W_{t_{i-1}}\right). \tag{5}$$

Definition 1 (G-Brownian motion). The expectation operator $\mathbb{E}$ (a nonlinear operator) defined above is called G-expectation, and the corresponding coordinate process W is called G-Brownian motion.

For $\vartheta \in L^{\text{lip}}(\mathscr{F}_T)$ and $1\leq r\leq\infty$, we denote by $L_G^r(\mathscr{F}_T)$ (resp. $L_G^r(\mathscr{F})$ as the completion of $L^{\text{lip}}(\mathscr{F}_T)$ (resp. $L^{\text{lip}}(\mathscr{F})$ under the Banach norm $\|\vartheta\|_r = (\mathbb{E}[|\vartheta|^r])^{1/r}$, for ϑ. According to Denis et al. [37], $L_G^1(\mathscr{F})$ can be written as the collection of all the quasi-continuous random vectors $X \in L^0(\Omega)$ with $\lim_{n\longrightarrow+\infty}\widehat{\mathbb{E}}[|X|_{\{|X|>n\}}] = 0$. Moreover, for all $X \in L_G^1((\mathscr{F})$, it holds that $\mathbb{E}(X) = \widehat{\mathbb{E}}[X]$.

Under the above preparation, we introduce W_t to be G-Brownian motion defined on the space of G-expectation $(\Omega, L^{\text{lip}}(\mathscr{F}), \widehat{\mathbb{E}})$ with quadratic variation [20]:

$$\langle W, W\rangle_s \coloneqq W_s^2 - 2\int_0^s W_\tau \mathrm{d}W_\tau, s \in [0,T]. \tag{6}$$

Definition 2 (see [37]). Let $\mathscr{D}(\Omega)$ be the Borel σ-algebra of Ω. The capacity $\widehat{c}(.)$ associated with $\mathbb{P}$, a weakly compact class of probability measures P defined on $(\Omega, \mathscr{D}(\Omega))$, is defined as

$$\widehat{c}(B) = \sup_{P\in\mathbb{P}} P(B), B \in \mathscr{D}(\Omega). \quad (7)$$

We take this lemma from [21].

Lemma 1. *Assume $\Upsilon \in L_G^1(F_T)$, then for some positive r and each $\beta > 0$, we have*

$$c(\{|\Upsilon(t)| > \beta\}) \leq \frac{\widehat{\mathbb{E}}|\Upsilon(t)|^r}{\beta^r}, \quad (8)$$

where $\widehat{\mathbb{E}}|\Upsilon(t)|^r < \infty$.

Definition 3. Assuming $r\geq 1$ and positive T, the space of simple processes $M_G^{r,0}([0,T])$ is defined as

$$M_G^{r,0}([0,T]) = \left\{ \eta_\tau(\omega) = \eta(\tau,\omega) = \sum_{i=1}^{N-1} \vartheta_{\tau_i}(\omega) I_{[\tau_{i-1},\tau_i]}(\tau); \vartheta_{\tau_i}(\omega) \in L_G^r(\mathscr{F}_{\tau_i}), \forall N\geq 1, \tau_i \in [0,T], i = 0,1\ldots, N-1 \right\}, \quad (9)$$

where $M_G^r([0,T])$ denotes the completion of $M_G^{r,0}([0,T])$ under this norm

$$\|\eta\|_{M_G^r([0,T])} := \frac{1}{T}\left(\int_0^T \widehat{\mathbb{E}}[\eta_s^r] ds\right)^{1/r}. \quad (10)$$

This lemma in [25] is needed.

Lemma 2. *Assume $r \geq 1$ and $\phi \in M_G^r([0,T])$, then for all $t_1, t_2 \in [0,T]$, $t_1 \leq t_2$, we have*

$$\widehat{\mathbb{E}}\left[\sup_{t_1\leq t\leq u\leq t_2} \left|\int_{t_1}^u \phi(s) d\langle W,W\rangle_s\right|^r\right] \leq |t_2 - t_1|^{r-1}\int_{t_1}^{t_2} \widehat{\mathbb{E}}|\phi(s)|^r ds. \quad (11)$$

Lemma 3 (see [34]). *Assuming $r\geq 1$ and $\phi \in M_G^2([0,T])$, we conclude*

$$\widehat{\mathbb{E}}\left[\sup_{0\leq v\leq T}\left|\int_0^v \phi(\tau) dW_\tau\right|^r\right] \leq C_r \widehat{\mathbb{E}}\left[\left|\int_0^T [\phi(\tau)]^2 d\langle W,W\rangle_\tau\right|^{r/2}\right]. \quad (12)$$

2.2. Model Preparation. This work focuses on MISDEGs interpreted as

$$\begin{cases} d\mathscr{X}(t) + \partial\varphi(\mathscr{X}(t)) \ni f(t,\mathscr{X}(t))dt + h(t,\mathscr{X}(t))d\langle W,W\rangle_t + \sigma(t,\mathscr{X}(t))dW(t), t \neq t_j, \\ \Delta\mathscr{X}(t_j) = \mathscr{X}(t_j^+) - \mathscr{X}(t_j^-) = I_j(\mathscr{X}(t_j^-)), t = t_j, j \in \mathbb{N}, \\ \mathscr{X}(0) = \mathscr{X}_0 \in \overline{\text{Dom}(\varphi)}, \end{cases} \quad (13)$$

where φ: $\mathbb{R}^d \longrightarrow (-\infty, +\infty]$ is a convex and lower semi-continuous function with the domain $\text{Dom}(\varphi) = \{\wp \in \mathbb{R}^d: \varphi(\wp) < \infty\}$ such that $0 \in \text{Int}(\text{Dom}(\varphi)) \neq \varnothing$, and $\varphi(\wp)\geq\varphi(0) = 0$, for every $\wp \in \mathbb{R}^d$. Functions f: $[0,\infty)\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$, h: $[0,\infty)\times\mathbb{R}^d \longrightarrow \mathbb{R}^{d\times m}$, and σ: $[0,\infty)\times\mathbb{R}^d \longrightarrow \mathbb{R}^{d\times m}$ are continuous. I_j: $\mathbb{R}^d \longrightarrow \mathbb{R}^d$ and impulsive moments t_j satisfy $0 < t_1 < t_2 < \cdots t_j < \cdots$, and $\lim_{j\longrightarrow+\infty} t_j = \infty$. $\mathscr{X}(t_j^-)$ and $\mathscr{X}(t_j^+)$ are the left and right limits for the continuous process $\mathscr{X}(t)$ at time t_j, respectively. $W(t)$ is a m-dimensional G-Brownian motion with quadratic variation $\langle W,W\rangle_t$.

Definition 4. A subdifferential operator $\partial\varphi(\varsigma) = \{\iota \in \mathbb{R}^d: \langle\iota, \upsilon - \varsigma\rangle + \varphi(\varsigma) \leq \varphi(\upsilon), \forall\upsilon \in \mathbb{R}^d\}$ with $\text{Dom}(\partial\varphi) = \{\varsigma \in \mathbb{R}^d: \partial\varphi(\varsigma) \neq \varnothing\}$

(i) It is called monotone if

$$\langle\iota^* - \upsilon^*, \iota - \upsilon\rangle \geq 0, \forall(\iota,\iota^*), (\upsilon,\upsilon^*) \in Gr(\partial\varphi), \quad (14)$$

where $Gr(\partial\varphi) = \{(\varsigma,\upsilon) \in \mathbb{R}^d \times \mathbb{R}^d: \varsigma \in \text{Dom}(\partial\varphi), \upsilon \in \partial\varphi(\varsigma)\}$ is the graph of the subdifferential operator $\partial\varphi$.

(ii) It is called maximal monotone if

$$(\iota,\iota^*) \in Gr(\partial\varphi) \Leftrightarrow \{\langle\iota^* - \upsilon^*, \iota - \upsilon\rangle \geq 0, \forall(\upsilon,\upsilon^*) \in Gr(\partial\varphi)\}. \quad (15)$$

We propose this definition of the solution to (13).

Definition 5. A couple of measurable and continuous stochastic processes $(\mathscr{X}, K)$ are named a solution to equation (13) if

(i) $\mathscr{X}(t) \in \overline{\text{Dom}(\varphi)}$ and $\mathscr{X}(0) = \mathscr{X}_0$, *q.s.*

(ii) For all $t\geq 0$, $K(t)$ is of total variation $|K(.)|_t^0 < \infty$ and $K(0) = 0$

(iii) For any $0 < t < \infty$, we have

$$\begin{aligned} \mathscr{X}(t) = \mathscr{X}_0 &+ \int_0^t f(s,\mathscr{X}(s))ds + \int_0^t h(s,\mathscr{X}(s))d\langle W,W\rangle_s \\ &+ \int_0^t \sigma(s,\mathscr{X}(s))dW(s) + \sum_{j=1}^\infty I_j(\mathscr{X}(t_j^-)) \\ &- K(t), q.s. \end{aligned} \quad (16)$$

(iv) $dK(t) \in \partial\varphi(\mathscr{X}(t))dt$, *q.s.*

The following is an important lemma from [34].

Lemma 4. *If the two pairs of stochastic processes* $(\mathscr{X}(t), K(t))$ *and* $(\acute{\mathscr{X}}(t), \acute{K}(t))$ *satisfy Definition 4, then*

$$\int_0^t \langle \mathscr{X}(s) - \acute{\mathscr{X}}(s), \mathrm{d}K(s) - \mathrm{d}\acute{K}(s)\rangle \geqslant 0. \tag{17}$$

3. Periodic Averaging Principle

Here, we focus on the periodic averaging method for (13). For $\varepsilon \in [0, \varepsilon_0]$ with fixed quantity $\varepsilon_0 (0 < \varepsilon_0 < (1/4))$, we consider these initial MISDEGs:

$$\begin{cases} \mathrm{d}\mathscr{X}_\varepsilon(t) = \varepsilon f(t, \mathscr{X}_\varepsilon(t))\mathrm{d}t + \sqrt{\varepsilon}\, h(t, \mathscr{X}_\varepsilon(t)) d\langle W, W\rangle_t + \sqrt{\varepsilon}\, \sigma(t, \mathscr{X}_\varepsilon(t))\mathrm{d}W(t) - \varepsilon \mathrm{d}K(t), t \neq t_j, \\ \Delta \mathscr{X}_\varepsilon(t_j) = \varepsilon I_j(\mathscr{X}_\varepsilon(t_j^-)), t = t_j, j \in \mathbb{N}, \\ \mathscr{X}_\varepsilon(0) = \mathscr{X}_0 \in \overline{\mathrm{Dom}(\varphi)}, \end{cases} \tag{18}$$

where f, h, σ are bounded T-periodic in the first argument. Moreover, impulsive moments are also periodic such that there exists a $l \in \mathbb{N}$ satisfying $0 \leqslant t_1 < t_2 < \cdots \infty < t_l < T$, and for each $j > l$, we obtain $t_j = t_{j-l} + T$ and $I_j = I_{j-l}$.

Assume the following simplified MSDEGs without impulses, then we get

$$\begin{cases} d\overline{\mathscr{X}}_\varepsilon(t) = \varepsilon\left(\overline{f}\left(\overline{\mathscr{X}}_\varepsilon(t)\right) + \overline{I}\left(\overline{\mathscr{X}}_\varepsilon(t)\right)\right)\mathrm{d}t + \sqrt{\varepsilon}\, \overline{h}\left(\overline{\mathscr{X}}_\varepsilon(t)\right) d\langle W, W\rangle_t + \sqrt{\varepsilon}\, \overline{\sigma}\left(\overline{\mathscr{X}}_\varepsilon(t)\right)\mathrm{d}W(t) - \varepsilon \mathrm{d}\overline{K}(t), \\ \overline{\mathscr{X}}_\varepsilon(0) = \overline{\mathscr{X}}_0 \in \overline{\mathrm{Dom}(\varphi)}, \end{cases} \tag{19}$$

and with $\overline{f}: \mathbb{R}^{\mathrm{d}} \longrightarrow \mathbb{R}^{\mathrm{d}}, \overline{h}: \mathbb{R}^{\mathrm{d}} \longrightarrow \mathbb{R}^{d\times m}, \overline{\sigma}: \mathbb{R}^{\mathrm{d}} \longrightarrow \mathbb{R}^{d\times m}$, and $\overline{I}: \mathbb{R}^{\mathrm{d}} \longrightarrow \mathbb{R}^{\mathrm{d}}$ are measurable functions defined as

$$\begin{aligned} \overline{f}(\wp) &= \frac{1}{T}\int_0^T f(s, \wp)\mathrm{d}s, \\ \overline{h}(\wp) &= \frac{1}{T}\int_0^T h(s, \wp)\mathrm{d}s, \\ \overline{\sigma}(\wp) &= \frac{1}{T}\int_0^T \sigma(s, \wp)\mathrm{d}s, \\ \overline{I}(\wp) &= \frac{1}{T}\sum_{j=1}^{l} I_j(\wp). \end{aligned} \tag{20}$$

For deriving our main findings, we bring the present conditions.

Hypothesis 1. Assume $\mathscr{G}(t, \wp): \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ where

(1a) For each fixed $z \in [0, +\infty)$, $\mathscr{G}(t, z)$ is locally integrable in t, and it is continuous, increasing, and concave in z for each fixed $t \geqslant 0$ and $\mathscr{G}(t, 0) = 0$.

(1b) For all $Z_1, Z_2 \in \mathbb{R}^{\mathrm{d}}$ and $t \geqslant 0$, we get

$$|f(t, Z_1) - f(t, Z_2)|^2 + |h(t, Z_1) - h(t, Z_2)|^2 + |\sigma(t, Z_1) - \sigma(t, Z_2)|^2 \leqslant \mathscr{G}\left(t, |Z_1 - Z_2|^2\right). \tag{21}$$

(1c) If a continuous and positive function $\mathscr{M}(\tau)$ derives

$$\begin{cases} \mathscr{M}(\tau) \leqslant \gamma \int_0^\tau \mathscr{G}(\nu, \mathscr{M}(\nu)) d\nu, \tau \in \mathbb{R}, \\ \mathscr{M}(0) = 0. \end{cases} \tag{22}$$

for any $\gamma > 0$, then $\mathscr{M}(\tau) \equiv 0 \forall \tau \in [0, \infty)$.

Hypothesis 2. For all $t \in [0, T]$, it follows that

$$|f(t, 0)|^2 + |h(t, 0)|^2 + |\sigma(t, 0)|^2 \leqslant L, \tag{23}$$

where $L > 0$ is a constant.

Hypothesis 3. For all $Z_1, Z_2 \in \mathbb{R}^{\mathrm{d}}$, we find two positive constants K_1, K_2 satisfying

$$\begin{aligned} \left|I_j(Z_1)\right|^2 &\leqslant K_1, \\ \left|I_j(Z_1) - I_j(Z_2)\right|^2 &\leqslant K_2 |Z_1 - Z_2|^2. \end{aligned} \tag{24}$$

Hypothesis 4. For each $t \in [0, \infty)$, there exist $Q > 0$ satisfying $|f(t, Z)|^2 \leqslant Q$, $|h(t, Z)|^2 \leqslant Q$, $|\sigma(t, Z)|^2 \leqslant Q$, $|\overline{f}(Z)|^2 \leqslant Q$, $|\overline{h}(Z)|^2 \leqslant Q$ and $|\overline{\sigma}(Z)|^2 \leqslant Q$.

Theorem 1. *Assume that Hypotheses 1–3 hold, then Equation (13) has a unique solution.*

Proof. According to the assumptions and use of the same argument of Propositions 3.1 and 3.2 and Theorem 1 in [34], it is easy to prove that Equation (13) has a unique solution. Here, we omit the detailed proof.

Theorem 2 argues that nonautonomous MSDEGs with impulses (18) can strongly be replaced by autonomous MSDEGs without impulses (19). □

Theorem 2. *Assume Hypotheses 1–4 hold and suppose $\mathcal{Z}_\varepsilon$ and $\overline{\mathcal{Z}}_\varepsilon$ are solutions for Equations (18) and (19), respectively, then, for any $\varepsilon_1 \in (0, \varepsilon_0]$ and $0 \le t \le \infty$, $M > 0$ exists, satisfying $|\mathcal{Z}_0 - \overline{\mathcal{Z}}_0| \le M_\varepsilon$ and*

$$\lim_{\varepsilon \longrightarrow 0} \widehat{\mathbb{E}} \sup_{0 \le s \le t} \left|\mathcal{Z}_\varepsilon(s) - \overline{\mathcal{Z}}_\varepsilon(s)\right|^2 = 0. \tag{25}$$

For all $\varepsilon \in (0, \varepsilon_1]$.

Proof. Applying the G-Itô formula, we have

$$\begin{aligned}
\left|\mathcal{Z}_\varepsilon(t) - \overline{\mathcal{Z}}_\varepsilon(t)\right|^2 = {} & \left|\mathcal{Z}_0 - \overline{\mathcal{Z}}_0\right|^2 + 2\varepsilon \int_0^t \langle \mathcal{Z}_\varepsilon(s) - \overline{\mathcal{Z}}_\varepsilon(s), f(s, \mathcal{Z}_\varepsilon(s)) - \overline{f}(\overline{\mathcal{Z}}_\varepsilon(s))\rangle \mathrm{d}s,\\
& + 2\sqrt{\varepsilon} \int_0^t \langle \mathcal{Z}_\varepsilon(s) - \overline{\mathcal{Z}}_\varepsilon(s), h(s, \mathcal{Z}_\varepsilon(s)) - \overline{h}(\overline{\mathcal{Z}}_\varepsilon(s))\rangle d\langle W, W\rangle_s + \varepsilon \int_0^t \left|\sigma(s, \mathcal{Z}_\varepsilon(s)) - \overline{\sigma}(\overline{\mathcal{Z}}_\varepsilon(s))\right|^2 \mathrm{d}\langle W, W\rangle_s,\\
& + 2\sqrt{\varepsilon} \int_0^t \langle \mathcal{Z}_\varepsilon(s) - \overline{\mathcal{Z}}_\varepsilon(s), \sigma(s, \mathcal{Z}_\varepsilon(s)) - \overline{\sigma}(\overline{\mathcal{Z}}_\varepsilon(s))\rangle \mathrm{d}W(s) - 2\varepsilon \int_0^t \langle \mathcal{Z}_\varepsilon(s) - \overline{\mathcal{Z}}_\varepsilon(s), \mathrm{d}K(s) - \mathrm{d}\overline{K}(s)\rangle,\\
& + 2\varepsilon \langle \mathcal{Z}_\varepsilon(t) - \overline{\mathcal{Z}}_\varepsilon(t), \sum_{j=1}^{\infty} I_j\left(\mathcal{Z}_\varepsilon\left(t_j^-\right)\right) - \int_0^t \overline{I}\left(\overline{\mathcal{Z}}_\varepsilon(s)\right)\mathrm{d}s\rangle.
\end{aligned} \tag{26}$$

It is obvious from Lemma 4 that

$$2\varepsilon \int_0^t \langle \mathcal{Z}_\varepsilon(s) - \overline{\mathcal{Z}}_\varepsilon(s), \mathrm{d}K(s) - \mathrm{d}\overline{K}(s)\rangle \geqslant 0. \tag{27}$$

Therefore, we have

$$\begin{aligned}
& \sup_{0 \leqslant s \leqslant t} \left|\mathcal{Z}_\varepsilon(s) - \overline{\mathcal{Z}}_\varepsilon(s)\right|^2,\\
& \leqslant \left|\mathcal{Z}_0 - \overline{\mathcal{Z}}_0\right|^2 + 2\varepsilon \sup_{0 \leqslant s \leqslant t} \left|\int_0^s \langle \mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau), f(\tau, \mathcal{Z}_\varepsilon(\tau)) - \overline{f}(\overline{\mathcal{Z}}_\varepsilon(\tau))\rangle \mathrm{d}\tau\right|,\\
& \quad + 2\sqrt{\varepsilon} \sup_{0 \leqslant s \leqslant t} \left|\int_0^s \langle \mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau), h(\tau, \mathcal{Z}_\varepsilon(\tau)) - \overline{h}(\overline{\mathcal{Z}}_\varepsilon(\tau))\rangle d\langle W, W\rangle_\tau\right|,\\
& \quad + \varepsilon \sup_{0 \leqslant s \leqslant t} \left|\int_0^s \left[\sigma(\tau, \mathcal{Z}_\varepsilon(\tau)) - \overline{\sigma}(\overline{\mathcal{Z}}_\varepsilon(\tau))\right]^2\right| d\langle W, W\rangle_\tau,\\
& \quad + 2\sqrt{\varepsilon} \sup_{0 \leqslant s \leqslant t} \left|\int_0^s \langle \mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau), \sigma(\tau, \mathcal{Z}_\varepsilon(\tau)) - \overline{\sigma}(\overline{\mathcal{Z}}_\varepsilon(\tau))\rangle \mathrm{d}W(\tau)\right|,\\
& \quad + 2\varepsilon \sup_{0 \leqslant s \leqslant t} \left|\langle \mathcal{Z}_\varepsilon(s) - \overline{\mathcal{Z}}_\varepsilon(s), \sum_{j=1}^{\infty} I_j\left(\mathcal{Z}_\varepsilon\left(t_j^-\right)\right) - \int_0^s \overline{I}\left(\overline{\mathcal{Z}}_\varepsilon(\tau)\right) d\tau\rangle\right| := \left|\mathcal{Z}_0 - \overline{\mathcal{Z}}_0\right|^2 + \sum_{i=1}^{5} J_i.
\end{aligned} \tag{28}$$

Now, the technique of plus and minus gives

$$
\begin{aligned}
\widehat{\mathbb{E}}\left|J_1\right| &= 2\varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\int_0^s\langle\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau), f\left(\tau,\mathscr{Z}_\varepsilon(\tau)\right)-\overline{f}\left(\overline{\mathscr{Z}}_\varepsilon(\tau)\right)\rangle\mathrm{d}\tau\right|,\\
&\leqslant 2\varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\int_0^s\langle\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau), f\left(\tau,\mathscr{Z}_\varepsilon(\tau)\right)-f\left(\tau,\overline{\mathscr{Z}}_\varepsilon(\tau)\right)\rangle\mathrm{d}\tau\right|,\\
&\quad + 2\varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\int_0^s\left|\langle\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau), f\left(\tau,\overline{\mathscr{Z}}_\varepsilon(\tau)\right)-\overline{f}\left(\overline{\mathscr{Z}}_\varepsilon(\tau)\right)\rangle\right|\mathrm{d}\tau \coloneqq J_{11}+J_{12}.
\end{aligned}
\tag{29}
$$

Due to Young's inequality and Hypothesis (1b), we conclude

$$
\begin{aligned}
J_{11} &\leqslant 2\varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\int_0^s\left|\langle\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau), f\left(\tau,\mathscr{Z}_\varepsilon(\tau)\right)-f\left(\tau,\overline{\mathscr{Z}}_\varepsilon(\tau)\right)\rangle\right|\mathrm{d}\tau,\\
&\leqslant \varepsilon^2\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\int_0^s\left|\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2\mathrm{d}\tau+\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\int_0^s\left|f\left(\tau,\mathscr{Z}_\varepsilon(\tau)\right)-f\left(\tau,\overline{\mathscr{Z}}_\varepsilon(\tau)\right)\right|^2\mathrm{d}\tau,\\
&\leqslant \varepsilon^2\widehat{\mathbb{E}}\int_0^t\left|\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2\mathrm{d}\tau+\widehat{\mathbb{E}}\int_0^t\mathscr{G}\left(\tau,\left|\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2\mathrm{d}\tau\right).
\end{aligned}
\tag{30}
$$

Letting q be large so that $qT\leqslant t$, we may obtain by Hypotheses (1b) and 4:

$$
\begin{aligned}
J_{12} &= 2\varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\int_0^s\left|\langle\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau), f\left(\tau,\overline{\mathscr{Z}}_\varepsilon(\tau)\right)-\overline{f}\left(\overline{\mathscr{Z}}_\varepsilon(\tau)\right)\rangle\right|d\tau,\\
&\leqslant \varepsilon\widehat{\mathbb{E}}\int_0^t\left|\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2\mathrm{d}\tau+\varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\int_0^s\left|f\left(\tau,\overline{\mathscr{Z}}_\varepsilon(\tau)\right)-\overline{f}\left(\overline{\mathscr{Z}}_\varepsilon(\tau)\right)\right|^2\mathrm{d}\tau,\\
&\leqslant \varepsilon\widehat{\mathbb{E}}\int_0^t\left|\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2\mathrm{d}\tau+\varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\sum_{l=1}^{q}\int_{(l-1)T}^{lT}\left|f\left(\tau,\overline{\mathscr{Z}}_\varepsilon(\tau)\right)-\overline{f}\left(\overline{\mathscr{Z}}_\varepsilon(\tau)\right)\right|^2\mathrm{d}\tau,\\
&\quad + \varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\int_{qT}^s\left|f\left(\tau,\overline{\mathscr{Z}}_\varepsilon(\tau)\right)-\overline{f}\left(\overline{\mathscr{Z}}_\varepsilon(\tau)\right)\right|^2\mathrm{d}\tau,\\
&\leqslant 3\varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\sum_{l=1}^{q}\int_{(l-1)T}^{lT}\left|f\left(\tau,\overline{\mathscr{Z}}_\varepsilon(\tau)\right)-f\left(\tau,\overline{\mathscr{Z}}_\varepsilon(lT)\right)\right|^2\mathrm{d}\tau,\\
&\quad + 3\varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\sum_{l=1}^{q}\int_{(l-1)T}^{lT}\left|f\left(\tau,\overline{\mathscr{Z}}_\varepsilon(lT)\right)-\overline{f}\left(\overline{\mathscr{Z}}_\varepsilon(lT)\right)\right|^2\mathrm{d}\tau,\\
&\quad + 3\varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\sum_{l=1}^{q}\int_{(l-1)T}^{lT}\left|\overline{f}\left(\overline{\mathscr{Z}}_\varepsilon(lT)\right)-\overline{f}\left(\overline{\mathscr{Z}}_\varepsilon(\tau)\right)\right|^2\mathrm{d}\tau,\\
&\quad + \varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\int_{qT}^s\left|f\left(\tau,\overline{\mathscr{Z}}_\varepsilon(\tau)\right)-\overline{f}\left(\overline{\mathscr{Z}}_\varepsilon(\tau)\right)\right|^2 d\tau+\varepsilon\widehat{\mathbb{E}}\int_0^t\left|\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2\mathrm{d}\tau,\\
&\leqslant \varepsilon\widehat{\mathbb{E}}\int_0^t\left|\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2 d\tau+6q\varepsilon\widehat{\mathbb{E}}\int_0^T\mathscr{G}\left(\tau,\left|\overline{\mathscr{Z}}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(iT)\right|^2\right)\mathrm{d}s+12q\varepsilon TQ+4\varepsilon QT.
\end{aligned}
\tag{31}
$$

For each $l = \{1, \ldots, q\}$.

Consequently, we deduce

$$\begin{aligned}\widehat{\mathbb{E}}|J_1| \leqslant & \varepsilon(1+\varepsilon)\widehat{\mathbb{E}}\int_0^t \left|\mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau)\right|^2 \mathrm{d}\tau + \widehat{\mathbb{E}}\int_0^t \mathcal{G}\left(\tau, \left|\mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau)\right|^2\right)\mathrm{d}\tau \\ & + 6q\varepsilon\widehat{\mathbb{E}}\int_0^T \mathcal{G}\left(\tau, \left|\overline{\mathcal{Z}}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(lT)\right|^2\right)\mathrm{d}\tau + 12q\varepsilon TQ + 4\varepsilon QT.\end{aligned} \tag{32}$$

For J_2, we get with the aid of Lemma 2:

$$\begin{aligned}\widehat{\mathbb{E}}|J_2| &= 2\sqrt{\varepsilon}\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\int_0^s \langle \mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau), h(\tau, \mathcal{Z}_\varepsilon(\tau)) - \overline{h}\left(\overline{\mathcal{Z}}_\varepsilon(\tau)\right)\rangle d\langle W, W\rangle_\tau\right|, \\ &\leqslant 2\sqrt{\varepsilon}C\widehat{\mathbb{E}}\int_0^t \left|\langle \mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau), h(\tau, \mathcal{Z}_\varepsilon(\tau)) - \overline{h}\left(\overline{\mathcal{Z}}_\varepsilon(\tau)\right)\rangle\right|\mathrm{d}\tau, \\ &\leqslant 2\sqrt{\varepsilon}C\widehat{\mathbb{E}}\int_0^t \left|\langle \mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau), h(\tau, \mathcal{Z}_\varepsilon(\tau)) - h\left(\tau, \overline{\mathcal{Z}}_\varepsilon(\tau)\right)\rangle\right|\mathrm{d}\tau, \\ &\quad + 2\sqrt{\varepsilon}C\widehat{\mathbb{E}}\int_0^t \left|\langle \mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau), h\left(\tau, \overline{\mathcal{Z}}_\varepsilon(\tau)\right) - \overline{h}\left(\overline{\mathcal{Z}}_\varepsilon(\tau)\right)\rangle\right|\mathrm{d}\tau \coloneqq J_{21} + J_{22}.\end{aligned} \tag{33}$$

Similar to J_{11}, hawse have

$$\begin{aligned}J_{21} &= 2\sqrt{\varepsilon}C\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\int_0^s \left|\langle \mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau), h(\tau, \mathcal{Z}_\varepsilon(\tau)) - h\left(\tau, \overline{\mathcal{Z}}_\varepsilon(\tau)\right)\rangle\right|\mathrm{d}\tau, \\ &\leqslant \varepsilon C\widehat{\mathbb{E}}\int_0^t \left|\mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau)\right|^2 \mathrm{d}u + C\widehat{\mathbb{E}}\int_0^t \left|h(\tau, \mathcal{Z}_\varepsilon(\tau)) - h\left(\tau, \overline{\mathcal{Z}}_\varepsilon(\tau)\right)\right|^2 \mathrm{d}\tau, \\ &\leqslant \varepsilon C\widehat{\mathbb{E}}\int_0^t \left|\mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau)\right|^2 \mathrm{d}u + C\widehat{\mathbb{E}}\int_0^t \mathcal{G}\left(\tau, \left|\mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau)\right|^2\right)\mathrm{d}\tau.\end{aligned} \tag{34}$$

According to J_{12}, J_{22} becomes

$$\begin{aligned}J_{22} \leqslant & \sqrt{\varepsilon}C\widehat{\mathbb{E}}\int_0^t \left|\mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau)\right|^2 \mathrm{d}\tau + \sqrt{\varepsilon}C\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\int_0^s \left|h\left(\tau, \overline{\mathcal{Z}}_\varepsilon(\tau)\right) - \overline{h}\left(\overline{\mathcal{Z}}_\varepsilon(\tau)\right)\right|^2 \mathrm{d}\tau, \\ \leqslant & \sqrt{\varepsilon}C\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\sum_{l=1}^q \int_{(l-1)T}^{lT} \left|h\left(\tau, \overline{\mathcal{Z}}_\varepsilon(\tau)\right) - \overline{h}\left(\overline{\mathcal{Z}}_\varepsilon(\tau)\right)\right|^2 \mathrm{d}\tau, \\ & + \sqrt{\varepsilon}C\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\int_{qT}^s \left|h\left(\tau, \overline{\mathcal{Z}}_\varepsilon(\tau)\right) - \overline{h}\left(\overline{\mathcal{Z}}_\varepsilon(\tau)\right)\right|^2 \mathrm{d}\tau + \sqrt{\varepsilon}C\widehat{\mathbb{E}}\int_0^t \left|\mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau)\right|^2 d\tau, \\ \leqslant & 3\sqrt{\varepsilon}C\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\sum_{l=1}^q \int_{(l-1)T}^{lT} \left|h\left(\tau, \overline{\mathcal{Z}}_\varepsilon(\tau)\right) - h\left(\tau, \overline{\mathcal{Z}}_\varepsilon(lT)\right)\right|^2 \mathrm{d}\tau, \\ & + 3\sqrt{\varepsilon}C\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\sum_{l=1}^q \int_{(l-1)T}^{lT} \left|h\left(\tau, \overline{\mathcal{Z}}_\varepsilon(lT)\right) - \overline{h}\left(\overline{\mathcal{Z}}_\varepsilon(lT)\right)\right|^2 \mathrm{d}\tau, \\ & + 3\sqrt{\varepsilon}C\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\sum_{l=1}^q \int_{(l-1)T}^{lT} \left|\overline{h}\left(\overline{\mathcal{Z}}_\varepsilon(lT)\right) - \overline{h}\left(\overline{\mathcal{Z}}_\varepsilon(\tau)\right)\right|^2 \mathrm{d}\tau, \\ & + \sqrt{\varepsilon}C\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\int_{qT}^s \left|h\left(\tau, \overline{\mathcal{Z}}_\varepsilon(\tau)\right) - \overline{h}\left(\overline{\mathcal{Z}}_\varepsilon(\tau)\right)\right|^2 d\tau + \sqrt{\varepsilon}C\widehat{\mathbb{E}}\int_0^t \left|\mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau)\right|^2 \mathrm{d}\tau, \\ \leqslant & 6\sqrt{\varepsilon}Cq\widehat{\mathbb{E}}\int_0^T \mathcal{G}\left(\tau, \left|\overline{\mathcal{Z}}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(lT)\right|^2\right)\mathrm{d}t + \sqrt{\varepsilon}C\widehat{\mathbb{E}}\int_0^t \left|\mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau)\right|^2 \mathrm{d}\tau + 12\sqrt{\varepsilon}CqTQ + 4\sqrt{\varepsilon}CQT.\end{aligned} \tag{35}$$

Then, we have

$$\widehat{\mathbb{E}}\left|J_2\right| \leqslant C(\sqrt{\varepsilon}+\varepsilon)\widehat{\mathbb{E}}\int_0^t\left|\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2 \mathrm{d}\tau + C\widehat{\mathbb{E}}\int_0^t \mathscr{G}\left(\tau,\left|\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2\right)\mathrm{d}\tau, + 6\sqrt{\varepsilon}Cq\widehat{\mathbb{E}}\int_0^T \mathscr{G}\left(\tau,\left|\overline{\mathscr{Z}}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(lT)\right|^2\right)\mathrm{d}\tau + 12\sqrt{\varepsilon}CqTQ + 4\sqrt{\varepsilon}CQT. \tag{36}$$

Similarly, it can be deduced that

$$\widehat{\mathbb{E}}\left|J_3\right| \leqslant 2\varepsilon C\widehat{\mathbb{E}}\int_0^t \mathscr{G}\left(\tau,\left|\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2\right)\mathrm{d}\tau + 24\varepsilon CqTQ + 8\varepsilon CQT, + 12\varepsilon Cq\widehat{\mathbb{E}}\int_0^T \mathscr{G}\left(\tau,\left|\overline{\mathscr{Z}}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(lT)\right|^2\right)\mathrm{d}\tau. \tag{37}$$

By Lemma 3 and the inequality of Young, it can be obtained that

$$\begin{aligned}\widehat{\mathbb{E}}\left|J_4\right| &= 2\sqrt{\varepsilon}\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\int_0^s\left\langle \mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau), \sigma(\tau,\mathscr{Z}_\varepsilon(\tau))-\overline{\sigma}\left(\overline{\mathscr{Z}}_\varepsilon(\tau)\right)\right\rangle \mathrm{d}W(\tau)\right|,\\ &\leqslant 2\sqrt{\varepsilon}C\widehat{\mathbb{E}}\left[\int_0^t\left|\left\langle Z_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau), \sigma(\tau,\mathscr{Z}_\varepsilon(\tau))-\overline{\sigma}\left(\overline{\mathscr{Z}}_\varepsilon(\tau)\right)\right\rangle\right|^2 d\langle W,W\rangle_\tau\right]^{1/2},\\ &\leqslant 2\sqrt{\varepsilon}C\widehat{\mathbb{E}}\left[\sup_{0\leqslant u\leqslant t}\left|\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau)\right|\left(\int_0^t\left|\sigma(\tau,\mathscr{Z}_\varepsilon(\tau))-\overline{\sigma}\left(\overline{\mathscr{Z}}_\varepsilon(\tau)\right)\right|^2 d\langle W,W\rangle_\tau\right)^{1/2}\right],\\ &\leqslant \sqrt{\varepsilon}\widehat{\mathbb{E}}\sup_{0\leqslant\tau\leqslant t}\left|\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2 + \sqrt{\varepsilon}C\widehat{\mathbb{E}}\int_0^t\left|\sigma(\tau,\mathscr{Z}_\varepsilon(\tau))-\overline{\sigma}\left(\overline{\mathscr{Z}}_\varepsilon(\tau)\right)\right|^2 d\langle W,W\rangle_\tau.\end{aligned} \tag{38}$$

Similar to (37), we get

$$\widehat{\mathbb{E}}\left|J_4\right| \leqslant \sqrt{\varepsilon}\widehat{\mathbb{E}}\sup_{0\leqslant\tau\leqslant t}\left|\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2 + 2\sqrt{\varepsilon}C\widehat{\mathbb{E}}\int_0^t \mathscr{G}\left(\tau,\left|\mathscr{Z}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2\right)\mathrm{d}\tau, +12\sqrt{\varepsilon}Cq\widehat{\mathbb{E}}\int_0^T \mathscr{G}\left(\tau,\left|\overline{\mathscr{Z}}_\varepsilon(\tau)-\overline{\mathscr{Z}}_\varepsilon(lT)\right|^2\right)\mathrm{d}\tau + 24\sqrt{\varepsilon}CqTQ + 8\sqrt{\varepsilon}CQT. \tag{39}$$

With respect to J_5, Young's inequality and Hypothesis 2 yield

$$\begin{aligned}
\widehat{\mathbb{E}}\left|J_5\right| &= 2\varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\left\langle \mathcal{Z}_\varepsilon(s)-\overline{\mathcal{Z}}_\varepsilon(s), \sum_{j=1}^{\infty} I_j\left(\mathcal{Z}_\varepsilon\left(\left(t_j\right)\right)\right) - \int_0^s \overline{I}\left(\overline{\mathcal{Z}}_\varepsilon(\tau)\right)\mathrm{d}\tau\right\rangle\right|,\\
&\leqslant \sqrt{\varepsilon}\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\mathcal{Z}_\varepsilon(s)-\overline{\mathcal{Z}}_\varepsilon(s)\right|^2 + \varepsilon\sqrt{\varepsilon}\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\sum_{j=1}^{\infty} I_i\left(\mathcal{Z}_\varepsilon\left(t_j\right)\right) - \int_0^s \overline{I}\left(\overline{\mathcal{Z}}_\varepsilon(\tau)\right)\mathrm{d}\tau\right|^2,\\
&\leqslant \sqrt{\varepsilon}\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\mathcal{Z}_\varepsilon(s)-\overline{\mathcal{Z}}_\varepsilon(s)\right|^2 + 2\varepsilon\sqrt{\varepsilon}l(q+1)\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\sum_{j=1}^{l}\left|I_j\left(\mathcal{Z}_\varepsilon\left(t_j\right)\right)\right|^2,\\
&\quad + 2\varepsilon\sqrt{\varepsilon}(q+1)\frac{1}{T^2}lt\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\sum_{j=1}^{l}\int_0^s\left|I_j\left(\overline{\mathcal{Z}}_\varepsilon(\tau)\right)\right|^2\mathrm{d}\tau,\\
&\leqslant \sqrt{\varepsilon}\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\mathcal{Z}_\varepsilon(s)-\overline{\mathcal{Z}}_\varepsilon(s)\right|^2 + 2\varepsilon\sqrt{\varepsilon}(q+1)l^2K_1 + 2\varepsilon\sqrt{\varepsilon}(q+1)l^2(q+1)^2K_1,\\
&\leqslant \sqrt{\varepsilon}\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\mathcal{Z}_\varepsilon(s)-\overline{\mathcal{Z}}_\varepsilon(s)\right|^2 + 2\varepsilon\sqrt{\varepsilon}(q+1)C'.
\end{aligned} \tag{40}$$

Taking expectation from (28) and combining with (32)–(40), we obtain

$$\begin{aligned}
&\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\mathcal{Z}_\varepsilon(s)-\overline{\mathcal{Z}}_\varepsilon(s)\right|^2,\\
&\leqslant \sqrt{\varepsilon}\left(\sqrt{\varepsilon}(1+\varepsilon)+C(1+\sqrt{\varepsilon})\right)\widehat{\mathbb{E}}\int_0^t\left|Z_\varepsilon(\tau)-\overline{Z}_\varepsilon(\tau)\right|^2\mathrm{d}\tau,\\
&\quad + (1+C+2C(\sqrt{\varepsilon}+\varepsilon))\widehat{\mathbb{E}}\int_0^t \mathcal{G}\left(\tau,\left|\mathcal{Z}_\varepsilon(\tau)-\overline{\mathcal{Z}}_\varepsilon(\tau)\right|^2\right)\mathrm{d}\tau,\\
&\quad + (\sqrt{\varepsilon}+\varepsilon)(6qC+12Cq)\widehat{\mathbb{E}}\int_0^T \mathcal{G}\left(\tau,\left|\overline{\mathcal{Z}}_\varepsilon(\tau)-\overline{\mathcal{Z}}_\varepsilon(lT)\right|^2\right)\mathrm{d}\tau,\\
&\quad + (\sqrt{\varepsilon}+\varepsilon)(12CqQT+24CqQT+4CQT+8CQT),\\
&\quad + 2\varepsilon\sqrt{\varepsilon}(q+1)C' + 2\sqrt{\varepsilon}\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\mathcal{Z}_\varepsilon(s)-\overline{\mathcal{Z}}_\varepsilon(s)\right|^2.
\end{aligned} \tag{41}$$

Thus, Jensen inequality yields

$$
\begin{aligned}
&\widehat{\mathbb{E}} \sup_{0\leqslant s\leqslant t} \left|\mathscr{Z}_\varepsilon(s) - \overline{\mathscr{Z}}_\varepsilon(s)\right|^2 \\
&\leqslant \frac{\sqrt{\varepsilon}\left(\sqrt{\varepsilon}\,(1+\varepsilon) + C\left(1+\sqrt{\varepsilon}\right)\right)}{1-2\sqrt{\varepsilon}} \int_0^t \widehat{\mathbb{E}} \sup_{0\leqslant\tau\leqslant s} \left|\mathscr{Z}_\varepsilon(\tau) - \overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2 \mathrm{d}s \\
&\quad + \frac{\left(1 + C + 2C\left(\sqrt{\varepsilon} + \varepsilon\right)\right)}{1-2\sqrt{\varepsilon}} \int_0^t \mathscr{G}\left(\tau, \widehat{\mathbb{E}} \sup_{0\leqslant\tau\leqslant s} \left|\mathscr{Z}_\varepsilon(\tau) - \overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2\right) \mathrm{d}s \\
&\quad + \frac{\left(\sqrt{\varepsilon} + \varepsilon\right) C_1}{1-2\sqrt{\varepsilon}} \int_0^T \mathscr{G}\left(\tau, \widehat{\mathbb{E}} \sup_{0\leqslant\tau\leqslant s} \left|\overline{\mathscr{Z}}_\varepsilon(\tau) - \overline{\mathscr{Z}}_\varepsilon(lT)\right|^2\right) \mathrm{d}s \\
&\quad + \frac{\left(\sqrt{\varepsilon} + \varepsilon\right) C_2 + 2\varepsilon\sqrt{\varepsilon}\,(q+1))C'}{1-2\sqrt{\varepsilon}},
\end{aligned}
\tag{42}
$$

where $C_1 = 6qC + 12Cq$ and $C_2 = 12CqQT + 24CqQT + 4CQT + 8CQT$.

Taking limit as $\varepsilon \longrightarrow 0$ in Equation (42), we obtain

$$
\lim_{\varepsilon\longrightarrow 0} \widehat{\mathbb{E}} \sup_{0\leqslant s\leqslant t} \left|\mathscr{Z}_\varepsilon(s) - \overline{\mathscr{Z}}_\varepsilon(s)\right|^2 \leqslant (1+C) \int_0^t \mathscr{G}\left(\tau, \lim_{\varepsilon\longrightarrow 0} \widehat{\mathbb{E}} \sup_{0\leqslant\tau\leqslant s} \left|\mathscr{Z}_\varepsilon(\tau) - \overline{\mathscr{Z}}_\varepsilon(\tau)\right|^2\right) \mathrm{d}s, \tag{43}
$$

which with the help of Hypothesis (1c) gives

$$
\lim_{\varepsilon\longrightarrow 0} \widehat{\mathbb{E}} \sup_{0\leqslant s\leqslant t} \left|\mathscr{Z}_\varepsilon(s) - \overline{\mathscr{Z}}_\varepsilon(s)\right|^2 = 0. \tag{44}
$$

Our proof is therefore completed. □

Theorem 3. *Suppose Hypotheses 1–4 hold, then, for any two positive constants θ_1 and θ_2, there is a number $\varepsilon_1 \in (0, \varepsilon_0]$ so that*

$$
\widehat{\mathbb{E}} \left|\mathscr{Z}_\varepsilon(t) - \overline{\mathscr{Z}}_\varepsilon(t)\right|^2 \leq \theta_1\sqrt{\varepsilon}, \forall t \in \left[0, \theta_2 \varepsilon^{-\frac{1}{2}}\left(1-\sqrt{\varepsilon}\right)\right], \tag{45}
$$

where $\varepsilon \in (0, \varepsilon_1]$.

Proof. Noticing that for every $t \in [(l-1)T, lT]$, we have

$$
\begin{aligned}
\overline{\mathscr{Z}}_\varepsilon(t) - \overline{\mathscr{Z}}_\varepsilon(lT) &= \varepsilon \int_{lT}^t \overline{f}\left(\overline{\mathscr{Z}}_\varepsilon(\tau)\right) d\tau + \sqrt{\varepsilon} \int_{lT}^t \overline{h}\left(\overline{\mathscr{Z}}_\varepsilon(\tau)\right) d\langle W, W\rangle_\tau, \\
&\quad + \sqrt{\varepsilon} \int_{lT}^t \overline{\sigma}\left(\overline{\mathscr{Z}}_\varepsilon(\tau)\right) \mathrm{d}W(\tau) - \varepsilon \int_{lT}^t d\overline{K}(\tau).
\end{aligned}
\tag{46}
$$

By G-Itô's formula, we imply

$$\begin{aligned}
&\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\overline{\mathscr{X}}_{\varepsilon}(s)-\overline{\mathscr{X}}_{\varepsilon}(lT)\right|^{2},\\
&=2\varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\int_{lT}^{s}\langle\overline{\mathscr{X}}_{\varepsilon}(\tau)-\overline{\mathscr{X}}_{\varepsilon}(lT),\overline{f}\left(\overline{\mathscr{X}}_{\varepsilon}(\tau)\right)\rangle\mathrm{d}\tau\right|,\\
&\quad+2\sqrt{\varepsilon}\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\int_{lT}^{s}\langle\overline{\mathscr{X}}_{\varepsilon}(\tau)-\overline{\mathscr{X}}_{\varepsilon}(lT),\overline{h}\left(\overline{\mathscr{X}}_{\varepsilon}(\tau)\right)\rangle d\langle W,W\rangle_{\tau}\right|,\\
&\quad+\varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\int_{lT}^{s}\left[\overline{\sigma}\left(\overline{\mathscr{X}}_{\varepsilon}(\tau)\right)\right]^{2}d\langle W,W\rangle_{\tau}\right|,\\
&\quad+2\sqrt{\varepsilon}\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\int_{lT}^{s}\langle\overline{\mathscr{X}}_{\varepsilon}(\tau)-\overline{\mathscr{X}}_{\varepsilon}(lT),\overline{\sigma}\left(\overline{\mathscr{X}}_{\varepsilon}(\tau)\right)\rangle\mathrm{d}W(\tau)\right|,\\
&\quad-2\varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\int_{lT}^{s}\langle\overline{\mathscr{X}}_{\varepsilon}(\tau)-\overline{\mathscr{X}}_{\varepsilon}(lT),\mathrm{d}\overline{K}(\tau)\rangle\right|.
\end{aligned}\tag{47}$$

Thanks to Young's inequality, Lemmas 2 and 3, and Hypothesis 4, we have

$$\begin{aligned}
&\widehat{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\overline{\mathscr{X}}_{\varepsilon}(s)-\overline{\mathscr{X}}_{\varepsilon}(lT)\right|^{2},\\
&\leqslant\varepsilon\widehat{\mathbb{E}}\int_{lT}^{t}\left|\overline{f}\left(\overline{\mathscr{X}}_{\varepsilon}(\tau)\right)\right|^{2}\mathrm{d}\tau+\varepsilon\widehat{\mathbb{E}}\int_{lT}^{t}\left|\overline{\mathscr{X}}_{\varepsilon}(\tau)-\overline{\mathscr{X}}_{\varepsilon}(lT)\right|^{2}\mathrm{d}\tau,\\
&\quad+\sqrt{\varepsilon}C\widehat{\mathbb{E}}\int_{lT}^{t}\left|\overline{h}\left(\overline{\mathscr{X}}_{\varepsilon}(\tau)\right)\right|^{2}\mathrm{d}\tau+\sqrt{\varepsilon}C\widehat{\mathbb{E}}\int_{lT}^{t}\left|\overline{\mathscr{X}}_{\varepsilon}(\tau)-\overline{\mathscr{X}}_{\varepsilon}(lT)\right|^{2}\mathrm{d}\tau,\\
&\quad+C\varepsilon\widehat{\mathbb{E}}\int_{lT}^{t}\left|\overline{\sigma}\left(\overline{\mathscr{X}}_{\varepsilon}(\tau)\right)\right|^{2}\mathrm{d}\tau+\sqrt{\varepsilon}C\widehat{\mathbb{E}}\int_{lT}^{t}\left|\overline{\mathscr{X}}_{\varepsilon}(\tau)-\overline{\mathscr{X}}_{\varepsilon}(lT)\right|^{2},\\
&\quad+C\sqrt{\varepsilon}\widehat{\mathbb{E}}\sup_{0\leqslant\tau\leqslant t}\left|\overline{\sigma}\left(\overline{\mathscr{X}}_{\varepsilon}(\tau)\right)\right|^{2}-2\varepsilon\widehat{\mathbb{E}}\left[\sup_{0\leqslant s\leqslant t}\left|\overline{\mathscr{X}}_{\varepsilon}(\tau)-\overline{\mathscr{X}}_{\varepsilon}(lT)\|\overline{K}\right|_{T}^{0}\right],\\
&\leqslant\varepsilon(q+1)QT+\sqrt{\varepsilon}CQT+\varepsilon CQT+\sqrt{\varepsilon}CQ,\\
&\quad+(\varepsilon+\sqrt{\varepsilon})C\int_{lT}^{t}\widehat{\mathbb{E}}\sup_{0\leqslant\tau\leqslant s}|\overline{\mathscr{X}}_{\varepsilon}(\tau)-\overline{\mathscr{X}}_{\varepsilon}(lT)|^{2}ds,\\
&\quad-\varepsilon\widehat{\mathbb{E}}\sup_{0\leqslant\tau\leqslant t}\left|\overline{\mathscr{X}}_{\varepsilon}(\tau)-\overline{\mathscr{X}}_{\varepsilon}(lT)\right|^{2}-\varepsilon\widehat{\mathbb{E}}\left(|\overline{K}|\|_{T}^{0}\right)^{2},\\
&\leqslant\frac{\varepsilon(q+1)QT+\sqrt{\varepsilon}CQT+\varepsilon CQT+\sqrt{\varepsilon}CQ}{1+\varepsilon},\\
&\quad+\frac{(\varepsilon+\sqrt{\varepsilon})C}{1+\varepsilon}\int_{lT}^{t}\widehat{\mathbb{E}}\sup_{0\leqslant\tau\leqslant s}\left|\overline{\mathscr{X}}_{\varepsilon}(\tau)-\overline{\mathscr{X}}_{\varepsilon}(lT)\right|^{2}\mathrm{d}s,\\
&\quad-\frac{\varepsilon}{1+\varepsilon}\widehat{\mathbb{E}}\left(|\overline{K}|_{T}^{0}\right)^{2}.
\end{aligned}\tag{48}$$

Therefore, Gronwall's inequality produces

$$\widehat{\mathbb{E}} \sup_{0\leqslant s\leqslant t} \left|\overline{\mathcal{Z}}_\varepsilon(s) - \overline{\mathcal{Z}}_\varepsilon(lT)\right|^2, \leqslant \frac{\varepsilon(q+1)QT + \sqrt{\varepsilon}CQT + \varepsilon CQT + \sqrt{\varepsilon}CQ}{\varepsilon+1}, \times e^{\frac{C(\varepsilon+\sqrt{\varepsilon})}{\varepsilon+1}t} = A. \quad (49)$$

Because of concavity of $\mathcal{G}(t,\wp)$, we find $\varrho(t)\geqslant 0$, $\nu(t)\geqslant 0$ so that

$$\mathcal{G}(t,\wp)\leqslant\varrho(t) + \nu(t)\wp, \int_0^T \varrho(t)\mathrm{d}t < \infty, \int_0^T \nu(t)\mathrm{d}t < \infty. \quad (50)$$

Inserting equations (49) and (50) into (42), we obtain

$$\begin{aligned}
&\widehat{\mathbb{E}} \sup_{0\leq s\leq t} \left|\mathcal{Z}_\varepsilon(s) - \overline{\mathcal{Z}}_\varepsilon(s)\right|^2, \\
&\leq \frac{\sqrt{\varepsilon}\left[C(\sqrt{\varepsilon}+1)(1+2\sup_{0\leq t\leq T}\nu(t)) + \sqrt{\varepsilon}(1+\varepsilon) + (1+C)(\sqrt{\varepsilon})^{-1}\sup_{0\leq t\leq T}\nu(t)\right]}{1-2\sqrt{\varepsilon}}, \\
&\times \int_0^t \widehat{\mathbb{E}} \sup_{0\leq\tau\leq s} \left|\mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau)\right|^2 \mathrm{d}s, \\
&+ \frac{\sqrt{\varepsilon}[(1+\sqrt{\varepsilon})(C_2 + 2Ct\sup_{0\leq t\leq T}\varrho(t) + C_1 t(\sup_{0\leq t\leq T}\varrho(t) + A\sup_{0\leq t\leq T}\varrho(t)))]}{1-2\sqrt{\varepsilon}}, \\
&+ \frac{\sqrt{\varepsilon}\left[2\varepsilon C'(q+1) + (1+C)t(\sqrt{\varepsilon})^{-1}\sup_{0\leq t\leq T}\varrho(t)\right]}{1-2\sqrt{\varepsilon}}, \\
&\leq \frac{\sqrt{\varepsilon}}{1-2\sqrt{\varepsilon}}\mathcal{A}_1 \int_0^t \widehat{\mathbb{E}} \sup_{0\leq\tau\leq s} \left|\mathcal{Z}_\varepsilon(\tau) - \overline{\mathcal{Z}}_\varepsilon(\tau)\right|^2 \mathrm{d}s + \frac{\sqrt{\varepsilon}}{1-2\sqrt{\varepsilon}}(\mathcal{A}_2 + \mathcal{A}_3).
\end{aligned} \quad (51)$$

By employing Gronwall's inequality, we find

$$\widehat{\mathbb{E}} \sup_{0\leqslant s\leqslant t} \left|\mathcal{Z}_\varepsilon(s) - \overline{\mathcal{Z}}_\varepsilon(s)\right|^2 \leqslant \frac{\sqrt{\varepsilon}}{1-2\sqrt{\varepsilon}}(\mathcal{A}_2 + \mathcal{A}_3)e^{\frac{\sqrt{\varepsilon}}{1-2\sqrt{\varepsilon}}\mathcal{A}_1 t}. \quad (52)$$

We choose $\theta_2 > 0$ such that for all $t \in [0, \theta_2\varepsilon^{-1/2}(1-\sqrt{\varepsilon})]\subseteq[0,\infty)$, and suppose $\theta_1 = \mathcal{A}_2 + \mathcal{A}_3/(1-2\sqrt{\varepsilon})e^{\mathcal{A}_1\theta_2}$, we may take $\varepsilon_1 \in (0,\varepsilon_0]$ in a sense that for any $\varepsilon \in (0,\varepsilon_1]$ and $0\leqslant t\leqslant\theta_2\varepsilon^{-1/2}(1-\sqrt{\varepsilon})$,

$$\widehat{\mathbb{E}} \sup_{0\leqslant t\leqslant\theta_2} \varepsilon^{-1/2}(1-\sqrt{\varepsilon})\left|Z_\varepsilon(t) - \overline{Z}_\varepsilon(t)\right|^2 \leqslant \theta_1\sqrt{\varepsilon}. \quad (53)$$

Hence, it is proved.

By Theorem 3, Theorem 4 gives the convergence in capacity between $\mathcal{Z}_\varepsilon(t)$ and $\overline{\mathcal{Z}}_\varepsilon(t)$. □

Theorem 4. *Suppose Hypotheses 1–4 are true, then, for arbitrarily small* $\delta > 0$, *there is* $\varepsilon_1 \in (0,\varepsilon_0]$ *so that for* $\varepsilon \in (0,\varepsilon_1]$

$$\lim_{\varepsilon\longrightarrow 0} \widehat{\mathrm{C}}\left(\sup_{0\leq t\leq\theta_2\varepsilon^{-\frac{1}{2}}(1-\sqrt{\varepsilon})} \left|\mathcal{Z}_\varepsilon(t) - \overline{\mathcal{Z}}_\varepsilon(t)\right| > \delta\right) = 0, \quad (54)$$

where Theorem 3 defines η_2.

Proof. For any $\delta > 0$, Lemma 1 and Theorem 3 give

$$\begin{aligned}
&\widehat{c}\left(\sup_t \in \left[0, \theta_2\varepsilon^{-1/2}(1-\sqrt{\varepsilon})\right]\left|\mathcal{Z}_\varepsilon(t) - \overline{\mathcal{Z}}_\varepsilon(t)\right| > \delta\right), \\
&\leqslant \frac{1}{\delta^2}\widehat{\mathbb{E}}\left(\sup_t \in \left[0, \theta_2\varepsilon^{-1/2}(1-\sqrt{\varepsilon})\right]\left|\mathcal{Z}_\varepsilon(t) - \overline{\mathcal{Z}}_\varepsilon(t)\right|^2\right) \\
&\leqslant \frac{\theta_1\sqrt{\varepsilon}}{\delta^2},
\end{aligned} \quad (55)$$

which tends to 0 as ε goes to 0. □

Remark 1. If $I_j(.) \equiv 0 (j \in \mathbb{N})$ in Equation (13), then the periodic averaging method for MSDEGs is recovered. Moreover, compared to the published work on the averaging principles for MSDEs [9–12], we started to present the periodic averaging principle to MSDEs with or without impulses.

4. Examples

Here, a couple of examples are brought to justify the theoretical method for MISDEGs.

Example 1. We refer to these one-dimensional linear MISDEGs as follows:

$$\begin{cases} \mathrm{d}\mathcal{Z}_\varepsilon(t) + \partial\varphi(\mathcal{Z}_\varepsilon(t)) \ni 2\varepsilon\cos^2(t)\mathcal{Z}_\varepsilon(t)\mathrm{d}t + 2\sqrt{\varepsilon}\,\sin(t)\mathcal{Z}_\varepsilon(t)d\langle W,W\rangle_t, \\ +\sqrt{\varepsilon}\,X_\varepsilon(t)\mathrm{d}W(t), t \neq t_j, \\ \Delta\mathcal{Z}_\varepsilon(t_j) = \varepsilon j\mathcal{Z}_\varepsilon(t_j^-), t = t_j, j \in \mathbb{N}, \\ \mathcal{Z}_\varepsilon(0) = \mathcal{Z}_0 \in \overline{\mathrm{Dom}(\varphi)}, \end{cases} \tag{56}$$

where $(W(t))_{t\geqslant 0}$ represents one-dimensional G-Brownian motion $W_1 \sim N(\{0\}\times[\underline{\sigma}^2,\overline{\sigma}^2])$. Let

$$f(t,\mathcal{Z}_\varepsilon) = 2\cos^2(t)\mathcal{Z}_\varepsilon, h(t,\mathcal{Z}_\varepsilon) = 2\sin(t)\mathcal{Z}_\varepsilon, \sigma(t,\mathcal{Z}_\varepsilon) = \mathcal{Z}_\varepsilon, I_j(\mathcal{Z}_\varepsilon) = j\mathcal{Z}_\varepsilon. \tag{57}$$

We define

$$\overline{f}(\overline{\mathcal{Z}}_\varepsilon) = \frac{1}{\pi}\int_0^\pi f(t,\overline{\mathcal{Z}}_\varepsilon)\mathrm{d}t = \overline{\mathcal{Z}}_\varepsilon, \overline{h}(\overline{\mathcal{Z}}_\varepsilon) = \frac{1}{\pi}\int_0^\pi h(t,\overline{\mathcal{Z}}_\varepsilon)\mathrm{d}t = \frac{4}{\pi}\overline{\mathcal{Z}}_\varepsilon. \tag{58}$$

And

$$\overline{\sigma}(\overline{\mathcal{Z}}_\varepsilon) = \frac{1}{\pi}\int_0^\pi \sigma(t,\overline{\mathcal{Z}}_\varepsilon)\mathrm{d}t = \overline{\mathcal{Z}}_\varepsilon, \overline{I}(\overline{\mathcal{Z}}_\varepsilon) = \frac{1}{\pi}\sum_{j=1}^{l} I_j(\overline{\mathcal{Z}}_\varepsilon) = \frac{l(1+l)}{2\pi}\overline{\mathcal{Z}}_\varepsilon. \tag{59}$$

The associated simplified equation is

$$\begin{cases} d\overline{\mathcal{Z}}_\varepsilon(t) + \overline{\partial\varphi}(\overline{\mathcal{Z}}_\varepsilon(t)) \ni \varepsilon\left(1 + \frac{l(l+1)}{2\pi}\right)\overline{\mathcal{Z}}_\varepsilon(t)\mathrm{d}t + \frac{4}{\pi}\sqrt{\varepsilon}\,\overline{\mathcal{Z}}_\varepsilon(t)d\langle W,W\rangle_t, \\ +\sqrt{\varepsilon}\,\overline{\mathcal{Z}}_\varepsilon(t)\mathrm{d}W(t), \\ \overline{\mathcal{Z}}_\varepsilon(0) = \overline{\mathcal{Z}}_0 \in \overline{\mathrm{Dom}(\varphi)}. \end{cases} \tag{60}$$

It seems that Equation (60) is linear MSDEGs with solution

$$\overline{\mathscr{Z}}_\varepsilon(t) = e^{\varepsilon\left(1+\frac{l(l+1)}{2\pi}\right)t+\left(\frac{4}{\pi}\sqrt{\varepsilon}-\frac{\varepsilon^2}{2}\right)\langle W,W\rangle_t+\sqrt{\varepsilon}W(t)}, \times\left[\overline{\mathscr{Z}}_0 - \varepsilon\int_0^t e^{-\varepsilon\left(1+\frac{l(l+1)}{2\pi}\right)s-\left(\frac{4}{\pi}\sqrt{\varepsilon}-\frac{\varepsilon^2}{2}\right)\langle W,W\rangle_s-\sqrt{\varepsilon}W(s)}\mathrm{d}\overline{K}(s)\right]. \tag{61}$$

It is not difficult to check that Hypotheses 1–4 are satisfied. Therefore, Theorems 2–4 are applicable, and the solutions of (56) and (60) are convergent in L^2-sense and capacity.

Example 2. We refer to these one-dimensional nonlinear MISDEGs as follows:

$$\begin{cases} d\mathscr{Z}_\varepsilon(t) + \partial\varphi(\mathscr{Z}_\varepsilon(t)) \ni \varepsilon\rho(t)\sin^2(t)\kappa(\mathscr{Z}_\varepsilon(t))\mathrm{d}t + \sqrt{\varepsilon}\,\rho(t)\cos^2(t)\kappa(\mathscr{Z}_\varepsilon(t))d\langle W,W\rangle_t, \\ +\sqrt{2\varepsilon}\,\rho(t)\sin(t)\cos(t)\kappa(\mathscr{Z}_\varepsilon(t))\mathrm{d}W(t), t \neq t_j, \\ \Delta\mathscr{Z}_\varepsilon(t_j) = \varepsilon j^2\kappa(\mathscr{Z}_\varepsilon(t_j^-)), t = t_j, j \in \mathbb{N}, \\ \mathscr{Z}_\varepsilon(0) = \mathscr{Z}_0 \in \overline{\mathrm{Dom}(\varphi)}, \end{cases} \tag{62}$$

where $(W(t))_{t\geqslant 0}$ represents one-dimensional G-Brownian motion $W_1 \sim N(\{0\}\times[\underline{\sigma}^2,\overline{\sigma}^2])$, $\rho^2(t) > 0$ is a locally integrable function, and $\kappa(.)$ is a continuous function satisfying

$$|\kappa(u) - \kappa(v)|^2 \leqslant \Gamma\left(|u-v|^2\right), u, v \in \mathbb{R}, \tag{63}$$

where $\Gamma(.)$ is a concave function defined as

$$\Gamma(z) = \begin{cases} 0, & z = 0, \\ cz\log\left(\wp^{-1}\right), & 0\leqslant z\leqslant\beta, \\ c\beta\log\left(\beta^{-1}\right), & z > \beta. \end{cases} \tag{64}$$

Or

$$\Gamma(z) = \begin{cases} 0, & z = 0, \\ cz\left(\log\left(z^{-1}\right)\right)^{\frac{1}{3}}\log\log\left(z^{-1}\right), & 0\leqslant z\leqslant\beta, \\ c\beta\left(\log\left(\beta^{-1}\right)\right)^{\frac{1}{3}}\log\log\left(\beta^{-1}\right), & z > \beta. \end{cases} \tag{65}$$

With $c > 0$, $0 < \beta < 1$ is sufficiently small, $\Gamma(0) = 0$, and $\int_{0+}^{\infty} 1/\Gamma(z)\mathrm{d}z = +\infty$. We define

$$\begin{aligned} \overline{f}\left(\overline{\mathscr{Z}}_\varepsilon\right) &= \frac{1}{\pi}\int_0^\pi \rho(t)\sin^2(t)\kappa\left(\overline{\mathscr{Z}}_\varepsilon\right)\mathrm{d}t = \frac{\rho(t)}{2}\kappa\left(\overline{\mathscr{Z}}_\varepsilon\right), \\ \overline{h}\left(\overline{\mathscr{Z}}_\varepsilon\right) &= \frac{1}{\pi}\int_0^\pi \rho(t)\cos^2(t)\kappa\left(\overline{\mathscr{Z}}_\varepsilon\right)\mathrm{d}t = \frac{\rho(t)}{2}\kappa\left(\overline{\mathscr{Z}}_\varepsilon\right), \\ \overline{\sigma}\left(\overline{\mathscr{Z}}_\varepsilon\right) &= \frac{\sqrt{2}}{\pi}\int_0^\pi \rho(t)\sin(t)\cos(t)\kappa\left(\overline{\mathscr{Z}}_\varepsilon(t)\right)\mathrm{d}t = \frac{-\sqrt{2}\rho(t)}{\pi}\kappa\left(\overline{\mathscr{Z}}_\varepsilon\right). \end{aligned} \tag{66}$$

We have

$$\begin{aligned} \overline{I}\left(\overline{\mathscr{Z}}_\varepsilon\right) &= \pi^{-1}\sum_{j=1}^{l} j^2\kappa\left(\overline{\mathscr{Z}}_\varepsilon\right), \\ &= \frac{l(l+1)(2l+1)}{6\pi}\kappa\left(\overline{\mathscr{Z}}_\varepsilon\right). \end{aligned} \tag{67}$$

Therefore, the associated simplified equation is

$$\begin{cases} d\overline{\mathscr{X}}_{\varepsilon}(t) + \overline{\partial\varphi}\left(\overline{\mathscr{X}}_{\varepsilon}(t)\right) \ni \dfrac{\varepsilon}{2}\left(\rho(t) + \dfrac{l(1+l)(2l+1)}{3\pi}\right)\kappa\left(\overline{\mathscr{X}}_{\varepsilon}(t)\right)\mathrm{d}t + \dfrac{\sqrt{\varepsilon}}{2}\rho(t)\kappa\left(\overline{\mathscr{X}}_{\varepsilon}(t)\right)d\langle W, W\rangle_t, \\ -\dfrac{\sqrt{2\varepsilon}}{\pi}\rho(t)\kappa\left(\overline{\mathscr{X}}_{\varepsilon}(t)\right)\mathrm{d}W(t), \\ \overline{\mathscr{X}}_{\varepsilon}(0) = \overline{\mathscr{X}}_0 \in \overline{\mathrm{Dom}(\varphi)}. \end{cases} \tag{68}$$

It is obvious that Equation (68) is not linear MSDEGs and that their analytical solution cannot be obtained. Note that

$$\begin{aligned} &\left|f(t,z_1) - f(t,z_2)\right|^2 + \left|h(t,z_1) - h(t,z_2)\right|^2 + \left|\sigma(t,z_1) - \sigma(t,z_2)\right|^2, \\ &= (\sin t)^4 + (\cos t)^4 + 2(\sin t)^2(\cos t)^2\rho^2(t)\left|\kappa(z_1) - \kappa(z_2)\right|^2, \\ &\leqslant \rho^2(t)\Gamma\left(\left|z_1 - z_2\right|^2\right). \end{aligned} \tag{69}$$

Therefore, Hypothesis 1 is satisfied, and it is easy to check that Hypotheses 2–4 are also satisfied. Hence, Theorems 2–4 hold, and the solutions of Equations (62) and (68) are convergent in the mean square sense and capacity.

5. Conclusion

In this paper, by focusing on MSDEs with impulses and G-Brownian motion, we establish a periodic averaging principle under non-Lipschitz conditions. We proved that the solutions to simplified MSDEGs without impulses converge to those of the initial MISDEGs in mean square sense and also in capacity. For future research, a two-time scale SDE driven by G-Brownian motion under non-Lipschitz conditions is interesting and important. It deserves to be further investigated in the future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All authors completed the paper together. All authors read and approved the final manuscript.

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