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Research Article

Identities on Changhee Polynomials Arising from λ-Sheffer Sequences

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In this paper, authors found a new and interesting identity between Changhee polynomials and some degenerate polynomials such as degenerate Bernoulli polynomials of the first and second kind, degenerate Euler polynomials, degenerate Daehee polynomials, degenerate Bell polynomials, degenerate Lah–Bell polynomials, and degenerate Frobenius–Euler polynomials and Mittag–Leffer polynomials by using λ-Sheffer sequences and λ-differential operators to find the coefficient polynomial when expressing the n-th Changhee polynomials as a linear combination of those degenerate polynomials. In addition, authors derive the inversion formulas of these identities.

1. Introduction

Umbral calculus from 1850 to 1970 consisted primarily of symbolic techniques for sequence manipulation, and its mathematical rigor left little room for demands. In the 1970s, Gian-Carlo Rota began building a completely rigid foundation for theories based on relatively modern ideas of linear functions, linear operators, and adjacency functions (see [1–4]). Umbral calculus contributed to the generalization of Lagrange inversion formula and has been applied in many fields such as combinatorial counting with linear recurrences and lattice path counting, graph theory using chromatic polynomials, probability theory, link invariant theory, statistics, topology, and physics (see [3]). It is being actively applied in various fields by researchers (see [1–16]).

In the past few years, many distinct umbral calculus types have begun to be studied (see [2, 4, 6, 10]). In particular, Kim–Kim defined the degenerate Sheffer sequences, λ-Sheffer sequence, a family of λ-linear functionals, and λ-differential operators as follows (see [2]).

Let $\mathbb{C}$ be the field of complex numbers:

$$\mathcal{F} = \left\{ f(t) = \sum_{n=0}^{\infty} a_k \frac{t^n}{n!} \middle| a_k \in \mathbb{C} \right\}, \tag{1}$$

and let

$$\begin{aligned} \mathbb{P} = \mathbb{C}[x] = \Bigg\{ \sum_{k=0}^{\infty} a_l x^k \Bigg| a_k \in \mathbb{C} \text{ with } a_k \\ = 0 \text{ for all but finite number of } k \Bigg\}. \end{aligned} \tag{2}$$

Let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$.

Then, each real number λ gives rise to the linear functional $\langle f(t) | \cdot \rangle_\lambda$ on $\mathbb{P}$, called λ-linear functional given by $f(t)$, which is defined by (see [2])

$$\left\langle f(t) | (x)_{n,\lambda} \right\rangle_\lambda = a_n, \ (n \geq 0), \tag{3}$$

and by linear extension where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$. From (3), we have

$$\left\langle t^k | (x)_{n,\lambda} \right\rangle_\lambda = n!\delta_{n,k}, \quad (n,k \geq 0), \tag{4}$$

where $\delta_{n,k}$ is Kronecker's symbol (see [2]).

For each real number λ and each positive integer k, Kim and Kim defined the differential operator on $\mathbb{P}$ in [2] as follows:

$$\left(t^k\right)_\lambda (x)_{n,\lambda} = \begin{cases} (n)_k (x)_{n-k,\lambda}, & \text{if } k \leq n, \\ 0, & \text{if } k > n, \end{cases} \tag{5}$$

and for any $f(t) = \sum_{k=0}^{\infty} a_k (t^k/k!) \in \mathcal{F}$,

$$(f(t))_\lambda (x)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} a_k (x)_{n-k,\lambda}. \tag{6}$$

In addition, they showed that for $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$,

$$\begin{aligned} \langle f(t)g(t) | p(x) \rangle_\lambda &= \langle g(t) | (f(t))_\lambda p(x) \rangle_\lambda \\ &= \langle f(t) | (g(t))_\lambda p(x) \rangle_\lambda. \end{aligned} \tag{7}$$

The order $o(f(t))$ of $f(t) \in \mathcal{F} - \{0\}$ is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is called invertible and such series has a multiplicative inverse $1/f(t)$ of $f(t)$. If $o(f(t)) = 1$, then $f(t)$ is called delta series and it has a compositional inverse $\overline{f}(t)$ of $f(t)$ with $\overline{f}(f(t)) = f(\overline{f}(t)) = t$ (see [1, 2, 12, 16]).

Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then, there exists a unique sequence $S_{n,\lambda}(x)$ ($\deg S_{n,\lambda}(x) = n$) of polynomials satisfying the orthogonality conditions (see [2])

$$\left\langle g(t)(f(t))^k | S_{n,\lambda}(x) \right\rangle_\lambda = n!\delta_{n,k}, \quad (n,k \geq 0). \tag{8}$$

Here, $S_{n,\lambda}(x)$ is called the λ-Sheffer sequence for $(g(t), f(t))$, which is denoted by $S_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$. The sequence $S_{n,\lambda}(x)$ is the λ-Sheffer sequence for $(g(t), f(t))$ if and only if

$$\frac{1}{g(\overline{f}(t))} e_\lambda^y(\overline{f}(t)) = \sum_{n=0}^{\infty} S_{n,\lambda}(y) \frac{t^n}{n!}, \tag{9}$$

for all $y \in \mathbb{C}$, where $\overline{f}(t)$ is the compositional inverse of $f(t)$ such that $f(\overline{f}(t)) = \overline{f}(f(t)) = t$ (see [1, 2, 12, 16]).

Let $S_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ and let $h(x) = \sum_{l=0}^{n} a_l S_{l,\lambda}(x) \in \mathbb{P}$. Then, by (8), we have

$$\begin{aligned} \left\langle g(t)(f(t))^k | h(x) \right\rangle_\lambda &= \sum_{l=0}^{n} a_l \left\langle g(t)(f(t))^k | S_{l,\lambda}(x) \right\rangle_\lambda \\ &= k! a_k, \end{aligned} \tag{10}$$

and thus we know that

$$a_k = \frac{1}{k!} \left\langle g(t)(f(t))^k \middle| h(x) \right\rangle_\lambda. \tag{11}$$

The following theorem is proved by Kim and Kim [2] and is a very useful tool for researching degenerate versions of special polynomials and numbers.

Theorem 1. *Let $s_{n,\lambda} \sim (g(t), f(t))_\lambda$, $r_{n,\lambda} = (h(t), l(t))_\lambda$. Then, we have*

$$s_{n,\lambda} = \sum_{k=0}^{n} c_{n,k} r_{k,\lambda}, \tag{12}$$

where

$$c_{n,k} = \frac{1}{k!} \left\langle \frac{h(\overline{f}(t))}{g(\overline{f}(t))} (l(\overline{f}(t)))^k \middle| (x)_{n,\lambda} \right\rangle_\lambda. \tag{13}$$

For $n \geq 0$, the Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$ and Stirling numbers of the second kind $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, respectively, are given by the following (see [11, 12, 17–20]):

$$(x)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k \text{ and } x^n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (x)_k. \tag{14}$$

For each positive integer k, it is well known that (see [11, 12, 17–20])

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{t^n}{n!}, \frac{1}{k!}\left(e^t - 1\right)^k = \sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{t^n}{n!}. \tag{15}$$

For any nonzero real number λ, the degenerate exponential function is defined by (see [1, 21–27])

$$e_\lambda^x(t) = (1+\lambda t)^{x/\lambda}, \quad e_\lambda(t) = (1+\lambda t)^{1/\lambda}, \tag{16}$$

Note that

$$\frac{e_\lambda(t) + 1}{2} = 1 + \frac{1}{2} \sum_{n=1}^{\infty} (1)_{n,\lambda} \frac{t^n}{n!}. \tag{17}$$

A study of degenerate versions of some special numbers and polynomials was initiated by Carlitz who found interesting relationships connected with important numbers in combinatorics, Bernoulli polynomials, and Eulerian polynomials (see [28]). In the past decades, the study of degenerate versions of various special polynomials or numbers has been studied by many researchers (see [1, 2, 21–27, 29–32]).

By using (16), the higher-order degenerate Bernoulli polynomials are defined as follows (see [1, 10, 12, 30, 33, 34]):

$$\sum_{n=0}^{\infty} B_{n,\lambda}^{(r)}(x) \frac{t^n}{n!} = \left(\frac{t}{e_\lambda(t) - 1} \right)^r e_\lambda^x(t). \tag{18}$$

When $x = 0$, $B_{n,\lambda}^{(r)}(0) = B_{n,\lambda}^{(r)}$ are called the higher-order degenerate Bernoulli numbers. In addition, when $r = 1$, we denote $B_{n,\lambda}^{(1)}(x) = B_{n,\lambda}(x)$.

On the other hand, Kim and Kim defined $\log_\lambda(t)$ called the degenerate logarithm function as the compositional inverse function of $e_\lambda(t)$ satisfying $\log_\lambda(e_\lambda(t)) = t$. Then, we have (see [1, 10, 22, 24, 32])

$$\log_\lambda(1+t)=\sum_{n=1}^{\infty}\lambda^{n-1}(1)_{n,1/\lambda}\frac{t^n}{n!}. \tag{19}$$

By using (19), the degenerate Bernoulli polynomials of the second kind are defined by the generating function to be (see [16])

$$\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}=\frac{t}{\log_\lambda(1+t)}e_\lambda^x(\log_\lambda(1+t)). \tag{20}$$

In the special case $x=0$, $\beta_{n,\lambda}=\beta_{n,\lambda}(0)$ are called the Bernoulli numbers of the second kind.

As degenerate version of the Stirling numbers of the first and second kind in (14), the degenerate Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_\lambda$ and the degenerate Stirling numbers of the second kind $\begin{Bmatrix} n \\ k \end{Bmatrix}_\lambda$ are, respectively, introduced by Kim–Kim (see [1, 2, 21, 22, 24, 26–30, 35, 36]) as follows:

$$\begin{aligned}\frac{1}{k!}(\log_\lambda(1+t))^k&=\sum_{n=k}^{\infty}\begin{bmatrix} n \\ k \end{bmatrix}_\lambda\frac{t^n}{n!}\text{ and }\frac{1}{k!}(e_\lambda(t)-1)^k\\&=\sum_{n=k}^{\infty}\begin{Bmatrix} n \\ k \end{Bmatrix}_\lambda\frac{t^n}{n!}.\end{aligned} \tag{21}$$

Let $(x)_n=\sum_{k=0}^{n}c_{n,k}(x)_{k,\lambda}$. Since

$$\begin{aligned}(x)_{k,\lambda n}&\sim(1,e_\lambda(t)-1)_\lambda,\\(x)_{n,\lambda}&\sim(1,t)_\lambda,\end{aligned} \tag{22}$$

by Theorem 1, we obtain

$$\begin{aligned}c_{n,k}&=\frac{1}{k!}\left\langle(\log_\lambda(1+t))^k\middle|(x)_{n,\lambda}\right\rangle_\lambda\\&=\sum_{l=k}^{\infty}S_{1,\lambda}(l,k)\frac{1}{l!}\left\langle t^l\middle|(x)_{n,\lambda}\right\rangle_\lambda\\&=\begin{bmatrix} n \\ k \end{bmatrix}_\lambda,\end{aligned} \tag{23}$$

and thus, we know that

$$(x)_n=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_\lambda(x)_{k,\lambda}, \tag{24}$$

where $(x)_0=1$, $(x)_n=x(x-1)\ldots(x-n+1)$, $(n\geq 1)$ is the falling factorial sequences. In the similar way, we also know that

$$(x)_{n,\lambda}=\sum_{k=0}^{n}\begin{Bmatrix} n \\ k \end{Bmatrix}_\lambda(x)_k. \tag{25}$$

The aim of this paper is to find some new and interesting identities related to the Changhee polynomials and some special polynomials by using λ-Sheffer sequences and λ-differential operators. In more detail, we find the coefficients which are also polynomials or numbers when the n-th Changhee polynomial is expressed as a linear combination of some degenerate special polynomials by using the λ-Sheffer sequences and λ-differential operators (see Theorems 2–10), and by using the λ-Sheffer sequences and the linear combinations of those polynomials (see Theorems 5–8 and 10), and derive the inversion formulas of these identities.

2. Changhee Polynomials Arising from λ-Sheffer Sequences

In this section, we find some relationships between the Changhee polynomials and some special polynomials arising from λ-Sheffer sequences.

The Changhee polynomials are given by

$$\sum_{n=0}^{\infty}Ch_n(x)\frac{t^n}{n!}=\frac{2}{2+t}(1+t)^x. \tag{26}$$

By (24) and (26), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}Ch_n(x)\frac{t^n}{n!}&=\left(\sum_{n=0}^{\infty}Ch_n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}Ch_{n-m}(x)_m\right)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\begin{bmatrix} m \\ k \end{bmatrix}_\lambda Ch_{n-m}(x)_{k,\lambda}\right)\frac{t^n}{n!},\end{aligned} \tag{27}$$

and, by (27), we have

$$Ch_n(x)=\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\begin{bmatrix} m \\ k \end{bmatrix}_\lambda Ch_{n-m}(x)_{k,\lambda},\quad(n\geq 0). \tag{28}$$

By (28), we compute the first few Changhee polynomials as follows:

$$\begin{aligned}Ch_0(x)&=1,\\Ch_1(x)&=x-\frac{1}{2},\\Ch_2(x)&=x^2-2x+\frac{1}{2},\\Ch_3(x)&=-x^3+\frac{9}{2}x^2-5x+\frac{3}{4},\\Ch_4(x)&=x^4-8x^3+20x^2-16x+\frac{3}{2},\\Ch_5(x)&=-x^5+\frac{25}{2}x^4-55x^3+100x^2-64x+\frac{15}{4}.\end{aligned} \tag{29}$$

In addition, graphs for some Changhee polynomials are shown in Figure 1.

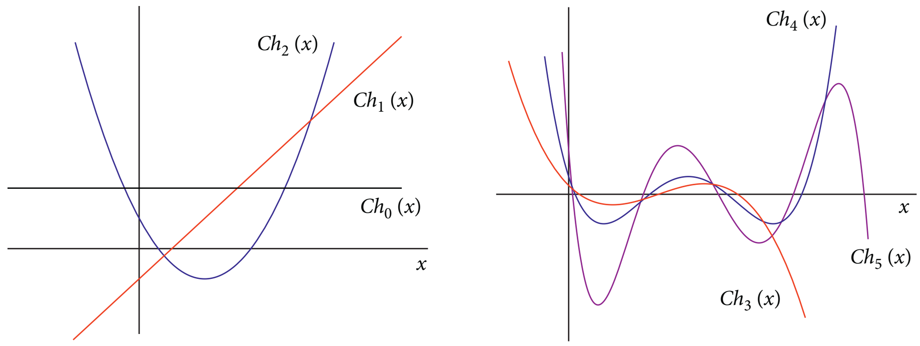

Figure 1: The shapes of Changhee polynomials $Ch_n(x)$.

Note that

$$\frac{1}{(t+2)^k} = (t+2)^{-k} = \sum_{l=0}^{\infty}\binom{-k}{l}t^l 2^{-k-l} = \sum_{l=0}^{\infty}\frac{\langle k\rangle_l(-1)^l}{2^{l+k}}\frac{t^l}{l!}, \tag{30}$$

where $\langle x\rangle_0 = 1$, $\langle x\rangle_n = x(x+1)(x+2)\dots(x+(n-1))$, $(n\geq 1)$.

Theorem 2. *For each nonnegative integer n, we have*

$$Ch_n(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\binom{n}{l}\begin{bmatrix} l \\ k\end{bmatrix}_\lambda Ch_{n-l}\right)(x)_{k,\lambda}. \tag{31}$$

As the inversion formula of (31), we have

$$(x)_{n,\lambda} = \sum_{k=0}^{n}\left(\begin{Bmatrix} n \\ k\end{Bmatrix}_\lambda + \frac{1}{2}\sum_{m=1}^{n-k}(1)_{m,\lambda}\binom{n}{m}\begin{Bmatrix} n-m \\ k\end{Bmatrix}_\lambda\right)Ch_k(x). \tag{32}$$

Proof. Let $Ch_n(x) = \sum_{k=0}^{n} c_{n,k}(x)_{k,\lambda}$. Since

$$Ch_n(x) \sim \left(\frac{e_\lambda(t)+1}{2}, e_\lambda(t)-1\right)_\lambda, \quad (x)_{n,\lambda} \sim (1,t)_\lambda, \tag{33}$$

by Theorem 1 and (30), we have

$$\begin{aligned} c_{n,k} &= \frac{1}{k!}\left\langle \frac{1}{((t+2)/2)}(\log_\lambda(1+t))^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \frac{2}{t+2}\middle| \left(\frac{1}{k!}(\log_\lambda(1+t))^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}\binom{n}{l}\begin{bmatrix} l \\ k\end{bmatrix}_\lambda \left\langle \frac{2}{t+2}\middle| (x)_{n-l,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}\binom{n}{l}\begin{bmatrix} l \\ k\end{bmatrix}_\lambda Ch_{n-l}. \end{aligned} \tag{34}$$

Conversely, we assume that $(x)_{n,\lambda} = \sum_{k=0}^{n} d_{n,k}Ch_k(x)$. By (6) and (17), we obtain

$$\begin{aligned} d_{n,k} &= \frac{1}{k!}\left\langle \frac{e_\lambda(t)+1}{2}(e_\lambda(t)-1)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \frac{1}{k!}(e_\lambda(t)-1)^k \middle| \left(\frac{e_\lambda(t)+1}{2}\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \frac{1}{k!}(e_\lambda(t)-1)^k \middle| \left(1+\frac{1}{2}\sum_{m=1}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \sum_{l=k}^{\infty}\begin{Bmatrix} l \\ k\end{Bmatrix}_\lambda \frac{t^l}{l!}\middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &\quad + \frac{1}{2}\sum_{m=1}^{n}\binom{n}{m}(1)_{m,\lambda}\left\langle \sum_{l=k}^{\infty}\begin{Bmatrix} l \\ k\end{Bmatrix}_\lambda \frac{t^l}{l!}\middle| (x)_{n-m,\lambda}\right\rangle_\lambda \\ &= \begin{Bmatrix} n \\ k\end{Bmatrix}_\lambda + \frac{1}{2}\sum_{m=1}^{n-k}\binom{n}{m}(1)_{m,\lambda}\begin{Bmatrix} n-m \\ k\end{Bmatrix}_\lambda. \end{aligned} \tag{35}$$

□

Since

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} &= \left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t) \\ &= \left(\sum_{n=0}^{\infty} B_{n,\lambda}^{(r)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}B_{n-m,\lambda}^{(r)}(x)_{m,\lambda}\right)\frac{t^n}{n}, \end{aligned} \tag{36}$$

and thus we know that

$$B_{n,\lambda}^{(r)}(x) = \sum_{m=0}^{n}\binom{n}{m}B_{n-m,\lambda}^{(r)}(x)_{m,\lambda}. \tag{37}$$

Theorem 3. *For each $n \geq 0$, we have*

$$Ch_n(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}\begin{bmatrix} l \\ k \end{bmatrix}_\lambda B_{m,\lambda}Ch_{n-l-m}\right) \cdot B_{k,\lambda}(x). \tag{38}$$

As the inversion formula of (38), we have

$$\begin{aligned} B_{n,\lambda}(x) &= \sum_{k=0}^{n}\left(\sum_{m=0}^{n}\binom{n}{m}B_{n-m,\lambda}\left(\begin{Bmatrix} m \\ k \end{Bmatrix}_\lambda + \frac{1}{2}\sum_{l=1}^{m}\binom{m}{l}\begin{Bmatrix} m-l \\ k \end{Bmatrix}_\lambda (1)_{l,\lambda}\right)\right)Ch_k(x) \\ &= \sum_{k=0}^{n}\left(\sum_{m=k}^{n}\binom{n}{m}\begin{Bmatrix} m \\ k \end{Bmatrix}_\lambda B_{n-m,\lambda} + \frac{1}{2}\sum_{l=1}^{n}\sum_{m=l}^{n-l}\binom{n}{l}\binom{n-l}{m}(1)_{l,\lambda}\begin{Bmatrix} m \\ k \end{Bmatrix}_\lambda B_{n-l-m,\lambda}\right)Ch_k(x). \end{aligned} \tag{39}$$

Proof. Let $Ch_n(x) = \sum_{k=0}^{n} c_{n,k}B_{k,\lambda}(x)$. Since

$$Ch_n(x) \sim \left(\frac{e_\lambda(t)+1}{2}, e_\lambda(t)-1\right)_\lambda, \quad B_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)-1}{t}, t\right)_\lambda, \tag{40}$$

by Theorem 1, we obtain

$$\begin{aligned} c_{n,k} &= \frac{1}{k!}\left\langle \frac{t/\log_\lambda(1+t)}{t+2/2}(\log_\lambda(1+t))^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \frac{2}{t+2}\frac{t}{\log_\lambda(1+t)} \middle| \left(\frac{1}{k!}(\log_\lambda(1+t))^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}\binom{n}{l}\begin{bmatrix} l \\ k \end{bmatrix}_\lambda \left\langle \frac{2}{t+2}\middle| \left(\frac{t}{\log_\lambda(1+t)}\right)_\lambda (x)_{n-l,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}\begin{bmatrix} l \\ k \end{bmatrix}_\lambda \beta_{m,\lambda}\left\langle \frac{2}{t+2}\middle| (x)_{n-l-m,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}\begin{bmatrix} l \\ k \end{bmatrix}_\lambda \beta_{m,\lambda}Ch_{n-l-m}. \end{aligned} \tag{41}$$

Conversely, let $B_{n,\lambda}(x) = \sum_{k=0}^{n} d_{n,k}Ch_k(x)$. By (8), (12), (15) and (37), we obtain

$$\begin{aligned} d_{n,k} &= \frac{1}{k!}\left\langle \frac{e_\lambda(t)+1}{2}(e_\lambda(t)-1)^k \middle| B_{n,\lambda}(x)\right\rangle_\lambda \\ &= \sum_{m=0}^{n}\binom{n}{m}B_{n-m,\lambda}\frac{1}{k!}\left\langle \frac{e_\lambda(t)+1}{2}(e_\lambda(t)-1)^k \middle| (x)_{m,\lambda}\right\rangle_\lambda \\ &= \sum_{m=0}^{n}\binom{n}{m}B_{n-m,\lambda}\frac{1}{k!}\left\langle (e_\lambda(t)-1)^k \middle| \left(\frac{e_\lambda(t)+1}{2}\right)_\lambda (x)_{m,\lambda}\right\rangle_\lambda \\ &= \sum_{m=0}^{n}\binom{n}{m}B_{n-m,\lambda}\frac{1}{k!}\left\langle (e_\lambda(t)-1)^k \middle| \left(1+\frac{1}{2}\sum_{l=1}^{\infty}(1)_{l,\lambda}\frac{t^l}{l!}\right)_\lambda (x)_{m,\lambda}\right\rangle_\lambda \\ &= \sum_{m=0}^{n}\binom{n}{m}B_{n-m,\lambda}\left\langle \frac{1}{k!}(e_\lambda(t)-1)^k \middle| (x)_{m,\lambda} + \frac{1}{2}\sum_{l=1}^{m}\binom{m}{l}(1)_{l,\lambda}(x)_{m-l,\lambda}\right\rangle_\lambda \\ &= \sum_{m=0}^{n}\binom{n}{m}B_{n-m,\lambda}\begin{Bmatrix} m \\ k \end{Bmatrix}_\lambda + \frac{1}{2}\sum_{m=0}^{n}\sum_{l=1}^{m}\binom{n}{m}\binom{m}{l}B_{n-m,\lambda}\begin{Bmatrix} m-l \\ k \end{Bmatrix}_\lambda (1)_{l,\lambda}. \end{aligned} \tag{42}$$

On the other hand, by Theorem 1, we have

$$\begin{aligned} d_{n,k} &= \frac{1}{k!}\left\langle \frac{e_\lambda(t)+1/2}{e_\lambda(t)-1/t}(e_\lambda(t)-1)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \frac{1}{k!}\left\langle \frac{t}{e_\lambda(t)-1}(e_\lambda(t)-1)^k \middle| \left(\frac{e_\lambda(t)+1}{2}\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\ &= \frac{1}{k!}\left\langle \frac{t}{e_\lambda(t)-1}(e_\lambda(t)-1)^k \middle| (x)_{n,\lambda}\right\rangle + \frac{1}{2}\sum_{l=1}^{n}\binom{n}{l}(1)_{l,\lambda}\frac{1}{k!}\left\langle \frac{t}{e_\lambda^t-1}(e_\lambda^t-1)^k \middle| (x)_{n-l,\lambda}\right\rangle_\lambda \\ &= \left\langle \frac{t}{e_\lambda(t)-1} \middle| \left(\frac{1}{k!}(e_\lambda(t)-1)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda + \frac{1}{2}\sum_{l=1}^{n}\binom{n}{l}(1)_{l,\lambda}\left\langle \frac{t}{e_\lambda(t)-1} \middle| \left(\frac{1}{k!}(e_\lambda(t)-1)^k\right)_\lambda (x)_{n-l,\lambda}\right\rangle_\lambda \\ &= \sum_{m=k}^{n}\binom{n}{m}\begin{Bmatrix} m \\ k \end{Bmatrix}_\lambda \left\langle \frac{t}{e_\lambda(t)-1} \middle| (x)_{n-m,\lambda}\right\rangle_\lambda + \frac{1}{2}\sum_{l=1}^{n}\sum_{m=k}^{n-l}\binom{n}{l}\binom{n-l}{m}(1)_{l,\lambda}\begin{Bmatrix} m \\ k \end{Bmatrix}_\lambda \left\langle \frac{t}{e_\lambda(t)-1} \middle| (x)_{n-l-m}\right\rangle_\lambda \\ &= \sum_{m=k}^{n}\binom{n}{m}\begin{Bmatrix} m \\ k \end{Bmatrix}B_{n-m,\lambda} + \frac{1}{2}\sum_{l=1}^{n}\sum_{m=k}^{n-l}\binom{n}{l}\binom{n-l}{m}(1)_{l,\lambda}\begin{Bmatrix} m \\ k \end{Bmatrix}B_{n-l-m,\lambda}, \end{aligned} \tag{43}$$

and hence our proofs are completed.

The degenerate Euler polynomials are defined by the generating function to be (see [28])

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\mathscr{E}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{44}$$

When $x=0$, $\mathscr{E}_{n,\lambda}=\mathscr{E}_{n,\lambda}(0)$ are called the degenerate Euler numbers.

By (44), we know that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{E}_{n,\lambda}(x)\frac{t^n}{n!}&=\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)\\&=\left(\sum_{n=0}^{\infty}\mathscr{E}_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathscr{E}_{n-m,\lambda}(x)_{m,\lambda}\right)\frac{t^n}{n!},\end{aligned} \tag{45}$$

and so

$$\mathscr{E}_{n,\lambda}(x)=\sum_{m=0}^{n}\binom{n}{m}\mathscr{E}_{n-m,\lambda}(x)_{m,\lambda}. \tag{46}$$

□

Theorem 4. *For each $n\geq 0$, we have*

$$Ch_n(x)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_\lambda\mathscr{E}_{k,\lambda}(x). \tag{47}$$

As the inversion formula of (47), we have

$$\begin{aligned}\mathscr{E}_{n,\lambda}(x)&=\sum_{k=0}^{n}\begin{Bmatrix}n\\k\end{Bmatrix}_\lambda Ch_k(x)\\&=\sum_{k=0}^{n}\left(\sum_{m=k}^{n}\binom{n}{m}\begin{Bmatrix}m\\k\end{Bmatrix}_\lambda\mathscr{E}_{n-m,\lambda}\right.\\&\left.+\frac{1}{2}\sum_{m=k+1}^{n}\sum_{l=k}^{m-1}\binom{n}{m}\binom{m}{l}\begin{Bmatrix}l\\k\end{Bmatrix}_\lambda\mathscr{E}_{n-m,\lambda}(1)_{m-l,\lambda}\right)Ch_k(x).\end{aligned} \tag{48}$$

Proof. Let $Ch_n(x)=\sum_{k=0}^{n}c_{n,k}\mathscr{E}_{k,\lambda}(x)$. Since

$$\begin{aligned}Ch_n(x)&\sim\left(\frac{e_\lambda(t)+1}{2},e_\lambda(t)-1\right)_\lambda,\\\mathscr{E}_{n,\lambda}(x)&\sim\left(\frac{e_\lambda(t)+1}{2},t\right)_\lambda,\end{aligned} \tag{49}$$

by Theorem 1, we obtain

$$\begin{aligned}c_{n,k}&=\frac{1}{k!}\left\langle\frac{t+2/2}{t+2/2}(\log_\lambda(1+t))^k\middle|(x)_{n,\lambda}\right\rangle_\lambda\\&=\left\langle\frac{1}{k!}(\log_\lambda(1+t))^k\middle|(x)_{n,\lambda}\right\rangle_\lambda=\left\langle\sum_{l=k}^{\infty}\begin{bmatrix}l\\k\end{bmatrix}_\lambda\frac{t^l}{l!}\middle|(x)_{n,\lambda}\right\rangle_\lambda\\&=\begin{bmatrix}n\\k\end{bmatrix}_\lambda.\end{aligned} \tag{50}$$

Conversely, we assume that $\mathscr{E}_{n,\lambda}(x)=\sum_{k=0}^{n}d_{n,k}Ch_k(x)$. Then,

$$\begin{aligned}d_{n,k}&=\frac{1}{k!}\left\langle\frac{e_\lambda(t)+1/2}{e_\lambda(t)+1/2}(e_\lambda(t)-1^k\middle|(x)_{n,\lambda}\right\rangle_\lambda\\&=\left\langle\frac{1}{k!}(e_\lambda(t)-1)^k\middle|(x)_{n,\lambda}\right\rangle_\lambda=\left\langle\sum_{l=k}^{\infty}\begin{Bmatrix}l\\k\end{Bmatrix}_\lambda\frac{t^l}{l!}\middle|(x)_{n,\lambda}\right\rangle_\lambda\\&=\begin{Bmatrix}n\\k\end{Bmatrix}_\lambda.\end{aligned} \tag{51}$$

On the other hand, by (11) and (46), we obtain

$$\begin{aligned}d_{n,k}&=\frac{1}{k!}\left\langle\frac{e_\lambda(t)+1}{2}(e_\lambda(t)-1)^k\middle|\mathscr{E}_{n,\lambda}(x)\right\rangle_\lambda\\&=\sum_{m=0}^{n}\binom{n}{m}\mathscr{E}_{n-m,\lambda}\left\langle\frac{e_\lambda(t)+1}{2}\middle|\left(\frac{1}{k!}(e_\lambda(t)-1)^k\right)_\lambda(x)_{m,\lambda}\right\rangle_\lambda\\&=\sum_{m=0}^{n}\sum_{l=k}^{m}\binom{n}{m}\binom{m}{l}\begin{Bmatrix}l\\k\end{Bmatrix}_\lambda\mathscr{E}_{n-m,\lambda}\left\langle\frac{e_\lambda(t)+1}{2}\middle|(x)_{m-l,\lambda}\right\rangle_\lambda\\&=\sum_{m=0}^{n}\sum_{l=k}^{m}\binom{n}{m}\binom{m}{l}\begin{Bmatrix}l\\k\end{Bmatrix}_\lambda\mathscr{E}_{n-m,\lambda}\left\langle1+\frac{1}{2}\sum_{a=1}^{\infty}(1)_{a,\lambda}\frac{t^a}{a!}\middle|(x)_{m-l,\lambda}\right\rangle_\lambda\\&=\sum_{m=0}^{n}\sum_{l=k}^{m}\binom{n}{m}\binom{m}{l}\begin{Bmatrix}l\\k\end{Bmatrix}_\lambda\mathscr{E}_{n-m,\lambda}\\&\quad\times\left(\left\langle1\middle|(x)_{m-l,\lambda}\right\rangle_\lambda+\frac{1}{2}\sum_{a=1}^{\infty}\frac{(1)_{a,\lambda}}{a!}\left\langle t^a\middle|(x)_{m-l,\lambda}\right\rangle_\lambda\right)\\&=\sum_{m=k}^{n}\binom{n}{m}\begin{Bmatrix}m\\k\end{Bmatrix}_\lambda\mathscr{E}_{n-m,\lambda}\\&\quad+\frac{1}{2}\sum_{m=k+1}^{n}\sum_{l=k}^{m-1}\binom{n}{m}\binom{m}{l}\begin{Bmatrix}l\\k\end{Bmatrix}_\lambda\mathscr{E}_{n-m,\lambda}(1)_{m-l,\lambda},\end{aligned} \tag{52}$$

and so our proofs are completed.

By (46), we compute the first few degenerate Euler polynomials as follows:

$$\begin{aligned}\mathscr{E}_{0,\lambda}(x)&=1,\\\mathscr{E}_{1,\lambda}(x)&=x-\frac{1}{2},\\\mathscr{E}_{2,\lambda}(x)&=x^2-(\lambda+1)x+\frac{1}{2}\lambda,\\\mathscr{E}_{3,\lambda}(x)&=-x^3+\left(3\lambda+\frac{3}{2}\right)x^2-\left(2\lambda^2+3\lambda\right)x+\lambda^2-\frac{1}{4},\\\mathscr{E}_{4,\lambda}(x)&=-x^4+(2+6\lambda)x^3-\left(11\lambda^2+9\lambda\right)x^2\\&\quad+\left(6\lambda^3+11\lambda^2-1\right)x-3\lambda^2+\frac{3}{2}\lambda.\end{aligned} \tag{53}$$

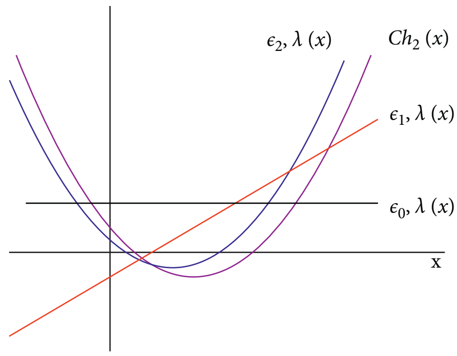

Figure 2: The shapes of $Ch_2(x)$, $\mathscr{E}_{0,\lambda}(x)$, $\mathscr{E}_{1,\lambda}(x)$, and $\mathscr{E}_{2,\lambda}(x)$ when $\lambda = 0.5$.

Although $Ch_2(x) = \sum_{l=0}^{2} a_l(x)\mathscr{E}_{l,\lambda}(x)$, it is difficult to find $a_i(x), i = 0, 1, 2$ through Figure 2. But by Theorem 4, we see that $a_i(x) = \begin{bmatrix} 2 \\ i \end{bmatrix}_{0.5}$, $i = 0, 1, 2$.

The degenerate Daehee polynomials are defined by the generating function to be

$$\frac{\log_\lambda(1+t)}{t} e_\lambda^x(\log_\lambda(1+t)) = \sum_{n=0}^{\infty} D_{n,\lambda}(x)\frac{t^n}{n!}. \tag{54}$$

In the special case of $x = 0$, $D_{n,\lambda} = D_{n,\lambda}(0)$ are called the degenerate Daehee numbers (see [32, 37]).

Note that

$$\begin{aligned}\sum_{n=0}^{\infty} D_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{\log_\lambda(1+t)}{t}(1+t)^x \\ &= \left(\sum_{n=0}^{\infty} D_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}D_{n-l,\lambda}(x)_l\right)\frac{t^n}{n!},\end{aligned} \tag{55}$$

and by (24), we have

$$D_{n,\lambda}(x) = \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}\begin{bmatrix} l \\ k \end{bmatrix}_\lambda D_{n-l,\lambda}(x)_{k,\lambda}. \tag{56}$$

□

Theorem 5. *For each nonnegative integer n, we have*

$$\begin{aligned}Ch_n(x) &= \sum_{k=0}^{n}\left(\binom{n}{k}\sum_{m=0}^{n-k}\binom{n-k}{m}Ch_m\beta_{n-k-m,\lambda}\right)D_{k,\lambda}(x) \\ &= \sum_{k=0}^{n}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{a=k}^{m}\binom{n}{l}\binom{m}{a}\right. \\ &\quad \left.\times\frac{(1)_{m-a+1,\lambda}Ch_{n-l}}{m-a+1}\begin{bmatrix} l \\ m \end{bmatrix}_\lambda\begin{Bmatrix} a \\ k \end{Bmatrix}_\lambda\right)D_{k,\lambda}(x).\end{aligned} \tag{57}$$

As the inversion formula of (57), we have

$$\begin{aligned}D_{n,\lambda}(x) &= \sum_{k=0}^{n}\left(\binom{n}{k}\left(D_{n-k,\lambda} + \frac{n-k}{2}D_{n-k-1,\lambda}\right)\right)Ch_k(x) \\ &= \sum_{k=0}^{n}\left\{\sum_{l=0}^{n}\sum_{r=0}^{l}\binom{n}{l}\begin{bmatrix} l \\ r \end{bmatrix}_\lambda D_{n-l,\lambda}\left(\begin{Bmatrix} r \\ k \end{Bmatrix}_\lambda\right.\right. \\ &\quad \left.\left.+\frac{1}{2}\sum_{a=1}^{r}\binom{r}{a}(1)_{a,\lambda}\begin{Bmatrix} r-a \\ k \end{Bmatrix}_\lambda\right)\right\}Ch_k(x).\end{aligned} \tag{58}$$

Proof. Let $Ch_n(x) = \sum_{k=0}^{n} c_{n,k}D_{k,\lambda}(x)$. Since

$$\begin{aligned}Ch_n(x) &\sim \left(\frac{e_\lambda(t)+1}{2}, e_\lambda(t)-1\right)_\lambda, \\ D_{n,\lambda}(x) &\sim \left(\frac{e_\lambda(t)-1}{t}, e_\lambda(t)-1\right)_\lambda,\end{aligned} \tag{59}$$

by Theorem 1, we have

$$\begin{aligned}c_{n,k} &= \frac{1}{k!}\left\langle \frac{t/\log_\lambda(1+t)}{t+2/2}t^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \frac{1}{k!}\left\langle \frac{t}{\log_\lambda(1+t)}\frac{2}{t+2}\middle| \left(t^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\ &= \binom{n}{k}\left\langle \frac{t}{\log_\lambda(1+t)}\frac{2}{t+2}\middle| (x)_{n-k,\lambda}\right\rangle_\lambda \\ &= \binom{n}{k}\left\langle \frac{t}{\log_\lambda(1+t)}\middle| \left(\frac{2}{t+2}\right)_\lambda (x)_{n-k,\lambda}\right\rangle_\lambda \\ &= \binom{n}{k}\sum_{m=0}^{n-k}\binom{n-k}{m}Ch_m\left\langle \frac{t}{\log_\lambda(1+t)}\middle| (x)_{n-k-m,\lambda}\right\rangle_\lambda \\ &= \binom{n}{k}\sum_{m=0}^{n-k}\binom{n-k}{m}Ch_m\beta_{n-k-m,\lambda}.\end{aligned} \tag{60}$$

On the other hand, by (11) and (28), we obtain

$$
\begin{aligned}
c_{n,k} &= \frac{1}{k!}\left\langle \frac{e_\lambda(t)-1}{t}(e_\lambda(t)-1)^k \middle| Ch_n(x)\right\rangle \\
&= \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}[lm]_\lambda Ch_{n-l}\left\langle \frac{e_\lambda(t)-1}{t} \middle| \left(\frac{1}{k!}(e_\lambda(t)-1)^k\right)_\lambda (x)_{m,\lambda}\right\rangle_\lambda \\
&= \sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{a=k}^{m}\binom{n}{l}\binom{m}{a}\begin{bmatrix} l \\ m\end{bmatrix}_\lambda \begin{Bmatrix} a \\ k\end{Bmatrix}_\lambda Ch_{n-l}\left\langle \frac{e_\lambda(t)-1}{t} \middle| (x)_{m-a,\lambda}\right\rangle_\lambda \\
&= \sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{a=k}^{m}\binom{n}{l}\binom{m}{a}\begin{bmatrix} l \\ m\end{bmatrix}_\lambda \begin{Bmatrix} a \\ k\end{Bmatrix}_\lambda Ch_{n-l} \\
&\quad \times \left\langle \sum_{b=0}^{\infty}\frac{(1)_{b+1,\lambda}}{b+1}\frac{t^b}{b!} \middle| (x)_{m-a,\lambda}\right\rangle_\lambda \\
&= \sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{a=k}^{m}\binom{n}{l}\binom{m}{a}\frac{(1)_{m-a+1,\lambda}Ch_{n-l}}{m-a+1}\begin{bmatrix} l \\ m\end{bmatrix}_\lambda \begin{bmatrix} a \\ k\end{bmatrix}_\lambda .
\end{aligned} \tag{61}
$$

Conversely, we assume that $D_{n,\lambda}(x) = \sum_{k=0}^{n} d_{n,k}Ch_k(x)$. Then, by Theorem 1 and (6), we obtain

$$
\begin{aligned}
d_{n,k} &= \frac{1}{k!}\left\langle \frac{t+2/2}{(t/\log_\lambda(1+t))}t^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
&= \frac{1}{k!}\left\langle \frac{t+2}{2}\frac{\log_\lambda(1+t)}{t} \middle| \left(t^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
&= \binom{n}{k}\left\langle \frac{\log_\lambda(1+t)}{t} \middle| \left(\frac{t+2}{2}\right)_\lambda (x)_{n-k,\lambda}\right\rangle_\lambda \\
&= \binom{n}{k}\left\langle \frac{\log_\lambda(1+t)}{t} \middle| (x)_{n-k,\lambda} + \frac{n-k}{2}(x)_{n-k-1,\lambda}\right\rangle_\lambda \\
&= \binom{n}{k}\left(\left\langle \frac{\log_\lambda(1+t)}{t} \middle| (x)_{n-k,\lambda}\right\rangle_\lambda + \frac{n-k}{2}\left\langle \frac{\log_\lambda(1+t)}{t} \middle| (x)_{n-k-1,\lambda}\right\rangle_\lambda\right) \\
&= \binom{n}{k}\left(D_{n-k,\lambda} + \frac{n-k}{2}D_{n-k-1,\lambda}\right).
\end{aligned} \tag{62}
$$

On the other hand, by (11) and (56), we obtain

$$
\begin{aligned}
d_{n,k} &= \frac{1}{k!}\left\langle \frac{e_\lambda(t)+1}{2}(e_\lambda(t)-1)^k \middle| D_{n,\lambda}(x)\right\rangle_\lambda \\
&= \frac{1}{k!}\sum_{l=0}^{n}\sum_{r=0}^{l}\binom{n}{l}\begin{bmatrix} l \\ r\end{bmatrix}_\lambda D_{n-l,\lambda}\left\langle \frac{e_\lambda(t)+1}{2}(e_\lambda(t)-1)^k \middle| (x)_{r,\lambda}\right\rangle_\lambda \\
&= \sum_{l=0}^{n}\sum_{r=0}^{l}\binom{n}{l}\begin{bmatrix} l \\ r\end{bmatrix}_\lambda D_{n-l,\lambda}\left\langle \frac{1}{k!}(e_\lambda(t)-1)^k \middle| \left(\frac{e_\lambda(t)+1}{2}\right)_\lambda (x)_{r,\lambda}\right\rangle_\lambda \\
&= \sum_{l=0}^{n}\sum_{r=0}^{l}\binom{n}{l}\begin{bmatrix} l \\ r\end{bmatrix}_\lambda D_{n-l,\lambda} \\
&\quad \times \left\langle \frac{1}{k!}(e_\lambda(t)-1)^k \middle| \left(1+\sum_{a=1}^{\infty}\frac{(1)_{a,\lambda}}{2}\frac{t^a}{a!}\right)_\lambda (x)_{r,\lambda}\right\rangle_\lambda \\
&= \sum_{l=0}^{n}\sum_{r=0}^{l}\binom{n}{l}\begin{bmatrix} l \\ r\end{bmatrix}_\lambda D_{n-l,\lambda} \\
&\quad \times \left\langle \sum_{b=k}^{\infty}\begin{Bmatrix} b \\ k\end{Bmatrix}_\lambda \frac{t^b}{b!} \middle| (x)_{r,\lambda} + \frac{1}{2}\sum_{a=1}^{r}\binom{r}{a}(1)_{a,\lambda}(x)_{r-a,\lambda}\right\rangle_\lambda \\
&= \sum_{l=0}^{n}\sum_{r=0}^{l}\binom{n}{l}\begin{bmatrix} l \\ r\end{bmatrix}_\lambda D_{n-l,\lambda}\left(\begin{Bmatrix} r \\ k\end{Bmatrix}_\lambda + \frac{1}{2}\sum_{a=1}^{r}\binom{r}{a}(1)_{a,\lambda}\begin{Bmatrix} r-a \\ k\end{Bmatrix}_\lambda\right).
\end{aligned} \tag{63}
$$

The degenerate Bell polynomials are defined by the generating function to be (see [1, 23])

$$
e_\lambda^x((e_\lambda(t)-1)) = \sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!}. \tag{64}
$$

Note that

$$
\begin{aligned}
\sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!} &= e_\lambda^x((e_\lambda(t)-1)) = \sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{1}{m!}(e_\lambda(t)-1)^m \\
&= \sum_{m=0}^{\infty}(x)_{m,\lambda}\sum_{l=m}^{\infty}\begin{Bmatrix} l \\ m\end{Bmatrix}_\lambda \frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\begin{Bmatrix} n \\ m\end{Bmatrix}_\lambda (x)_{m,\lambda}\right)\frac{t^n}{n!},
\end{aligned} \tag{65}
$$

and thus

$$
Bel_{n,\lambda}(x) = \sum_{m=0}^{n} n\backslash bracem_\lambda(x)_{m,\lambda}. \tag{66}
$$

In addition, we know that

$$\begin{aligned}\frac{1}{k!}\left(\log_\lambda\left(1+\log_\lambda(1+t)\right)\right)^k &= \sum_{l=k}^{\infty}\begin{bmatrix} l \\ k \end{bmatrix}_\lambda \frac{1}{l!}\left(\log_\lambda(1+t)\right)^l \\ &= \sum_{l=k}^{\infty}\sum_{m=l}^{\infty}\begin{bmatrix} l \\ k \end{bmatrix}_\lambda \begin{bmatrix} m \\ l \end{bmatrix}_\lambda \frac{t^m}{m!} \\ &= \sum_{l=k}^{\infty}\sum_{m=k}^{l}\begin{bmatrix} m \\ k \end{bmatrix}_\lambda \begin{bmatrix} l \\ m \end{bmatrix}_\lambda \frac{t^l}{l!}.\end{aligned} \tag{67}$$

$$\begin{aligned}\frac{e_\lambda\left(e_\lambda(t)-1\right)+1}{2} &= \frac{1}{2}\left(1+\sum_{a=0}^{\infty}(1)_{a,\lambda}\frac{1}{a!}\left(e_\lambda(t)-1^{a}\right)\right) \\ &= 1+\frac{1}{2}\sum_{a=1}^{\infty}\sum_{b=a}^{\infty}(1)_{a,\lambda}\left\{\begin{matrix} b \\ a \end{matrix}\right\}_\lambda \frac{t^b}{b!} \\ &= 1+\frac{1}{2}\sum_{a=1}^{\infty}\sum_{b=1}^{a}(1)_{b,\lambda}\left\{\begin{matrix} a \\ b \end{matrix}\right\}_\lambda \frac{t^a}{a!}.\end{aligned} \tag{68}$$

□

Theorem 6. *For each nonnegative integer n, we have*

$$Ch_n(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{m=k}^{l}\binom{n}{l}\begin{bmatrix} l \\ k \end{bmatrix}_\lambda \begin{bmatrix} l \\ m \end{bmatrix}_\lambda Ch_{n-l}\right)Bel_{k,\lambda}(x). \tag{69}$$

As the inversion formula of (69), we have

$$\begin{aligned}Bel_{n,\lambda}(x) &= \sum_{k=0}^{n}\left(\sum_{m=0}^{n-k}\left\{\begin{matrix} m+k \\ k \end{matrix}\right\}_\lambda \left\{\begin{matrix} n \\ m+k \end{matrix}\right\}_\lambda \right. \\ &\left.\quad +\frac{1}{2}\sum_{l=k}^{n-1}\sum_{m=0}^{l-k}\sum_{b=1}^{n-l}(1)_{b,\lambda}\binom{n}{l}\left\{\begin{matrix} m+k \\ k \end{matrix}\right\}_\lambda \left\{\begin{matrix} l \\ m+k \end{matrix}\right\}_\lambda \left\{\begin{matrix} n-l \\ b \end{matrix}\right\}_\lambda\right)Ch_k(x) \\ &= \sum_{k=0}^{n}\left(\sum_{m=k}^{n}\left\{\begin{matrix} n \\ m \end{matrix}\right\}_\lambda \left\{\begin{matrix} m \\ k \end{matrix}\right\}_\lambda \right. \\ &\left.\quad +\frac{1}{2}\sum_{m=k+1}^{n}\sum_{l=k}^{m-1}\binom{m}{l}\left\{\begin{matrix} n \\ m \end{matrix}\right\}_\lambda \left\{\begin{matrix} l \\ k \end{matrix}\right\}_\lambda (1)_{m-l,\lambda}\right)Ch_k(x).\end{aligned} \tag{70}$$

Proof. Let $Ch_n(x) = \sum_{k=0}^{n} c_{n,k}Bel_{k,\lambda}(x)$. Since

$$\begin{aligned}Ch_n(x) &\sim \left(\frac{e_\lambda(t)+1}{2}, e_\lambda(t)-1\right)_\lambda, \\ Bel_{n,\lambda}(x) &\sim \left(1, \log_\lambda(1+t)\right)_\lambda,\end{aligned} \tag{71}$$

by Theorem 1and (67), we obtain

$$\begin{aligned}c_{n,k} &= \frac{1}{k!}\left\langle \frac{1}{((t+2)/2)}\left(\log_\lambda\left(1+\log_\lambda(1+t)\right)\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \frac{2}{t+2} \middle| \left(\frac{1}{k!}\left(\log_\lambda\left(1+\log_\lambda(1+t)\right)\right)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \frac{2}{t+2} \middle| \left(\sum_{l=k}^{\infty}\sum_{m=k}^{l}\begin{bmatrix} m \\ k \end{bmatrix}_\lambda \begin{bmatrix} l \\ m \end{bmatrix}_\lambda \frac{t^l}{l!}\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}\sum_{m=k}^{l}\binom{n}{l}\begin{bmatrix} m \\ k \end{bmatrix}_\lambda \begin{bmatrix} l \\ m \end{bmatrix}_\lambda \left\langle \frac{2}{t+2} \middle| (x)_{n-l,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}\sum_{m=k}^{l}\binom{n}{l}\begin{bmatrix} m \\ k \end{bmatrix}_\lambda \begin{bmatrix} l \\ m \end{bmatrix}_\lambda Ch_{n-l}.\end{aligned} \tag{72}$$

Conversely, we assume that $Bel_{n,\lambda}(x) = \sum_{k=0}^{n} d_{n,k}Ch_k(x)$. By (68), we obtain

$$\begin{aligned}d_{n,k} &= \frac{1}{k!}\left\langle \frac{e_\lambda\left(e_\lambda(t)-1\right)+1}{2}\left(e_\lambda\left(e_\lambda(t)-1\right)-1\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \frac{e_\lambda\left(e_\lambda(t)-1\right)+1}{2} \middle| \left(\frac{1}{k!}\left(e_\lambda\left(e_\lambda(t)-1\right)-1\right)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \frac{e_\lambda\left(e_\lambda(t)-1\right)+1}{2} \middle| \left(\sum_{l=k}^{\infty}\left\{\begin{matrix} l \\ k \end{matrix}\right\}_\lambda \frac{1}{l!}\left(e_\lambda(t)-1\right)^l\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \frac{e_\lambda\left(e_\lambda(t)-1\right)+1}{2} \middle| \left(\sum_{l=k}^{\infty}\sum_{m=l}^{\infty}\left\{\begin{matrix} l \\ k \end{matrix}\right\}_\lambda \left\{\begin{matrix} m \\ l \end{matrix}\right\}_\lambda \frac{t^m}{m!}\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}\sum_{m=0}^{l-k}\binom{n}{l}\left\{\begin{matrix} m+k \\ k \end{matrix}\right\}_\lambda \left\{\begin{matrix} l \\ m+k \end{matrix}\right\}_\lambda \left\langle \frac{e_\lambda\left(e_\lambda(t)-1\right)+1}{2} \middle| (x)_{n-l,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}\sum_{m=0}^{l-k}\binom{n}{l}\left\{\begin{matrix} m+k \\ k \end{matrix}\right\}_\lambda \left\{\begin{matrix} l \\ m+k \end{matrix}\right\}_\lambda \\ &\quad\times\left(\left\langle 1 \middle| (x)_{n-l,\lambda}\right\rangle + \frac{1}{2}\left\langle \sum_{a=1}^{\infty}\sum_{b=1}^{a}(1)_{b,\lambda}\left\{\begin{matrix} a \\ b \end{matrix}\right\}_\lambda \frac{t^a}{a!} \middle| (x)_{n-l,\lambda}\right\rangle_\lambda\right) \\ &= \sum_{m=0}^{n-k}\left\{\begin{matrix} m+k \\ k \end{matrix}\right\}_\lambda \left\{\begin{matrix} n \\ m+k \end{matrix}\right\}_\lambda \\ &\quad+\frac{1}{2}\sum_{l=k}^{n-1}\sum_{m=0}^{l-k}\sum_{b=1}^{n-l}(1)_{b,\lambda}\binom{n}{l}\left\{\begin{matrix} m+k \\ k \end{matrix}\right\}_\lambda \left\{\begin{matrix} l \\ m+k \end{matrix}\right\}_\lambda \left\{\begin{matrix} n-l \\ b \end{matrix}\right\}_\lambda.\end{aligned} \tag{73}$$

On the other hand, by (11) and (66), we have

$$
\begin{aligned}
d_{n,k} &= \frac{1}{k!}\left\langle \frac{e_\lambda(t)+1}{2}\left(e_\lambda(t)-1\right)^k \middle| Bel_{n,\lambda}(x)\right\rangle_\lambda \\
&= \frac{1}{k!}\sum_{m=0}^{n}\left\{ \begin{matrix} n \\ m \end{matrix}\right\}_\lambda \left\langle \frac{e_\lambda(t)+1}{2}\left(e_\lambda(t)-1\right)^k \middle| (x)_{m,\lambda}\right\rangle_\lambda \\
&= \sum_{m=0}^{n}\left\{ \begin{matrix} n \\ m \end{matrix}\right\}_\lambda \left\langle \frac{e_\lambda(t)+1}{2} \middle| \left(\frac{1}{k!}\left(e_\lambda(t)-1\right)^k\right)_\lambda (x)_{m,\lambda}\right\rangle_\lambda \\
&= \sum_{m=0}^{n}\sum_{l=k}^{m}\binom{m}{l}\left\{ \begin{matrix} n \\ m \end{matrix}\right\}_\lambda \left\{ \begin{matrix} l \\ k \end{matrix}\right\}_\lambda \left\langle \frac{e_\lambda(t)+1}{2} \middle| (x)_{m-l,\lambda}\right\rangle_\lambda \\
&= \sum_{m=0}^{n}\sum_{l=k}^{m}\binom{m}{l}\left\{ \begin{matrix} n \\ m \end{matrix}\right\}_\lambda \left\{ \begin{matrix} l \\ k \end{matrix}\right\}_\lambda \left\langle 1+\frac{1}{2}\sum_{a=1}^{\infty}(1)_{a,\lambda}\frac{t^a}{a!} \middle| (x)_{m-l,\lambda}\right\rangle_\lambda \\
&= \sum_{m=0}^{n}\sum_{l=k}^{m}\binom{m}{l}\left\{ \begin{matrix} n \\ m \end{matrix}\right\}_\lambda \left\{ \begin{matrix} l \\ k \end{matrix}\right\}_\lambda \\
&\quad \cdot\left(\left\langle 1 \middle| (x)_{m-l,\lambda}\right\rangle_\lambda + \frac{1}{2}\sum_{a=1}^{\infty}\frac{(1)_{a,\lambda}}{a!}\left\langle t^a \middle| (x)_{m-l,\lambda}\right\rangle_\lambda\right) \\
&= \sum_{m=k}^{n}\left\{ \begin{matrix} n \\ m \end{matrix}\right\}_\lambda \left\{ \begin{matrix} m \\ k \end{matrix}\right\}_\lambda + \frac{1}{2}\sum_{m=k+1}^{n}\sum_{l=k}^{m-1}\binom{m}{l}\left\{ \begin{matrix} n \\ m \end{matrix}\right\}_\lambda \left\{ \begin{matrix} l \\ k \end{matrix}\right\}_\lambda (1)_{m-l,\lambda},
\end{aligned}
\tag{74}
$$

and thus our proofs are completed.

The unsigned Lah number $L(n,k)$ counts the number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets and has the explicit formula (see [1, 20, 23, 38, 39])

$$
L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}. \tag{75}
$$

By (75), we can derive the generating function of $L(n,k)$ to be (see [1, 20, 23, 38, 39])

$$
\frac{1}{k!}\left(\frac{t}{1-t}\right)^k = \sum_{n=k}^{\infty} L(n,k)\frac{t^n}{n!},\ (k \geq 0). \tag{76}
$$

Recently, Kim–Kim introduced the degenerate Lah–Bell polynomials as follows (see [1, 20]):

$$
e_\lambda^x\left(\frac{1}{1-t}-1\right) = \sum_{n=0}^{\infty} B_{n,\lambda}^L(x)\frac{t^n}{n!}. \tag{77}
$$

In the special case of $x=1$, $B_n^L = B_n^L(1)$ are called Lah–Bell numbers. Note that n-th Lah–Bell number B_n^L $(n \geq 0)$ is the number of ways a set of n elements can be partitioned into nonempty linearly ordered subsets. By (77), we can derive the following:

$$
\begin{aligned}
\sum_{n=0}^{\infty} B_{n,\lambda}^L(x)\frac{t^n}{n!} &= e_\lambda^x\left(\frac{t}{1-t}\right) \\
&= \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{1}{n!}\left(\frac{t}{1-t}\right)^n \\
&= \sum_{n=0}^{\infty}(x)_{n,\lambda}\sum_{m=n}^{\infty}L(m,n)\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}L(n,m)(x)_{m,\lambda}\right)\frac{t^n}{n!},
\end{aligned}
\tag{78}
$$

and thus, we obtain

$$
B_{n,\lambda}^L(x) = \sum_{m=0}^{n} L(n,m)(x)_{m,\lambda}. \tag{79}
$$

□

Theorem 7. *For each nonnegative integer n, we have*

$$
Ch_n(x) = \sum_{k=0}^{n}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}(-1)^{m-k}\binom{n}{l}\binom{m}{k}\langle k\rangle_{m-k}\left[\begin{matrix} l \\ m \end{matrix}\right]_\lambda Ch_{n-l}\right)B_k^L(x). \tag{80}
$$

As the inversion formula of (80), we have

$$
\begin{aligned}
B_n^L(x) &= \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\left\{ \begin{matrix} l \\ k \end{matrix}\right\}_\lambda L(n,l)\right. \\
&\quad \left. +\frac{1}{2}\sum_{m=k}^{n-1}\sum_{l=k}^{m}\sum_{b=1}^{n-m}\binom{n}{m}(1)_{b,\lambda}\left\{ \begin{matrix} l \\ k \end{matrix}\right\}_\lambda (m,l)L(n-m,b)\right)Ch_k(x) \\
&= \sum_{k=0}^{n}\left(\sum_{m=k}^{n}\left\{ \begin{matrix} m \\ k \end{matrix}\right\}_\lambda L(n,m)\right. \\
&\quad \left. +\frac{1}{2}\sum_{m=0}^{n}\sum_{l=k}^{m-1}\binom{m}{l}(1)_{m-l,\lambda}\left\{ \begin{matrix} l \\ k \end{matrix}\right\}_\lambda L(n,m)\right)Ch_k(x).
\end{aligned}
\tag{81}
$$

Proof. Let $Ch_n(x) = \sum_{k=0}^{n} c_{n,k}B_k^L(x)$. Since

$$
\begin{aligned}
Ch_n(x) &\sim \left(\frac{e_\lambda(t)+1}{2}, e_\lambda(t)-1\right)_\lambda, \\
B_n^L(x) &\sim \left(1, \frac{t}{1+t}\right)_\lambda,
\end{aligned}
\tag{82}
$$

by (11) and (28), we obtain

$$\begin{aligned}
c_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{t}{1+t}\right)^k \middle| Ch_n(x) \right\rangle_\lambda \\
&= \frac{1}{k!}\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\begin{bmatrix} l \\ m \end{bmatrix}_\lambda Ch_{n-l}\left\langle \left(\frac{t}{1+t}\right)^k \middle| (x)_{m,\lambda} \right\rangle_\lambda \\
&= \frac{1}{k!}\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\begin{bmatrix} l \\ m \end{bmatrix}_\lambda Ch_{n-l}\left\langle (1+t)^{-k} \middle| \left(t^k\right)_\lambda (x)_{m,\lambda} \right\rangle_\lambda \\
&= \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\binom{m}{k}\begin{bmatrix} l \\ m \end{bmatrix}_\lambda Ch_{n-l}\left\langle \sum_{a=0}^{\infty}\frac{(-1)^a\langle k\rangle_a}{a!}t^a \middle| (x)_{m-k,\lambda} \right\rangle_\lambda \\
&= \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\binom{m}{k}(-1)^{m-k}\langle k\rangle_{m-k}\begin{bmatrix} l \\ m \end{bmatrix}_\lambda Ch_{n-l}.
\end{aligned} \tag{83}$$

Conversely, we assume that $B^L_{n,\lambda}(x) = \sum_{k=0}^{n} d_{n,k}Ch_k(x)$. Then,

$$\begin{aligned}
d_{n,k} &= \frac{1}{k!}\left\langle \frac{e_\lambda(t/1-t)+1}{2}\left(e_\lambda(t/1-t)-1\right)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\
&= \left\langle \frac{e_\lambda(t/1-t)+1}{2} \middle| \left(\frac{1}{k!}\left(e_\lambda((t/1-t))-1\right)^k\right)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\
&= \left\langle \frac{e_\lambda((t/1-t))+1}{2} \middle| \left(\sum_{l=k}^{\infty}\begin{Bmatrix} l \\ k \end{Bmatrix}_\lambda \frac{1}{l!}((t/1-t))^l\right)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\
&= \left\langle \frac{e_\lambda((t/1-t))+1}{2} \middle| \left(\sum_{l=k}^{\infty}\sum_{m=l}^{\infty}\begin{Bmatrix} l \\ k \end{Bmatrix}_\lambda L(m,l)\frac{t^m}{m!}\right)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\
&= \sum_{m=k}^{n}\sum_{l=k}^{m}\binom{n}{m}\begin{Bmatrix} l \\ k \end{Bmatrix}_\lambda L(m,l)\left\langle \frac{e_\lambda((t/1-t))+1}{2} \middle| (x)_{n-m,\lambda} \right\rangle_\lambda \\
&= \sum_{m=k}^{n}\sum_{l=k}^{m}\binom{n}{m}\begin{Bmatrix} l \\ k \end{Bmatrix}_\lambda L(m,l) \\
&\quad \left\langle 1+\frac{1}{2}\sum_{a=1}^{\infty}\sum_{b=a}^{\infty}(1)_{a,\lambda}L(b,a)\frac{t^b}{b!} \middle| (x)_{n-m,\lambda} \right\rangle_\lambda \\
&= \sum_{m=k}^{n}\sum_{l=k}^{m}\binom{n}{m}\begin{Bmatrix} l \\ k \end{Bmatrix}_\lambda L(m,l) \\
&\quad \times\left(\left\langle 1 \middle| (x)_{n-m,\lambda} \right\rangle_\lambda + \frac{1}{2}\left\langle \sum_{a=1}^{\infty}\sum_{b=1}^{a}(1)_{b,\lambda}L(a,b)\frac{t^a}{a!} \middle| (x)_{n-m,\lambda} \right\rangle_\lambda\right) \\
&= \sum_{l=k}^{n}\begin{Bmatrix} l \\ k \end{Bmatrix}_\lambda L(n,l) \\
&\quad + \frac{1}{2}\sum_{m=k}^{n-1}\sum_{l=k}^{m}\sum_{b=1}^{n-m}\binom{n}{m}(1)_{b,\lambda}\begin{Bmatrix} l \\ k \end{Bmatrix}_\lambda L(m,l)L(n-m,b).
\end{aligned} \tag{84}$$

On the other hand, by (11) and (79), we obtain

$$\begin{aligned}
d_{n,k} &= \frac{1}{k!}\left\langle \frac{e_\lambda(t)+1}{2}\left(e_\lambda(t)-1\right)^k \middle| B^L_{n,\lambda}(x) \right\rangle \\
&= \frac{1}{k!}\sum_{m=0}^{n}L(n,m)\left\langle \frac{e_\lambda(t)+1}{2}\left(e_\lambda(t)-1\right)^k \middle| (x)_{m,\lambda} \right\rangle_\lambda \\
&= \sum_{m=0}^{n}L(n,m)\left\langle \frac{e_\lambda(t)+1}{2} \middle| \left(\frac{1}{k!}\left(e_\lambda(t)-1\right)^k\right)_\lambda (x)_{m,\lambda} \right\rangle_\lambda \\
&= \sum_{m=0}^{n}\sum_{l=k}^{m}\binom{m}{l}L(n,m)\begin{Bmatrix} l \\ k \end{Bmatrix}_\lambda \left\langle \frac{e_\lambda(t)+1}{2} \middle| (x)_{m-l,\lambda} \right\rangle_\lambda \\
&= \sum_{m=0}^{n}\sum_{l=k}^{m}\binom{m}{l}L(n,m)\begin{Bmatrix} l \\ k \end{Bmatrix}_\lambda \left\langle 1+\frac{1}{2}\sum_{a=1}^{\infty}(1)_{a,\lambda}\frac{t^a}{a!} \middle| (x)_{m-l,\lambda} \right\rangle_\lambda \\
&= \sum_{m=0}^{n}\sum_{l=k}^{m}\binom{m}{l}L(n,m)\begin{Bmatrix} l \\ k \end{Bmatrix}_\lambda \\
&\quad \times\left(\left\langle 1 \middle| (x)_{m-l,\lambda} \right\rangle_\lambda + \frac{1}{2}\left\langle \sum_{a=1}^{\infty}\frac{(1)_{a,\lambda}}{a!}t^a \middle| (x)_{m-l,\lambda} \right\rangle_\lambda\right) \\
&= \sum_{m=k}^{n}\begin{Bmatrix} m \\ k \end{Bmatrix}_\lambda L(n,m) \\
&\quad + \frac{1}{2}\sum_{m=0}^{n}\sum_{l=k}^{m-1}\binom{m}{l}(1)_{m-l,\lambda}\begin{Bmatrix} l \\ k \end{Bmatrix}_\lambda L(n,m),
\end{aligned} \tag{85}$$

and so our proofs are completed.

By the definition of the degenerate Bernoulli polynomials of the second kind, we note that

$$\begin{aligned}
\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{t}{\log_\lambda(1+t)}e_\lambda^x\left(\log_\lambda(1+t)\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\begin{bmatrix} m \\ l \end{bmatrix}_\lambda \beta_{n-m,\lambda}(x)_{l,\lambda}\right)\frac{t^n}{n!},
\end{aligned} \tag{86}$$

and thus, we obtain

$$\beta_{n,\lambda}(x) = \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\begin{bmatrix} m \\ l \end{bmatrix}_\lambda \beta_{n-m,\lambda}(x)_{l,\lambda}. \tag{87}$$

□

Theorem 8. *For each $n \geq 0$, we have*

$$Ch_n(x) = \left(\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}Ch_l D_{n-k-l,\lambda}\right)\beta_{k,\lambda}(x). \tag{88}$$

As the inversion formula of (88), we have

$$\beta_{n,\lambda}(x)=\sum_{k=0}^{n}\left(\binom{n}{k}\beta_{n-k,\lambda}+\binom{n}{k}\frac{n-k}{2}\beta_{n-k-1,\lambda}\right)Ch_k(x)$$
$$=\sum_{k=0}^{n}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\beta_{n-m,\lambda}\left\{\begin{matrix}m\\ l\end{matrix}\right\}_\lambda\left\{\begin{matrix}l\\ k\end{matrix}\right\}_\lambda\right.$$
$$\left.+\frac{1}{2}\sum_{m=0}^{n}\sum_{l=k+1}^{m}\sum_{a=k}^{l-1}\binom{n}{m}\binom{l}{a}\beta_{n-m,\lambda}\left[\begin{matrix}m\\ l\end{matrix}\right]_\lambda\left\{\begin{matrix}a\\ k\end{matrix}\right\}_\lambda(1)_{l-a,\lambda}\right)Ch_k(x). \quad (89)$$

Proof. Let $Ch_n(x)=\sum_{k=0}^{n}c_{n,k}\beta_{k,\lambda}(x)$. Since

$$Ch_n(x)\sim\left(\frac{e_\lambda(t)+1}{2},e_\lambda(t)-1\right)_\lambda,$$
$$\beta_{n,\lambda}(x)\sim\left(\frac{t}{e_\lambda(t)-1},e_\lambda(t)-1\right)_\lambda, \quad (90)$$

by Theorem 1, we obtain

$$c_{n,k}=\frac{1}{k!}\left\langle\frac{\log_\lambda(1+t)/t}{t+2/2}t^k\middle|(x)_{n,\lambda}\right\rangle_\lambda$$
$$=\frac{1}{k!}\left\langle\frac{2}{t+2}\frac{\log_\lambda(1+t)}{t}\middle|\left(t^k\right)_\lambda(x)_{n,\lambda}\right\rangle_\lambda$$
$$=\binom{n}{k}\left\langle\frac{2}{t+2}\frac{\log_\lambda(1+t)}{t}\middle|(x)_{n-k,\lambda}\right\rangle_\lambda$$
$$=\binom{n}{k}\left\langle\frac{\log_\lambda(1+t)}{t}\middle|\left(\frac{2}{t+2}\right)_\lambda(x)_{n-k,\lambda}\right\rangle_\lambda \quad (91)$$
$$=\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}Ch_l\left\langle\frac{\log_\lambda(1+t)}{t}\middle|(x)_{n-k-l,\lambda}\right\rangle_\lambda$$
$$=\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}Ch_lD_{n-k-l,\lambda}.$$

Conversely, we assume that $\beta_{n,\lambda}(x)=\sum_{k=0}^{n}d_{n,k}Ch_k(x)$. Then,

$$d_{n,k}=\frac{1}{k!}\left\langle\frac{t+2/2}{\log_\lambda(1+t)/t}t^k\middle|(x)_{n,\lambda}\right\rangle_\lambda$$
$$=\frac{1}{k!}\left\langle\frac{t}{\log_\lambda(1+t)}\frac{t+2}{2}\middle|\left(t^k\right)_\lambda(x)_{n,\lambda}\right\rangle_\lambda$$
$$=\binom{n}{k}\left\langle\frac{t}{\log_\lambda(1+t)}\middle|\left(1+\frac{1}{2}t\right)(x)_{n-k,\lambda}\right\rangle_\lambda \quad (92)$$
$$=\binom{n}{k}\left\langle\sum_{m=0}^{\infty}\beta_{m,\lambda}\frac{t^m}{m!}\middle|(x)_{n-k,\lambda}+\frac{n-k}{2}(x)_{n-k-1,\lambda}\right\rangle_\lambda$$
$$=\binom{n}{k}\beta_{n-k,\lambda}+\binom{n}{k}\frac{n-k}{2}\beta_{n-k-1,\lambda}.$$

On the other hand, by (11) and (87), we obtain

$$d_{n,k}=\frac{1}{k!}\left\langle\frac{e_\lambda(t)+1}{2}(e_\lambda(t)-1)^k\middle|\beta_{n,\lambda}(x)\right\rangle_\lambda$$
$$=\frac{1}{k!}\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\beta_{n-m,\lambda}\left\{\begin{matrix}m\\ l\end{matrix}\right\}_\lambda\left\langle\frac{e_\lambda(t)+1}{2}(e_\lambda(t)-1)^k\middle|(x)_{l,\lambda}\right\rangle_\lambda$$
$$=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\beta_{n-m,\lambda}\left\{\begin{matrix}m\\ l\end{matrix}\right\}_\lambda\left\langle\frac{e_\lambda(t)+1}{2}\middle|\left(\frac{1}{k!}(e_\lambda(t)-1)^k\right)_\lambda(x)_{l,\lambda}\right\rangle_\lambda$$
$$=\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{a=k}^{l}\binom{n}{m}\binom{l}{a}\beta_{n-m,\lambda}\left\{\begin{matrix}m\\ l\end{matrix}\right\}_\lambda\left\{\begin{matrix}a\\ k\end{matrix}\right\}_\lambda\left\langle\frac{e_\lambda(t)+1}{2}\middle|(x)_{l-a,\lambda}\right\rangle_\lambda$$
$$=\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{a=k}^{l}\binom{n}{m}\binom{l}{a}\beta_{n-m,\lambda}\left\{\begin{matrix}m\\ l\end{matrix}\right\}_\lambda\left\{\begin{matrix}a\\ k\end{matrix}\right\}_\lambda$$
$$\left\langle 1+\frac{1}{2}\sum_{b=1}^{\infty}(1)_{b,\lambda}\frac{t^b}{b!}\middle|(x)_{l-a,\lambda}\right\rangle_\lambda$$
$$=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\beta_{n-m,\lambda}\left\{\begin{matrix}m\\ l\end{matrix}\right\}_\lambda\left\{\begin{matrix}l\\ k\end{matrix}\right\}_\lambda$$
$$+\frac{1}{2}\sum_{m=0}^{n}\sum_{l=k+1}^{m}\sum_{a=k}^{l-1}\binom{n}{m}\binom{l}{a}\beta_{n-m,\lambda}\left\{\begin{matrix}m\\ l\end{matrix}\right\}_\lambda\left\{\begin{matrix}a\\ k\end{matrix}\right\}_\lambda(1)_{l-a,\lambda}, \quad (93)$$

and thus our proofs are completed.

The Mittag–Leffler polynomials are defined by the generating function to be see ([14, 34, 36])

$$\left(\frac{1+t}{1-t}\right)^x=\sum_{n=0}^{\infty}M_n(x)\frac{t^n}{n!}. \quad (94)$$

□

Theorem 9. *For each nonnegative integer n, we have*

$$Ch_n(x)=\sum_{k=0}^{n}\left(\binom{n}{k}\frac{(-1)^{n-k}\langle k+1\rangle_{n-k}}{2^n}\right)M_k(x). \quad (95)$$

As the inversion formula of (95), we have

$$M_n(x)=\sum_{k=0}^{n}\left(2^k\sum_{l=k}^{n}\binom{n}{l}L(l,k)(n-l)!\right)Ch_k(x). \quad (96)$$

Proof. Let $Ch_n(x)=\sum_{k=0}^{n}c_{n,k}M_k(x)$. Since

$$Ch_n(x)\sim\left(\frac{e_\lambda(t)+1}{2},e_\lambda(t)-1\right)_\lambda,$$
$$M_n(x)\sim\left(1,\frac{e_\lambda(t)-1}{e_\lambda(t)+1}\right)_\lambda, \quad (97)$$

by Theorem 1 and (30), we obtain

$$c_{n,k}=\frac{1}{k!}\left\langle\frac{1}{t+2/2}\left(\frac{t}{t+2}\right)^k\middle|(x)_{n,\lambda}\right\rangle_\lambda=\frac{1}{k!}\left\langle\frac{2t^k}{(t+2)^{k+1}}\middle|(x)_{n,\lambda}\right\rangle_\lambda$$
$$=\frac{1}{k!}\left\langle\sum_{m=0}^{\infty}\frac{(-1)^m\langle k+1\rangle_m}{2^{k+m}}\frac{t^m}{m!}\middle|\left(t^k\right)_\lambda(x)_{n,\lambda}\right\rangle_\lambda$$
$$=\binom{n}{k}\left\langle\sum_{m=0}^{\infty}\frac{(-1)^m\langle k+1\rangle_m}{2^{k+m}}\frac{t^m}{m!}\middle|(x)_{n-k,\lambda}\right\rangle_\lambda$$
$$=\binom{n}{k}\frac{(-1)^{n-k}\langle k+1\rangle_{n-k}}{2^n}. \tag{98}$$

Conversely, we assume that $M_n(x)=\sum_{k=0}^{n}d_{n,k}Ch_k(x)$. Then, by (76), we obtain

$$d_{n,k}=\frac{1}{k!}\left\langle\frac{e_\lambda(\log_\lambda 1+t/1-t)+1}{2}\left(e_\lambda\left(\log_\lambda\frac{1+t}{1-t}\right)-1\right)^k\middle|(x)_{n,\lambda}\right\rangle_\lambda$$
$$=\frac{1}{2k!}\left\langle\frac{2}{1-t}\left(\frac{2t}{1-t}\right)^k\middle|(x)_{n,\lambda}\right\rangle_\lambda$$
$$=2^k\left\langle\frac{1}{1-t}\middle|\left(\frac{1}{k!}\left(\frac{t}{1-t}\right)^k\right)_\lambda(x)_{n,\lambda}\right\rangle_\lambda$$
$$=2^k\sum_{l=k}^{n}\binom{n}{l}L(l,k)\left\langle\frac{1}{1-t}\middle|(x)_{n-l,\lambda}\right\rangle_\lambda$$
$$=2^k\sum_{l=k}^{n}\binom{n}{l}L(l,k)\left\langle\sum_{m=0}^{\infty}\langle 1\rangle_m\frac{t^m}{m!}\middle|(x)_{n-l,\lambda}\right\rangle_\lambda$$
$$=2^k\sum_{l=k}^{n}\binom{n}{l}L(l,k)(n-l)!, \tag{99}$$

and so our proofs are completed.

The degenerate Frobenius–Euler polynomials of order α are defined by the generating function to be (see [1, 2, 31, 40])

$$\left(\frac{1-u}{e_\lambda(t)-u}\right)^\alpha e_\lambda^x(t)=\sum_{n=0}^{\infty}h_{n,\lambda}^{(\alpha)}(x|u)\frac{t^n}{n!},\quad (u\neq 1,\ u\in\mathbb{C}),\ (r\geq 0). \tag{100}$$

In the special case $x=0$, $h_{n,\lambda}^{(\alpha)}(u)=h_{n,\lambda}^{(\alpha)}(0|u)$ are called the degenerate Frobenius–Euler numbers of order α. By the definition of the Frobenius–Euler polynomials of order α, we note that

$$\sum_{n=0}^{\infty}h_{n,\lambda}^{(\alpha)}(x|u)\frac{t^n}{n!}=\left(\frac{1-u}{e_\lambda(t)-u}\right)^\alpha e_\lambda^x(t)$$
$$=\left(\sum_{n=0}^{\infty}h_{n,\lambda}^{(\alpha)}(u)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}\right) \tag{101}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}h_{n-l,\lambda}^{(\alpha)}(u)(x)_{l,\lambda}\right)\frac{t^n}{n!},$$

and thus we have

$$h_{n,\lambda}^{(\alpha)}(x|u)=\sum_{l=0}^{n}\binom{n}{l}h_{n-l,\lambda}^{(\alpha)}(u)(x)_{l,\lambda}. \tag{102}$$

Furthermore, we see that

$$(e_\lambda(t)-u)^{-\alpha}=((e_\lambda(t)-1)+(1-u))^{-\alpha}$$
$$=\sum_{a=0}^{\infty}\binom{-\alpha}{a}(1-u)^{-\alpha-a}(e_\lambda(t)-1)^a$$
$$=\sum_{a=0}^{\infty}(-1)^a\langle\alpha\rangle_a(1-u)^{-\alpha-a}\frac{1}{a!}(e_\lambda(t)-1)^a$$
$$=\sum_{m=0}^{\infty}\sum_{a=0}^{m}\frac{(-1)^a\langle\alpha\rangle_a}{(1-u)^{\alpha+a}}\left\{\begin{matrix}m\\a\end{matrix}\right\}_\lambda\frac{t^m}{m!}. \tag{103}$$

□

Theorem 10. *For each nonnegative integer n, we have*

$$Ch_n(x)=\sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\binom{\alpha}{m}\frac{(n-l)_m}{(1-u)^m}\left\{\begin{matrix}l\\k\end{matrix}\right\}_\lambda Ch_{n-l-m}\right)h_{k,\lambda}^{(\alpha)}(x|u). \tag{104}$$

As the inversion formula of (104), we have

$$h_{n,\lambda}^{(\alpha)}(x)=\sum_{k=0}^{n}\left(\sum_{m=k}^{n}\sum_{b=0}^{n-m}\binom{n}{m}\frac{(-1)^b\langle\alpha\rangle_b}{(1-u)^b}\left\{\begin{matrix}m\\k\end{matrix}\right\}_\lambda\left\{\begin{matrix}n-m\\b\end{matrix}\right\}_\lambda\right.$$
$$+\frac{1}{2}\sum_{m=k}^{n-1}\sum_{a=0}^{n-m-1}\sum_{b=0}^{a}\binom{n}{m}\binom{n-m}{a}$$
$$\left.\times\frac{(-1)^b\langle\alpha\rangle_b}{(1-u)^b}\{mk\}_\lambda\left\{\begin{matrix}a\\b\end{matrix}\right\}_\lambda(1)_{n-m-a,\lambda}\right)Ch_k(x)$$
$$=\sum_{k=0}^{n}\left(\sum_{l=0}^{n}\binom{n}{l}\left\{\begin{matrix}l\\k\end{matrix}\right\}_\lambda h_{n-l,\lambda}^{(\alpha)}(u)\right.$$
$$\left.+\frac{1}{2}\sum_{l=k+1}^{n}\sum_{m=k}^{l-1}\binom{n}{l}\binom{l}{m}\left\{\begin{matrix}m\\k\end{matrix}\right\}_\lambda h_{n-l,\lambda}^{(\alpha)}(u)(1)_{l-m,\lambda}\right)Ch_k(x). \tag{105}$$

Proof. Let $Ch_n(x)=\sum_{k=0}^{n}c_{n,k}h_{k,\lambda}^{(\alpha)}(x|u)$. Since

$$Ch_n(x)\sim\left(\frac{e_\lambda(t)+1}{2},e_\lambda(t)-1\right)_\lambda,$$
$$h_{n,\lambda}^{(\alpha)}(x|u)\sim\left(\left(\frac{e_\lambda(t)-u}{1-u}\right)^\alpha,t\right)_\lambda, \tag{106}$$

by Theorem 1, we obtain

$$c_{n,k}=\frac{1}{k!}\left\langle\frac{((1+t)-u/1-u)^\alpha}{t+2/2}(\log_\lambda(1+t))^k\middle|(x)_{n,\lambda}\right\rangle_\lambda$$
$$=\left\langle\frac{2}{t+2}\left(\frac{t+(1-u)}{1-u}\right)^\alpha\middle|\left(\frac{1}{k!}(\log_\lambda(1+t))^k\right)_\lambda(x)_{n,\lambda}\right\rangle_\lambda$$
$$=\frac{1}{(1-u)^\alpha}\sum_{l=k}^{n}\binom{n}{l}\begin{bmatrix}l\\k\end{bmatrix}_\lambda\left\langle\frac{2}{t+2}(t+(1-u))^\alpha\middle|(x)_{n-l,\lambda}\right\rangle_\lambda$$
$$=\frac{1}{(1-u)^\alpha}\sum_{l=k}^{n}\binom{n}{l}\begin{bmatrix}l\\k\end{bmatrix}_\lambda\left\langle\frac{2}{t+2}\middle|((t+(1-u))^\alpha)_\lambda(x)_{n-l,\lambda}\right\rangle_\lambda$$
$$=\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\binom{\alpha}{m}\frac{\begin{bmatrix}l\\k\end{bmatrix}_\lambda(n-l)_m}{(1-u)^m}\left\langle\frac{2}{t+2}\middle|(x)_{n-l-m,\lambda}\right\rangle_\lambda$$
$$=\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\binom{\alpha}{m}\frac{(n-l)_m}{(1-u)^m}\begin{bmatrix}l\\k\end{bmatrix}_\lambda Ch_{n-l-m}. \tag{107}$$

Conversely, we assume that $h_{n,\lambda}^{(\alpha)}(x|u)=\sum_{k=0}^{n}d_{n,k}Ch_k(x)$. Then, by Theorem 1 and (103), we have

$$d_{n,k}=\frac{1}{k!}\left\langle\frac{e_\lambda(t)+1/2}{(e_\lambda(t)-u/1-u)^\alpha}(e_\lambda(t)-1)^k\middle|(x)_{n,\lambda}\right\rangle_\lambda$$
$$=\left\langle\left(\frac{1-u}{e_\lambda(t)-u}\right)^\alpha\frac{e_\lambda(t)+1}{2}\middle|\left(\frac{1}{k!}(e_\lambda(t)-1)^k\right)_\lambda(x)_{n,\lambda}\right\rangle_\lambda$$
$$=\sum_{m=k}^{n}\binom{n}{m}\left\{\begin{matrix}m\\k\end{matrix}\right\}_\lambda\left\langle\left(\frac{1-u}{e_\lambda(t)-u}\right)^\alpha\frac{e_\lambda(t)+1}{2}\middle|(x)_{n-m,\lambda}\right\rangle_\lambda$$
$$=(1-u)^\alpha\sum_{m=k}^{n}\binom{n}{m}\left\{\begin{matrix}m\\k\end{matrix}\right\}_\lambda$$
$$\left\langle\frac{e_\lambda(t)+1}{2}\middle|((e_\lambda(t)-u)^{-\alpha})_\lambda(x)_{n-m,\lambda}\right\rangle_\lambda$$
$$=\sum_{m=k}^{n}\sum_{a=0}^{n-m}\sum_{b=0}^{a}\binom{n}{m}\binom{n-m}{a}\frac{(-1)^b\langle\alpha\rangle_b}{(1-u)^b}\left\{\begin{matrix}m\\k\end{matrix}\right\}_\lambda$$
$$\{ab\}_\lambda\left\langle\frac{e_\lambda(t)+1}{2}\middle|(x)_{n-m-a,\lambda}\right\rangle_\lambda$$
$$=\sum_{m=k}^{n}\sum_{a=0}^{n-m}\sum_{b=0}^{a}\binom{n}{m}\binom{n-m}{a}\frac{(-1)^b\langle\alpha\rangle_b}{(1-u)^b}\left\{\begin{matrix}m\\k\end{matrix}\right\}_\lambda\left\{\begin{matrix}a\\b\end{matrix}\right\}_\lambda$$
$$\times\left\langle1+\frac{1}{2}\sum_{l=1}^{\infty}(1)_{l,\lambda}\frac{t^l}{l!}\middle|(x)_{n-m-a,\lambda}\right\rangle_\lambda$$
$$=\sum_{m=k}^{n}\sum_{b=0}^{n-m}\binom{n}{m}\frac{(-1)^b\langle\alpha\rangle_b}{(1-u)^b}\left\{\begin{matrix}m\\k\end{matrix}\right\}_\lambda\left\{\begin{matrix}n-m\\b\end{matrix}\right\}_\lambda$$
$$+\frac{1}{2}\sum_{m=k}^{n-1}\sum_{a=0}^{n-m-1}\sum_{b=0}^{a}\binom{n}{m}\binom{n-m}{a}\frac{(-1)^b\langle\alpha\rangle_b}{(1-u)^b}$$
$$\left\{\begin{matrix}m\\k\end{matrix}\right\}_\lambda\left\{\begin{matrix}a\\b\end{matrix}\right\}_\lambda(1)_{n-m-a,\lambda}. \tag{108}$$

In addition, by (11) and (102), we obtain

$$
\begin{aligned}
d_{n,k} &= \frac{1}{k!}\left\langle \frac{e_\lambda(t)+1}{2}(e_\lambda(t)-1)^k \middle| h_{n,\lambda}^{(\alpha)}(x|u)\right\rangle_\lambda \\
&= \frac{1}{k!}\sum_{l=0}^{n}\binom{n}{l} h_{n-l,\lambda}^{(\alpha)}(u)\left\langle \frac{e_\lambda(t)+1}{2}(e_\lambda(t)-1)^k \middle| (x)_{l,\lambda}\right\rangle_\lambda \\
&= \sum_{l=0}^{n}\binom{n}{l} h_{n-l,\lambda}^{(\alpha)}(u)\left\langle \frac{e_\lambda(t)+1}{2} \middle| \left(\frac{1}{k!}(e_\lambda(t)-1)^k\right)_\lambda (x)_{l,\lambda}\right\rangle_\lambda \\
&= \sum_{l=0}^{n}\sum_{m=k}^{l}\binom{n}{l}\binom{l}{m}\left\{ \begin{matrix} m \\ k \end{matrix} \right\}_\lambda h_{n-l,\lambda}^{(\alpha)}(u)\left\langle \frac{e_\lambda(t)+1}{2} \middle| (x)_{l-m,\lambda}\right\rangle_\lambda \\
&= \sum_{l=0}^{n}\sum_{m=k}^{l}\binom{n}{l}\binom{l}{m}\left\{ \begin{matrix} m \\ k \end{matrix} \right\}_\lambda h_{n-l,\lambda}^{(\alpha)}(u) \\
&\quad \left\langle 1+\frac{1}{2}\sum_{a=1}^{\infty}(1)_{a,\lambda}\frac{t^a}{a!} \middle| (x)_{l-m,\lambda}\right\rangle_\lambda \\
&= \sum_{l=0}^{n}\binom{n}{l}\left\{ \begin{matrix} l \\ k \end{matrix} \right\}_\lambda h_{n-l,\lambda}^{(\alpha)}(u) \\
&\quad + \frac{1}{2}\sum_{l=k+1}^{n}\sum_{m=k}^{l-1}\binom{n}{l}\binom{l}{m}\left\{ \begin{matrix} m \\ k \end{matrix} \right\}_\lambda h_{n-l,\lambda}^{(\alpha)}(u)(1)_{l-m,\lambda}.
\end{aligned}
\tag{109}
$$

□

3. Conclusion

In this paper, we studied the Changhee polynomials related to the lambda falling factorial (Theorem 2), the degenerate Bernoulli polynomials (Theorem 3), the degenerate Euler polynomials (Theorem 4), the degenerate Daehee polynomials (Theorem 5), the degenerate Bell polynomials (Theorem 6), the degenerate Lah–Bell polynomials (Theorem 7), the degenerate Bernoulli polynomials of the second kind (Theorem 8), the Mittag–Leffler polynomials (Theorem 9), and the degenerate Frobenius–Euler polynomials (Theorem 10) by finding the coefficients which are also polynomials or numbers when the n-th Changhee polynomial is expressed as a linear combination of those degenerate special polynomials by using the λ-Sheffer sequences and λ-differential operators. In addition, we derive the inversion formulas of these identities.

Umbral calculus has been applied in many fields such as combinatorial counting with linear recurrences and lattice path counting, graph theory using chromatic polynomials, probability theory, link invariant theory, statistics, topology, and physics. It is being actively applied in various fields by researchers. As one of our future projects, we would like to continue to study degenerate versions of certain special polynomials and numbers by using λ-Umbral calculus. [41].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

TK and JWP conceived the framework and structured the whole manuscript. TK and JWP wrote the paper. BMK and TAR checked the results of the manuscript. All authors read and approved the final paper.

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