Hindawi
Complexity
Volume 2022, Article ID 5715922, 19 pages
https://doi.org/10.1155/2022/5715922

Research Article

Qualitative Analysis of a Spatiotemporal Prey-Predator Model with Additive Allee Effect and Fear Effect

Changcheng Ke,[1] Ming Yi,[1] and Yanfeng Guo[1,2]

[1]*School of Mathematics and Physics, China University of Geosciences, Wuhan, Hubei 430074, China*
[2]*School of Science, Guangxi University of Science and Technology, Liuzhou, Guangxi 545006, China*

Correspondence should be addressed to Yanfeng Guo; guoyan_feng@163.com

Received 24 February 2022; Revised 18 March 2022; Accepted 3 May 2022; Published 2 June 2022

Academic Editor: Jawad Ahmad

A diffusive predator-prey system with both the additive Allee effect and the fear effect in the prey subject to Neumann boundary conditions is considered in this paper. Firstly, non-negative and non-trivial solution a priori estimations are shown. Furthermore, for specific parameter ranges, the absence of non-constant positive solutions is demonstrated. Secondly, we use the linearized theory to investigate the stability of non-negative constant solutions. The spatially homogeneous and non-homogeneous periodic solutions, as well as non-constant steady state solutions, are next investigated by using Allee effect parameters as the bifurcation parameter. Finally, numerical simulation is used to illustrate some theoretical results.

1. Introduction

The biodynamics of ecosystems are current hot issues in biology and ecology. The intense effort to understand the pattern formation and mechanisms of spatial diffusion during the late 20th century, especially in the context of biological and ecological contexts, has gradually raised more and more concerns. Especially, in biochemical reactions characterized by interactions of different species, the study on predator-prey types has been studied widely in [1–4].

Recently, Allee effect, which was initially introduced by Allee in 1931 [5], has been studied extensively [6–10]. With the development of the theory for reaction-diffusion equations, many scholars have done many mathematical research to better describe the relationship between different species. Especially, introducing the Allee effect into the model makes the dynamic behavior of the model closer to reality. The spatiotemporal complexity of a delayed predator-prey model with double Allee effect was given by [11]. In [12], P. J. Pal and S. Tapan consider a system with a double Allee effect in prey population growth, which are very sensitive to parameter perturbations and position of initial conditions. H. Molla and S. Sarwardi developed a predator-prey model that combines these phenomena, considering variable prey refuge with additive Allee effect on the prey species, and also investigated the appearance of Hopf bifurcations in a neighborhood of the unique interior equilibrium point of the dynamical system [13]. The rich behaviour of the dynamics suggests that both prey refuge and a strong Allee affect are important factors in ecological complexity. For a reaction-diffusion system with double Allee effect induced by fear factors subject to Neumann boundary conditions, for details, please refer to [2]. The dynamical behavior of a reaction-diffusion-advection model with weak Allee effect type growth has been studied in [9]. Han and Dai investigated the spatiotemporal pattern formation and selection driven by nonlinear cross-diffusion of a toxic-phytoplankton-zooplankton model with Allee effect. By taking cross-diffusion rate as bifurcation parameter, amplitude equations under nonlinear cross-diffusion are derived that describe the spatiotemporal dynamics [14].

Some researchers have indicated that predators can not only capture prey directly but also affect the behavior of prey, even that it could affect the prey more influential than predation [15, 16]. In fact, all animals show various kinds of antipredator responses, such as feeling of fear, habitat changes, vigilance, foraging, and different physiological changes ([17–21]).

The cost of fear is objective, and it should be taken into consideration when establishing predation and predation models. For example, Jana et al. [22] have explored the influence of habitat complexity on a predator-prey system under fear effect by incorporating self-diffusion. Tiwari et al. analyzed a predator-prey interaction model with Beddington-DeAngelis functional response (BDFR) and incorporating the cost of fear into prey reproduction. For the spatial system, the Hopf bifurcation around the interior equilibrium, stability of homogeneous steady state, direction, and stability of spatially homogeneous periodic orbits have been established [23]. For a plankton-fish model with both the zooplankton refuge and the fear effect, the local and global dynamics of such a model have been investigated in [24]. Moreover, the investigation in [25] has revealed the threshold behavior of a stochastic predator-prey system with fear effect, prey refuge, and non-constant mortality rate. Sasmal and Takeuchi studied the dynamics of a prey-predator interaction model using Monod–Haldane type functional response and provided detailed mathematical results, including basic dynamical properties, existence of positive equilibria, asymptotic stability of all equilibria, Hopf bifurcation, direction, and stability of bifurcated periodic solutions [26]. Furthermore, they also investigated the role of predation fear and its carry-over effects in the prey-predator model. Basic dynamical properties, as well as the global stability of each equilibrium, have been discussed [27].

Allee effect comes in different forms, including multiplicative Allee effect and additive Allee effect. Furthermore, Dennis [6] first proposed the equation incorporating additive Allee effect:

$$\frac{\mathrm{du}}{\mathrm{dt}} = ru\left(1 - \frac{u}{k} - \frac{m}{u+a}\right), \tag{1}$$

where m and a are constants, which reflect the degree of Allee effect; $m/u + a$ denotes the additive Allee effect; r is the intrinsic growth rate of prey; k presents capacity. We note that if $0 < m < a$, then (1) has the weak Allee effect and if $m > a$, then it has the strong Allee effect.

Motivated by the previous works above, we further consider the following reaction-diffusion system with fear effect and additive Allee effect:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + ru\left(1 - u - \dfrac{m}{u+a}\right)\dfrac{1}{1+fv} - buv, x \in \Omega, t > 0, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + cbuv - dv, x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = 0, x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0, x \in \Omega, \\ v(x,0) = v_0(x) \geq 0, x \in \Omega, \end{cases} \tag{2}$$

where Δ is the Laplace operator on domains. $d_1 > 0, d_2 > 0$ meanss the diffusion coefficients. The homogeneous Neumann boundary condition is imposed so that there is no population flow across the boundary, ν denotes the outward normal to the boundary $\partial\Omega$. u, v stand for the density of the prey and predator, respectively; m and a are constants, which reflect the degree of Allee effect; f is a constant, which reflects the degree of fear effect; $1/1 + fv$ and $m/u + a$ denote the fear effect and additive Allee effect, respectively; b represents the modified capture rate; c is the conversion coefficient; r is the intrinsic growth rate of prey; d is the death rate of predator. Then, the steady-state system corresponding to (2) is

$$\begin{cases} d_1 \Delta u + ru\left(1 - u - \dfrac{m}{u+a}\right)\dfrac{1}{1+fv} - buv = 0, x \in \Omega, \\ d_2 \Delta v + cbuv - dv = 0, x \in \Omega, \\ \partial_\nu u = \partial_\nu v = 0, x \in \partial\Omega. \end{cases} \tag{3}$$

The remainder of the paper is structured as follows. In Section 2, we carry out a priori estimates for (3) and the requirements for the nonexistence of non-constant positive solutions. In Section 3, we consider the stability of non-negative constant steady state solutions for system (3). In Section 4, we demonstrate the existence of Hopf bifurcation and steady state bifurcation. In Section 5, we show how the parameters affect the dynamical behavior of the system. Furthermore, we verify the analysis results with the numerical simulation results. In section 6, the paper ends with some conclusions.

2. Preliminaries

In this section, we first present some properties of equilibrium solutions of (3) including a priori estimates. Then, we discuss the nonexistence of non-constant positive solutions for certain parameter range. It is an essential part for analysis of the existence of non-constant positive steady states and the global bifurcation. We first recall the maximum principle in [28].

Lemma 1 (see [28]). *We suppose that $F(x, w) \in C(\overline{\Omega} \times R)$. If $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies*

$$\begin{cases} \Delta w(x) + F(x, w(x)) \geq 0, x \in \Omega, \\ \dfrac{\partial w}{\partial \nu} \leq 0, x \in \partial\Omega, \end{cases} \tag{4}$$

and $w(x_0) = max_{\overline{\Omega}} w$, then $F(x_0, w(x_0)) \geq 0$. Similarly, if the two inequalities are reversed and $w(x_0) = \min_{\overline{\Omega}} w$, then $F(x_0, w(x_0)) \leq 0$.

We note that Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary. Let $\lambda_i, i = 0, 1, 2, \ldots$ be the eigenvalues of $-\Delta$ under Neumann boundary condition.

By Lemma 1, we have a prior estimates as follows:

Theorem 1. *Let* $(u(x)), v(x)$ *be non-negative and non-trivial solution of (3); we assume that* $cr/(1+a)d\,(1+a)^4/4 - m + cd_1/d_2 > 0$. *Then,* $(u(x)), v(x)$*satisfies*

$$0 < u(x) \le 1, 0 < v \le \frac{cr}{(1+a)d}\left(\frac{(1+a)^2}{4} - m\right) + \frac{cd_1}{d_2}, \tag{5}$$

where $d, r, c, d_1, d_2, a, m > 0$.

Proof. From the strong maximum principle, we have $u > 0$ and $v > 0$. Then, by Lemma 1, it follows $0 < u(x) \le 1$. The first equation of (3) is multiplied by c and adding the two equations of (3), we obtain

$$\begin{aligned}-(cd_1\Delta u + d_2\Delta v) &= cru\left(1 - u - \frac{m}{u+a}\right)\frac{1}{1+fv} - \mathrm{d}v\\ &= cr\frac{u}{u+a}((1-u)(u+a)-m)\frac{1}{1+fv} - \mathrm{d}v, \le cr\frac{u}{u+a}\left(\frac{(1+a)^2}{4} - m\right)\frac{1}{1+fv} - dv,\\ &\le \frac{cr}{1+a}\left(\frac{(1+a)^2}{4} - m\right) - dv, \le \frac{cr}{1+a}\left(\frac{(1+a)^2}{4} - m\right) + \frac{dd_1c}{d_2} - \frac{e}{d_2}(cd_1u + d_2v),\end{aligned} \tag{6}$$

which leads to

$$\begin{aligned}\Delta(cd_1u + d_2v) + \frac{cr}{1+a}\left(\frac{(1+a)^2}{4} - m\right) + \frac{\mathrm{dd}_1c}{\mathrm{d}_2}\\ - \frac{\mathrm{d}}{\mathrm{d}_2}(cd_1u + d_2v) \ge 0,\end{aligned} \tag{7}$$

under the condition of $m \le a$. Then, by Lemma 1, we obtain

$$cd_1u + d_2v \le \frac{cr\mathrm{d}_2}{(1+a)\mathrm{d}}\left(\frac{(1+a)^2}{4} - m\right) + cd_1, \tag{8}$$

which implies

$$v \le \frac{cr}{(1+a)\mathrm{d}}\left(\frac{(1+a)^2}{4} - m\right) + \frac{\mathrm{cd}_1}{\mathrm{d}_2}. \tag{9}$$

□

Theorem 2. *For any fixed* $d, r, a, b, c, f > 0$, *there exists* $d^*(r, b, c, d, m, a, f, \Omega)$ *such that if* $\min\{d_1, d_2\} > d^*$, *then (3) has no non-constant positive solution.*

Proof. Let (u, v) be a non-negative solution of (3). We denote (10) as

$$\begin{aligned}\overline{u} &= |\Omega|^{-1}\int_\Omega u dx,\\ \overline{v} &= |\Omega|^{-1}\int_\Omega v dx,\\ F(u) &= ru\left(1 - u - \frac{m}{u+a}\right).\end{aligned} \tag{10}$$

Then,

$$\int_\Omega (u - \overline{u})dx = \int_\Omega (v - \overline{v})dx = 0. \tag{11}$$

Multiplying the first equation of (3) by $u - \overline{u}$ and integrating on Ω , applying Theorem 1 that

$$\begin{aligned}d_1\int_\Omega |\nabla(u-\overline{u})|^2 dx &= \int_\Omega (u-\overline{u})F(u)\frac{1}{1+fv}dx - \int_\Omega buv(u-\overline{u})dx\\ &= \int_\Omega (u-\overline{u})\left(F(u)\frac{1}{1+fv} - F(\overline{u})\frac{1}{1+f\overline{v}}\right)dx - \int_\Omega bv(u-\overline{u})^2dx - \int_\Omega bv\overline{u}(u-\overline{u})dx\\ &\le \int_\Omega (u-\overline{u})\left(F(u)\frac{1}{1+fv} - F(\overline{u})\frac{1}{1+fv} + F(\overline{u})\frac{1}{1+fv} - F(\overline{u})\frac{1}{1+f\overline{v}}\right) - \int_\Omega bv\overline{u}(u-\overline{u})dx\\ &\le r\left(1 + \frac{m}{a^2}\right)\int_\Omega (u-\overline{u})^2dx + \int_\Omega fF(\overline{u})\frac{(u-\overline{u})(v-\overline{v})}{(1+fv)(1+f\overline{v})}dx - \int_\Omega bv\overline{u}(u-\overline{u})dx.\end{aligned} \tag{12}$$

Similarly, multiplying the second equation of (3) by $v - \overline{v}$, we obtain

$$
\begin{aligned}
d_2\int_\Omega |\nabla(v-\overline{v})|^2dx &= cb\int_\Omega uv(v-\overline{v})dx - \int_\Omega dv(v-\overline{v})dx\\
&= cb\int_\Omega uv(v-\overline{v})dx - \int_\Omega d(v-\overline{v})(v-\overline{v})dx \le cb\int_\Omega (uv(v-\overline{v}) - u\overline{v}(v-\overline{v}) + u\overline{v}(v-\overline{v}) - \overline{uv}(v-\overline{v}))dx\\
&= cb\int_\Omega \left(u(v-\overline{v})^2 + (u-\overline{u})(v-\overline{v})\overline{v}\right)dx \le cb\int_\Omega (v-\overline{v})^2dx + cb\int_\Omega u\overline{v}(v-\overline{v})dx.
\end{aligned}
\tag{13}
$$

Multiplying the first equation of (3) by c, added to the second equations of (3), and integrating on Ω, we obtain

$$
-\int_\Omega (cd_1\Delta u + d_2\Delta v)dx = \int_\Omega \left(\mathrm{cru}\left(1-u-\frac{m}{u+a}\right)\frac{1}{1+fv} - dv\right)dx.
\tag{14}
$$

Subject it to the boundary conditions, we have

$$
d\int_\Omega vdx = \int_\Omega cru\left(1-u-\frac{m}{u+a}\right)\frac{1}{1+fv}\mathrm{d}x \le |\Omega|\frac{cr}{4}.
\tag{15}
$$

Hence,

$$
\overline{v} = \frac{1}{|\Omega|}\int_\Omega vdx \le \frac{cr}{4\mathrm{d}}.
\tag{16}
$$

By and (16), it follows from Theorem 1 and Young inequality that

$$
\begin{aligned}
\int_\Omega u\overline{v}(v-\overline{v})dx &= \int_\Omega \overline{v}(u-\overline{u})(v-\overline{v}) \le \frac{cr}{4d}\int_\Omega |u-\overline{u}||v-\overline{v}|dx\\
&\le \frac{cr}{8d}\int_\Omega (u-\overline{u})^2dx + \frac{cr}{8d}\int_\Omega (v-\overline{v})^2dx.
\end{aligned}
\tag{17}
$$

Similarly, we have

$$
\int_\Omega -v\overline{u}(u-\overline{u})dx = \int_\Omega (\overline{v}-v)\overline{u}(u-\overline{u})dx \le \frac{1}{2}\int_\Omega (u-\overline{u})^2dx + \frac{1}{2}\int_\Omega (v-\overline{v})^2dx,
\tag{18}
$$

$$
\begin{aligned}
\int_\Omega fF(\overline{u})\frac{(u-\overline{u})(\overline{v}-v)}{(1+fv)(1+f\overline{v})}dx &= \int_\Omega fr\overline{u}\left(1-\overline{u}-\frac{m}{\overline{u}+a}\right)\frac{(u-\overline{u})(\overline{v}-v)}{(1+fv)(1+f\overline{v})}dx \le \left(\frac{rf}{8}+\frac{mrf}{2a}\right)\int_\Omega (u-\overline{u})^2dx\\
&\quad + \left(\frac{rf}{8}+\frac{mrf}{2a}\right)\int_\Omega (v-\overline{v})^2dx.
\end{aligned}
\tag{19}
$$

From (12), (13), (16)–(19) and the Poincaré inequality, we obtain that

$$
\begin{aligned}
&d_1\int_\Omega |\nabla(u-\overline{u})|^2dx + d_2\int_\Omega |\nabla(v-\overline{v})|^2dx\\
&\le \frac{1}{\lambda_1}\left(A\int_\Omega |\nabla(u-\overline{u})|^2dx + B\int_\Omega |\nabla(v-\overline{v})|^2\right)dx,
\end{aligned}
\tag{20}
$$

where

$$
\begin{aligned}
A &= r\left(1+\frac{m}{a^2}\right) + \frac{bc^2r}{8\mathrm{d}} + \frac{rf}{8} + \frac{mrf}{2a} + \frac{1}{2},\\
B &= \frac{bc^2r}{8\mathrm{d}} + \frac{rf}{8} + \frac{mrf}{2a} + \frac{1}{2} + cb.
\end{aligned}
\tag{21}
$$

This shows that if

$$
\min\{d_1, d_2\} > \frac{1}{\lambda_1}\max\{A, B\}:\ = d^*,
\tag{22}
$$

then

$$
\nabla(u-\overline{u}) = \nabla(v-\overline{v}) = 0.
\tag{23}
$$

and (u, v) must be a constant solution. □

3. Non-Negative Constant Steady-State Solutions

In this section, the stability of non-negative constant steady state solutions of (3) will be investigated by the standard linearization theory. By [17], under particular situations, (3) has the non-negative constant steady state solutions as follows.

(1) the trivial solution $E_0 = (0, 0)$ always exists.
(2) if $a \in (0, 1)$, there is no boundary constant solution when $a < (a+1)^2/4 < m$.
(3) if $a \in (0, 1)$, then $E_1(1-a/2, 0)$ is unique boundary equilibria when $a < m = (a+1)^2/4$.
(4) if $a \in (0, 1)$, there is two boundary constant solution $E_2(1-a-\sqrt{(a+1)^2-4m}/2, 0)$ and $E_3(1-a+\sqrt{(a+1)^2-4m}/2, 0)$ when $a < m < (a+1)^2/4$.
(5) if $a \in (0, 1)$, there is unique boundary constant solution E_3 under the condition of $0 < m \le a < (a+1)^2/4$.
(6) if $a = 1$, there is unique boundary constant solution $E_4(\sqrt{1-m}, 0)$ only when $0 < m < 1$.

(7) if $a > 1$, there is unique boundary constant solution $E_5(1 - a + \sqrt{(a+1)^2 - 4m}/2, 0)$ when $0 < m < a < (a+1)^2/4$.

(8) there is unique positive constant solution $E^*(d/cb, -b + \sqrt{\Delta}/2bf) = (u^*, v^*)$ with $\Delta = b^2 + 4bfr(1 - u^* - m/u^* + a)$ when $1 - u^* - m/u^* + a > 0$.

Under the no-flux boundary condition, $-\Delta$ has eigenvalues $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots$ and $\lim_{i\longrightarrow\infty}\lambda_i = \infty$. Let $X(\lambda_i)$ be the eigenspace generated by the eigenfunctions corresponding to λ_i. Let m_i be the algebraic multiplicity of λ_i. Let $\phi_{ij}(i \ge 0, 1 \le j \le m_i)$ be the normalized eigenfunctions corresponding to λ_i. Then, the set $\{\phi_{ij}(1 \le j \le m_i)\}$ forms a complete orthonormal basis in $L^2(\Omega)$.

Next, we consider the stability of constant steady state solutions.

Theorem 3. *For all constants $a, b, c, d, r, f, d_1, d_2 > 0$, we have that*

(1) For trivial solution E_0, if $m > a$, then E_0 is locally asymptotically stable; if $m < a$, then E_0 is unstable

(2) If $0 < a < m = (a+1)^2/4 < 1$, then E_1 is unstable

(3) If $0 < a < m < (a+1)^2/4 < 1$, then E_2 is unstable

(4) If $cbu_j - d < 0$, $E_j(j = 3, 4, 5)$ is stable and if $cbu_j - d > 0$, E_j is unstable

(5) E^* exists if and only if $-m/a + u^* - u^* + 1 > 0$. If $a > \sqrt{m} - u^*$, then E^* is stable. If $a < \sqrt{m} - u^*$, then E^* is unstable

Proof. We rewrite (3) as

$$\begin{cases} d_1\Delta u + F_1(u, v) = 0, x \in \Omega, \\ d_2\Delta v + F_2(u, v) = 0, kx \in \Omega, \\ \partial_\nu u = \partial_\nu v = 0, x \in \partial\Omega. \end{cases} \tag{24}$$

The linearization matrix of (3) at a constant solution $E = (u_0, V_0)$ can be expressed by

$$J = \begin{pmatrix} \partial_u F_1(u_0, v_0) + d_1\Delta & \partial_v F_1(u_0, v_0) \\ \partial_u F_2(u_0, v_0) & \partial_v F_2(u_0, v_0) + d_2\Delta \end{pmatrix}. \tag{25}$$

where

$$\begin{aligned} \partial_u F_1(u_0, v_0) &= -bv + \frac{ru\left(-1 + m/(a+u)^2\right)}{1 + fv} \\ &\quad + \frac{r(1 - u - m/a + u)}{1 + fv}, \\ \partial_v F_1(u_0, v_0) &= -bu - \frac{fru(1 - u - m/a + u)}{(1 + fv)^2}, \\ \partial_u F_2(u_0, v_0) &= cbv, \\ \partial_v F_2(u_0, v_0) &= cbu - d. \end{aligned} \tag{26}$$

We define that $X_{ij} = \{a \cdot \phi_{ij}: a \in R^2\}$, $X_i = \oplus_{j=1}^{m_i} X_{ij}$, and $X = \oplus_{i=1}^{\infty} X_i$. Let $(\Phi(x), \Psi(x))$ be a pair of eigenfunction of J corresponding to an eigenvalue λ. Then, we have

$$J\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = \begin{pmatrix} f_u + d_1\Delta f_v \\ g_u g_v + d_2\Delta \end{pmatrix}\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = \lambda\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}. \tag{27}$$

We set

$$\begin{aligned} \Phi &= \sum_{0 \le i \le \infty, 1 \le j \le m_i} a_{ij}\phi_{ij}, \\ \Psi &= \sum_{0 \le i \le \infty, 1 \le j \le m_i} b_{ij}\psi_{ij}. \end{aligned} \tag{28}$$

Then, we obtain

$$\begin{aligned} &\sum_{0 \le i \le \infty, 1 \le j \le m_i} \begin{pmatrix} f_u + d_1\lambda_i & f_v \\ g_u & g_v + d_2\lambda_i \end{pmatrix}\begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix} \\ \phi_{ij} &= \sum_{0 \le i \le \infty, 1 \le j \le m_i} P_i\begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix} \\ \phi_{ij} &= \lambda\begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix}\phi_{ij}. \end{aligned} \tag{29}$$

From the chapter 5 of [29, 30], we know that if all the eigenvalues of J have negative real parts, then the constant solution E is locally asymptotically stable; J is unstable if there is an eigenvalue of J with positive real part; if all the eigenvalues have non-positive real parts while some eigenvalues have zero real parts, then the stability of E cannot be determined by the linearization. Furthermore, λ is an eigenvalue of J if and only if λ is an eigenvalue of the matrix $\lambda I - P_i$ for some $i \ge 0$. We have.

$$|\lambda I - P_i| = \lambda^2 - T_i\lambda + D_i, \tag{30}$$

where

$$\begin{aligned} T_i &= -(d_1 + d_2)\lambda_i + f_u + g_v, \\ D_i &= d_1 d_2\lambda_i^2 - (d_2 f_u + d_1 g_v)\lambda_i + f_u g_v - f_v g_u. \end{aligned} \tag{31}$$

(1) For trival solution $E_0 = (0, 0)$,

$$\begin{aligned} T_i &= -(d_1 + d_2)\lambda_i + r\left(1 - \frac{m}{a}\right) - d, \\ D_i &= d_1 d_2\lambda_i^2 + \left(dd_1 + r\left(\frac{m}{a} - 1\right)d_2\right)\lambda_i + dr\left(\frac{m}{a} - 1\right). \end{aligned} \tag{32}$$

If $m > a$, then for all eigenvalues λ, we have $T_i < 0$ and $D_i > 0$, which leads to $\mathrm{Re}\lambda < 0$. Hence, E_0 is locally asymptotically stable. If $m < a$, then for $i = 0$, there exists a positive eigenvalue $r(m/a - 1)$, which implies that E_0 is unstable. In addition, if $m = a = 1$, E_0 is stable, else if $m = a \ne 1$, E_0 is unstable.

(2) For $E_1(1 - a/2, 0)$, with $0 < a < m = (a+1)^2/4 < 1$,

$$T_i = -(d_1+d_2)\lambda_i + \frac{cb(a-1)}{2} - d,$$
$$D_i = d_1 d_2 \lambda_i^2 + d_1\left(\mathrm{d} + \frac{bc(1-a)}{2}\right)\lambda_i. \tag{33}$$

For corresponding ordinary system, E_1 is unstable, so for any $d_1, d_2 > 0$, E_1 is unstable.

(3) For $E_2(1-a-\sqrt{(a+1)^2-4m}/2, 0) = (u_2, v_2)$, with $0 < a < m < (a+1)^2/4 < 1$,

$$T_i = -(d_1+d_2)\lambda_i + ru_2\left(-1+\frac{m}{(u_2+a)^2}\right) + cbu_2 - d,$$
$$D_i = d_1 d_2 \lambda_i^2 - \left(d_2 ru_2\left(-1+\frac{m}{(u_2+a)^2}\right) + d_1(cbu_2-d)\right)\lambda_i + d_1 d_2 (cbu_2-d)\left(-ru_2+\frac{mru_2}{(u_2+a)^2}\right). \tag{34}$$

For $i=0$, there exists a positive eigenvalue $ru_2(-1+m/(u_2+a)^2)$. So, E_2 is always unstable.

(4) For $j = 3,4,5$, E_j is stable when $cbu_j - d < 0$, and in this case, $T_i < 0, D_i > 0$ for any $i \geq 0$. Additionally, in other cases, E_j is unstable, so for any $d_1, d_2 > 0$, E_j is unstable.

$$T_i = -(d_1+d_2)\lambda_i + ru_i\left(-1+\frac{m}{(u_i+a)^2}\right) + cbu_i - d,$$
$$D_i = d_1 d_2 \lambda_i^2 - \left(d_2 ru_i\left(-1+\frac{m}{(u_i+a)^2}\right) + d_1(cbu_i-d)\right)\lambda_i + ru_i\left(-1+\frac{m}{(u_i+a)^2}\right)(cbu_i-d). \tag{35}$$

(5) For positive constant solution, $E^*(u^*, v^*) = (d/cb, 1/2bf(-b+\sqrt{b^2+4bfr(1-u^*-m/u^*+a)}))$. The Jacobi matrix of (3) at E^* is

$$J(E^*) = \begin{pmatrix} \left(\frac{m}{(a+u^*)^2}-1\right)\frac{ru^*}{fv^*+1} - \mathrm{d}_1\lambda_i & \left(-\frac{m}{a+u^*}-u^*+1\right)\frac{fru^*}{(fv^*+1)^2} - bu^* \\ cbv^* & -d_2\lambda_i \end{pmatrix}. \tag{36}$$

It is noted that

$$T_i = -(d_1+d_2)\lambda_i + ru^*\left(\frac{m}{(a+u^*)^2}-1\right)\frac{1}{fv^*+1},$$
$$D_i = d_1 d_2 \lambda_i^2 - \left(d_2 ru^*\left(\frac{m}{(a+u^*)^2}-1\right)\frac{1}{fv^*+1}\right)\lambda_i + cbv^*\left(\left(-\frac{m}{a+u^*}-u^*+1\right)\frac{fru^*}{(fv^*+1)^2} + bu^*\right). \tag{37}$$

For E^* exists if and only if $-m/a+u^*-u^*+1>0$, so it is easy to conclude that $T_i<0, D_i>0$ if $m/(a+u^*)^2 - 1 < 0\,(a > \sqrt{m} - u^*)$, which implies that E^* is stable. If $a < \sqrt{m} - u^*$, for $i=0$, we obtain that $T_i > 0$ and $D_i > 0$, so it follows that there exist two of the eigenvalues with positive real parts, which implies that E^* is unstable. □

4. Existence of Non-Constant Positive Solutions

In this section, we consider the existence of non-constant positive solutions to (3) in $\Omega = [0, l\pi]$. First, the existence of spatially homogeneous and non-homogeneous periodic solutions is studied by taking m as the bifurcation parameter. Then, the structure and the stability of the bifurcation solutions that bifurcate from (u^*, v^*) are shown. From Theorem 3, the stability of (u^*, v^*) is determined by the trace and determinant of J. Furthermore, we will restrict $-m/a + u^* - u^* + 1 > 0$. To put out our discussion into the context of the Hopf bifurcation, we convert (3) into the following system by $\tilde{u} = u - u^*$ and $\tilde{v} = v - v^*$ and drop " $\sim$ " for simplicity. We have

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \dfrac{\partial^2 u}{\partial x^2} + r\left(u + u^*\right)\left(1 - \left(u + u^*\right) - \dfrac{m}{\left(u + u^*\right) + a}\right)\dfrac{1}{1 + f\left(v + v^*\right)} \\ - b\left(u + u^*\right)\left(v + v^*\right), x \in (0, l\pi), t > 0, \dfrac{\partial v}{\partial t} = d_2 \dfrac{\partial^2 v}{\partial x^2} + cb\left(u + u^*\right)\left(v + v^*\right) - d\left(v + v^*\right), x \in (0, l\pi), t > 0, \\ \partial_\nu u(0, t) = \partial_\nu u(l\pi, t) = \partial_\nu v(0, t) = \partial_\nu v(l\pi, t) = 0, t > 0. \end{cases} \tag{38}$$

Firstly, we define the real-valued Sobolev space

$$\mathcal{X} = \left\{(u, v) \in H^2([0, l\pi]) \times H^2([0, l\pi]): \frac{\partial u(0, t)}{\partial x} = \frac{\partial u(l\pi, t)}{\partial x} = \frac{\partial v(0, t)}{\partial x} = \frac{\partial v(l\pi, t)}{\partial x} = 0\right\}, \tag{39}$$

and the corresponding complexification space is given by $\mathcal{X}_C = \mathcal{X} \oplus i\mathcal{X} = \{a + ib: a, b \in \mathcal{X}\}$.

The linearized operator of the steady state system of (39) evaluated at $(m, 0, 0)$ is

$$L(m) = \left(\frac{ru^*\left(m/\left(a + u^*\right)^2 - 1\right)1/fv^* + 1 + d_1\partial^2/\partial x^2 - fru^*\left(-m/a + u^* - u^* + 1\right)1/\left(fv^* + 1\right)^2 - bu^*}{cbv^* d_2\partial^2/\partial x^2}\right), \tag{40}$$

where $\mathcal{X}_C$ is the domain of $L(m)$.

The adjoint operator of $L(m)$ is defined by

$$L^*(m) = \left(\frac{ru^*\left(m/\left(a + u^*\right)^2 - 1\right)1/fv^* + 1 + d_1\partial^2/\partial x^2 cbv^*}{-fru^*\left(-m/a + u^* - u^* + 1\right)1/\left(fv^* + 1\right)^2 - bu^* d_2\partial^2/\partial x^2}\right), \tag{41}$$

where the domain of $L^*(m)$ is $\mathcal{X}_C$.

The following condition in [31] is crucial to ensure that the Hopf bifurcation occurs.

(H1) There exists a neighborhood $\mathcal{O}$ of m_0 such that for $m \in \mathcal{O}$, $L(m)$ has a pair of complex, simple, conjugate eigenvalues $\alpha(m) \pm i\omega(m)$, continuously differentiable in m, with $\alpha(m_0) = 0, \omega(m_0) = \omega_0 > 0$, and $\alpha'(m_0) \neq 0$, all other eigenvalues of $L(m)$ have non-zero real parts for $m \in \mathcal{O}$.

Motived by [31], we apply the Hopf bifurcation theory to analyze our system. For the eigenvalue problem

$$\begin{aligned} -\varphi'' &= \lambda\varphi, x \in (0, l\pi), \\ \varphi'(0) &= \varphi'(l\pi) = 0, \end{aligned} \tag{42}$$

we know that the corresponding (42) eigenvalues are $\lambda_n = n^2/l^2$ $(n = 0, 1, \ldots)$, with corresponding eigenfunctions $\varphi_n(x) = \cos nx/l$. Let

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos\frac{nx}{l}, \tag{43}$$

be a pair of eigenfunctions of $L(m)$ corresponding to an eigenvalue $\rho(m)$, that is, $L(m)(\phi,\psi)^T = \rho(m)(\phi,\psi)^T$. By a straightforward analysis, we have

$$L_n(m)\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \rho(m)\begin{pmatrix} a_n \\ b_n \end{pmatrix}, n = 0, 1, \ldots, \quad (44)$$

where

$$L_n(m) = \left(\frac{ru^*\left(m/(a+u^*)^2 - 1\right)1/fv^* + 1 - d_1 n^2/l^2 - fru^*(-m/a+u^* - u^* + 1)1/(fv^*+1)^2 - bu^*}{cbv^* - d_2 n^2/l^2}\right). \quad (45)$$

Hence, the eigenvalues of $L(m)$ are given by the eigenvalues of $L_n(m)$, $(n = 0, 1, \ldots)$. The characteristic equation of $L_n(m)$ is

$$\rho^2 - T_n(m)\rho + D_n(m) = 0, \quad n = 0, 1, \ldots, \quad (46)$$

where

$$\begin{cases} T_n(m) = -\dfrac{(d_1+d_2)n^2}{l^2} + ru^*\left(\dfrac{m}{(a+u^*)^2} - 1\right)\dfrac{1}{fv^*+1}, \\ D_n(m) = \dfrac{d_1 d_2 n^4}{l^4} - \left(d_2 ru^*\left(\dfrac{m}{(a+u^*)^2} - 1\right)\dfrac{1}{fv^*+1}\right)\dfrac{n^2}{l^2} + cbv^*\left(fru^*\left(-\dfrac{m}{a+u^*} - u^* + 1\right)\dfrac{1}{(fv^*+1)^2} + bu^*\right). \end{cases} \quad (47)$$

$$T_0 = ru^*\left(\frac{m}{(a+u^*)^2} - 1\right)\frac{1}{fv^*+1},$$
$$D_0 = cbv^*\left[fru^*\left(-\frac{m}{a+u^*} - u^* + 1\right)\frac{1}{(fv^*+1)^2} + bu^*\right] > 0. \quad (48)$$

Therefore, the eigenvalues are determined by

$$\rho(m) = \frac{T_n(m) \pm \sqrt{T_n^2(m) - 4D_n(m)}}{2}, \quad n = 0, 1, \ldots \quad (49)$$

If the condition (H1) holds, $L(a)$ has a pair of simple purely imaginary $\pm i\omega_0$ at $a = a_0$, if and only if there exists a unique $n \in N$ such that $\pm i\omega_0$ are the purely imaginary eigenvalues of $L_n(m)$. The related eigenvector is denoted by $q = q_n = (a_n, b_n)^T \cos nx/l$, with $a_n, b_n \in C$, such that $L(m_0)q = i\omega_0 q$.

We identify the Hopf bifurcation point m_0 which satisfies the condition (H1): there exists $n \in N$ such that

$$T_n(m_0) = 0, D_n(m_0) > 0, T_j(m_0) \neq 0, D_j(m_0) \neq 0 \text{ for } j \neq n, \quad (50)$$

and for the unique pair of complex eigenvalues near the imaginary axis $\alpha(m) \pm i\omega(m)$

$$\alpha'(m_0) \neq 0. \quad (51)$$

It is easy to obtain $T_n(m) < 0$ and $D_n(m) > 0$ if $0 < m < (a+u^*)^2$, which implies that the steady state (u^*, v^*) is locally asymptotically stable. Hence, any potential bifurcation points must be in the interval $[(a+u^*)^2, (a+u^*)(1-u^*))$. This means that $u^* < 1 - a/2$ is essential for bifurcation condition. For any Hopf bifurcation point m_0 in $[(a+u^*)^2, (a+u^*)(1-u^*))$, $\alpha(m) \pm i\omega(m)$ are the eigenvalues of $L_n(m)$, where

$$\alpha(m) = \frac{ru^*\left(m/(a+u^*)^2 - 1\right)1/fv^* + 1}{2} - \frac{(d_1+d_2)n^2}{2l^2},$$
$$\omega(m) = \sqrt{D_n(m) - \alpha^2(m)}, \quad (52)$$

$$\alpha'(m) > 0, \quad (53)$$

for m in $[(a+u^*)^2, (a+u^*)(1-u^*))$. Hence, the transversality condition is always satisfied.

From the discussion above, the determination of Hopf bifurcation points reduces to describing the set

$$\Gamma := \left\{ m \in \left[(a+u^*)^2, (a+u^*)(1-u^*) \right) : \text{for some } n \in N, \text{which satisfies the condition } (H1) \right\}. \tag{54}$$

when a set of parameters $d_1, d_2, l, a, b, c, d, f, r$ are given.

In the following, for $d_1, d_2, a, b, c, d, f, r > 0$ and $0 < m < (a+u^*)(1-u^*)$ fixed, we choose l appropriately. $m^H = (a+u^*)^2$ is always an element of Γ for any $l > 0$ because of $T_0(m_0^H) = 0, T_j(m_0^H) < 0$ for any $j > 1$, and $D_k(m_0^H) > 0$ for any $k \in N$. This corresponds to the Hopf bifurcation of spatially homogeneous periodic solution. Apparently, m_0^H is also the unique value m for the Hopf bifurcation of spatially homogeneous periodic solution for any $l > 0$.

In the following, we search for spatially non-homogeneous Hopf bifurcation points for $n \geq 1$. As $T_0(m_0^H) = 0$ and $T_0'(m) > 0$ for $m \in [m_0^H, (a+u^*)(1-u^*))$, we obtain that $0 < T_0(m) < T_0((a+u^*)(1-u^*)) = ru^*(1-a-2u^*)/((a+u^*)(fv^*+1)) := M_*$ for $m \in (m_0^H, (a+u^*)(1-u^*))$. We define

$$l_n = n\sqrt{\frac{d_1+d_2}{M_*}}, n \in N^*. \tag{55}$$

Then for $l_n < l \leq l_{n+1}$, and $1 \leq j \leq n$, we derive the root of $T_0(m) = (d_1+d_2)j^2/l^2$ as m_j^H such that $m_0^H < m_j^H < (a+u^*)(1-u^*)$. Moreover, by $T_0'(m) > 0$ in $[m_0^H, (a+u^*)(1-u^*))$, we derive (56) and (57)

$$0 < m_0^H < m_1^H < m_2^H < \ldots < m_n^H < (a+u^*)(1-u^*) \tag{56}$$

$$T_j(m_j^H) = 0, T_i(m_j^H) \neq 0 \text{for} i \neq j. \tag{57}$$

Since $D_j(m_j^H) > 0$, now we discuss a condition to verify $D_n(m_j^H) \neq 0$ for $j \neq n$. For $m \in [m_0^H, (a+u^*)(1-u^*))$, we have

$$D_i(m) = \frac{d_1 d_2 i^4}{l^4} - d_2 T_0(m)\frac{i^2}{l^2} + D_0(m) := \tau\left(\frac{i^2}{l^2}\right). \tag{58}$$

The quadratic function $\tau(i^2/l^2)$ is positive for all $l \in R$ if the discriminant of $\tau(i^2/l^2) = 0$ is negative, which means that (60)

$$d_2^2 T_0^2(m) - 4d_1 d_2 D_0 = d_2^2\left(ru^*\left(\frac{m}{(a+u^*)^2} - 1\right)\frac{1}{fv^*+1}\right)^2, -4d_1 d_2 cbv^*\left(fru^*\left(-\frac{m}{a+u^*} - u^* + 1\right)\frac{1}{(fv^*+1)^2} + bu^*\right). \tag{59}$$

We note that

$$\overline{f} = \left(\frac{r^2 d_2^2 u^{*4}\left(m/(a+u^*)^2 - 1\right)^4}{16 d_1^2 c^2 (1-u^* - m/u^* + a)^2} - b^2\right) \tag{60}$$

$$\frac{1}{4br(1-u^* - m/u^* + a)}.$$

For $(a+u^*)^2 \leq m < (a+u^*)(1-u^*)$, we can choose $f > \overline{f}$ such that the discriminant of $\tau(i^2/l^2) = 0$ is negative. Then, $\tau(i^2/l^2) > 0$ for $i \in N$ such that $D_i(m_j^H) > 0$.

We summarize our analysis above and apply Theorem 2 in [31]. The existence of both spatially homogeneous and non-homogeneous periodic solutions bifurcation from (u^*, v^*) can be obtained as follows:

Theorem 4. *For any l in $(l_n, l_{n+1}]$ and $f > \overline{f}$, system (2) undergoes Hopf bifurcation at each $m = m_j^H (1 \leq j \leq n)$. Moreover, the bifurcation periodic solutions near $(m, u, v) = (m_j^H, u^*, v^*)$ can be parameterized as $(m(s), u(s), v(s))$ so that $m(s) \in L^\infty$ in the form of $m(s) = m_j^H + o(s)$ for $s \in (0, \delta)$ for some small $\delta > 0$, (61) and (62)*

$$\begin{cases} u(s)(t,x) & = u^* + s\left(a_n e^{i2\pi t/T(s)} + \overline{a}_n e^{-i2\pi t/T(s)}\right)\cos\frac{nx}{l} + o(s^2), \\ v(s)(t,x) & = v^* + s\left(b_n e^{i2\pi t/T(s)} + \overline{b}_n e^{-i2\pi t/T(s)}\right)\cos\frac{nx}{l} + o(s^2), \end{cases} \tag{61}$$

where (a_n, b_n) is the corresponding eigenvector, and

$$T(s)=\frac{2\pi}{\omega_0}\left(1+\tau_2 s^2\right)+o\left(s^4\right),$$

$$\tau_2=-\frac{1}{\omega_0}\left(\operatorname{Im}\left(c_1\left(m_j^H\right)\right)-\frac{Re\left(c_1\left(m_j^H\right)\right)}{\alpha'\left(m_j^H\right)}\omega_0'\left(m_j^H\right)\right), \tag{62}$$

$$T''(0)=\frac{4\pi}{\omega_0}\tau_2=-\frac{4\pi}{\omega_0^2}\left(\operatorname{Im}\left(c_1\left(m_j^H\right)\right)-\frac{Re\left(c_1\left(m_j^H\right)\right)}{\alpha'\left(m_j^H\right)}\omega_0'\left(m_j^H\right)\right). \tag{63}$$

Furthermore, we notice that

(1) The bifurcating periodic orbits from $m=m_0^H$ are spatially homogeneous, which coincide with the periodic orbits of the corresponding ODE system

(2) The bifurcating periodic orbits from $m=m_j^H$ are spatially non-homogeneous.

Then, we consider the direction and stability of spatially homogeneous Hopf bifurcation.

Theorem 5. *For system (2), if all other eigenvalues of $L_n(m_0^H)$ have negative real parts and $Re(c_1(m_0^H))<0$ (resp. >0), the spatially homogeneous periodic solutions bifurcating from $m=m_0^H$ are locally asymptotically stable (resp. unstable). Moreover, the Hopf bifurcation at m_0^H is supercritical (resp. subcritical) if $1/\alpha'(m_0^H)Re(c_1(m_0^H))<0$ (resp. >0).*

Proof. Here, the notations and calculations in [31] are used in the same way. For the sake of simplicity, we denote $1-u^*-m/u^*+a$ by M. Then, we introduce

$$\begin{cases} q:=\begin{pmatrix} a_0 \\ b_0 \end{pmatrix}=\begin{pmatrix} 1 \\ --cb+c\sqrt{b^2+4bfrM}/2f\omega_0 i \end{pmatrix}, \\ q^*:=\begin{pmatrix} a_0^* \\ b_0^* \end{pmatrix}=\begin{pmatrix} 1/2\pi l \\ -\omega_0 f/\left(-cb+c\sqrt{b^2+4bfrM}\right)\pi li \end{pmatrix}, \end{cases} \tag{64}$$

such that $\langle q^*,q\rangle=1$, $\langle q^*,\overline{q}\rangle=0$, $L(m_0^H)q=i\omega_0 q$ and $L^*(m_0^H)q^*=-i\omega_0 q^*$, where

$$\omega_0=2b\frac{\sqrt{u^* cM\sqrt{b^2+4bfrM}}}{b+\sqrt{b^2+4bfrM}}. \tag{65}$$

And $\langle u,v\rangle=\int_0^{l\pi}\overline{u}^T v dx$ denotes the inner product in $L^2(0,l\pi)\times L^2(0,l\pi)$. Then, we get the derivatives at (u^*,v^*,m_0^H) as follows:

$$\begin{aligned}
f_{uu}&=-\frac{2r\left(a^3+3a^2u^*-am+3au^{*2}+u^{*3}\right)}{(a+u^*)^3(fv^*+1)}, g_{uu}=0,\\
f_{uv}&=\frac{fr\left(a^2(2u^*-1)+a(m+2u^*(2u^*-1))+u^{*2}(2u^*-1)\right)}{(a+u^*)^2(fv^*+1)^2}-b, g_{uv}=cb,\\
f_{vv}&=\frac{2f^2ru^*M}{(1+fv^*)^3}, g_{vv}=0,\\
f_{uuu}&=-\frac{6amr}{(a+u^*)^4(1+fv^*)}, g_{uuu}=0,\\
f_{uuv}&=\frac{2fr\left(a^3+3a^2u^*-am+3au^{*2}+u^{*3}\right)}{(a+u^*)^3(fv^*+1)^2}, g_{uuv}=0,\\
f_{uvv}&=-\frac{2f^2r\left(a^2(2u^*-1)+a(m+2u^*(2u^*-1))+u^{*2}(2u^*-1)\right)}{(a+u^*)^2(fv^*+1)^3}, g_{uvv}=0,\\
f_{vvv}&=-\frac{6f^3ru^*M}{(1+fv^*)^4}, g_{vvv}=0.
\end{aligned} \tag{66}$$

In addition, we note

$$Q_{qq}=\begin{pmatrix} c_n \\ d_n \end{pmatrix}\cos^2\frac{nx}{l},$$
$$Q_{q\bar{q}}=\begin{pmatrix} e_n \\ f_n \end{pmatrix}\cos^2\frac{nx}{l}, \tag{67}$$
$$Q_{qq\bar{q}}=\begin{pmatrix} g_n \\ h_n \end{pmatrix}\cos^3\frac{nx}{l},$$

where $c_n, d_n, e_n, f_n, g_n, h_n$ are defined as the same with [31].

$$\begin{aligned}
c_n &= f_{uu}a_n^2 + 2f_{uv}a_nb_n + f_{vv}b_n^2,\\
d_n &= g_{uu}a_n^2 + 2g_{uv}a_nb_n + g_{vv}b_n^2,\\
e_n &= f_{uu}|a_n|^2 + f_{uv}\left(a_n\bar{b}_n + \bar{a}_nb_n\right) + f_{vv}|b_n|^2,\\
f_n &= g_{uu}|a_n|^2 + g_{uv}\left(a_n\bar{b}_n + \bar{a}_nb_n\right) + g_{vv}|b_n|^2,\\
g_n &= f_{uuu}|a_n|^2a_n + f_{uuv}\left(2|a_n|^2b_n + a_n^2\bar{b}_n\right) + f_{uvv}\left(2|b_n|^2a_n + b_n^2\bar{a}_n\right) + f_{vvv}|b_n|^2b_n,\\
h_n &= g_{uuu}|a_n|^2a_n + g_{uuv}\left(2|a_n|^2b_n + a_n^2\bar{b}_n\right) + g_{uvv}\left(2|b_n|^2a_n + b_n^2\bar{a}_n\right) + g_{vvv}|b_n|^2b_n.
\end{aligned} \tag{68}$$

For $n = 0$, by calculation, we derive (69)

$$\begin{aligned}
c_0 &= -\frac{2cf^2r^3M^2}{(fv^*+1)^3\sqrt{4bfrM+b^2}} + \frac{2r\left(m/(a+u^*)^2-1\right)}{fv^*+1} - \frac{2mru^*}{(a+u^*)^3(fv^*+1)}\\
&\quad - \frac{i2crM\left(-frM/(fv^*+1)^2 - fru^*\left(m/(a+u^*)^2-1\right)/(fv^*+1)^2 - b\right)}{\sqrt{cu^*M\sqrt{4bfrM+b^2}}},\\
d_0 &= -\frac{i2c^2brM}{\sqrt{cu^*M\sqrt{b(4frM+b)}}},\\
e_0 &= \frac{2cf^2r^3M^2}{(fv^*+1)^3\sqrt{4bfrM+b^2}} - \frac{2rmu^*}{(a+u^*)^3(fv^*+1)} + \frac{2r\left(m-(a+u^*)^2\right)}{(a+u^*)^2(1+fv^*)},\\
g_0 &= \frac{r}{32(fv^*+1)^4}\left(\frac{192ic^3f^3r^3u^*M^4}{\left(cu^*M\sqrt{b^2+4bfrM}\right)^{3/2}} + \frac{64cf^2r^2M^2\left(ma+(2u^*-1)(a+u^*)^2\right)(fv^*+1)}{u^*(a+u^*)((a+u^*)(u^*-1)+m)\sqrt{b^2+4bfrM}}\right.\\
&\quad \left. + \frac{64icfrM\left((a+u^*)^3-am\right)(fv^*+1)^2}{(a+u^*)^3\sqrt{cu^*M\sqrt{b^2+4bfrM}}} + \frac{192mu^*(fv^*+1)^3}{(a+u^*)^4} - \frac{192m(fv^*+1)^3}{(a+u^*)^3}\right),\\
f_0 &= h_0 = 0.
\end{aligned} \tag{69}$$

Then, we can obtain (70)

$$
\begin{aligned}
\langle q^*, Q_{qq}\rangle &= bc - \frac{cf^2r^3M^2}{(fv^*+1)^3\sqrt{b^2+4bfrM}} - \frac{mru^*}{(a+u^*)^3(fv^*+1)} + \frac{r\left(m-(a+u^*)^2\right)}{(a+u^*)^2(1+fv^*)} \\
&\quad + \frac{icrM\left(b-\left(fr\left((a+u^*)^2(2u^*-1)+am\right)/(a+u^*)^2(1+fv^*)^2\right)\right)}{\sqrt{cu^*M\sqrt{b^2+4bfrM}}}, \\
\langle q^*, Q_{q\overline{q}}\rangle &= \frac{cf^2r^3M^2}{(fv^*+1)^3\sqrt{b^2+4bfrM}} - \frac{mru^*}{(a+u^*)^3(fv^*+1)} + \frac{r\left(m-(a+u^*)^2\right)}{(a+u^*)^2(fv^*+1)}, \\
\langle \overline{q}^*, Q_{qq}\rangle &= -bc - \frac{cf^2r^3M^2}{(fv^*+1)^3\sqrt{b^2+4bfrM}} - \frac{mru^*}{(a+u^*)^3(fv^*+1)} + \frac{r\left(m-(a+u^*)^2\right)}{(a+u^*)^2(1+fv^*)} \\
&\quad + \frac{icrM\left(b-\left(fr\left((a+u^*)^2(2u^*-1)+am\right)/(a+u^*)^2(1+fv^*)^2\right)\right)}{\sqrt{cu^*M\sqrt{b^2+4bfrM}}}, \\
\langle \overline{q}^*, Q_{q\overline{q}}\rangle &= \frac{cf^2r^3M^2}{(fv^*+1)^3\sqrt{b^2+4bfrM}} - \frac{mru^*}{(a+u^*)^3(fv^*+1)} + \frac{r\left(m-(a+u^*)^2\right)}{(a+u^*)^2(fv^*+1)}. \\
\langle \overline{q}^*, C_{qq\overline{q}}\rangle &= \frac{r}{64(fv^*+1)^4}\left(\frac{192ic^3f^3r^3u^*M^4}{\left(cu^*M\sqrt{b^2+4bfrM}\right)^{3/2}} + \frac{64cf^2r^2M^2\left(ma+(2u^*-1)(a+u^*)^2\right)(fv^*+1)}{u^*(a+u^*)((a+u^*)(u^*-1)+m)\sqrt{b^2+4bfrM}}\right. \\
&\quad \left. + \frac{64icfrM\left((a+u^*)^3-am\right)(fv^*+1)^2}{(a+u^*)^3\sqrt{cu^*M\sqrt{b^2+4bfrM}}} + \frac{192mu^*(fv^*+1)^3}{(a+u^*)^4} - \frac{192m(fv^*+1)^3}{(a+u^*)^3}\right).
\end{aligned}
\tag{70}
$$

And, we note

$$
\begin{aligned}
\omega_{20} &= \left[2i\omega_0 I - L\left(m_0^H\right)\right]^{-1}H_{20}, \\
\omega_{11} &= -\left[L\left(m_0^H\right)\right]^{-1}H_{11}, \\
H_{20} &= \begin{pmatrix} c_0 \\ d_0 \end{pmatrix} - \langle q^*, Q_{qq}\rangle \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} - \langle \overline{q}^*, Q_{qq}\rangle \begin{pmatrix} \overline{a}_0 \\ \overline{b}_0 \end{pmatrix} = 0, \\
H_{11} &= \begin{pmatrix} e_0 \\ f_0 \end{pmatrix} - 0\langle q^*, Q_{q\overline{q}}\rangle \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} - \langle \overline{q}^*, Q_{q\overline{q}}\rangle \begin{pmatrix} \overline{a}_0 \\ \overline{b}_0 \end{pmatrix} = 0.
\end{aligned}
\tag{71}
$$

Hence, $\omega_{20} = \omega_{11} = 0$, $\langle q^*, Q_{\omega_{20}\overline{q}}\rangle = \langle q^*, Q_{\omega_{11}q}\rangle = 0$. By further calculation, we obtain that

$$\mathrm{Re}\left(c_1\left(m_0^H\right)\right) = \mathrm{Re}\left\{\frac{i}{2\omega_0}\langle q^*, Q_{qq}\rangle \cdot \langle q^*, Q_{q\bar{q}}\rangle + \frac{1}{2}\langle q^*, C_{qq\bar{q}}\rangle\right\}$$

$$= -\frac{1}{2\omega_0}\left(\frac{crM\left(b - fr\left((a+u^*)^2(2u^*-1)+am\right)/(a+u^*)^2(1+fv^*)^2\right)}{\sqrt{cu^*M\sqrt{b^2+4bfrM}}}\right.$$

$$\cdot\left(\frac{cf^2r^3M^2}{(fv^*+1)^3\sqrt{b^2+4bfrM}} - \frac{mru^*}{(a+u^*)^3(fv^*+1)} + \frac{r\left(m-(a+u^*)^2\right)}{(a+u^*)^2(fv^*+1)}\right)$$

$$\left. + \frac{1}{128(1+fv^*)^4}\left(\frac{64cf^2r^2M^2\left(ma+(2u^*-1)(a+u^*)^2\right)(fv^*+1)}{u^*(a+u^*)((a+u^*)(u^*-1)+m)\sqrt{b^2+4bfrM}} + \frac{192mu^*(fv^*+1)^3}{(a+u^*)^4} - \frac{192m(fv^*+1)^3}{(a+u^*)^3}\right)\right). \tag{72}$$

From above analysis, we know that $\alpha'(m_0^H) > 0$. Hence, by Theorem 2 in [31], the bifurcating solutions bifurcated from (m_0^H, u^*, v^*) are locally asymptotically stable (resp. unstable) if $\mathrm{Re}(c_1(m_0^H)) < 0$ (resp. > 0) and $T_j(m_0^H) < 0$, $D_j(m_0^H) > 0$ for $j \geq 1$, and the Hopf bifurcation at m_0^H is supercritical (resp. subcritical) if $1/\alpha'(m_0^H)\mathrm{Re}(c_1(m_0^H)) < 0$ (resp. > 0). The proof is complete.

Inspired by [31, 32], we take m as the bifurcation parameter and also restrict $(a+u^*)^2 \leq m < (a+u^*)(1-u^*)$. We suppose that $\Omega = (0, l\pi)$. The non-negative steady state solutions of (72) satisfy the elliptic problem corresponding to

$$\begin{cases} d_1\dfrac{\partial^2 u}{\partial x^2} + r(u+u^*)\left(1-(u+u^*) - \dfrac{m}{(u+u^*)+a}\right)\dfrac{1}{1+f(v+v^*)} - b(u+u^*)(v+v^*) = 0, x \in (0, l\pi), t > 0, \\ d_2\dfrac{\partial^2 v}{\partial x^2} + cb(u+u^*)(v+v^*) - d(v+v^*) = 0, x \in (0, l\pi), t > 0, \partial_\nu u(0,t) = \partial_\nu u(l\pi, t) = \partial_\nu v(0,t) = \partial_\nu v(l\pi,t) = 0. \end{cases} \tag{73}$$

From Theorem 3, we know that (u^*, v^*) is locally asymptotically stable for $0 < m < (a+u^*)^2$ and unstable for $(a+u^*)^2 < m < (a+u^*)(1-u^*)$.

The steady state bifurcation point m_0 satisfies the steady state bifurcation condition (H2) in [31]:

$$D_n(m_0) = 0, T_n(m_0) \neq 0, \text{and } T_j(m_0) \neq 0, D_j(m_0) \neq 0 \text{ for } j \neq n \in N_0. \tag{74}$$

$$\frac{\mathrm{d}}{\mathrm{d}m}D_n(m_0) \neq 0. \tag{75}$$

It means that the potential steady state bifurcation points m shall satisfy conditions (74) and (75). Recall that for $m \in (0, m_0^H]$, $T_n(m) \leq 0$ and $D_n(m) > 0$. Then, any potential bifurcation point m_0 must be in the interval $(m_0^H, (a+u^*)(1-u^*))$. Hence, the steady state bifurcation points reduces to the set

$$\Lambda: = \left\{m \in \left(m_0^H, (a+u^*)(1-u^*)\right): \text{for some } n \in N, (74) \text{ and } (75) \text{are satisfied}\right\}, \tag{76}$$

when a set of parameters $(d_1, d_2, a, b, c, d, f, r, l)$ are fixed.

Recall that $D_n(m) = d_1 d_2 \rho^2 - d_2 T_0(m)\rho + D_0$, where $\rho = n^2/l^2$. By solving $D_n(m) = 0$, we have

$$\rho = \rho_\pm(m): = \frac{\mathrm{d}_2 T_0(m) \pm \sqrt{\mathrm{d}_2^2 T_0^2(m) - 4\mathrm{d}_1\mathrm{d}_2 D_0(m)}}{2\mathrm{d}_1\mathrm{d}_2}. \tag{77}$$

We define that

$$z(m) = d_2 r^2 u^{*2}\left(\frac{m}{(a+u^*)^2} - 1\right)^2 - 4d_1 cr\left(1 - u^* - \frac{m}{u^*+a}\right)\sqrt{b^2 + 4bfr\left(1 - u^* - \frac{m}{u^*+a}\right)},$$

$$B(m) = d_2^2 T_0^2(m) - 4d_1 d_2 D_0(m) = \frac{4b^2 d_2 u^*}{\left(b + \sqrt{b^2 + 4bfr(1 - u^* - m/u^* + a)}\right)^2} z(m). \tag{78}$$

For $z(m)$, we have $z'(m) > 0$ and $z(a+u^*)^2 < 0$, $z((a+u^*)(1-u^*)) > 0$ for $(a+u^*)^2 \le m < (a+u^*)(1-u^*)$.

Hence, there exists a unique root of $z(m) = 0$ denoted by m^B, which implies that $B(m^B) = 0$ and $\rho_\pm(m) > 0$ exists only for $m^B \le m < (a+u^*)(1-u^*)$. Therefore, the potential steady state bifurcation points reduces to the set (79).

$$\Theta:= \left\{m \in \left[m^B, (a+u^*)(1-u^*)\right): \text{for some } n \in N, (74) \text{ and } (75) \text{are satisfied}\right\}. \tag{79}$$

Then, the properties of $\rho_\pm(m)$ can be summarized as follows: □

Lemma 2. *We assume that $d_1, d_2, f > 0, u^* < 1 - a/2$. Then, for any $m \in [m^B, (a+u^*)(1-u^*))$, $\rho_\pm(m)$ exists. Moreover, $\rho_+(m)$ is increasing and $\rho_-(m)$ is decreasing.*

$$\lim_{m \longrightarrow m^B} \rho_+(m) = \lim_{m \longrightarrow m^B} \rho_-(m) = \frac{T_0(m^B)}{2d_1},$$

$$\lim_{m \longrightarrow m^B} \rho_+'(m) = +\infty, \quad \lim_{m \longrightarrow m^B} \rho_-'(m) = -\infty,$$

$$\rho_+((a+u^*)(1-u^*)) = \frac{1}{d_1} ru^*\left(\frac{1-u^*}{a+u^*} - 1\right), \tag{80}$$

$$\rho_-((a+u^*)(1-u^*)) = 0.$$

Proof. The first limit equation is trivial, so we omit here. We mainly analyze the monotonicity result on $\rho_\pm((a+u^*)(1-u^*))$ with respect to m for $m \in (m^B, (a+u^*)(1-u^*)]$.

Differentiating $D_n(m)$ with respect to m, it follows that

$$2d_1 d_2 \rho_\pm(m)\rho_\pm{}'(m) - d_2 T_0(m)\rho_\pm(m) - d_2 T_0 \rho_\pm{}'(m) + D_0' = 0. \tag{81}$$

Hence, $\rho_\pm{}'(m) = d_2 T_0'(m)\rho_\pm(m) - D_0'(m)/2d_1 d_2 \rho_\pm - d_2 T_0(m)$. It is easy to get $2d_1 d_2 \rho_+(m) - d_2 T_0(m) > 0$ and $2d_1 d_2 \rho_-(m) - d_2 T_0(m) < 0$ from (77). In addition, by calculation, we obtain that for $m \in (m^B, (1-u^*)(a+u^*))$, $d_2 T_0'(m)\rho_\pm(m) - D_0'(m) > 0$. The proof is completed.

It follows from Lemma 1 that the curve $(m, \rho_\pm)$ forms a smooth connected curve which connects $(m, \rho) = (m^B, T_0(m^B)/2d_1)$, $((1-u^*)(a+u^*), 1/d_1 ru^*(1-u^*/a+u^*-1)$, and $(m, \rho) = (m, 0)$.

By the properties of $\rho_\pm$, if

$$0 < \frac{n^2}{l^2} < \frac{1}{d_1} ru^*\left(\frac{1-u^*}{a+u^*} - 1\right), \tag{82}$$

then there exists $m_n^B \in [m^B, (1-u^*)(a+u^*))$ such that $\rho_+(m_n^B) = n^2/l^2$ or $\rho_-(m_n^B) = n^2/l^2$, and thus $D_n(m_n^B) = 0$. We define $\bar{l} = n/\sqrt{1/d_1 ru^*(1-u^*/a+u^*-1)}$. Then, for any $l > \bar{l}$, there exists a m_n^B such that $D_n(m_n^B) = 0$.

Next, we verify $\mathrm{d}D_n(m_n^B)/\mathrm{d}m \neq 0$. We recall that $D_n'(m) = -d_2 n^2/l^2 T_0'(m) + D_0'(m)$. Moreover, we know that $T_0'(m) > 0$ and $D_0'(m) < 0$. It follows that $\mathrm{d}D_n(m_n^B)/\mathrm{d}m < 0$. □

5. Numerical Simulations

In this section, in order to reveal the influence of fear effect, Allee effect, and other factors on the predator-prey model, a numerical method is used to analyze the effect of parameters on the asymptotic behavior of system (2) so as to verify and supplement the theoretical results mentioned before.

In Figure 1, we choose $d_1 = 0.1, d_2 = 0.1, a = 0.5, b = 1, c = 1, d = 0.2, r = 1, f = 15$. Varying the parameter m and choosing the initial data near (u^*, v^*), we indicate the following numerical results on the effects of parameter m:

(1) Take $0 < m = 0.3 < a$. Since $m < (d/bc + a)^2$, $(u^*, v^*) \approx (0.2000, 0.1275)$ is locally asymptotically stable by Theorem 3. The simulation results indicate that system (2) converges to the equilibrium (see Figures 1(a) and 1(b)).

(2) Taking $m = 0$, there is no Allee effect on prey. Since $m < (d/bc + a)^2$, $(u^*, v^*) \approx (0.2000, 0.2000)$ is locally asymptotically stable by Theorem 3. The simulation results indicate that system (2) converges to the equilibrium (see Figures 1(c) and 1(d)).

(3) Taking $0 < (d/bc + a)^2 < m = 0.495 < a < (d/bc + a)(1 - d/bc)$, it satisfies the condition of the weak Allee effect and Hopf bifurcation condition by Theorem 4. The simulation results indicate that system (2) undergoes Hopf bifurcation (see Figures 1(e) and 1(f)).

(4) Taking $0 < (d/bc + a)^2 < a < m = 0.51 < (d/bc + a)(1 - d/bc)$, it satisfies the condition of the strong Allee effect and Hopf bifurcation condition by

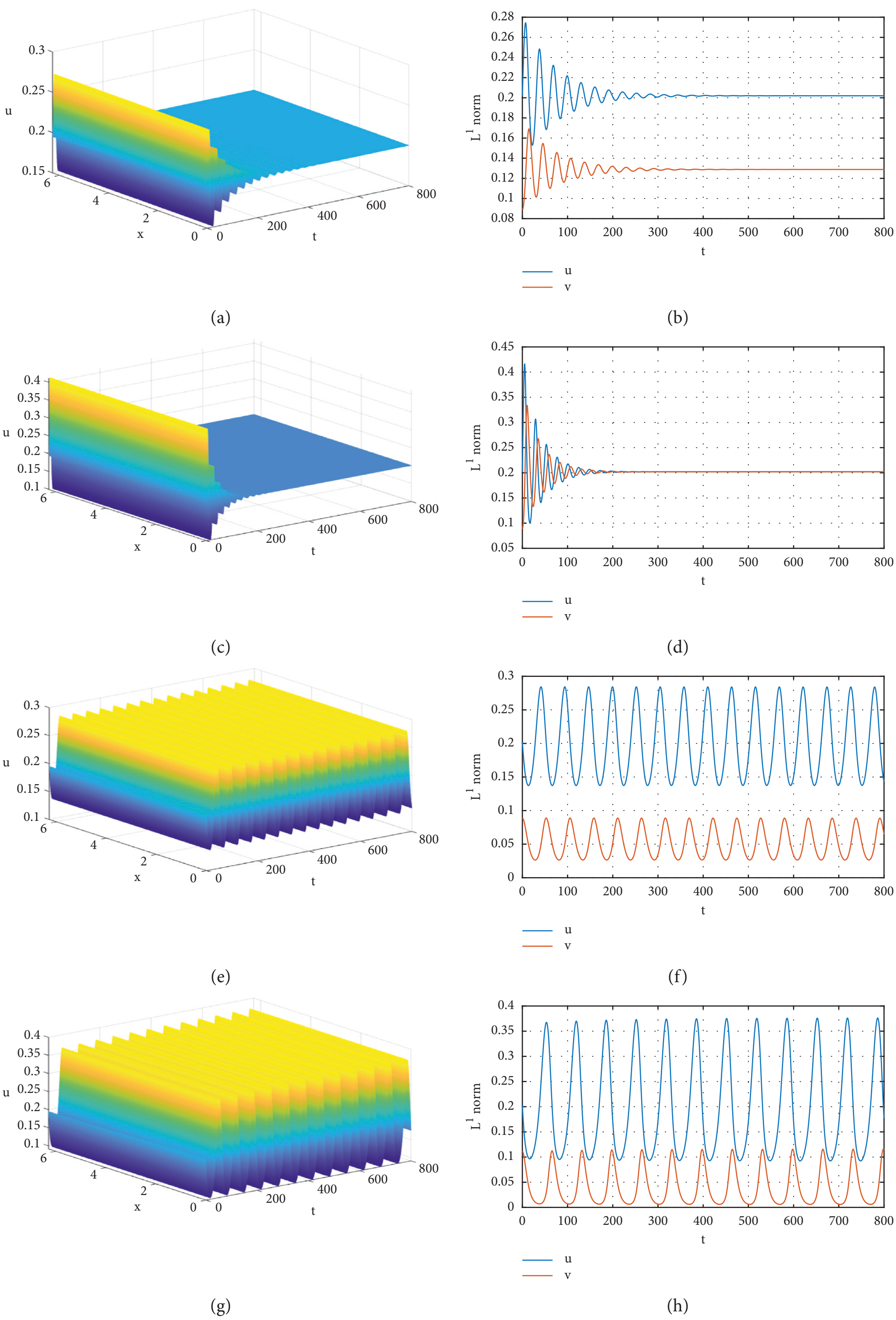

Figure 1: The effects of parameter m for $T = 800, l = 2$. The values of parameter m are as follows: (a, b) $m = 0.3$; (c, d) $m = 0$; (e, f) $m = 0.495$; (g, h) $m = 0.51$. The right column is the L^1 norm of u and v.

Theorem 4. The simulation results indicate that system (2) undergoes Hopf bifurcation (see Figure 1(g) and 1(h)).

In Figure 2, we choose $d_1 = 0.1, d_2 = 0.1, a = 0.5, b = 1, c = 1, d = 0.2, r = 1, m = 0.51$. Varying the parameter f and choosing the initial data near (u^*, v^*), it

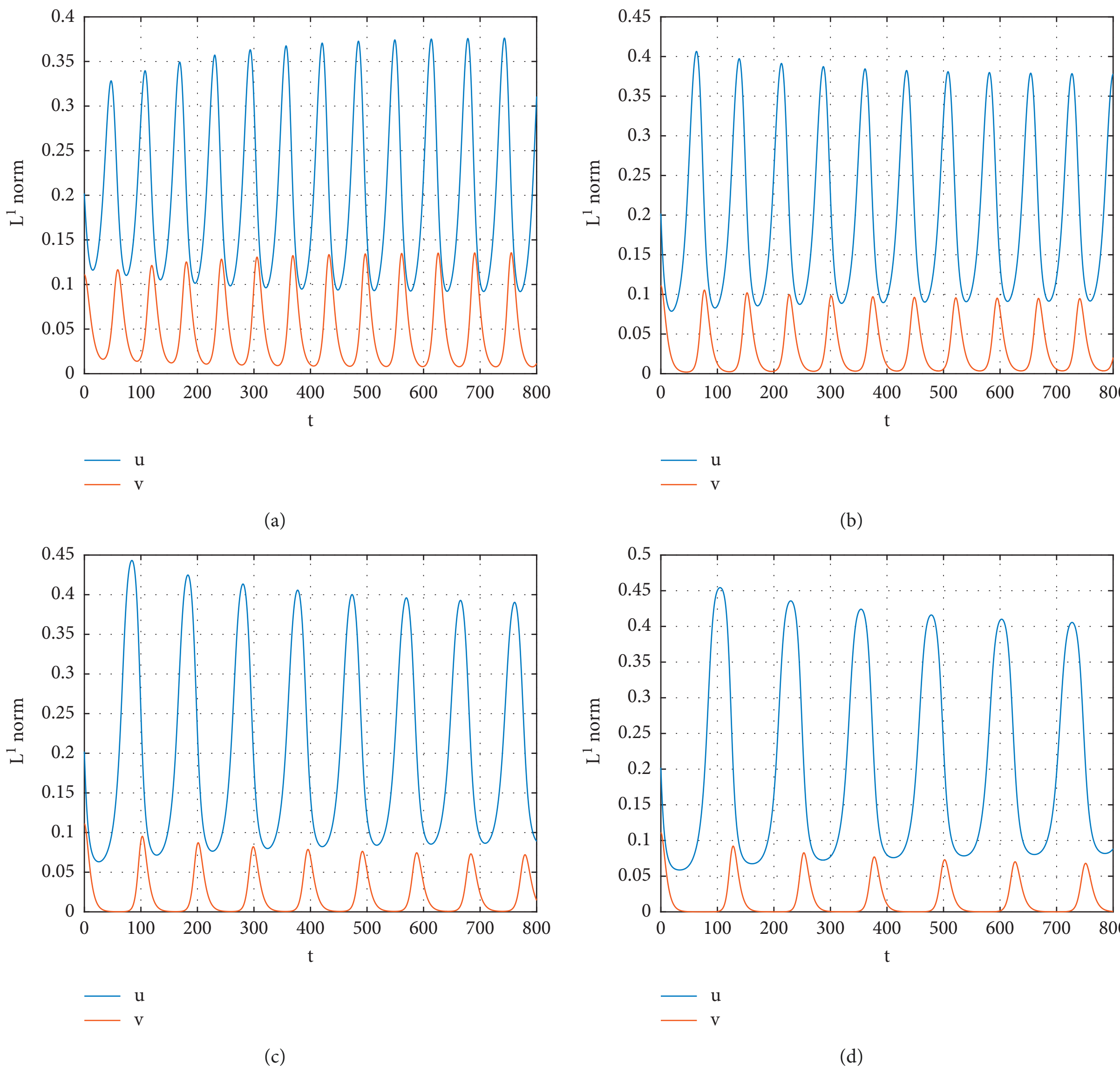

FIGURE 2: The effects of parameter f for $T = 800$, $l = 2$. The values of parameter f are as follows: (a) $f = 5$; (b) $f = 50$; (c) $f = 500$; (d) $f = 5000$.

shows that when f is small, system (2) has obviously periodic oscillation (see Figure 2(a)). When f increases, the maximum L^1 norms of u are almost the same. However, the maximum L^1 norms of v decrease with f increasing (see Figures 2(a)–2(d)). This means that the fear has a negative impact on predators. Moreover, the period of periodic solutions becomes larger as f increases. (see Figures 2(a)–2(d)).

In Figure 3, we choose $d_1 = 0.1, d_2 = 0.1, a = 0.5, b = 1, c = 1, r = 1, f = 15, m = 0.51$. Varying the parameter d and choosing the initial data near (u^*, v^*), it indicates that when d increases, the period of periodic solution is decreasing. Furthermore, with d increasing, the amplitude of periodic solutions is also decreasing. (see Figures 3(a)–3(c)). As d continues to increase, system (2) converges to an equilibrium (see Figure 3(d)).

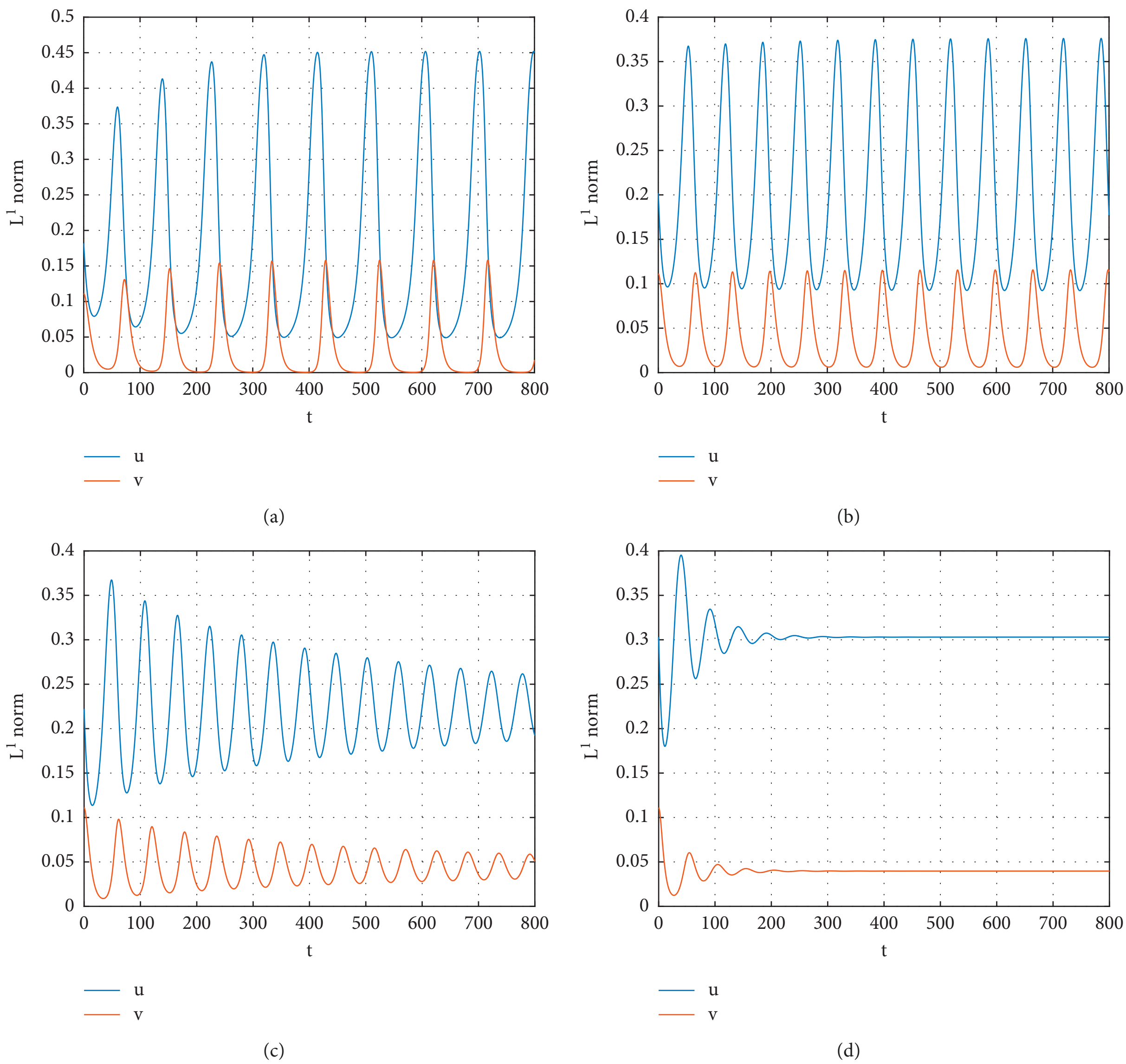

Figure 3: The effects of parameter d for $T = 800, l = 2$. The values of parameter d are as follows: (a) $d = 0.18$; (b) $d = 0.2$; (c) $d = 0.22$; (d) $d = 0.3$.

6. Conclusion

In this paper, a diffusive predator-prey model with additive Allee effects induced by fear factors is considered, in which prey can represent the antipredator behavior due to fear factors. Analytical results indicate that the upper bound of v depends on the diffusion rates d_1, d_2, the death rate of the predators d, the Allee effects parameters m, a, the conversion rate c, and the level of fear f. There exists d^*, which depends on $a, b, c, d, r, m, f, \Omega$, such that if $\min\{d_1, d_2\} > d^*$, the system have only constant positive solution. Furthermore, the dynamic behavior near $E^* = (u^*, v^*)$ is of more concerned. Then, we indicate the existence of non-constant positive solutions. Taking m as a bifurcation parameter, system undergoes Hopf bifurcation at each $m = m_j^H$ $(0 \leq j \leq n)$. Furthermore, for $d_1, d_2, a, b, c, d, r, f > 0$, $0 < m < (1 - d/bc)(d/bc + a)$ are fixed, and there is a smooth curve Γ_n of non-constant positive solutions bifurcating from (u^*, v^*).

We observe that the Allee effect is essential to the dynamical behavior of system (2) by numerical simulations. The amplitude in the strong Allee effect is increasing with m increasing. On the other hand, numerical simulations reveal that the fear effect have an impact on the dynamical behavior of system (2). With the fear effect increasing, the period of periodic solutions is increasing, but the maximum L_1 norm of u is almost the same. On the contrary, the maximum L_1 norm of v decreases with f increasing. From a biological standpoint, the prey survives by adopting antipredator behavior as a result of the fear effect, and the predator is impacted by the prey's antipredator behavior. At last, we show how the death rate d affects system (2) with the strong

Allee effect. As d increases, the amplitude of periodic solution is decreasing, and the period of periodic solution is also decreasing.

It is extremely important to construct animal interaction models in the incorporation of these different types of factors. Considering the different ways of introducing the Allee effect and the interaction with the fear effect, further analysis of the bifurcating solutions of (2) remains a challenging problem. From our discussion before, we conjecture that Turing-Hopf bifurcation, Hopf-Hopf bifurcation is likely to exist in the system, which reveals more complex dynamic behavior and potential biological significance.

Data Availability

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation, to any qualified researcher.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This paper was supported by National Natural Science Foundation of China (Nos. 11861013, 11771444), and the Fundamental Research Funds for the Central Universities, China University of Geosciences (Wuhan) (No. 2018061).

References

[1] Y. Dong, S. Li, and S. Zhang, "Hopf bifurcation in a reaction-diffusion model with Degn-Harrison reaction scheme," *Nonlinear Analysis: Real World Applications*, vol. 33, pp. 284–297, 2017.

[2] Y. Shi, J. Wu, and Q. Cao, "Analysis on a diffusive multiple Allee effects predator-prey model induced by fear factors," *Nonlinear Analysis: Real World Applications*, vol. 59, Article ID 103249, 2021.

[3] Y. e. Wang, J. Wu, and Y. Jia, "Steady-state bifurcation for a biological depletion model," *International Journal of Bifurcation and Chaos*, vol. 26, no. 04, Article ID 1650066, 2016.

[4] L. Yang and S. Zhong, "Dynamics of a diffusive predator-prey model with modified Leslie-Gower schemes and additive Allee effect," *Computational and Applied Mathematics*, vol. 34, no. 2, pp. 671–690, 2015.

[5] W. C. Allee, *Animal Aggregations: A Study in General Sociology*, University of Chicago Press, Chicago, Illinois, 1931.

[6] B. Dennis, "Allee effects: population growth, critical density, and the chance of extinction," *Natural Resource Modeling*, vol. 3, no. 4, pp. 481–538, 1989.

[7] M. J. Wittmann, H. Stuis, and D. Metzler, "Genetic Allee effects and their interaction with ecological Allee effects," *Journal of Animal Ecology*, vol. 87, no. 1, pp. 11–23, 2018.

[8] J. Jiao and C. Chen, "Bogdanov-Takens bifurcation analysis of a delayed predator-prey system with double Allee effect," *Nonlinear Dynamics*, vol. 104, no. 2, pp. 1697–1707, 2021.

[9] Y. Wang, J. Shi, and J. Wang, "Persistence and extinction of population in reaction-diffusion-advection model with strong Allee effect growth," *Journal of Mathematical Biology*, vol. 78, no. 7, pp. 2093–2140, 2019.

[10] C. Zhang and H. Yuan, "Pattern formation in a variable diffusion predator–prey model with additive Allee effect," *Mathematical Methods in the Applied Sciences*, vol. 43, no. 7, pp. 4023–4035, 2020.

[11] B. Xie, Z. Wang, Y. Xue, and Z. Zhang, "The dynamics of a delayed predator-prey model with double Allee effect," *Discrete Dynamics in Nature and Society*, vol. 2015, pp. 1–8, Article ID 102597, 2015.

[12] P. J. Pal and T. Saha, "Qualitative analysis of a predator-prey system with double Allee effect in prey," *Chaos, Solitons & Fractals*, vol. 73, pp. 36–63, 2015.

[13] H. Molla, S. Sarwardi, S. R. Smith, and M. Haque, "Dynamics of adding variable prey refuge and an Allee effect to a predator-prey model," *Alexandria Engineering Journal*, vol. 61, no. 6, pp. 4175–4188, 2022.

[14] R. Han and B. Dai, "Spatiotemporal pattern formation and selection induced by nonlinear cross-diffusion in a toxic-phytoplankton-zooplankton model with Allee effect," *Nonlinear Analysis: Real World Applications*, vol. 45, no. 6, pp. 822–853, 2019.

[15] S. Creel and D. Christianson, "Relationships between direct predation and risk effects," *Trends in Ecology & Evolution*, vol. 23, no. 4, pp. 194–201, 2008.

[16] W. Cresswell, "Predation in bird populations," *Journal of Ornithology*, vol. 152, no. S1, pp. 251–263, 2011.

[17] L. Lai, Z. Zhu, and F. Chen, "Stability and bifurcation in a predator-prey model with the additive allee effect and the fear effect," *Mathematics*, vol. 8, no. 8, 1280 pages, 2020.

[18] K. H. Elliott, G. S. Betini, and D. R. Norris, "Fear creates an Allee effect: experimental evidence from seasonal populations," *Proc Biol Sci*, vol. 284, no. 1857, Article ID 20170878, 2017.

[19] M. Clinchy, M. J. Sheriff, and L. Y. Zanette, "Predator-induced stress and the ecology of fear," *Functional Ecology*, vol. 27, no. 1, pp. 56–65, 2013.

[20] S. D. Peacor, B. L. Peckarsky, G. C. Trussell, and J. R. Vonesh, "Costs of predator-induced phenotypic plasticity: a graphical model for predicting the contribution of nonconsumptive and consumptive effects of predators on prey," *Oecologia*, vol. 171, no. 1, pp. 1–10, 2013.

[21] X. Wang, L. Zanette, and X. Zou, "Modelling the fear effect in predator-prey interactions," *Journal of Mathematical Biology*, vol. 73, no. 5, pp. 1179–1204, 2016.

[22] D. Jana, S. Batabyal, and M. Lakshmanan, "Self-diffusion-driven pattern formation in prey-predator system with complex habitat under fear effect," *The European Physical Journal Plus*, vol. 135, no. 11, pp. 884–942, 2020.

[23] V. Tiwari, J. P. Tripathi, S. Mishra, and R. K. Upadhyay, "Modeling the fear effect and stability of non-equilibrium patterns in mutually interfering predator-prey systems," *Applied Mathematics and Computation*, vol. 371, Article ID 124948, 2020.

[24] R. P. Kaur, A. Sharma, and A. K. Sharma, "Impact of fear effect on plankton-fish system dynamics incorporating zooplankton refuge," *Chaos, Solitons & Fractals*, vol. 143, Article ID 110563, 2021.

[25] H. Qi and X. Meng, "Threshold behavior of a stochastic predator-prey system with prey refuge and fear effect," *Applied Mathematics Letters*, vol. 113, Article ID 106846, 2021.

[26] S. K. Sasmal and Y. Takeuchi, "Dynamics of a predator-prey system with fear and group defense," *Journal of Mathematical*

Analysis and Applications, vol. 481, no. 1, Article ID 123471, 2020.

[27] S. K. Sasmal and Y. Takeuchi, "Modeling the Allee effects induced by cost of predation fear and its carry-over effects," *Journal of Mathematical Analysis and Applications*, vol. 505, no. 2, Article ID 125485, 2022.

[28] W. M. Ni and M. Tang, "Turing patterns in the Lengyel-Epstein system for the CIMA reaction," *Transactions of the American Mathematical Society*, vol. 357, no. 10, pp. 3953–3969, 2005.

[29] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*Springer-Verlag, Berlin, NY, USA, 2006.

[30] B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge Press, England UK, 1981.

[31] F. Yi, J. Wei, and J. Shi, "Bifurcation and spatiotemporal patterns in a homogeneous diffusive predator-prey system," *Journal of Differential Equations*, vol. 246, no. 5, pp. 1944–1977, 2009.

[32] J. Wang, J. Shi, and J. Wei, "Dynamics and pattern formation in a diffusive predator-prey system with strong Allee effect in prey," *Journal of Differential Equations*, vol. 251, no. 4-5, pp. 1276–1304, 2011.