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Research Article

Power Option Pricing Based on Time-Fractional Model and Triangular Interval Type-2 Fuzzy Numbers

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The problem of generalizing the power option-pricing model to incorporate more empirical features becomes an urgent and necessary event. A new power option pricing method is designed for the financial market uncertainty that simultaneously involves randomness and fuzziness. The randomness in market uncertainty is modeled by a time-fractional diffusion model, which describes trend memory in underlying asset prices. The fuzziness in market uncertainty is characterized by a triangular interval type-2 fuzzy numbers, which better captures the fuzziness of underlying asset prices. Considering the decision-maker's subjective judgment, we show the price mean value with the possibility-necessity weight and pessimistic-optimistic index under the type-2 fuzzy environment. We develop the power option pricing model with the time-fractional diffusion model and the triangular interval type-2 fuzzy numbers. Furthermore, the analytic solutions of pricing call power option and put power option are obtained and verified by the variational iterative reconstruction method. Our study shows that power option pricing, which adopts the time-fractional model and the triangular interval type-2 fuzzy numbers, can better capture the trend memory and double fuzziness of the real market. In addition, a numerical example is provided to illustrate that the power option means the value is decreasing with respect to the pessimistic-optimistic index and is fluctuating with respect to the possibility-necessity weight index.

1. Introduction

In the wake of the maturity of financial markets and the diversification of products, the leverage characteristic of the power option has attracted a lot of attention from investors. An appropriate option pricing model is one of the primary conditions for the realization of future option value. The Black-Scholes (BS) model is proposed in the process framework of continuous time by Black and Scholes [1]. They study the option formula under the assumption that the prices of underlying assets follow the geometric Brownian motion. Jiang [2] reduces the BS equation to a heat equation and presents another view of obtaining the analytic solution of the option price. Besides, Jodar et al. [3] and Bohner and Zheng [4] solve the BS equation through different methods. With the increasing application of fractional order differential equations, BS equations have been explored in the form of fractional order. Wyss [5] derives the time fractional order BS equation and calculates the price of European call options. By combining Itô lemma with fractional Taylor series, see Jumarie [6], two fractional-order BS equations are derived from the stochastic differential equation with fractional order Brownian motion in Jumarie [7]. The fraction in the equations describes the memory effect of the underlying assets on noise. Considering the time-fractional order differential equation satisfied by the underlying asset, Li et al. [8] construct a fractional stochastic differential equation with classical Brownian motion and apply it to option pricing. Farhadi et al. [9] adopt a new time-fractional-order BS equation to calculate the price of European call options based on Li et al. [8]. The approximate solution is obtained by the homotopy perturbation method in Kumar et al. [10]; the variation iteration method in Ahmad et al. [11]; and the finite difference method in Song and Wang [12]. Under the fractional jump-diffusion model, Zhao et al. [13] study the N-fold compound option pricing with technical risk. The fractional models are more consistent with the real situation than the financial markets. Since some of the properties of the fractional derivative are very complicated in comparison to the classical ones, it is very useful to find the solutions of some generic equations (1).

In the option pricing formula, the parameters put into the formula are generally regarded as the ones estimated from accurate real-valued data. However, the volatility of the underlying asset price is affected by market uncertainty. Besides, there are errors between the estimated values of the parameters and the actual ones. Since option prices are used to trade future strike prices, it is not appropriate to choose the current stock price as a future one. Therefore, there are many fuzzy factors affecting the option prices. It is hard for us to obtain these factors exactly. Fortunately, fuzzy set theory see Zedeh [14] and fuzzy random variable theory see Puri and Ralescu [15] are effective methods to deal with such problems in option pricing. Moreover, Wu [16] prices European options with the fuzzy pattern of the B-S formula. Based on the fractional Brownian motion, Zhao et al. [17] study the N-Fold compound option fuzzy pricing. In Gaussian type-2 fuzzy environments, Pramanik et al. [18] present two mathematical models representing imprecise capacitated fixed-charge transportation problems for a two-stage supply chain network. Moreover, Bera et al. [19] present two novel MCDM techniques in interval type-2 fuzzy environments capable of handling uncertain subjective and objective factors simultaneously for the selection of efficient suppliers in real-life applications. Based on T2 linguistic fuzzy logic, Dey and Jana [20] study the evaluation of the convincing ability through presentation skills of preservice management wizards. Ejegwa and Agbetayo [21] introduce a similarity-distance decision-making technique via intuitionistic fuzzy pairs. In Atanassov [22], a new topological operator over intuitionistic fuzzy sets is defined, some of its properties are studied, and some open problems related to it are given.

The related research on option pricing is divided into several categories. One of the pricing models employs the fractional-order BS formula to calculate the option prices. The differential equation with fractional order in space position describes the long memory of stock prices. Besides the long memory, the trend memory in the pricing process is also an important factor influencing option prices. In addition, in a complicated financial system, it is difficult for participants to estimate the parameters of the pricing formula accurately due to market fluctuations, inadequate information, and human errors. For example, stock prices fluctuate around \$7, which is difficult to describe from the perspective of probability theory and has obvious fuzzy characteristics. Therefore, the uncertainty of the financial market cannot be fully characterized just by randomness. It contains at least another aspect, fuzziness, which is also an important factor for option prices. Moreover, the satisfaction levels of investors are fuzzy rather than accurate in the actual market. In other words, the confidence level of a price is also a fuzzy number. For example, for a given stock price of 17, investors usually give a statement of satisfaction level, general satisfaction, or dissatisfaction, but not a 1-level satisfaction, a 0.6-level satisfaction, or a 0.3-level satisfaction.

In this paper, the trend memory of stock prices, two fuzzy characteristics of the financial market, and the subjective judgments of decision makers are considered in power option pricing. To reflect the trend memory of stock prices, this paper first adopts the partial differential equation with time-fractional ordering to calculate the option prices. As far as we know, this is the first time we have presented the pricing formulas of the call power option and the put power option in the time-fractional environment. To better capture price fuzziness and satisfaction fuzziness in the financial market, we introduce a triangular interval type-2 fuzzy set into the time fractional pricing model and obtain fuzzy pricing formulas for power options. Considering the decision-makers' subjective judgment, the price mean value with the possibility-necessity weight and pessimistic-optimistic index are given. Compared with the traditional power option pricing models, our innovative pricing models make the application of power options more consistent with the actual financial markets by taking into account the trend memory of stock prices, the diversification of market uncertainty, and subjective factors. Particularly, the fuzzy pricing model can be reduced to the traditional option pricing models under certain degenerative conditions.

The main contribution of this paper is that we first develop the power option pricing model with the time-fractional diffusion process and the triangular interval type-2 fuzzy numbers. Furthermore, the analytic solutions of the pricing call power option and put power option are achieved and verified by the variational iterative reconstruction method. Moreover, the valuation and properties of the formulas are analyzed under some reasonable assumptions. A numerical example is provided to illustrate that the power option's mean value is affected by the decision makers' subjective judgment. The pricing formula makes the application of the power option more consistent with the real markets than before.

The construction of this paper is as follows: in Section 2, using the reconstruction variational iteration method, we derive a closed-form solution for the power option when the underlying asset price process follows a time-fractional order BS model. This pricing formula broadens the reach of the power option. In Section 3, by introducing the interval type-2 fuzzy set and the decision maker's subjective judgment, we further present the fuzzy pricing formula and the price mean value formula of the power option.

2. Pricing Formula Based on the Time-Fractional Model

A European call power option is a call option with an expiration date T, exercise price K and terminal value $V(T,S)=\max\left\{S_T^{\beta}-K,0\right\}$. Here, the parameter β is the power index of asset price S_T. Under the risk-neutral measure, the value of the European call power option at the time t is

$$V(t,S)=e^{-r(T-t)}E_t^{Q}\left(\max\left\{S_T^{\beta}-K,0\right\}\right). \tag{1}$$

Suppose that the underlying asset price S_t satisfies a fractional-order stochastic differential equation as follows:

$$d^aS_t = \mu S_t dt^a + \sigma S_t dB_t, \tag{2}$$

where μ and σ are expected return rate and volatility, respectively, $a = 2H$ is time-fractional order, B_t is a standard Brownian motion. H is the Hurst index, which describes the memory effects of stock prices. Here, we only consider a case that $0.5 < a \leq 1$. The equation of option value $V(t,S)$ satisfies

$$\frac{\partial^a V}{\partial t^a} = \left(rV - \frac{\sigma^2 S^2}{2\Gamma^2(1+a)}\frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S}\right)\frac{t^{1-a}}{\Gamma(2-a)}, \tag{3}$$

where r is the risk-free interest rate. Suppose that $V(t,S)$ is rewritten as

$$V(t,S) = e^{-r(T-t)}U(t,S). \tag{4}$$

Which is employed to solve (3). Input (4) into (3), we eliminate the rV-term on the both sides of the equality, and then we obtain a simplified equation

$$\frac{\partial^a U}{\partial t^a} = -\left(\frac{\sigma^2 S^2}{2\Gamma^2(1+a)}\frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S}\right)\frac{t^{1-a}}{\Gamma(2-a)}. \tag{5}$$

Replacing S with e^x in (5), we have

$$\frac{\partial^a U}{\partial t^a} = \left[\left(\frac{\sigma^2}{2\Gamma^2(1+a)} - r\right)\frac{\partial U}{\partial x} - \frac{\sigma^2}{2\Gamma^2(1+a)}\frac{\partial^2 U}{\partial x^2}\right]\frac{t^{1-a}}{\Gamma}. \tag{6}$$

In order to obtain a general solution of (6), we construct an auxiliary equation as follows:

$$\frac{\partial^a \widetilde{U}}{\partial t^a} - \left(\frac{\sigma^2}{2\Gamma^2(1+a)} - r\right)\frac{t^{1-a}}{\Gamma(2-a)}\frac{\partial \widetilde{U}}{\partial x} = 0. \tag{7}$$

From the theorem of the auxiliary system, the general solution of (7) satisfies the equality as follows:

$$\frac{dt^a}{1} = \frac{d^a x}{\left(r - \sigma^2/2\Gamma^2(1+a)\right)t^{1-a}/\Gamma(2-a)} = \frac{d\widetilde{U}^a}{0}, \tag{8}$$

where zero indicates that $\widetilde{U}$ has constant characteristics, see Jumarie [6]. It is derived from the eigenvalue method of the hyperbolic equation, where the denominator changes along the direction of the characteristic line. Then, based on the properties of fractional calculus, the general solution of (8) can be expressed as follows:

$$\widetilde{U}(x,t) = \widetilde{F}\left(x - \left(r - \frac{\sigma^2}{2\Gamma^2(1+a)}\right)t\right), \tag{9}$$

where $\widetilde{F}$ is an arbitrary function (9), which implies that $U(x,t)$ has the form

$$U(S,t) = R(m,t), \tag{10}$$

where $m = x + (r - \sigma^2/2\Gamma^2(1+a))(T-t)$. From (6) and (10), the fractional order equation of R satisfies

$$\frac{\partial^a R}{\partial t^a} = -\eta t^{1-a}\frac{\partial^2 R}{\partial m^2}, \tag{11}$$

where $\eta = \sigma^2/2\Gamma^2(1+a)\Gamma(2-a)$ and $R(m,T) = \max\{e^{m\beta} - K, 0\}$, which is a terminal condition. From $m = x + (r - \sigma^2/2\Gamma^2(1+a))(T-t)$, it follows that

$$\frac{\partial^a R}{\partial t^a} = -(T-\tau)^{1-a}\tau^{a-1}\frac{\partial^a R}{\partial \tau^a}, \tag{12}$$

where $\tau = T - t$. Comparing (9) and (10), we obtain

$$\frac{\partial^a R}{\partial \tau^a} = \eta\tau^{1-a}\frac{\partial^2 R}{\partial m^2}. \tag{13}$$

With an initial condition $R(m,0) = \max\{e^{m\beta} - K, 0\}$. By performing the Laplace transformation on the both sides of equation (11), for the time variable τ, we have

$$L\{R(m,\tau)\} = \frac{1}{S}R(m,0) + \frac{1}{S^a}L\left\{\eta\tau^{1-a}\frac{\partial^2 R}{\partial m^2}\right\}. \tag{14}$$

Subsequently, applying the inverse Laplace transformation and convolution theorem to equation (12), we obtain the following equation:

$$R(m,\tau) = R(m,0) + \frac{\eta}{\Gamma(a)}\int_0^\tau (\tau-\xi)^{a-1}\xi^{1-a}\frac{\partial^2 R(m,\xi)}{\partial m^2}d\xi. \tag{15}$$

From the equality above, the iteration formula of $R(m,\tau)$ can be constructed through the reconstruction of the variational iteration method as follows:

$$R_n(m,\tau) = R_0(m,\tau) + \frac{\eta}{\Gamma(a)}\int_0^\tau (\tau-\xi)^{a-1}\xi^{1-a}\frac{\partial^2 R_{n-1}(m,\xi)}{\partial m^2}d\xi, \tag{16}$$

where $R_0(m,\tau) = \max\{e^{m\beta} - K, 0\}$ and $n = 1,2,3,\cdots$. For $\beta > -1$, by using the integral formula

$$\frac{1}{\Gamma(a)}\int_\phi^x (x-t)^{a-1}(t-\phi)^\beta dt = \frac{\Gamma(\beta+1)}{\Gamma(a+\beta+1)}(x-\phi)^{a+\beta}. \tag{17}$$

The approximation of $R_n(m,\tau)$ can be given as follows:

$$R_n(m,\tau) = \max\left\{e^{m\beta} - K, 0\right\} + e^{m\beta}\sum_{j=1}^{n}\frac{\Gamma(2-a)\cdots\Gamma(j+1-a)}{\Gamma(2)\cdots\Gamma(j+1)}\left(\beta^2\eta\tau\right)^j. \tag{18}$$

Finally, by applying the solution theorem of the fractional differential equation, we obtain that $\lim_{n\to\infty} R_n(m,\tau) = R(m,\tau)$, which is an exact solution of (13), and the solution is given as follows:

$$R = \max\left\{e^{m\beta} - K, 0\right\} + e^{m\beta}\sum_{j=1}^{\infty}\frac{\Gamma(2-a)\cdots\Gamma(j+1-a)}{\Gamma(2)\cdots\Gamma(j+1)}\left(\beta^2\eta\tau\right)^j. \tag{19}$$

Taking the variable transformation $\tau = T - t$, $S = e^x$, $m = x + (r - \sigma^2/2\Gamma^2(1+a))(T-t)$, $V(s,t) = e^{-r(T-t)}U(S,t)$ and $U(S,t) = R(S,t)$, from (19), we obtain the solution as follows:

$$\begin{aligned} V(S,t) = {} & \max\left\{S^\beta e^{[(\beta-1)r-\sigma^2/2\Gamma^2(1+a)\beta](T-t)} - Ke^{-r(T-t)}, 0\right\} \\ & + S^\beta e^{[(\beta-1)r-\sigma^2/2\Gamma^2(1+a)\beta](T-t)} \\ & \cdot \sum_{j=1}^{\infty} \frac{\Gamma(2-a)\cdots\Gamma(j+1-a)}{\Gamma(2)\cdots\Gamma(j+1)}\left[\beta^2\eta(T-t)\right]^j, \end{aligned} \tag{20}$$

where $\eta = \sigma^2/2\Gamma^2(1+a)\Gamma(2-a)$. The formula (20) is called the time-fractional Black-Scholes formula of the call power option. Moreover, using the same method, we obtain the price of the put power option

$$\begin{aligned} P(S,t) = {} & \max\left\{Ke^{-r(T-t)} - S^\beta e^{[(\beta-1)r-\sigma^2/2\Gamma^2(1+a)\beta](T-t)}, 0\right\} \\ & - S^\beta e^{[(\beta-1)r-\sigma^2/2\Gamma^2(1+a)\beta](T-t)} \sum_{j=1}^{\infty} \\ & \cdot \frac{\Gamma(2-a)\cdots\Gamma(j+1-a)}{\Gamma(2)\cdots\Gamma(j+1)}\left[\beta^2\eta(T-t)\right]^j. \end{aligned} \tag{21}$$

Equation (21) is called the time-fractional Black-Scholes formula of the put power option.

3. Fuzzy Pricing in the Interval Type-2 Fuzzy Framework

3.1. Definitions for the Triangular Interval Type-2 Fuzzy Number. We construct a mixed fractional stochastic differential equation to describe the uncertainty of the underlying asset price. However, in the highly complicated financial system, participants find it difficult to record the dynamic underlying asset price accurately due to market fluctuations, inadequate information, and human errors. To better capture these fuzzy characteristics in the financial market, a triangular interval type-2 fuzzy set is introduced into the mixed fractional pricing model with a jump of the N-fold compound option.

For clarity, the theory, definitions, and some basic operations of a triangular-interval type-2 fuzzy set are provided from the basic type-1 fuzzy set. Firstly, we show the definitions of type 1 fuzzy sets as follows:

Definition 1. Zadeh [23] Let X be a concentration of objects x, then a fuzzy set $\widetilde{A} \in X$ is a set of ordered pairs as follows:

$$\widetilde{A} = \{(x, \mu_A(x) | \forall x \in X\}. \tag{22}$$

Here, $\mu_A(x)$ is called the membership function for the type-1 fuzzy set $\widetilde{A}$. Particularly, a fuzzy set in $\mathbb{R}$ is called a fuzzy number $\widetilde{A}$ with a subnormal or normal, fuzzy convex, and continuous membership function of bounded support.

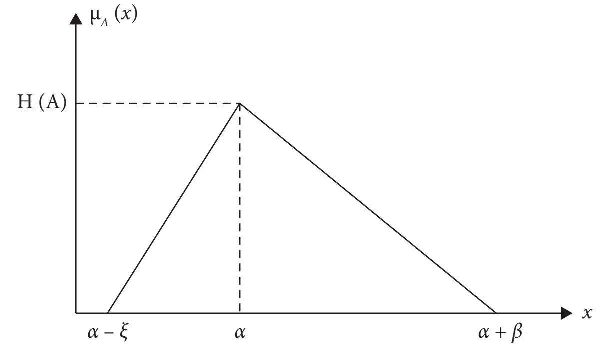

Figure 1: Representation of subnormal/normal of a triangular type-1 fuzzy set.

Generally, a triangular type-1 fuzzy number with subnormal or normal segments can be depicted as in Figure 1. Let $\widetilde{A}$ be a triangular type-1 fuzzy number with the following elements: $\widetilde{A} = (a, \xi, \beta; H(A))$, where ξ represents left-width, β represents right-width and $H(A)$ denotes the height element for a, with $\xi > 0$, $\beta > 0$ and $0 \le H(A) \le 1$. If $H(A) = 1$ then, the fuzzy set is a normal triangular type-1 fuzzy number, in other cases it is a subnormal triangular type-1 fuzzy number.

From the basic type-1 fuzzy set, the definitions of interval type-2 fuzzy set and interval type-2 fuzzy number are shown in definitions 2 and 3.

Definition 2. Mendel et al. [24] let a type-2 fuzzy set $\widetilde{\widetilde{A}}$ be in the universe of discourse X; it can be characterized by a type-2 membership function $\mu_{\widetilde{\widetilde{A}}}(x,u)$, shown as follows:

$$\widetilde{\widetilde{A}} = \left\{\left((x,u), \mu_{\widetilde{\widetilde{A}}}(x,u)\right) | \forall x \in X, \forall u \in J_x \subseteq [0,1]\right\}. \tag{23}$$

In which $0 \le \mu_{\widetilde{\widetilde{A}}}(x,u) \le 1$. $\widetilde{\widetilde{A}}$ can also be expressed as follows:

$$\widetilde{\widetilde{A}} = \int_{x\in X}\int_{u\in J_x} \mu_{\widetilde{\widetilde{A}}} \frac{(x,u)}{(x,u)}, \tag{24}$$

where $\int\int$ states the union over all admissible x and u. Additionally, the type-2 fuzzy set $\widetilde{\widetilde{A}}$ is an interval type-2 fuzzy set when all $\mu_{\widetilde{\widetilde{A}}}(x,u) = 1$. It means that an interval type-2 fuzzy set can be considered to be a special case of a type-2 fuzzy set and can be represented by

$$\widetilde{\widetilde{A}} = \int_{x\in X}\int_{u\in J_x} \frac{1}{(x,u)}. \tag{25}$$

Definition 3. Wang and Li [25] Let an interval type-2 fuzzy set $\widetilde{\widetilde{A}} = (\widetilde{A}^U, \widetilde{A}^U)$, where $\widetilde{A}^U$ and $\widetilde{A}^L$ are the upper and lower membership functions, respectively. An interval type-2 fuzzy set might be called an interval type-2 fuzzy number if $\widetilde{A}^U$ and $\widetilde{A}^L$ are fuzzy numbers

A triangular interval type-2 fuzzy number is introduced as follows:

$$\widetilde{\widetilde{A}} = \left(\widetilde{A}^U, \widetilde{A}^L\right) = \left(a, \xi, \beta, H\left(\widetilde{A}^U\right); b, \gamma, \delta, H\left(\widetilde{A}^L\right)\right). \tag{26}$$

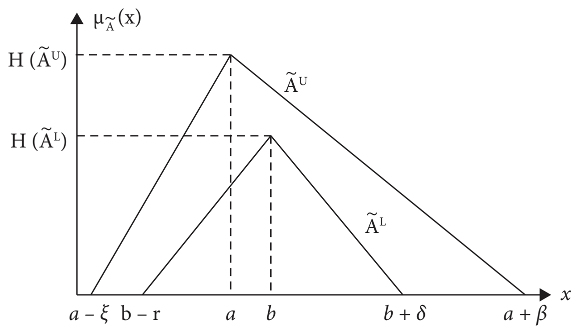

FIGURE 2: A triangular interval type-2 fuzzy number.

Here, $\widetilde{A}^U$ and $\widetilde{A}^L$ are the upper and lower membership functions, respectively. a and b are reference points for the triangular interval type-2 fuzzy number. Moreover, ξ, β, γ, and δ are the left and right widths of the related membership functions, respectively. Let $H(\widetilde{A}^U)$ be the membership value of the element a in the upper triangular membership function $\widetilde{A}^U$. Similarly, denote by $H(\widetilde{A}^L)$ the membership value of the element b in the lower triangular membership function $\widetilde{A}^L$. For clarity, a triangular interval type-2 fuzzy number with a subnormal or normal segment is depicted as in Figure 2. The uncertainty is shown by the region located between the lower and upper membership functions, which is called the footprint of uncertainty in Mendel et al. [24].

The arithmetic operations of the triangular interval type-2 fuzzy numbers are briefly introduced. Let $\widetilde{\widetilde{A}}_1$ and $\widetilde{\widetilde{A}}_2$ be a two triangular interval type-2 fuzzy numbers, given as follows:

$$\begin{aligned}\widetilde{\widetilde{A}}_1 &= \left(\widetilde{A_1}^U, \widetilde{A_1}^L\right)\\ &= \left(a_1, \xi_1, \beta_1, H\left(\widetilde{A_1}^U\right); b_1, \gamma_1, \delta_1, H\left(\widetilde{A_1}^L\right)\right).\end{aligned} \tag{27}$$

and

$$\begin{aligned}\widetilde{\widetilde{A}}_2 &= \left(\widetilde{A_2}^U, \widetilde{A_2}^L\right)\\ &= \left(a_2, \xi_2, \beta_2, H\left(\widetilde{A_2}^U\right); b_2, \gamma_2, \delta_2, H\left(\widetilde{A_2}^L\right)\right).\end{aligned} \tag{28}$$

Define "$\oplus$," "$\ominus$" and "$\otimes$" as the fuzzy arithmetic operations between two triangular interval type-2 fuzzy numbers $\widetilde{\widetilde{A}}_1$ and $\widetilde{\widetilde{A}}_2$. Utilizing the extension principle, the addition, subtraction, and multiplication of any two triangular interval type-2 fuzzy numbers are defined as follows:

$$\begin{aligned}\widetilde{\widetilde{A}}_1 \oplus \widetilde{\widetilde{A}}_2 &= \left(\widetilde{A_1}^U, \widetilde{A_1}^L\right) \oplus \left(\widetilde{A_2}^U, \widetilde{A_2}^L\right)\\ &:= \left(a_1 + a_2, \xi_1 + \xi_2, \beta_1 + \beta_2, \min\left(H\left(\widetilde{A_1}^U\right), H\left(\widetilde{A_2}^U\right)\right);\right.\\ &\qquad \left. b_1 + b_2, \gamma_1 + \gamma_2, \delta_1 + \delta_2, \min\left(H\left(\widetilde{A_1}^L\right), H\left(\widetilde{A_2}^L\right)\right)\right)\\ \widetilde{\widetilde{A}}_1 \ominus \widetilde{\widetilde{A}}_2 &= \left(\widetilde{A_1}^U, \widetilde{A_1}^L\right) \ominus \left(\widetilde{A_2}^U, \widetilde{A_2}^L\right)\\ &:= \left(a_1 a_2, \xi_1 + \beta_2, \beta_1 + \xi_2, \min\left(H\left(\widetilde{A_1}^U\right), H\left(\widetilde{A_2}^U\right)\right); b_1 - b_2, \gamma_1 + \delta_2, \delta_1 + \gamma_2, \min\left(H\left(\widetilde{A_1}^L\right), H\left(\widetilde{A_2}^L\right)\right)\right)\\ K \otimes \widetilde{\widetilde{A}}_1 &= K \otimes \left(\widetilde{A_1}^U, \widetilde{A_1}^L\right)\\ &:= \left(Ka_1, K\xi_1, K\beta_1, H\left(\widetilde{A_1}^U\right); Kb_1, K\gamma_1, K\delta_1, H\left(\widetilde{A_1}^L\right)\right).\end{aligned} \tag{29}$$

$$\begin{aligned}\widetilde{\widetilde{A}}_1 \otimes \widetilde{\widetilde{A}}_2 &= \left(\widetilde{A_1}^U, \widetilde{A_1}^L\right) \otimes \left(\widetilde{A_2}^U, \widetilde{A_2}^L\right)\\ &:= \left(a_1 a_2, a_1\xi_2 + a_2\xi_1 - \xi_1\xi_2, a_1\beta_2 + a_2\beta_1 + \beta_1\beta_2, \min\left(H\left(\widetilde{A_1}^U\right), H\left(\widetilde{A_2}^U\right)\right);\right.\\ &\qquad \left. b_1 b_2, b_1\gamma_2 + b_2\gamma_1 - \gamma_1\gamma_2, b_1\delta_2 + b_2\delta_1 + \delta_1\delta_2, \min\left(H\left(\widetilde{A_1}^L\right), H\left(\widetilde{A_2}^L\right)\right)\right).\end{aligned} \tag{30}$$

Where K is a scalar value which is greater than 0.

3.2. Mean Value and Variance of the Triangular Interval Type-2 Fuzzy Number. In financial markets, an investor's decision depends not only on objective models but also on their own subjective factors. Therefore, it is necessary to include the investor's subjective judgment in the compound option pricing model. Taking account of the decision maker's subjective judgment, the mean value $E(\cdot)$ of a triangular interval type-2 fuzzy number, which incorporates the possibility-necessity weight ν and the pessimistic-optimistic index λ is introduced. For convenience, we first give the definition of the mean value of a triangular interval type-2 fuzzy number.

Definition 4. Let $\tilde{\tilde{A}} = (\tilde{A}^U, \tilde{A}^L) = (a, \xi, \beta, H(\tilde{A}^U); b, \gamma, \delta, H(\tilde{A}^L))$ be a triangular interval type-2 fuzzy number. Then, the mean value with the possibility-necessity weight ν and the pessimistic-optimistic index λ is represented by

$$\begin{aligned} E(\tilde{\tilde{A}}) &= \nu E_P(\tilde{\tilde{A}}) + (1-\nu)E_N(\tilde{\tilde{A}}) \\ &= \int_0^{H(\tilde{A}^U)} (\nu + 2(1-\nu)(1-\alpha))G(\tilde{A}_\alpha^U)\,d\alpha \\ &\quad + \int_0^{H(\tilde{A}^L)} (\nu + 2(1-\nu)(1-\alpha))G(\tilde{A}_\alpha^L)\,d\alpha, \end{aligned} \tag{31}$$

where

$$\tilde{A}_\alpha^U = \left[a - \xi\left(1 - \frac{\alpha}{H(\tilde{A}^U)}\right), a + \beta\left(1 - \frac{\alpha}{H(\tilde{A}^U)}\right)\right]. \tag{32}$$

and

$$\tilde{A}_\alpha^L = \left[b - \gamma\left(1 - \frac{\alpha}{H(\tilde{A}^L)}\right), b + \delta\left(1 - \frac{\alpha}{H(\tilde{A}^L)}\right)\right]. \tag{33}$$

are the α-levels of the upper fuzzy number $\tilde{A}^U$ and the lower fuzzy number $\tilde{A}^L$, respectively. Furthermore, the λ-weighting function $G(\cdot)$ is given by

$$G([a,b]) := \lambda a + (1-\lambda)b. \tag{34}$$

In addition,

$$E_P(\tilde{\tilde{A}}) = \int_0^{H(\tilde{A}^U)} G(\tilde{A}_\alpha^U)\,d\alpha + \int_0^{H(\tilde{A}^L)} G(\tilde{A}_\alpha^L)\,d\alpha. \tag{35}$$

and

$$\begin{aligned} E_N(\tilde{\tilde{A}}) &= \int_0^{H(\tilde{A}^U)} 2(1-\alpha)G(\tilde{A}_\alpha^U)\,d\alpha \\ &\quad + \int_0^{H(\tilde{A}^L)} 2(1-\alpha)G(\tilde{A}_\alpha^L)\,d\alpha. \end{aligned} \tag{36}$$

are the possibility mean value and the necessity mean value of the triangular interval type-2 fuzzy number $\tilde{\tilde{A}}$, respectively. Moreover,

$$\begin{aligned} E_C(\tilde{\tilde{A}}) &= \int_0^{H(\tilde{A}^U)} \frac{2}{3}(2-\alpha)G(\tilde{A}_\alpha^U)\,d\alpha \\ &\quad + \int_0^{H(\tilde{A}^L)} \frac{2}{3}(2-\alpha)G(\tilde{A}_\alpha^L)\,d\alpha. \end{aligned} \tag{37}$$

is the credibility mean value of the triangular interval type-2 fuzzy number $\tilde{\tilde{A}}$.

According to the defining formulas of the possibility mean value $E_P(\cdot)$, necessity mean value $E_N(\cdot)$ and credibility mean value $E_C(\cdot)$, it is obvious that

$$E_C(\tilde{\tilde{A}}) = \frac{2}{3}E_P(\tilde{\tilde{A}}) + \frac{1}{3}E_N(\tilde{\tilde{A}}). \tag{38}$$

for the given type-2 fuzzy number $\tilde{\tilde{A}}$.

Compared with the singular possibility mean value $E_P(\cdot)$, necessity mean value $E_N(\cdot)$ and credibility mean value $E_C(\cdot)$, the mean value $E(\cdot)$ can better reflect the diversities of subjective judgment and market fuzziness. The mean value of N-fold compound option price can be calculated by taking the corresponding parameters into the formula. In addition, the possibility mean value $E_P(\cdot)$, necessity mean value $E_N(\cdot)$ and credibility mean value $E_C(\cdot)$ are represented by the mean value $E(\cdot)$ with the corresponding possibility-necessity weights $\nu = 0, \nu = 2/3$ and $\nu = 1$, respectively. That is to say, we can calculate a deterministic value of an interval type-2 fuzzy number by the mean value formula, which accommodates the diversity of subjective judgment, market fuzziness, and decision makers' choice. For clarity, this paper illustrates detailed steps for how to calculate a deterministic value (that is, a mean value) of an interval type-2 fuzzy number as follows:

Step 1. For given the interval type-2 fuzzy number $\tilde{\tilde{A}} = (a, \xi, \beta, H(\tilde{A}^U); b, \gamma, \delta, H(\tilde{A}^L))$, we first need to calculate the α-levels of the upper fuzzy number

$$\tilde{A}_\alpha^U = \left[a - \xi\left(1 - \frac{\alpha}{H(\tilde{A}^U)}\right), a + \beta\left(1 - \frac{\alpha}{H(\tilde{A}^U)}\right)\right]. \tag{39}$$

and the α-levels of the lower fuzzy number

$$\tilde{A}_\alpha^L = \left[b - \gamma\left(1 - \frac{\alpha}{H(\tilde{A}^L)}\right), b + \delta\left(1 - \frac{\alpha}{H(\tilde{A}^L)}\right)\right]. \tag{40}$$

Step 2. For given the pessimistic-optimistic index λ, we then calculate the λ-weighting function $G(\tilde{A}_\alpha^U)$

$$\begin{aligned} G(\tilde{A}_\alpha^U) &= \lambda\left(a - \xi\left(1 - \frac{\alpha}{H(\tilde{A}^U)}\right)\right) \\ &\quad + (1-\lambda)\left(a + \beta\left(1 - \frac{\alpha}{H(\tilde{A}^U)}\right)\right). \end{aligned} \tag{41}$$

and the λ-weighting function $G(\tilde{A}_\alpha^L)$

$$\begin{aligned} G(\tilde{A}_\alpha^L) &= \lambda\left(b - \gamma\left(1 - \frac{\alpha}{H(\tilde{A}^L)}\right)\right) \\ &\quad + (1-\lambda)\left(b + \delta\left(1 - \frac{\alpha}{H(\tilde{A}^L)}\right)\right). \end{aligned} \tag{42}$$

Step 3. For given the possibility-necessity weight ν, we finally calculate the mean value $E(\tilde{\tilde{A}})$ that is the deterministic value of an interval type-2 fuzzy number $\tilde{\tilde{A}}$

$$E(\tilde{\tilde{A}}) = \int_0^{H(\tilde{A}^U)} (\nu + 2(1-\nu)(1-\alpha))G(\tilde{A}_\alpha^U)d\alpha + \int_0^{H(\tilde{A}^L)} (\nu + 2(1-\nu)(1-\alpha))G(\tilde{A}_\alpha^L)d\alpha. \tag{43}$$

Without losing generality, this paper further takes a numerical example to present the deterministic values of an interval type-2 fuzzy number under the different subjective judgments.

Example 1. In this example, the mean value of an interval type-2 fuzzy number is considered. Similar to the parameter settings in Zhao et al. [13], we assume that the possibility-necessity weight $\nu = 1/3, 1/2, 2/3, 1$ and the pessimistic-optimistic index $\lambda = 1/3, 1/2, 2/3, 1$, with the triangular interval type-2 fuzzy number $\tilde{\tilde{C}} = (80, 10, 11, 1; 70, 9, 8, 0.9)$.

For the given possibility-necessity weight and pessimistic-optimistic index, the mean value of the interval type-2 fuzzy number is listed in Table 1. The deterministic value accommodates the double fuzziness and subjective judgment, which is more consistent with the real investment market. As shown in the table, the values are decreasing with respect to the pessimistic-optimistic index, which reflects the pessimistic degree of the decision-makers. In other words, the greater the pessimistic degree, the lower the mean value of the prices.

The variance of an interval type-2 fuzzy number is the arithmetic average of the upper and lower membership functions' variances, which is proved in Carlsson et al. [27]. For convenience, we give the definition of variance of a triangular interval type-2 fuzzy number based on the studies in Cagri Tolga [26].

Table 1: The mean value for different subjective judgments.

ν	λ	$E(\tilde{\tilde{C}})$
1/3	1/3	82.4780
1/3	1/2	80.3383
1/3	2/3	78.1986
1/3	1	73.9193
1/2	1/3	82.3760
1/2	1/2	80.3337
1/2	2/3	78.2915
1/2	1	74.2070
2/3	1/3	82.2739
2/3	1/2	80.3291
2/3	2/3	78.3843
2/3	1	74.4946
1	1/3	82.0699
1	1/2	80.3199
1	2/3	78.5699
1	1	75.0699

Definition 5. Let $\tilde{\tilde{A}} = (\tilde{A}^U, \tilde{A}^L) = (a, \xi, \beta, H(\tilde{A}^U); b, \gamma, \delta, H(\tilde{A}^L))$ be a triangular interval type-2 fuzzy number. Then, the variance of the triangular interval type-2 fuzzy number is defined as

$$\sigma^2(\tilde{\tilde{A}}) = \frac{\sigma^2(\tilde{A}^U) + \sigma^2(\tilde{A}^L)}{2}, \tag{44}$$

where

$$\sigma^2(\tilde{A}^U) = \frac{1}{2}\int_0^{H(\tilde{A}^U)} \left[a + \beta\left(1 - \frac{\alpha}{H(\tilde{A}^U)}\right) - a + \xi\left(1 - \frac{\alpha}{H(\tilde{A}^U)}\right)\right]^2 \alpha d\alpha, \tag{45}$$

$$\sigma^2(\tilde{A}^L) = \frac{1}{2}\int_0^{H(\tilde{A}^L)} \left[b + \delta\left(1 - \frac{\alpha}{H(\tilde{A}^L)}\right) - b + \gamma\left(1 - \frac{\alpha}{H(\tilde{A}^L)}\right)\right]^2 \alpha d\alpha. \tag{46}$$

Are the possibilistic variances of the upper and lower triangular membership function $\tilde{A}^U$ and the lower triangular membership function $\tilde{A}^L$.

3.3. Option Pricing with Triangular Interval Type-2 Fuzzy Numbers.

Considering the randomness and fuzziness of market uncertainty, the time-fractional stochastic model cannot comprehensively characterize the process of determining underlying asset price. Considering the fact that the satisfaction level of investors with a given price is fuzzy rather than accurate in the actual market, the type-1 fuzzy method cannot comprehensively describe the fuzziness in the market price. Moreover, in a fuzzy environment, investors are concerned about the most likely value, the maximum value, and the minimum value, which can be better converted into a triangular fuzzy number. The market fuzziness is characterized by a triangular interval type-2 fuzzy numbers, which meet flexibility, consistency, and accuracy in the actual market.

By assuming that both stock prices and strike prices are the triangular interval type-2 fuzzy numbers, this paper constructs a power option fuzzy pricing formula based on the research mentioned above. In the interval type-2 fuzzy framework, the fuzzy pricing formula $\tilde{\tilde{C}}_t$ of the call power option is given by

$$\tilde{\tilde{C}}_t\left(\tilde{\tilde{S}}_t, T-t, K, r, \sigma\right) = \max\left\{\tilde{\tilde{S}}_t^{\beta} e^d - \tilde{\tilde{K}} e^{-r(T-t)}, 0\right\} + \tilde{\tilde{S}}_t^{\beta} e^d \sum_{j=1}^{\infty} \frac{\Gamma(2-a)\cdots\Gamma(j+1-a)}{\Gamma(2)\cdots\Gamma(j+1)}\left[\beta^2 \eta (T-t)\right]^j, \quad (47)$$

where the Gamma functions $\Gamma(\cdot)$ in the type-2 fuzzy pricing formula are all sourced from the time-fractional Black-Scholes formula and

$$\sigma := \frac{\sigma\left(\tilde{\tilde{S}}_t\right)}{E_P\left(\tilde{\tilde{S}}_t\right)}. \quad (48)$$

indicates the possibilistic standard deviation of the stock price, which can be calculated by the variance formula (44) and the possibility mean value formula (35). Moreover,

$$\begin{aligned}
\tilde{\tilde{S}}_t &= \left(\tilde{S}_t^U, \tilde{S}_t^L\right) = \left(S_{at}, \xi, \beta, H\left(\tilde{S}_t^U\right); S_{bt}, \gamma, \delta, H\left(\tilde{S}_t^L\right)\right),\\
\tilde{\tilde{K}} &= \left(\tilde{K}^U, \tilde{K}^L\right) = \left(K_a, \eta, \theta, H\left(\tilde{K}^U\right); K_b, \phi, \chi, H\left(\tilde{K}^L\right)\right),\\
d &= \left[(\beta-1)r - \frac{\sigma^2}{2\Gamma^2(1+a)}\beta\right](T-t),
\end{aligned} \quad (49)$$

$$\eta = \frac{\sigma^2}{2\Gamma^2(1+a)\Gamma(2-a)}. \quad (50)$$

Similarly, the fuzzy pricing formula $\tilde{\tilde{P}}_t$ of put power option is given by

$$\tilde{\tilde{P}}_t\left(\tilde{\tilde{S}}_t, t\right) = \max\left\{\tilde{\tilde{K}} e^{-r(T-t)} - \tilde{\tilde{S}}_t^{\beta} e^d, 0\right\} - \tilde{\tilde{S}}_t^{\beta} e^d \sum_{j=1}^{\infty} \frac{\Gamma(2-a)\cdots\Gamma(j+1-a)}{\Gamma(2)\cdots\Gamma(j+1)}\left[\beta^2 \eta (T-t)\right]^j. \quad (51)$$

Actually, option prices depend not only on objective pricing models but also on decision-makers subjective factors. It is reasonable and necessary to cover the subjective judgment in the power option fuzzy pricing model. From the knowledge of fuzzy set theory, it is clearly known that the fuzzy price is an interval type-2 fuzzy number. However, it is not convenient for investors to make a quick decision when faced with a fuzzy price. A quick decision is pivotal to seizing the investment opportunity in the volatile financial market. By using the mean value formula and variance formula of the triangular interval type-2 fuzzy number, the price mean value of the call power option is

$$\begin{aligned}
E\left(\tilde{\tilde{C}}_t\right) &= \nu E_P\left(\tilde{\tilde{C}}_t\right) + (1-\nu)E_N\left(\tilde{\tilde{C}}_t\right)\\
&= \int_0^{H\left(\tilde{C}_t^U\right)} (\nu + 2(1-\nu)(1-\alpha))G\left(\widetilde{(C_t)}_\alpha^U\right)d\alpha\\
&\quad + \int_0^{H\left(\tilde{C}_t^L\right)} (\nu + 2(1-\nu)(1-\alpha))G\left(\widetilde{(C_t)}_\alpha^L\right)d\alpha,
\end{aligned} \quad (52)$$

where

$$\tilde{\tilde{C}}_t = \left(\tilde{C}_t^U, \tilde{C}_t^L\right). \quad (53)$$

is the fuzzy call power option price that is a triangular interval type-2 fuzzy number, which is calculated by pricing formula (47) in the interval type-2 fuzzy framework.

Similarly, the price mean value of the put power option is

$$\begin{aligned}
E\left(\tilde{\tilde{P}}_t\right) &= \nu E_P\left(\tilde{\tilde{P}}_t\right) + (1-\nu)E_N\left(\tilde{\tilde{P}}_t\right)\\
&= \int_0^{H\left(\tilde{P}_t^U\right)} (\nu + 2(1-\nu)(1-\alpha))G\left(\widetilde{(P_t)}_\alpha^U\right)d\alpha\\
&\quad + \int_0^{H\left(\tilde{P}_t^L\right)} (\nu + 2(1-\nu)(1-\alpha))G\left(\widetilde{(P_t)}_\alpha^L\right)d\alpha,
\end{aligned} \quad (54)$$

where

$$\tilde{\tilde{P}}_t = \left(\tilde{P}_t^U, \tilde{P}_t^L\right). \quad (55)$$

is the fuzzy put power option price that is a triangular interval type-2 fuzzy number, which is calculated by pricing (51).

4. Conclusions

By introducing the triangular interval type-2 fuzzy set theory into the time-fractional stochastic financial model, we provide an investigation of the power option pricing model with trend memory of stock prices and fuzzy uncertainty in financial markets. The time-fractional order BS formulas of the power option are derived from the reconstruction variational iteration method. Compared with the traditional power option pricing models, which adopt the classical Brownian motion to model the stochastic in the financial market, our pricing model with the time-fractional can better describe the trend memory of stock prices. Moreover, utilizing the arithmetic operations, mean value formula, and variance formula of the triangular interval type-2 fuzzy stock price and strick price, we obtain the power option fuzzy pricing formula. In addition, this paper also provides the price mean value of a power option, which incorporates the decision-makers subjective factors and makes the application of compound options more consistent with the actual financial markets. Compared with the traditional option fuzzy pricing models, adopting the type-1 fuzzy set theory to model the fuzziness in underlying asset prices, our power option fuzzy price and the mean value, which adopt the triangular interval type-2 fuzzy numbers, can better capture the double fuzziness, that is, the price fuzziness and the satisfaction fuzziness [28].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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