Hindawi
Complexity
Volume 2022, Article ID 5920189, 34 pages
https://doi.org/10.1155/2022/5920189

Research Article

Applications of Cubic Schweizer–Sklar Power Heronian Mean to Multiple Attribute Decision-Making

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Received 1 November 2021; Revised 21 January 2022; Accepted 19 July 2022; Published 8 November 2022

Academic Editor: Hiroki Sayama

The cubic set (CS) is a basic simplification of several fuzzy notions, including fuzzy set (FS), interval-valued FS (IVFS), and intuitionistic FS (IFS). By the degrees of IVFS and FS, CS exposes fuzzy judgement, and this is a much more potent mathematical approach for dealing with information that is unclear, ambiguous, or indistinguishable. The article provides many innovative operational laws for cubic numbers (CNs) drawn on the Schweizer–Sklar (SS) t-norm (SSTN), and the SS t-conorm (SSTCN), as well as several desired properties of these operational laws. We also plan to emphasise on the cubic Schweizer–Sklar power Heronian mean (CSSPHM) operator, as well as the cubic Schweizer–Sklar power geometric Heronian mean (CSSPGHM) operator, in order to maintain the supremacy of the power aggregation (PA) operators that seize the complications of the unsuitable information and Heronian mean (HM) operators that contemplate the interrelationship between the input data being aggregated. A novel multiple attribute decision-making (MADM) model is anticipated for these freshly launched aggregation operators (AOs). Finally, a numerical example of enterprise resource planning is used to validate the approach's relevance and usefulness. There is also a comparison with existing decision-making models.

1. Introduction

Zadeh [1] proposed the fuzzy set (FS) as a procedure for expressing and transmitting precariousness and ambiguity. Since its inception, FS has attracted significant attention from intellectuals all over the world, who have calculated its factual and theoretical characteristics. Economic and business [2–4], genetic algorithms [5, 6], and supply chain management [7, 8] etc., are some of the most recent academic attempts at the theory and implementations of FSs. Ensuring the insertion of the notion of FS, various modifications of FSs were predicted, namely interval-valued FS [9], which explained the membership degree (MD) as a subclass of [0, 1] and Atanassov's intuitionistic fuzzy set (AIFS) [10], which clarified the MD and nonmembership degree (NMD) as a single number in the [0, 1], with the constraint that sum of the two degrees must be less or equal to 1. As a consequence, IFS goes farther into explaining uncertainty and unreliability than FS. The attractive scenario occurs when the MDs of such an object is expressed in the form of IVFS and FS. Under such settings, the conformist IFS is unable to manage such data. To handle the aforementioned situation, Jun et al. [11] initiated the perception of the cubic set (CS). The aforementioned sets are special cases of CS. Mahmood et al. [12] proposed the concept of CNs and initiate some weighted aggregation operators

(AOs) and apply these AOs to resolve multiple attribute decision-making (MADM) problems under a cubic environment.

One of the key elements in the MADM process is the AOs. The AOs can blend many real numbers into a single one. Various AOs have different properties, namely, the PA operator offered by Yager [13], have the ability to remove the negative effects of uncomfortable information from last ranking results, and have been enlarged by numerous researchers from all over the world to figure out how to deal with different situations. Xu [14] enlarged the ordinary PA operator and delivered the IF power aggregation operator and implemented it in multiple attribute group decision-making (MAGDM), which can minimize the effects of inaccurate information. Some AOs, such as the Bonferroni mean (BM) operators [15], Heronian Mean (HM) operators [16], and Muirhead mean (MM) operators [17], as well as the Maclaurin symmetric mean (MSM) operator [18], ended up taking the connection between input arguments into consideration. BM and HM can take into account the connection between two input arguments, but MSM and MM operators can take into account the connection between any number of input opinions. These AOs were later extended to deal with a wide range of ambiguous circumstances [19–23].

The majority of AOs use algebraic T-norm (TN) and T-conorm (TCN) to aggregate CNs. Currently, Ayub et al. [24] have presented a set of cubic fuzzy Dombi AOs that have been implemented on Dombi [25] TN and TCN and utilized to resolve MADM issues in a cubic fuzzy context. Dombi TN and Dombi TCN, as well as other TN and TCN, such as algebraic, Einstein, Hamacher, and Frank, are simplified in Archimedean TN (ATN) and Archimedean TCN (ATCN). On a generic parameter, Dombi TN and TCN outperform generic TN and TCN, providing more flexibility in the input dataset. Fahmi et al. [26], anticipated Einstein AOs for CNs and apply these AOs to solve MADM problems unde cubic information. Wan [27] and Wan and Dong [28] developed some power average/geometric operators for trapezoidal intuitionistic fuzzy (trIF) numbers and apply them to solve MAGDM problems under a trIF environment. Wan and Yi [29] initiated PA operators for trIFNs using strict t-norm and strict t-conorm. CS was further extended by Ali et al. [30] who introduced the concept of neutrosophic cubic set and give its applications in pattern recognition.

Schweizer–Sklar (SS), TN (SSTN), and Schweizer–Sklar TCN (SSTCN) [31] are thorough ATN and ATCN instances, similar to the TN and TCN mentioned above. Because they contain a parameter that may be changed, SSTN and SSTCN are more flexible and superior to the prior techniques. Despite this, the majority of SS research has been on identifying the underlying theory and forms of SSTN and SSTCN [32, 33]. Recently, SS operational laws (OLs) were anticipated for interval-valued IFS (IVIFS) and IFS by Liu et al. [34] and Zhang [35], respectively, and predicted various power aggregation operators for these fuzzy structures. On the basis of SS OLs, Wang and Liu [36] projected MSM operators for IFS and apply them to resolve MADM problems. Liu et al. [37] also proposed SS OLs for single-valued neutrosophic (SVN) elements, as well as a variety of SS prioritised AOs for dealing with MADM issues in an SVN context. Zhang et al. [38] predicted and used certain MM operators for SVNS identified on SS OLs to solve MADM issues. By capturing the variable parameter from $[0, \infty]$, Nagarajan et al. [39] developed a couple of SS OLs for interval neutrosophic set (INS). For IN numbers, they anticipated various WA/WG AOs implemented on these SS OLs. The COPRAS was enhanced by Rong et al. [40], who predicted a new MAGDM technique based on SS OLs.

From the above literature, it has been observed that the existing aggregation operators for CNs have only the capacity of removing the effect of awkward data or have the capacity of taking interrelationships among input arguments and a generic parameter. Yet, there are no such aggregation operators for CNs, which have the capacity of removing the effect of awkward data, can consider the interrelationship among input CNs, and also consist of the generic parameter. It has been observed that studies on various implementations of fuzzy MADM AOs depending on SS OLs have been published rapidly. Yet, no one has attempted to define cubic SS OLs and merge them with a power HM operator to deal with cubic information. As a consequence, we propose the following:

(1) The SS operations are considerably more adaptable and superior than the prior methods in terms of a variable parameter.
(2) Fortunately, there are many MADM difficulties in which the characteristics are linked, and many existing AOs can only alleviate such scenarios when the attributes are in the shape of real integers or other fuzzy formations.
(3) In the current situation, no such AOs exist which are drawn on SS OLs. In response to this limitation, we combined PA and HM operators with SS OLs to address MADM problems utilising cubic information.

The subsequent are the urgencies and contributions of this effort as a result of important impacts from earlier studies as follows:

(1) Developing innovative SS ALs for CNs, describing their basic features, and using them in SS ALs that anticipate CSS power HM operators, CSS power geometric HM operators, and their weighted form
(2) Examining the commencing AOs' basic features and exceptional cases
(3) Expecting the deployment of a MADM model on these commencing AOs
(4) Assessing enterprise resource planning (ERP) applications using a MADM model
(5) Confirming the feasibility and appropriateness of the launched MADM model

This paper is structured in the following way to achieve these goals. Section 2 introduces a variety of key concepts such as CSS, score and accuracy functions, PA, and HM operators. In Section 3, we look at a few SS OLs for CNs with general parameters that take values from $[-\infty, 0]$. Section 4 introduces the CSSPHM and CSSPGHM operators, as well as their weighted variants, and examines limited properties and detailed instances of the proposed AOs. In Section 5, a novel MADM model is established on these new AOs. A numerical example of enterprise resource planning is provided in order to verify the unassailability and compensations of the initiated approach. Finally, in Section 6, a brief conclusion is provided.

2. Preliminaries

In this portion, various essential conceptions namely, cubic set (CS), the Heronian mean (HM) operator, and their basic characteristics are reviewed.

2.1. The Cubic Set and Its Operational Laws

Definition 1 (see [11]). Let U be a universe of discourse set. A CS is classified and mathematically indicated as follows:

$$\mathrm{CS} = \{\langle u'', I(u''), f(u'')\rangle \text{ for all } u'' \in U\}, \tag{1}$$

where $I('') = [I^L(''), I^U('')]$ and $f('')$ are IVFS and FS, respectively. For computational affluence, we shall label a cubic number (CN) by the ordered pair $cn = \langle [I^L, I^U], f\rangle$, where $[I^L, I^U]$ and f are IVFN and FN, respectively. If $f \in [I^L, I^U]$, then it is said to be an internal cubic number and if $f \notin [I^L, I^U]$, then it is said to be an external cubic number.

The OLs for CS were classified by Jun [11] and are established below as follows:

Definition 2 (see [11]). Let CS_1 and CS_2 be any two CSS. Then,

$$\begin{aligned}
&\mathrm{CS}_1 \subseteq \mathrm{CS}_2 \text{ iff } I_1^L('') \le I_2^L(''),\\
&I_1^U('') \le I_2^U(''),\\
&f_1('') \ge f_2(''), \quad \text{for all } '' \in U,\\
&\mathrm{CS}_1 = \mathrm{CS}_2 \text{ iff } I_1^L('') = I_2^L(''),\\
&I_1^U('') = I_2^U(''),\\
&f_1('') = f_2(''),\\
&\mathrm{CS}_1 \cup \mathrm{CS}_2 = \left\{\left\langle '', \left[\max\left(I_1^L(''), I_2^L('')\right), \max\left(I_1^L(''), I_2^U('')\right)\right], \min\left(f_1(''), f_2('')\right)\right\rangle \text{ for all } '' \in U\right\},\\
&\mathrm{CS}_1 \cap \mathrm{CS}_2 = \left\{\left\langle '', \left[\min\left(I_1^L(''), I_2^L('')\right), \min\left(I_1^L(''), I_2^U('')\right)\right], \max\left(f_1(''), f_2('')\right)\right\rangle \text{ for all } '' \in U\right\}.
\end{aligned} \tag{2}$$

For the comparison of two CNs cn_1 and cn_2 the score, accuracy functions, and comparison rules are designated as follows:

$$\mathrm{SoF}(\mathrm{cn}_1) = \frac{\left(I_1^L('') + I_1^L('') - f_1('')\right)}{3}, \tag{3}$$

$$\mathrm{ArY}(\mathrm{cn}_1) = \frac{\left(I_1^L('') + I_1^U('') + f_1('')\right)}{3}. \tag{4}$$

For comparison of two CNs, the comparison rules are listed below.

(i) If $\mathrm{SoF}(\mathrm{cn}_1) > \mathrm{SoF}(\mathrm{cn}_2)$, then cn_1 is superior to cn_2 and is labelled by $\mathrm{cn}_1 > \mathrm{cn}_2$

(ii) If $\mathrm{SoF}(\mathrm{cn}_1) = \mathrm{SoF}(\mathrm{cn}_2)$ and $\mathrm{ArY}(\mathrm{cn}_1) > \mathrm{ArY}(\mathrm{cn}_2)$, then cn_1 is superior to cn_2 and is labelled by $\mathrm{cn}_1 > \mathrm{cn}_2$

(iii) If $\mathrm{SoF}(\mathrm{cn}_1) = \mathrm{SoF}(\mathrm{cn}_2)$ and $\mathrm{ArY}(\mathrm{cn}_1) = \mathrm{ArY}(\mathrm{cn}_2)$, then cn_1 is same to cn_2 and is labelled by $\mathrm{cn}_1 = \mathrm{cn}_2$

Definition 3 (see [12]). Let the two CNs be $\mathrm{cn}_1 = \langle [I_1^L, I_1^U], f_1\rangle$ and $\mathrm{cn}_2 = \langle [I_2^L, I_2^U], f_2\rangle$. Then, the OLs for CNs are identified as go after:

$$\begin{aligned}
\mathrm{cn}_1 \oplus \mathrm{cn}_2 &= \left\langle \left[I_1^L + I_2^L - I_1^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U\right], f_1 f_2\right\rangle;\\
\mathrm{cn}_1 \otimes \mathrm{cn}_2 &= \left\langle \left[I_1^L I_2^L, I_1^U I_2^U\right], f_1 + f_2 - f_1 f_2\right\rangle;\\
\mathfrak{I}\mathrm{cn}_1 &= \left\langle \left[1 - \left(1 - I_1^L\right)^{\mathfrak{I}}, 1 - \left(1 - I_1^U\right)^{\mathfrak{I}}\right], f_1^{\mathfrak{I}}\right\rangle; \quad \mathfrak{I} > 0,\\
\mathrm{cn}_1^{\mathfrak{I}} &= \left\langle \left[\left(I_1^L\right)^{\mathfrak{I}}, \left(I_1^U\right)^{\mathfrak{I}}\right], 1 - \left(1 - f_1\right)^{\mathfrak{I}}\right\rangle; \quad \mathfrak{I} > 0,\\
\mathrm{cn}_1^c &= \left\langle \left[1 - I_1^U, 1 - I_1^L\right], 1 - f_1\right\rangle.
\end{aligned} \tag{5}$$

Definition 4. Assume that the two CNs be $\mathrm{cn}_1 = \langle [I_1^L, I_1^U], f_1\rangle$ and $\mathrm{cn}_2 = \langle [I_2^L, i_2^U], f_2\rangle$. Then, the normalized Hamming distance among cn_1 and cn_2 is labelled as go after:

$$\overline{\overline{\mathrm{DNE}}}(\mathrm{cn}_1, \mathrm{cn}_2) = \frac{1}{3}\left(\left|I_1^L - I_2^L\right| + \left|I_1^U - I_2^U\right| + \left|f_1 - f_2\right|\right). \tag{6}$$

2.2. The PA Operator.

Yagar [13] originated the acuity of the PA operator which is the vital AOs. The PA operator concentrated a variety of ineffectual consequences of inanely high or awkwardly low sentiments stated by professionals. The anticipated PA operator can merge a set of crisp numbers where the weighting vector is simply on the input information and is classified as go after.

Definition 5 (see [13]). Let $\ddot{U}_u$ $(u = 1, 2, \ldots, \Theta)$ be a faction of nonnegative real numbers. The PA operator is a function delineated by

$$\mathrm{PA}(\ddot{U}_1, \ddot{U}_2, \ldots, \ddot{U}_\Theta) = \frac{\sum_{u=1}^{\Theta} (1 + T(\ddot{U}_u))\ddot{U}_u}{\sum_{u=1}^{\Theta} (1 + T(\ddot{U}_u))}, \tag{7}$$

where

$$T(\ddot{U}_u) = \sum_{\substack{v=1 \\ v \neq u}}^{\Theta} \mathrm{Sup}(\ddot{U}_u, \ddot{U}_v), \tag{8}$$

and $\mathrm{Sup}(\ddot{U}_u, \ddot{U}_v)$ is the support degree for $\ddot{U}_u$ from $\ddot{U}_v$, which meets the following axioms. (1) $\mathrm{Sup}(\ddot{U}_u, \ddot{U}_v) \in [0, 1]$, (2) $\mathrm{Sup}(\ddot{U}_u, \ddot{U}_v) = \mathrm{Sup}(\ddot{U}_v, \ddot{U}_u)$, and (3) $\mathrm{Sup}(\ddot{U}_u, \ddot{U}_v) \geq \mathrm{Sup}(\mathfrak{A}, \mathfrak{M})$, if $|\ddot{U}_u - \ddot{U}_v| < |\mathfrak{A} - \mathfrak{M}|$.

2.3. Heronian Mean (HM) Operator. HM [16] operator is one of the substantial tools for aggregation, which can exemplify the interrelations of the input elements, and is demarcated as go after.

Definition 6 (see [16]). Let $\nabla = [0, 1]$, $p, q \geq 0$, $\mathrm{HM}^{p,q}$: $\nabla^m \longrightarrow \nabla$, if $\mathrm{HM}^{p,q}$ satisfies

$$\mathrm{HM}^{p,q}(bc_1, bc_2, \ldots, bc_m) = \left(\frac{2}{m^2 + m} \sum_{\mathrm{G}=1}^{m} \sum_{\mathrm{H}=i}^{m} bc_{\mathrm{G}}^{p} bc_{\mathrm{H}}^{p} \right)^{(1/p+q)}. \tag{9}$$

Then, the mapping $\mathrm{HM}^{p,q}$ is suspected to be an HM operator with constraints. The HM operator should certify the qualities of idempotency, boundedness, and monotonicity.

3. Schweizer–Sklar Operational Laws for Cubic Numbers

In this portion, the SS OLs are commenced for CNs based on SSTN and SSTCN, and numerous underlying characteristics of SS OLs for CNs are explored.

The SSTN and SSTCN [28, 29] are recognized as go after:

$$\begin{aligned} T_{SrSr}(\ddot{U}, \Theta) &= \left(\ddot{U}^{Ж} + \tilde{O}^{Ж} - 1 \right)^{(1/Ж)}, \\ T_{SrSr}^{*}(\ddot{U}, \Theta) &= 1 - \left((1 - \ddot{U})^{Ж} + (1 - \tilde{O})^{Ж} - 1 \right)^{(1/Ж)}, \end{aligned} \tag{10}$$

where $Ж < 0, \ddot{U}, \Theta \in [0, 1]$.

Additionally, when $Ж = 0$, we have $T_{Ж}(\ddot{U}, \Theta) = \ddot{U}\Theta$ and $T_{Ж}^{*}(\ddot{U}, \Theta) = \ddot{U} + \Theta - \ddot{U}\Theta$. That is, SS TN and TCN reduce to algebraic TN and TCN.

Now, based on SSTN $T_{Ж}(\ddot{U}, \Theta)$ and TCN $T_{Ж}^{*}(\ddot{U}, \Theta)$, we can permit the following definition of SRSR ALS of CNs.

Definition 7. Let the three CNs be $\mathrm{cn} = \langle [I^L, I^U], f \rangle$, $\mathrm{cn}_1 = \langle [I_1^L, I_1^U], f_1 \rangle$, and $\mathrm{cn}_2 = \langle [I_2^L, I_2^U], f_2 \rangle$. Then, the SS ALS for CNs are classified as follows:

$$\mathrm{cn}_1 \oplus \mathrm{cn}_2 = \left\langle \left[1 - \left(\left(1 - I_1^L\right)^{Ж} + \left(1 - I_2^L\right)^{Ж} - 1 \right)^{(1/Ж)}, 1 - \left(\left(1 - I_1^U\right)^{Ж} + \left(1 - I_2^U\right)^{Ж} - 1 \right)^{(1/Ж)} \right], \left((f_1)^{Ж} + (f_2)^{Ж} - 1 \right)^{(1/Ж)} \right\rangle, \tag{11}$$

$$\mathrm{cn}_1 \otimes \mathrm{cn}_2 = \langle \left\langle \left[\left(\left(I_1^L\right)^{Ж} + \left(I_2^L\right)^{Ж} - 1 \right)^{(1/Ж)}, \left(\left(I_1^U\right)^{Ж} + \left(I_2^U\right)^{Ж} - 1 \right)^{(1/Ж)} \right], 1 - \left((1 - f_1)^{Ж} + (1 - f_2)^{Ж} - 1 \right)^{(1/Ж)} \right\rangle, \tag{12}$$

$$\varsigma \mathrm{cn} = \left\langle \left[1 - \left(\varsigma \left(1 - I^L\right)^{Ж} - (\varsigma - 1) \right)^{(1/Ж)}, 1 - \left(\varsigma \left(1 - I^U\right)^{Ж} - (\varsigma - 1) \right)^{(1/Ж)} \right], \left(\varsigma f^{Ж} - (\varsigma - 1) \right)^{(1/Ж)} \right\rangle, \tag{13}$$

$$\mathrm{cn}^{\varsigma} = \left\langle \left[\left(\varsigma \left(I^L\right)^{Ж} - (\varsigma - 1) \right)^{(1/Ж)}, \left(\varsigma \left(I^U\right)^{Ж} - (\varsigma - 1) \right)^{(1/Ж)} \right], \left(1 - \left(\varsigma (1 - f)^{Ж} - (\varsigma - 1) \right)^{(1/Ж)} \right) \right\rangle. \tag{14}$$

Moreover, some worthy properties of the operational laws can be easily achieved.

Theorem 1. *Let $cn = \langle [I^L, I^U], f \rangle, cn_1 = \langle [I_1^L, I_1^U], f_1 \rangle$ and $cn_2 = \langle [I_2^L, I_2^U], f_2 \rangle$ be any three CNs. Then,*

$$\begin{aligned}
\mathrm{cn}_1 \oplus_{SS} \mathrm{cn}_2 &= \mathrm{cn}_2 \oplus_{SS} \mathrm{cn}_1; \\
\mathrm{cn}_1 \otimes_{SS} \mathrm{cn}_2 &= \mathrm{cn}_2 \otimes_{SS} \mathrm{cn}_1; \\
\lambda\left(\mathrm{cn}_1 \oplus_{SS} \mathrm{cn}_2\right) &= \lambda \mathrm{cn}_1 \oplus_{SS} \lambda \mathrm{cn}_2, \quad \lambda \geq 0; \\
\lambda_1 \mathrm{cn} \oplus_{SS} \lambda_2 \mathrm{cn} &= \left(\lambda_1 + \lambda_2\right)\mathrm{cn}; \quad \lambda_1, \lambda_2 \geq 0; \\
\mathrm{cn}^{\lambda_1} \otimes_{SS} \mathrm{cn}^{\lambda_2} &= (\mathrm{cn})^{\lambda_1+\lambda_2}, \quad \lambda_1, \lambda_2 \geq 0; \\
\mathrm{cn}_1^{\lambda} \otimes_{SS} \mathrm{cn}_2^{\lambda} &= \left(\mathrm{cn}_1 \otimes_{SS} \mathrm{cn}_2\right)^{\lambda}, \quad \lambda \geq 0.
\end{aligned} \tag{15}$$

Proof. The proof of 1 and 2 are easy, so we can only prove the remaining formulas.

$$\begin{aligned}
\lambda\left(\mathrm{cn}_1 \oplus_{SS} \mathrm{cn}_2\right) = &\left\langle \left(\left[1 - \left(\left(1 - I_1^L\right)^{Ж} + \left(1 - I_2^L\right)^{Ж} - 1 \right)^{(1/Ж)}, 1 - \left(\left(1 - I_1^U\right)^{Ж} + \left(1 - I_2^U\right)^{Ж} - 1 \right)^{(1/Ж)} \right], \right.\right. \\
&\left. \cdot \left(\left(f_1\right)^{Ж} + \left(f_2\right)^{Ж} - 1 \right)^{(1/Ж)} \right\rangle \cdot \lambda \\
= &\left\langle \left(\left[1 - \left(\lambda\left(\left(1 - I_1^L\right)^{Ж} + \left(1 - I_2^L\right)^{Ж} - 1 \right) - (\lambda - 1) \right)^{(1/Ж)}, 1 - \left(\lambda\left(\left(1 - I_1^U\right)^{Ж} + \left(1 - I_2^U\right)^{Ж} - 1 \right) - (\lambda - 1) \right)^{(1/Ж)} \right.\right.\right. \\
&\left.\left. \cdot \left(\lambda\left(\left(f_1\right)^{Ж} + \left(f_2\right)^{Ж} - 1 \right) - (\lambda - 1) \right)^{(1/Ж)} \right], \right\rangle, \\
= &\left\langle \left(\left[1 - \left(\lambda\left(1 - I_1^L\right)^{Ж} + \lambda\left(1 - I_2^L\right)^{Ж} - \lambda - (\lambda - 1) \right)^{(1/Ж)}, 1 \right.\right.\right. \\
&\left. - \left(\lambda\left(1 - I_1^U\right)^{Ж} + \lambda\left(1 - I_2^U\right)^{Ж} - \lambda - (\lambda - 1) \right)^{(1/Ж)} \right], \left(\lambda\left(f_1\right)^{Ж} + \lambda\left(f_2\right)^{Ж} - \lambda - (\lambda - 1) \right)^{(1/Ж)} \right\rangle, \\
= &\left\langle \left(\left[1 - \left(\lambda\left(1 - I_1^L\right)^{Ж} + \lambda\left(1 - I_2^L\right)^{Ж} - 2\lambda + 1 \right)^{(1/Ж)}, 1 - \left(\lambda\left(1 - I_1^U\right)^{Ж} + \lambda\left(1 - I_2^U\right)^{Ж} - 2\lambda + 1 \right)^{(1/Ж)} \right], \right.\right. \\
&\left. \left(\lambda\left(f_1\right)^{Ж} + \lambda\left(f_2\right)^{Ж} - 2\lambda + 1 \right)^{(1/Ж)} \right\rangle.
\end{aligned} \tag{16}$$

In the meantime, we can acquire that

$$\begin{aligned}
\lambda \mathrm{cn}_1 \oplus_{SS} \lambda \mathrm{cn}_2 = &\left\langle \left(\left[1 - \left(\lambda\left(1 - I_1^L\right)^{Ж} - (\lambda - 1) + \lambda\left(1 - I_2^L\right)^{Ж} - (\lambda - 1) - 1 \right)^{(1/Ж)}, 1 \right.\right.\right. \\
&\left. - \left(\lambda\left(1 - I_1^U\right)^{Ж} - (\lambda - 1) + \lambda\left(1 - I_2^U\right)^{Ж} - (\lambda - 1) - 1 \right)^{(1/Ж)} \right], \left(\lambda f_1^{Ж} - (\lambda - 1) + \lambda f_2^{Ж} - (\lambda - 1) - 1 \right)^{1/Ж} \right\rangle, \\
= &\left\langle \left(\left[1 - \left(\lambda\left(1 - I_1^L\right)^{Ж} + \lambda\left(1 - I_2^L\right)^{Ж} - 2(\lambda - 1) - 1 \right)^{1/Ж}, 1 \right.\right.\right. \\
&\left. - \left(\lambda\left(1 - I_1^U\right)^{Ж} + \lambda\left(1 - I_2^U\right)^{Ж} - 2(\lambda - 1) - 1 \right)^{1/Ж} \right], \left(\lambda f_1^{Ж} + \lambda f_2^{Ж} - 2(\lambda - 1) - 1 \right)^{1/Ж} \right\rangle, \\
= &\left\langle \left(\left[1 - \left(\lambda\left(1 - I_1^L\right)^{Ж} + \lambda\left(1 - I_2^L\right)^{Ж} - 2\lambda + 1 \right)^{1/Ж}, 1 \right.\right.\right. \\
&\left. - \left(\lambda\left(1 - I_1^U\right)^{Ж} + \lambda\left(1 - I_2^U\right)^{Ж} - 2\lambda + 1 \right)^{1/Ж} \right], \left(\lambda f_1^{Ж} + \lambda f_2^{Ж} - 2\lambda + 1 \right)^{1/Ж} \right\rangle, \\
= &\lambda\left(\mathrm{cn}_1 \oplus_{SS} \mathrm{cn}_2\right).
\end{aligned} \tag{17}$$

Therefore, $\lambda(\mathrm{cn}_1 \oplus_{SS} \mathrm{cn}_2) = \lambda \mathrm{cn}_1 \oplus_{SS} \lambda \mathrm{cn}_2, \quad \lambda \geq 0;$ holds.

$$
\begin{aligned}
&\lambda_1 \mathrm{cn} \oplus_{SS} \lambda_2 \mathrm{cn} \\
&= \Bigg\langle \Bigg(\Bigg[1 - \left(\left(\lambda_1 \left(1 - I^L\right)^{Ж} - (\lambda_1 - 1) \right) + \left(\lambda_2 \left(1 - I^L\right)^{Ж} - (\lambda_2 - 1) \right) - 1 \right)^{1/Ж}, 1 \\
&- \left(\left(\lambda_1 \left(1 - I^U\right)^{Ж} - (\lambda_1 - 1) \right) + \left(\lambda_2 \left(1 - I^U\right)^{Ж} - (\lambda_2 - 1) \right) - 1 \right)^{1/Ж} \Bigg]^{1/Ж}, \left(\left(\lambda_1 f^{Ж} - (\lambda_1 - 1) \right) + \left(\lambda_2 f^{Ж} - (\lambda_2 - 1) \right) - 1 \right) \Bigg\rangle \\
&= \Bigg\langle \Bigg(\Bigg[1 - \left(\overline{\lambda}_1 \left(1 - I^L\right)^{Ж} + \lambda_2 \left(1 - I^L\right)^{Ж} - (\lambda_1 + \lambda_2) + 1 \right)^{1/Ж} \cdot 1 \\
&- \left(\lambda_1 \left(1 - I^U\right)^{Ж} + \lambda_2 \left(1 - I^U\right)^{Ж} - (\lambda_1 + \lambda_2) + 1 \right)^{1/Ж} \Bigg] \left(\lambda_1 f^{Ж} + \lambda_2 f^{Ж} - (\lambda_1 + \lambda_2) + 1 \right)^{1/Ж} \Bigg\rangle, \\
&= \Bigg\langle \Bigg(\Bigg[1 - \left((\lambda_1 + \lambda_2) \left(1 - I^L\right)^{Ж} - (\lambda_1 + \lambda_2) + 1 \right)^{1/Ж}, 1 - \left((\lambda_1 + \lambda_2) \left(1 - I^U\right)^{Ж} - (\lambda_1 + \lambda_2) + 1 \right)^{1/Ж} \Bigg], \\
&\left((\lambda_1 + \lambda_2) f^{Ж} - (\lambda_1 + \lambda_2) + 1 \right)^{1/Ж} \Bigg\rangle = (\lambda_1 + \lambda_2) \mathrm{cn}.
\end{aligned} \tag{18}
$$

The proofs of the other two parts are the same as the above two parts. Therefore, the proofs are omitted here. □

4. The Cubic Schweizer–Sklar Power HM Operators

4.1. The CSSPHM and CSSPGHM Operators. In this segment, numerous new AOs namely, CSSPHM and CSSPGHM operators are anticipated by combining HM and GHM operators with PA operators to anticipate.

Definition 8. Let $\mathrm{cn}_{ì} = \langle [I_{ì}^L, I_{ì}^U], f_{ì} \rangle, \quad (ì = 1, 2, \ldots, \bar{a})$ be a faction of CNs, and then the cubic Schweizer–Sklar power HM (CSSPHM) operator is clarified as follows:

$$
\mathrm{CSSPHM}^{Ü,ϴ}(\mathrm{cn}_1, \mathrm{cn}_2, \ldots, \mathrm{cn}_{\bar{a}}) = \left(\frac{2}{\bar{a}(\bar{a}+1)} \left(\sum_{\substack{ì,j=1 \\ j=ì}}^{\bar{a}} \left(\frac{\bar{a}(1 + T(\mathrm{cn}_{ì}))}{\sum_{z=1}^{\bar{a}} (1 + T(\mathrm{cn}_z))} \mathrm{cn}_{ì} \right)^{Ü} \otimes_{SS} \left(\frac{\bar{a}\left(1 + T\left(\mathrm{cn}_j\right)\right)}{\sum_{z=1}^{\bar{a}} (1 + T(\mathrm{cn}_z))} \mathrm{cn}_j \right)^{Õ} \right) \right)^{1/(Ü+ϴ)}, \tag{19}
$$

where $Ü, ϴ \geq 0, \mathrm{T}(\mathrm{cn}_{ì}) = \sum_{\substack{j=1 \\ j \neq ì}}^{\bar{a}} \sup(\mathrm{cn}_{ì}, \mathrm{cn}_j), \sup(\mathrm{cn}_{ì}, \mathrm{cn}_j) = 1 - \overline{\overline{\mathrm{DNE}}}(\mathrm{cn}_i, \mathrm{cn}_j)$, and $\overline{\overline{\mathrm{DNE}}}(\mathrm{cn}_i, \mathrm{cn}_j)$ can be figured by (6).

Let $\ddot{Y}_{ì} = (1 + T(\mathrm{cn}_i)) / \sum_{z=1}^{\bar{a}} (1 + T(\mathrm{cn}_z))$, then the definition of CSSPHM operator is correspondent to the following type:

$$
\mathrm{CSSPHM}^{Ü,ϴ}(\mathrm{cn}_1, \mathrm{cn}_2, \ldots, \mathrm{cn}_{\dagger}) = \left(\frac{2}{\bar{a}(\bar{a}+1)} \left(\sum_{\substack{i,j=1 \\ j=i}}^{\bar{a}} \left(\bar{a} \ddot{Y}_{ì} \mathrm{cn}_i \right)^{Õ} \otimes_{SrSr} \left(\bar{a} \ddot{Y}_j \mathrm{cn}_j \right)^{Ü} \right) \right)^{1/(Ü+ϴ)}. \tag{20}
$$

Theorem 2. *Let* $\ddot{U} \geq 0, \text{Ө} \geq 0$ *and* G, H *grab no more than one value of 0 at a time,* $cn_i = \langle [I_i^L, I_i^U], f_i \rangle$ *be a set of CNs. Then, exploiting the CSSPHM operator, their merged values are CNs, and*

$$\text{CSSPHM}^{\ddot{U},\text{Ө}}(\text{cn}_1, \text{cn}_2, \ldots, \text{cn}_\dagger)$$

$$= \left\langle \begin{bmatrix} \left(\frac{1}{\ddot{U}+\text{Ө}} \left(1 - \left(\frac{2}{\bar{a}(\bar{a}+1)} \sum_{\substack{i,j=1 \\ j=i}}^{\bar{a}} \left(1 - \left(\ddot{U}\left(1 - \left(\bar{a}\ddot{Y}_i (1 - I_i^L)^{\aleph} - (\bar{a}\ddot{Y}_i - 1) \right)^{1/\text{Ж}} \right)^{\text{Ж}} + \text{Ө}\left(1 - \left(\bar{a}\ddot{Y}_j (1 - I_j^L)^{\text{Ж}} - (\bar{a}\ddot{Y}_j - 1) \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \ddot{U} - +\text{Ө} \right)^{1/\text{Ж}} \right)^{\text{Ж}} \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \left(\frac{1}{\ddot{U}+\text{Ө}} - 1 \right) \right)^{1/\text{Ж}}, \\ \left(\frac{1}{\ddot{U}+\text{Ө}} \left(1 - \left(\frac{2}{\bar{a}(\bar{a}+1)} \sum_{\substack{i,j=1 \\ j=i}}^{\bar{a}} \left(1 - \left(\ddot{U}\left(1 - \left(\bar{a}\ddot{Y}_i (1 - I_i^U)^{\text{Ж}} - (\bar{a}\ddot{Y}_i - 1) \right)^{1/\text{Ж}} \right)^{\text{Ж}} + \text{Ө}\left(1 - \left(\bar{a}\ddot{Y}_j (1 - I_j^U)^{\text{Ж}} - (\bar{a}\ddot{Y}_j - 1) \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right)^{1/\text{Ж}} \right)^{\text{Ж}} \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \left(\frac{1}{\ddot{U}+\text{Ө}} - 1 \right) \right)^{1/\text{Ж}} \end{bmatrix}, \right\rangle.$$

$$\left(1 - \left(\frac{1}{\ddot{U}+\text{Ө}} \left(1 - \left(\frac{2}{\bar{a}(\bar{a}+1)} \sum_{\substack{i,j=1 \\ j=i}}^{\bar{a}} \left(1 - \left(\ddot{U}\left(1 - \left(\bar{a}\ddot{Y}_i f_i^{\text{Ж}} - (\bar{a}\ddot{Y}_i - 1) \right)^{1/\text{Ж}} \right)^{\text{Ж}} + \text{Ө}\left(1 - \left(\bar{a}\ddot{Y}_j f_j^{\text{Ж}} - (\text{n}\Diamond_j - 1) \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right)^{1/\text{Ж}} \right)^{\text{Ж}} \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \left(\frac{1}{\ddot{U}+\text{Ө}} - 1 \right) \right)^{1/\text{Ж}} \right) \tag{21}$$

Proof. Firstly, we need to prove the following equation:

$$\sum_{\substack{i,j=1 \\ j=i}}^{\bar{a}} (\bar{a}\ddot{Y}_i \text{cn}_i)^{\ddot{U}} \otimes_{SS} (\bar{a}\ddot{Y}_j \text{cn}_j)^{\tilde{O}}$$

$$= \left\langle \begin{bmatrix} 1 - \left(\sum_{\substack{i,j=1 \\ j=i}}^{\bar{a}} \left(1 - \left(\ddot{U}\left(1 - \left(\bar{a}\ddot{Y}_i (1 - I_i^L)^{\text{Ж}} - (\bar{a}\ddot{Y}_i - 1) \right)^{1/\text{Ж}} \right)^{\text{Ж}} + \text{Ө}\left(1 - \left(\bar{a}\ddot{Y}_j (1 - I_j^L)^{\text{Ж}} - (\bar{a}\ddot{Y}_j - 1) \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \left(\frac{\bar{a}(\bar{a}+1)}{2} - 1 \right) \right)^{1/\text{Ж}}, \\ 1 - \left(\sum_{\substack{i,j=1 \\ j=i}}^{\bar{a}} \left(1 - \left(\ddot{U}\left(1 - \left(\bar{a}\ddot{Y}_i (1 - I_i^U)^{\text{Ж}} - (\bar{a}\ddot{Y}_i - 1) \right)^{1/\text{Ж}} \right)^{\text{Ж}} + \text{Ө}\left(1 - \left(\bar{a}\ddot{Y}_j (1 - I_j^U)^{\text{Ж}} - (\bar{a}\ddot{Y}_j - 1) \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \left(\frac{\bar{a}(\bar{a}+1)}{2} - 1 \right) \right)^{1/\text{Ж}} \end{bmatrix}, \right\rangle.$$

$$\left(\sum_{\substack{i,j=1 \\ j=i}}^{\bar{a}} \left(1 - \left(\ddot{U}\left(1 - \left(\bar{a}\ddot{Y}_i f_i^{\text{Ж}} - (\bar{a}\ddot{Y}_i - 1) \right)^{1/\text{Ж}} \right)^{\text{Ж}} + \text{Ө}\left(1 - \left(\bar{a}\ddot{Y}_j f_j^{\text{Ж}} - (\bar{a}\ddot{Y}_j - 1) \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \left(\frac{\bar{a}(\bar{a}+1)}{2} - 1 \right) \right)^{1/\text{Ж}} \tag{22}$$

By SS OLs defined in Definition 7, we get

$$\bar{a}\ddot{Y}_i cn_i = \left\langle \left[1 - \left(\bar{a}\ddot{Y}_i (1 - I_i^L)^{\text{Ж}} - (\bar{a}\ddot{Y}_i - 1) \right)^{1/\text{Ж}}, 1 - \left(\bar{a}\ddot{Y}_i (1 - I_i^U)^{\text{Ж}} - (\bar{a}\ddot{Y}_i - 1) \right)^{1/\text{Ж}} \right], \left(\bar{a}\ddot{Y}_i f_i^{\text{Ж}} - (\bar{a}\ddot{Y}_i - 1) \right)^{1/\text{Ж}} \right\rangle, \tag{23}$$

and

$$(\bar{a}\ddot{Y}_i cn_i)^{\ddot{U}} = \left\langle \begin{array}{c} \left[\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i\left(1-I_i^L\right)^{\text{Ж}} - \left(\bar{a}\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} - (\ddot{U}-1)\right)^{1/\text{Ж}}, \left(\tilde{O}\left(1-\left(\bar{a}\ddot{Y}_i\left(1-I_i^U\right)^{\text{Ж}}-\left(\bar{a}\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-(\ddot{U}-1)\right)^{1/\text{Ж}}\right], \\ 1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i f_i^{\text{Ж}}-\left(\bar{a}\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-(\ddot{U}-1)\right)^{1/\text{Ж}}\end{array}\right\rangle. \tag{24}$$

Similarly,

$$\begin{aligned} \bar{a}\ddot{Y}_j cn_j &= \left\langle \left[1-\left(\bar{a}\ddot{Y}_j\left(1-I_j^L\right)^{\text{Ж}}-\left(\bar{a}\ddot{Y}_j-1\right)\right)^{1/\text{Ж}}, 1-\left(\bar{a}\ddot{Y}_j\left(1-I_j^U\right)^{\text{Ж}}-\left(\bar{a}\ddot{Y}_j-1\right)\right)^{1/\text{Ж}}\right], \left(\bar{a}\ddot{Y}_j f_j^{\text{Ж}}-\left(\bar{a}\ddot{Y}_j-1\right)\right)^{1/\text{Ж}}\right\rangle, \\ \left(\bar{a}\ddot{Y}_j cn_j\right)^{\tilde{O}} &= \left\langle \begin{array}{c} \left[\left(\tilde{O}\left(1-\left(\bar{a}\ddot{Y}_j\left(1-I_j^L\right)^{\text{Ж}}-\left(\bar{a}\ddot{Y}_j-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-(\tilde{O}-1)\right)^{1/\text{Ж}}, \left(\tilde{O}\left(1-\left(\bar{a}\ddot{Y}_j\left(1-I_j^U\right)^{\text{Ж}}-\left(\bar{a}\ddot{Y}_j-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-(\tilde{O}-1)\right)^{1/\text{Ж}}\right], \\ 1-\left(\tilde{O}\left(1-\left(\bar{a}\ddot{Y}_j f_j^{\text{Ж}}-\left(\bar{a}\ddot{Y}_j-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-(\tilde{O}-1)\right)^{1/\text{Ж}}\end{array}\right\rangle. \end{aligned} \tag{25}$$

So,

$$\begin{aligned} &(\bar{a}\ddot{Y}_i cn_i)^{\ddot{U}} \otimes_{SS} \left(\bar{a}\ddot{Y}_j cn_j\right)^{\tilde{O}} \\ &= \left\langle \begin{array}{c} \left[\begin{array}{c} \left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i\left(1-I_i^L\right)^{\text{Ж}}-\left(\bar{a}\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} + \Theta\left(1-\left(\bar{a}\ddot{Y}_j\left(1-I_j^L\right)^{\text{Ж}}-\left(\bar{a}\ddot{Y}_j-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} - \ddot{U} - \Theta + 1\right)^{1/\text{Ж}}, \\ \left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i\left(1-I_i^U\right)^{\text{Ж}}-\left(\bar{a}\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} + \Theta\left(1-\left(\bar{a}\ddot{Y}_j\left(1-I_j^U\right)^{\text{Ж}}-\left(\bar{a}\ddot{Y}_j-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} - \ddot{U} - \Theta + 1\right)^{1/\text{Ж}} \end{array}\right], \\ 1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i f_i^{\text{Ж}}-\left(\bar{a}\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} + \Theta\left(1-\left(\bar{a}\ddot{Y}_j f_j^{\text{Ж}}-\left(\bar{a}\ddot{Y}_j-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} - \ddot{U} - \Theta + 1\right)^{1/\text{Ж}} \end{array}\right\rangle, \end{aligned} \tag{26}$$

(1) When $\bar{a} = 2$ by Equation (17) and Equation (25), we have

$$\begin{aligned} &\sum_{\substack{i,j=1 \\ j=i}}^{2} (\bar{a}\ddot{Y}_i cn_i)^{\ddot{U}} \otimes_{SS} \left(\bar{a}\ddot{Y}_j cn_j\right)^{\tilde{O}} \\ &= \left((2\ddot{Y}_1 cn_1)^{\ddot{U}} \otimes_{SS} (2\ddot{Y}_1 cn_1)^{\tilde{O}} \right) \oplus_{SS} \left((2\ddot{Y}_1 cn_1)^{\ddot{U}} \otimes_{SS} (2\ddot{Y}_2 cn_2)^{\tilde{O}} \right) \oplus_{SS} \left((2\ddot{Y}_2 cn_2)^{\ddot{U}} \otimes_{SS} (2\ddot{Y}_2 cn_2)^{\tilde{O}} \right) \\ &= \left[\begin{array}{c} \left(\ddot{U}\left(1-\left(2\ddot{Y}_1\left(1-I_1^L\right)^{\text{Ж}}-\left(2\ddot{Y}_1-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} + \Theta\left(1-\left(2\ddot{Y}_1\left(1-I_1^L\right)^{\text{Ж}}-\left(2\ddot{Y}_1-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} - \ddot{U} - \Theta + 1\right)^{1/\text{Ж}}, \\ \left(\ddot{U}\left(1-\left(2\ddot{Y}_1\left(1-I_1^U\right)^{\text{Ж}}-\left(2\ddot{Y}_1-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} + \Theta\left(1-\left(2\ddot{Y}_1\left(1-I_1^U\right)^{\text{Ж}}-\left(2\ddot{Y}_1-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} - \ddot{U} - \Theta + 1\right)^{1/\text{Ж}} \end{array}\right], \\ &\cdot 1-\left(\ddot{U}\left(1-\left(2\ddot{Y}_1 f_1^{\text{Ж}}-\left(2\ddot{Y}_1-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} + \Theta\left(1-\left(2\ddot{Y}_1 f_1^{\text{Ж}}-\left(2\ddot{Y}_1-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} - \ddot{U} - \Theta + 1\right)^{1/\text{Ж}} \rangle \oplus_{SS} \\ &\cdot \left[\begin{array}{c} \left(\ddot{U}\left(1-\left(2\ddot{Y}_1\left(1-I_1^L\right)^{\text{Ж}}-\left(2\ddot{Y}_1-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} + \Theta\left(1-\left(2\ddot{Y}_2\left(1-I_2^L\right)^{\text{Ж}}-\left(2\ddot{Y}_2-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} - \ddot{U} - \Theta + 1\right)^{1/\text{Ж}} \\ \left(\ddot{U}\left(1-\left(2\ddot{Y}_1\left(1-I_1^U\right)^{\text{Ж}}-\left(2\ddot{Y}_1-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} + \Theta\left(1-\left(2\ddot{Y}_2\left(1-I_2^U\right)^{\text{Ж}}-\left(2\ddot{Y}_2-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} - \ddot{U} - \Theta + 1\right)^{1/\text{Ж}} \end{array}\right], \end{aligned}$$

$$\cdot\, 1-\left(\ddot{U}\left(1-\left(2\ddot{Y}_1 f_1^{Ж}-(2\ddot{Y}_1-1)\right)^{1/Ж}\right)^{Ж}+\Theta\left(1-\left(2\ddot{Y}_2 f_2^{Ж}-(2\ddot{Y}_2-1)\right)^{1/Ж}\right)^{Ж}-\ddot{U}-\Theta+1\right)^{1/Ж}\rangle\oplus_{SS}$$

$$\cdot\left[\begin{array}{l}\left(\ddot{U}\left(1-\left(2\ddot{Y}_2\left(1-I_2^L\right)^{Ж}-(2\ddot{Y}_2-1)\right)^{1/Ж}\right)^{Ж}+\Theta\left(1-\left(2\ddot{Y}_2\left(1-I_2^L\right)^{Ж}-(2\ddot{Y}_2-1)\right)^{1/Ж}\right)^{Ж}-\ddot{U}-\Theta+1\right)^{1/Ж}\\ \left(\ddot{U}\left(1-\left(2\ddot{Y}_2\left(1-I_2^U\right)^{Ж}-(2\ddot{Y}_2-1)\right)^{1/Ж}\right)^{Ж}+\Theta\left(1-\left(2\ddot{Y}_2\left(1-I_2^U\right)^{Ж}-(2\ddot{Y}_2-1)\right)^{1/Ж}\right)^{Ж}-\ddot{U}-\Theta+1\right)^{1/Ж}\end{array}\right],$$

$$\cdot\, 1-\left(\ddot{U}\left(1-\left(2\ddot{Y}_2 f_2^{Ж}-(2\ddot{Y}_2-1)\right)^{1/Ж}\right)^{Ж}+\Theta\left(1-\left(2\ddot{Y}_2 f_2^{Ж}-(2\ddot{Y}_2-1)\right)^{1/Ж}\right)^{Ж}-\ddot{U}-\Theta+1\right)^{1/Ж}\rangle. \tag{27}$$

By using Equation (17), we get

$$\begin{aligned}&\sum_{\substack{i,j=1\\ j=i}}^{2}\left(\bar{a}\ddot{Y}_i cn_i\right)^{\ddot{U}}\otimes_{SS}\left(\bar{a}\ddot{Y}_j cn_j\right)^{\tilde{O}}\\ &=\left\langle\left[\begin{array}{l}1-\left(\sum_{\substack{i,j=1\\ j=i}}^{2}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i\left(1-I_i^L\right)^{Ж}-\left(\bar{a}\ddot{Y}_i-1\right)\right)^{1/Ж}\right)^{Ж}+\Theta\left(1-\left(\bar{a}\ddot{Y}_j\left(1-I_j^L\right)^{Ж}-\left(\bar{a}\ddot{Y}_j-1\right)\right)^{1/Ж}\right)^{Ж}-\ddot{U}-\Theta+1\right)^{\frac{1}{Ж}}\right)^{Ж}-\left(\frac{\bar{a}(\bar{a}+1)}{2}-1\right)\right)^{1/Ж},\\ 1-\left(\sum_{\substack{i,j=1\\ j=i}}^{2}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i\left(1-I_i^U\right)^{Ж}-\left(\bar{a}\ddot{Y}_i-1\right)\right)^{1/Ж}\right)^{Ж}+\Theta\left(1-\left(\bar{a}\ddot{Y}_j\left(1-I_j^U\right)^{Ж}-\left(\bar{a}\ddot{Y}_j-1\right)\right)^{1/Ж}\right)^{Ж}-\ddot{U}-\Theta+1\right)^{1/Ж}\right)^{Ж}-\left(\frac{\bar{a}(\bar{a}+1)}{2}-1\right)\right)^{1/Ж}\end{array}\right],\right.\\ &\left.\left(\sum_{\substack{i,j=1\\ j=i}}^{2}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i f_i^{Ж}-\left(\bar{a}\ddot{Y}_i-1\right)\right)^{1/Ж}\right)^{Ж}+\Theta\left(1-\left(\bar{a}\ddot{Y}_j f_j^{Ж}-\left(\bar{a}\ddot{Y}_j-1\right)\right)^{1/Ж}\right)^{Ж}-\ddot{U}-\Theta+1\right)^{1/Ж}\right)^{Ж}-\left(\frac{\bar{a}(\bar{a}+1)}{2}-1\right)\right)^{1/Ж}\right\rangle.\end{aligned} \tag{28}$$

That is (22) is true $\bar{a}=2$.

(2) Let us assume that Equation (24) is true $\bar{a}=z$.

$$\begin{aligned}&\sum_{\substack{i,j=1\\ j=i}}^{z}\left(\bar{a}\ddot{Y}_i cn_i\right)^{\ddot{U}}\otimes_{SS}\left(\bar{a}\ddot{Y}_j cn_j\right)^{\tilde{O}}\\ &=\left\langle\left[\begin{array}{l}1-\left(\sum_{\substack{i,j=1\\ j=i}}^{z}\left(1-\left(\ddot{U}\left(1-\left(z\ddot{Y}_i\left(1-I_i^L\right)^{Ж}-\left(z\ddot{Y}_i-1\right)\right)^{1/Ж}\right)^{Ж}+\Theta\left(1-\left(z\ddot{Y}_j\left(1-I_j^L\right)^{Ж}-\left(z\ddot{Y}_j-1\right)\right)^{1/Ж}\right)^{Ж}-\ddot{U}-\Theta+1\right)^{1/Ж}\right)^{Ж}-\left(\frac{z(z+1)}{2}-1\right)\right)^{1/Ж},\\ 1-\left(\sum_{\substack{i,j=1\\ j=i}}^{z}\left(1-\left(\ddot{U}\left(1-\left(z\ddot{Y}_i\left(1-I_i^U\right)^{Ж}-\left(z\ddot{Y}_i-1\right)\right)^{1/Ж}\right)^{Ж}+\Theta\left(1-\left(z\ddot{Y}_j\left(1-I_j^U\right)^{Ж}-\left(z\ddot{Y}_j-1\right)\right)^{1/Ж}\right)^{Ж}-\ddot{U}-\Theta+1\right)^{1/Ж}\right)^{Ж}-\left(\frac{z(z+1)}{2}-1\right)\right)^{1/Ж}\end{array}\right],\right.\\ &\left.\left(\sum_{\substack{i,j=1\\ j=i}}^{z}\left(1-\left(\ddot{U}\left(1-\left(z\ddot{Y}_i f_i^{Ж}-\left(z\ddot{Y}_i-1\right)\right)^{1/Ж}\right)^{Ж}+\Theta\left(1-\left(z\ddot{Y}_j f_j^{Ж}-\left(z\ddot{Y}_j-1\right)\right)^{1/Ж}\right)^{Ж}-\ddot{U}-\Theta+1\right)^{1/Ж}\right)^{Ж}-\left(\frac{z(z+1)}{2}-1\right)\right)^{1/Ж}\right\rangle.\end{aligned} \tag{29}$$

Furthermore, when $\overline{\mathrm{a}} = z + 1$

$$\begin{aligned}\sum_{\substack{i,j=1\\ j=i}}^{z+1} \left(\overline{\mathrm{a}}\ddot{Y}_i cn_i\right)^{\ddot{U}} \otimes_{SS} \left(\overline{\mathrm{a}}\ddot{Y}_j cn_j\right)^{\tilde{O}} = {} & \sum_{\substack{i,j=1\\ j=i}}^{z+1} \left(\left((z+1)\ddot{Y}_i cn_i\right)^{\ddot{U}} \otimes_{SS} \left((z+1)\ddot{Y}_j cn_j\right)^{\tilde{O}} \right) \\ & \oplus_{SS} \sum_{i=1}^{z} \left(\left((z+1)\ddot{Y}_i cn_i\right)^{\ddot{U}} \otimes_{SS} \left((z+1)\ddot{Y}_{z+1} cn_{z+1}\right)^{\tilde{O}} \right) \\ & \oplus_{SS} \left(\left((z+1)\ddot{Y}_{z+1} cn_{z+1}\right)^{\ddot{U}} \otimes_{SS} \left((z+1)\ddot{Y}_{z+1} \mathrm{cn}_{z+1}\right)^{\tilde{O}} \right).\end{aligned} \tag{30}$$

Firstly, we will show that

$$\begin{aligned}&\sum_{i=1}^{z} \left(\left((z+1)\ddot{Y}_i \mathrm{cn}_i\right)^{\ddot{U}} \otimes_{SS} \left((z+1)\ddot{Y}_{z+1} \mathrm{cn}_{z+1}\right)^{\tilde{O}} \right) \\ &= \left\langle \begin{bmatrix} 1 - \left(\sum_{\substack{i,j=1\\ j=i}}^{z} \left(1 - \left(\ddot{U}\left(1 - \left((z+1)\ddot{Y}_i\left(1 - \mathrm{I}_i^{\mathrm{L}}\right)^{\text{Ж}} - \left((z+1)\ddot{Y}_i - 1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} + \text{Ө}\left(1 - \left((z+1)\ddot{Y}_{z+1}\left(1 - \mathrm{I}_{z+1}^{\mathrm{L}}\right)^{\text{Ж}} - \left((z+1)\ddot{Y}_{z+1} - 1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \left(\frac{(z+1)(z+2)}{2} - 1 \right) \right)^{1/\text{Ж}}, \\ 1 - \left(\sum_{\substack{i,j=1\\ j=i}}^{z} \left(1 - \left(\ddot{U}\left(1 - \left((z+1)\ddot{Y}_i\left(1 - \mathrm{I}_i^{\mathrm{U}}\right)^{\text{Ж}} - \left((z+1)\ddot{Y}_i - 1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} + \text{Ө}\left(1 - \left((z+1)\ddot{Y}_{z+1}\left(1 - \mathrm{I}_{z+1}^{\mathrm{U}}\right)^{\text{Ж}} - \left((z+1)\ddot{Y}_{z+1} - 1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \left(\frac{(z+1)(z+2)}{2} - 1 \right) \right)^{1/\text{Ж}} \end{bmatrix}, \right\rangle. \\ &\left(\sum_{\substack{i,j=1\\ j=i}}^{z} \left(1 - \left(\ddot{U}\left(1 - \left((z+1)\ddot{Y}_i \mathrm{f}_i^{\text{Ж}} - \left((z+1)\ddot{Y}_i - 1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} + \text{Ө}\left(1 - \left((z+1)\ddot{Y}_{z+1} \mathrm{f}_{z+1}^{\text{Ж}} - \left((z+1)\ddot{Y}_{z+1} - 1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \left(\frac{(z+1)(z+2)}{2} - 1 \right) \right)^{1/\text{Ж}}\end{aligned} \tag{31}$$

We shall prove (31), on mathematical induction on z.

(i) For $z = 2$, we have

$$\begin{aligned}&\sum_{i=1}^{2} \left(\left((z+1)\ddot{Y}_i \mathrm{cn}_i\right)^{\ddot{U}} \otimes_{SS} \left((z+1)\ddot{Y}_{z+1} \mathrm{cn}_{z+1}\right)^{\tilde{O}} \right) \\ &= \left(\left(3\ddot{Y}_1 \mathrm{cn}_1\right)^{\ddot{U}} \otimes_{SS} \left(3\ddot{Y}_3 \mathrm{cn}_3\right)^{\tilde{O}} \right) \oplus_{SS} \left(\left(3\ddot{Y}_2 \mathrm{cn}_2\right)^{\ddot{U}} \otimes_{SS} \left(3\ddot{Y}_3 \mathrm{cn}_3\right)^{\tilde{O}} \right) \\ &= \begin{bmatrix} \left(\ddot{U}\left(1 - \left(3\ddot{Y}_1\left(1 - \mathrm{I}_1^{\mathrm{L}}\right)^{\text{Ж}} - \left(3\ddot{Y}_1 - 1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} + \text{Ө}\left(1 - \left(3\ddot{Y}_3\left(1 - \mathrm{I}_3^{\mathrm{L}}\right)^{\text{Ж}} - \left(3\ddot{Y}_3 - 1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right)^{1/\text{Ж}}, \\ \left(\ddot{U}\left(1 - \left(3\ddot{Y}_1\left(1 - \mathrm{I}_1^{\mathrm{U}}\right)^{\text{Ж}} - \left(2\ddot{Y}_1 - 1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} + \text{Ө}\left(1 - \left(3\ddot{Y}_3\left(1 - \mathrm{I}_3^{\mathrm{U}}\right)^{\text{Ж}} - \left(3\ddot{Y}_3 - 1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right)^{1/\text{Ж}} \end{bmatrix}, \\ &\cdot 1 - \left(\ddot{U}\left(1 - \left(3\ddot{Y}_1 \mathrm{f}_1^{\text{Ж}} - \left(2\ddot{Y}_1 - 1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} + \text{Ө}\left(1 - \left(3\ddot{Y}_3 \mathrm{f}_3^{\text{Ж}} - \left(3\ddot{Y}_3 - 1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right)^{1/\text{Ж}} \rangle \oplus_{SS}\end{aligned}$$

$$\cdot\begin{bmatrix} \left(\ddot{U}\left(1-\left(3\ddot{Y}_2\left(1-I_2^L\right)^{\text{Ж}}-\left(3\ddot{Y}_2-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(3\ddot{Y}_3\left(1-I_3^L\right)^{\text{Ж}}-\left(3\ddot{Y}_3-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}, \\ \left(\ddot{U}\left(1-\left(3\ddot{Y}_2\left(1-I_2^U\right)^{\text{Ж}}-\left(3\ddot{Y}_2-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(3\ddot{Y}_3\left(1-I_3^U\right)^{\text{Ж}}-\left(3\ddot{Y}_3-1\right)\right)^{\frac{1}{\text{Ж}}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}} \end{bmatrix},$$

$$\cdot\, 1-\left(\ddot{U}\left(1-\left(3\ddot{Y}_2 f_2^{\text{Ж}}-\left(3\ddot{Y}_2-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(3\ddot{Y}_3 f_3^{\text{Ж}}-\left(3\ddot{Y}_3-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\Bigg\rangle. \tag{32}$$

By using (11), we get

$$=\left\langle\begin{bmatrix} 1-\left(\sum_{i=1}^{2}\left(1-\left(\ddot{U}\left(1-\left(3\ddot{Y}_i\left(1-I_i^L\right)^{\text{Ж}}-\left(3\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(3\ddot{Y}_3\left(1-I_3^L\right)^{\text{Ж}}-\left(3\ddot{Y}_3-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{3(3+1)}{2}-1\right)\right)^{1/\text{Ж}}, \\ 1-\left(\sum_{i=1}^{2}\left(1-\left(\ddot{U}\left(1-\left(3\ddot{Y}_i\left(1-I_i^U\right)^{\text{Ж}}-\left(3\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(3\ddot{Y}_3\left(1-I_3^U\right)^{\text{Ж}}-\left(3\ddot{Y}_3-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{3(3+1)}{2}-1\right)\right)^{1/\text{Ж}} \end{bmatrix},\right\rangle.$$

$$\left(\sum_{i=1}^{2}\left(1-\left(\ddot{U}\left(1-\left(3\ddot{Y}_i f_i^{\text{Ж}}-\left(3\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(3\ddot{Y}_3 f_3^{\text{Ж}}-\left(3\ddot{Y}_3-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{3(3+1)}{2}-1\right)\right)^{1/\text{Ж}} \tag{33}$$

(ii) Suppose that Equation (28) is true $z=\text{ő}$.

$$\sum_{i=1}^{\text{ő}}\left(((\text{ő}+1)\ddot{Y}_i cn_i)^{\ddot{U}}\otimes_{SS}((\text{ő}+1)\ddot{Y}_{\text{ő}+1}cn_{\text{ő}+1})^{\tilde{O}}\right)$$

$$=\left\langle\begin{bmatrix} 1-\left(\sum_{i=1}^{\text{ő}}\left(1-\left(\ddot{U}\left(1-\left((\text{ő}+1)\ddot{Y}_i\left(1-I_i^L\right)^{\text{Ж}}-((\text{ő}+1)\ddot{Y}_i-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left((\text{ő}+1)\ddot{Y}_{\text{ő}+1}\left(1-I_{\text{ő}+1}^L\right)^{\text{Ж}}-((\text{ő}+1)\ddot{Y}_{\text{ő}+1}-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{(\text{ő}+1)(\text{ő}+2)}{2}-1\right)\right)^{1/\text{Ж}}, \\ 1-\left(\sum_{i=1}^{\text{ő}}\left(1-\left(\ddot{U}\left(1-\left((\text{ő}+1)\ddot{Y}_i\left(1-I_i^U\right)^{\text{Ж}}-((\text{ő}+1)\ddot{Y}_i-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left((\text{ő}+1)\ddot{Y}_{\text{ő}+1}\left(1-I_{\text{ő}+1}^U\right)^{\text{Ж}}-((\text{ő}+1)\ddot{Y}_{\text{ő}+1}-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{(\text{ő}+1)(\text{ő}+2)}{2}-1\right)\right)^{1/\text{Ж}} \end{bmatrix},\right\rangle.$$

$$\left(\sum_{i=1}^{\text{ő}}\left(1-\left(\ddot{U}\left(1-\left((\text{ő}+1)\ddot{Y}_i f_i^{\text{Ж}}-((\text{ő}+1)\ddot{Y}_i-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left((\text{ő}+1)\ddot{Y}_{\text{ő}+1}f_{\text{ő}+1}^{\text{Ж}}-((\text{ő}+1)\ddot{Y}_{\text{ő}+1}-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{(\text{ő}+1)(\text{ő}+2)}{2}-1\right)\right)^{1/\text{Ж}} \tag{34}$$

Then, when $z = ő + 1$, we have

$$
\begin{aligned}
&\sum_{i=1}^{ő+1}\Big(((ő+2)Ÿ_i cn_i)^{Ü} \otimes_{SS} ((ő+2)Ÿ_{ő+2}cn_{ő+2})^{Õ}\Big) \\
&= \sum_{l=1}^{ő+1}\Big(((ő+2)Ÿ_i cn_i)^{Ü} \otimes_{SS} ((ő+2)Ÿ_{ő+2}cn_{ő+2})^{Õ}\Big) \oplus_{SS} \Big(((ő+2)Ÿ_{ő+1}cn_{ő+1})^{Ü} \otimes_{SS} ((ő+2)Ÿ_{ő+2}cn_{ő+2})^{Õ}\Big) \\
&= \left\langle \begin{bmatrix} 1-\left(\sum_{i=1}^{ő}\left(1-\left(Ü\left(1-\left((ő+2)Ÿ_i\left(1-I_i^L\right)^{Ж}-((ő+2)Ÿ_i-1)\right)^{1/Ж}\right)^{Ж}+Θ\left(1-\left((ő+2)Ÿ_{ő+2}\left(1-I_{ő+2}^L\right)^{Ж}-((ő+2)Ÿ_{ő+2}-1)\right)^{1/Ж}\right)^{Ж}-Ü-Θ+1\right)^{1/Ж}\right)^{Ж}-\left(\frac{(ő+2)(ő+3)}{2}-1\right)\right)^{1/Ж}, \\ 1-\left(\sum_{i=1}^{ő}\left(1-\left(Ü\left(1-\left((ő+2)Ÿ_i\left(1-I_i^U\right)^{Ж}-((ő+2)Ÿ_i-1)\right)^{1/Ж}\right)^{Ж}+Θ\left(1-\left((ő+2)Ÿ_{ő+2}\left(1-I_{ő+2}^U\right)^{Ж}-((ő+2)Ÿ_{ő+2}-1)\right)^{1/Ж}\right)^{Ж}-Ü-Θ+1\right)^{1/Ж}\right)^{Ж}-\left(\frac{(ő+2)(ő+3)}{2}-1\right)\right)^{1/Ж} \end{bmatrix}, \right. \\
&\qquad \left. \left(\sum_{i=1}^{ő}\left(1-\left(Ü\left(1-\left((ő+2)Ÿ_i f_i^{Ж}-((ő+2)Ÿ_i-1)\right)^{1/Ж}\right)^{Ж}+Θ\left(1-\left((ő+2)Ÿ_{ő+1} f_{ő+1}^{Ж}-((ő+2)Ÿ_{ő+1}-1)\right)^{1/Ж}\right)^{Ж}-Ü-Θ+1\right)^{1/Ж}\right)^{Ж}-\left(\frac{(ő+2)(ő+3)}{2}-1\right)\right)^{1/Ж} \right\rangle \oplus_{SS} \\
&\cdot \left\langle \begin{bmatrix} \left(Ü\left(1-\left((ő+2)Ÿ_{ő+1}\left(1-I_{ő+1}^L\right)^{Ж}-((ő+2)Ÿ_{ő+1}-1)\right)^{1/Ж}\right)^{Ж}+Θ\left(1-\left((ő+2)Ÿ_{ő+2}\left(1-I_{ő+2}^L\right)^{Ж}-((ő+2)Ÿ_{ő+2}-1)\right)^{1/Ж}\right)^{Ж}-Ü-Θ+1\right)^{1/Ж}, \\ \left(Ü\left(1-\left((ő+2)Ÿ_{ő+1}\left(1-I_{ő+1}^U\right)^{Ж}-((ő+2)Ÿ_{ő+1}-1)\right)^{1/Ж}\right)^{Ж}+Θ\left(1-\left((ő+2)Ÿ_{ő+2}\left(1-I_{ő+2}^U\right)^{Ж}-((ő+2)Ÿ_{ő+1}-1)\right)^{1/Ж}\right)^{Ж}-Ü-Θ+1\right)^{1/Ж} \end{bmatrix}, \right. \\
&\cdot \left. 1-\left(Ü\left(1-\left((ő+2)Ÿ_{ő+1} f_{ő+1}^{Ж}-((ő+2)Ÿ_{ő+1}-1)\right)^{1/Ж}\right)^{Ж}+Θ\left(1-\left((ő+2)Ÿ_{ő+2} f_{ő+2}^{Ж}-((ő+2)Ÿ_{ő+2}-1)\right)^{1/Ж}\right)^{Ж}-Ü-Θ+1\right)^{1/Ж} \right\rangle. \\
&= \left\langle \begin{bmatrix} 1-\left(\sum_{i=1}^{ő+1}\left(1-\left(Ü\left(1-\left((ő+2)Ÿ_i\left(1-I_i^L\right)^{Ж}-((ő+2)Ÿ_i-1)\right)^{1/Ж}\right)^{Ж}+Θ\left(1-\left((ő+2)Ÿ_{ő+2}\left(1-I_{ő+2}^L\right)^{Ж}-((ő+2)Ÿ_{ő+2}-1)\right)^{1/Ж}\right)^{Ж}-Ü-Θ+1\right)^{1/Ж}\right)^{Ж}-\left(\frac{(ő+2)(ő+3)}{2}-1\right)\right)^{1/Ж}, \\ 1-\left(\sum_{i=1}^{ő+1}\left(1-\left(Ü\left(1-\left((ő+2)Ÿ_i\left(1-I_i^U\right)^{Ж}-((ő+2)Ÿ_i-1)\right)^{1/Ж}\right)^{Ж}+Θ\left(1-\left((ő+2)Ÿ_{ő+2}\left(1-I_{ő+2}^U\right)^{Ж}-((ő+2)Ÿ_{ő+2}-1)\right)^{1/Ж}\right)^{Ж}-Ü-Θ+1\right)^{1/Ж}\right)^{Ж}-\left(\frac{(ő+2)(ő+3)}{2}-1\right)\right)^{1/Ж} \end{bmatrix}, \right. \\
&\qquad \left. \left(\sum_{i=1}^{ő+1}\left(1-\left(Ü\left(1-\left((ő+2)Ÿ_i f_i^{Ж}-((ő+2)Ÿ_i-1)\right)^{1/Ж}\right)^{Ж}+Θ\left(1-\left((ő+2)Ÿ_{ő+2} f_{ő+2}^{Ж}-((ő+2)Ÿ_{ő+2}-1)\right)^{1/Ж}\right)^{Ж}-Ü-Θ+1\right)^{1/Ж}\right)^{Ж}-\left(\frac{(ő+2)(ő+3)}{2}-1\right)\right)^{1/Ж} \right\rangle.
\end{aligned}
\tag{35}
$$

Therefore, (31) is true $z = ő + 1$. Hence, (31) is true for all z. Similarly, we can prove other parts of (30). So, (30) becomes

$$
\begin{aligned}
&\sum_{\substack{i,j=1 \\ j=i}}^{z+1} ((z+1)Ÿ_i cn_i)^{Ü} \otimes_{SS} \left((z+1)Ÿ_j cn_j\right)^{Õ} \\
&= \left\langle \begin{bmatrix} 1-\left(\sum_{\substack{i,j=1 \\ j=i}}^{z+1}\left(1-\left(Ü\left(1-\left((z+1)Ÿ_i\left(1-I_i^L\right)^{Ж}-((z+1)Ÿ_i-1)\right)^{1/Ж}\right)^{Ж}+Θ\left(1-\left((z+1)Ÿ_j\left(1-I_j^L\right)^{Ж}-((z+1)Ÿ_j-1)\right)^{1/Ж}\right)^{Ж}-Ü-Θ+1\right)^{1/Ж}\right)^{Ж}-\left(\frac{(z+1)(z+2)}{2}-1\right)\right)^{1/Ж}, \\ 1-\left(\sum_{\substack{i,j=1 \\ j=i}}^{z+1}\left(1-\left(Ü\left(1-\left((z+1)Ÿ_i\left(1-I_i^U\right)^{Ж}-((z+1)Ÿ_i-1)\right)^{1/Ж}\right)^{Ж}+Θ\left(1-\left((z+1)Ÿ_j\left(1-I_j^U\right)^{Ж}-((z+1)Ÿ_j-1)\right)^{1/Ж}\right)^{Ж}-Ü-Θ+1\right)^{1/Ж}\right)^{Ж}-\left(\frac{(z+1)(z+2)}{2}-1\right)\right)^{1/Ж} \end{bmatrix}, \right. \\
&\qquad \left. \left(\sum_{\substack{i,j=1 \\ j=i}}^{z+1}\left(1-\left(Ü\left(1-\left((z+1)Ÿ_i f_i^{Ж}-((z+1)Ÿ_i-1)\right)^{1/Ж}\right)^{Ж}+Θ\left(1-\left((z+1)Ÿ_j f_j^{Ж}-((z+1)Ÿ_j-1)\right)^{1/Ж}\right)^{Ж}-Ü-Θ+1\right)^{1/Ж}\right)^{Ж}-\left(\frac{(z+1)(z+2)}{2}-1\right)\right)^{1/Ж} \right\rangle.
\end{aligned}
\tag{36}
$$

Hence, (22) is true $\bar{a} = z + 1$. Therefore, (22) is true for all $\dagger$.

By (22), we can say that (14), is right. From (22), and the SS operational laws for CNs, we have

$$\frac{2}{\bar{a}(\bar{a}+1)}\left(\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}} (\bar{a}\ddot{Y}_i cn_i)^{\ddot{U}} \otimes_{SS} (\bar{a}\ddot{Y}_j cn_j)^{\tilde{O}}\right)$$
$$= \left\langle \begin{bmatrix} 1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i(1-I_i^{L})^{\text{Ж}}-(\bar{a}\ddot{Y}_i-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(\bar{a}\ddot{Y}_j(1-I_j^{L})^{\text{Ж}}-(\bar{a}\ddot{Y}_j-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}, \\ 1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i(1-I_i^{U})^{\text{Ж}}-(\bar{a}\ddot{Y}_i-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(\bar{a}\ddot{Y}_j(1-I_j^{U})^{\text{Ж}}-(\bar{a}\ddot{Y}_j-1)\right)^{\frac{1}{\text{Ж}}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}} \end{bmatrix}, \left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i f_i^{\text{Ж}}-(\bar{a}\ddot{Y}_i-1)\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(\bar{a}\ddot{Y}_j f_j^{\text{Ж}}-(\bar{a}\ddot{Y}_j-1)\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}} \right\rangle. \tag{37}$$

Furthermore,

$$\left(\frac{2}{\bar{a}(\bar{a}+1)}\left(\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}} (\bar{a}\ddot{Y}_i cn_i)^{\ddot{U}} \otimes_{SS} (\bar{a}\ddot{Y}_j cn_j)^{\hat{O}}\right)\right)^{1/(\ddot{U}+\text{Ѳ})}$$
$$= \left\langle \begin{bmatrix} \left(\left(\frac{1}{\ddot{U}+\text{Ѳ}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i(1-I_i^{L})^{\text{Ж}}-(\bar{a}\ddot{Y}_i-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(\bar{a}\ddot{Y}_j(1-I_j^{L})^{\text{Ж}}-(\bar{a}\ddot{Y}_j-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}+\text{Ѳ}}-1\right)\right)^{1/\text{Ж}}, \\ \left(\left(\frac{1}{\ddot{U}+\text{Ѳ}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i(1-I_i^{U})^{\text{Ж}}-(\bar{a}\ddot{Y}_i-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(\bar{a}\ddot{Y}_j(1-I_j^{U})^{\text{Ж}}-(\bar{a}\ddot{Y}_j-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}+\text{Ѳ}}-1\right)\right)^{1/\text{Ж}} \end{bmatrix}, 1-\left(\frac{1}{\ddot{U}+\text{Ѳ}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i f_i^{\text{Ж}}-(\bar{a}\ddot{Y}_i-1)\right)\right)^{\frac{1}{\text{Ж}}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(\bar{a}\ddot{Y}_j f_j^{\text{Ж}}-(\bar{a}\ddot{Y}_j-1)\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}+\text{Ѳ}}-1\right)\right)^{1/\aleph} \right\rangle. \tag{38}$$

Hence,

$$\mathrm{CSSPHM}^{\ddot{U},\text{Ѳ}}(cn_1, cn_2, \ldots, cn_{\dagger})$$
$$= \left\langle \begin{bmatrix} \left(\left(\frac{1}{\ddot{U}+\text{Ѳ}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i(1-I_i^{L})^{\text{Ж}}-(\bar{a}\ddot{Y}_i-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(\bar{a}\ddot{Y}_j(1-I_j^{L})^{\text{Ж}}-(\bar{a}\ddot{Y}_j-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}+\text{Ѳ}}-1\right)\right)^{1/\text{Ж}}, \\ \left(\left(\frac{1}{\ddot{U}+\text{Ѳ}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i(1-I_i^{U})^{\text{Ж}}-(\bar{a}\ddot{Y}_i-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(\bar{a}\ddot{Y}_j(1-I_j^{U})^{\text{Ж}}-(\bar{a}\ddot{Y}_j-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}+\text{Ѳ}}-1\right)\right)^{1/\text{Ж}} \end{bmatrix}, 1-\left(\frac{1}{\ddot{U}+\text{Ѳ}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i f_i^{\text{Ж}}-(\bar{a}\ddot{Y}_i-1)\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(\bar{a}\ddot{Y}_j f_j^{\text{Ж}}-(\bar{a}\ddot{Y}_j-1)\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}+\text{Ѳ}}-1\right)\right)^{1/\text{Ж}} \right\rangle \tag{39}$$

Now, we confer some advantageous possessions of the projected CSrSrPHOM operator. □

Theorem 3 (idempotency). *Let $\ddot{U} \geq 0$, $\text{Ө} \geq 0$ and $\ddot{U}, \text{Ө}$ capture no more than one value of 0 at a time, $cn_i = \langle [I_i^L, I_i^U], f_i \rangle$ $(i = 1, 2, \ldots, \bar{a})$ be a faction of CNs, and $cn_i = \langle [I_i^L, I_i^U], fs_i \rangle = cn = \langle [I^L, I^U], f \rangle$ $(i = 1, 2, \ldots, \bar{a})$. Then,*

$$\mathrm{CSSPHM}^{\ddot{U},\text{Ө}}(\mathrm{cn}_1, \mathrm{cn}_2, \ldots, \mathrm{cn}_{\dagger}) = \mathrm{cn} = \left\langle \left[I^L, I^U \right], f \right\rangle. \tag{40}$$

Proof. Since $cn_i = \langle [I_i^L, I_i^U], f_i \rangle = cn = \langle [I^L, I^U], f \rangle$ $(i = 1, 2, \ldots, \bar{a})$, we have $\sup(\mathrm{cn}_g, \mathrm{cn}_z) = 1$ $(\forall\, g, z = 1, 2, \ldots, \bar{a})$ so $\diamond_g = (1/\bar{a})$ $(\forall g = 1, 2, \ldots, \bar{a})$ and $\bar{a}\ddot{Y}_g = \bar{a}(1/\bar{a}) = 1$ $(\forall g = 1, 2, \ldots, \bar{a})$.

Then,

$$\begin{aligned}
&\mathrm{CSSPHM}^{\ddot{U},\text{Ө}}(\mathrm{cn}_1, \mathrm{cn}_2, \ldots, \mathrm{cn}_{\bar{a}}) \\
&= \mathrm{CSSPHM}^{\ddot{U},\text{Ө}}(\mathrm{cn}, \mathrm{cn}, \ldots, \mathrm{cn}) \\
&= \left\langle \left[\begin{array}{l}
\left(\frac{1}{\ddot{U}+\text{Ө}} \left(1 - \left(\frac{2}{\bar{a}(\bar{a}+1)} \sum_{\substack{i,j=1 \\ j\neq i}}^{\bar{a}} \left(1 - \left(\ddot{U}\left(1 - \left((1-\mathrm{I}^{\mathrm{L}})^{\text{Ж}} \right)^{1/\text{Ж}} \right)^{\text{Ж}} + \text{Ө}\left(1 - \left((1-\mathrm{I}^{\mathrm{L}})^{\text{Ж}} \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right)^{1/\text{Ж}} \right)^{\text{Ж}} \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \left(\frac{1}{\ddot{U}+\text{Ө}} - 1 \right) \right)^{1/\text{Ж}}, \\
\left(\frac{1}{\ddot{U}+\text{Ө}} \left(1 - \left(\frac{2}{\bar{a}(\bar{a}+1)} \sum_{\substack{i,j=1 \\ j\neq i}}^{\bar{a}} \left(1 - \left(\ddot{U}\left(1 - \left((1-\mathrm{I}^{\mathrm{U}})^{\text{Ж}} \right)^{1/\text{Ж}} \right)^{\text{Ж}} + \text{Ө}\left(1 - \left((1-\mathrm{I}^{\mathrm{U}})^{\text{Ж}} \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right)^{1/\text{Ж}} \right)^{\text{Ж}} \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \left(\frac{1}{\ddot{U}+\text{Ө}} - 1 \right) \right)^{1/\text{Ж}}
\end{array} \right], \right. \\
&\qquad \left. 1 - \left(\frac{1}{\ddot{U}+\text{Ө}} \left(1 - \left(\frac{2}{\bar{a}(\bar{a}+1)} \sum_{\substack{i,j=1 \\ j\neq i}}^{\bar{a}} \left(1 - \left(\ddot{U}(1-\mathrm{f})^{\text{Ж}} + \text{Ө}(1-\mathrm{f})^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right)^{1/\text{Ж}} \right)^{\text{Ж}} \right)^{1/\text{Ж}} \right)^{\text{Ж}} - \left(\frac{1}{\ddot{U}+\text{Ө}} - 1 \right) \right)^{1/\text{Ж}} \right\rangle \\
&= \left\langle \left[\begin{array}{l}
\left(\frac{1}{\ddot{U}+\text{Ө}} \left((\ddot{U}+\text{Ө})(\mathrm{I}^{\mathrm{L}})^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right) - \frac{1}{\ddot{U}+\text{Ө}} + 1 \right)^{1/\text{Ж}}, \\
\left(\frac{1}{\ddot{U}+\text{Ө}} \left((\ddot{U}+\text{Ө})(\mathrm{I}^{\mathrm{U}})^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right) - \frac{1}{\ddot{U}+\text{Ө}} + 1 \right)^{1/\text{Ж}}
\end{array} \right], \right. \\
&\qquad \left. 1 - \left(\frac{1}{\ddot{U}+\text{Ө}} \left((\ddot{U}+\text{Ө})(1-\mathrm{f})^{\text{Ж}} - \ddot{U} - \text{Ө} + 1 \right) - \frac{1}{\ddot{U}+\text{Ө}} + 1 \right)^{1/\text{Ж}} \right\rangle \\
&= \left\langle \left[I^L, I^U \right], f \right\rangle = \mathrm{cn}.
\end{aligned} \tag{41}$$

□

Theorem 4. *(commutativity). Let $(cn_1', cn_2', \ldots, cn_{\bar{a}}')$ be any permutation of $(cn_1, cn_2, \ldots, cn_{\bar{a}})$. Then,*

$$\mathrm{CSSPHM}^{\ddot{U},\text{Ө}}\left(\mathrm{cn}_1', \mathrm{cn}_2', \ldots, \mathrm{cn}_{\bar{a}}' \right) = \mathrm{CSSPHM}^{\ddot{U},\text{Ө}}(\mathrm{cn}_1, \mathrm{cn}_2, \ldots, \mathrm{cn}_{\bar{a}}). \tag{42}$$

Proof. Since $(\mathrm{cn}_1', \mathrm{cn}_2', \ldots, \mathrm{cn}_{\bar{a}}')$ is any permutation of $(\mathrm{cn}_1, \mathrm{cn}_2, \ldots, \mathrm{cn}_{\bar{a}})$,

$$\mathrm{CSSPHM}^{\ddot{U},\text{Ө}}(\mathrm{cn}_1, \mathrm{cn}_2, \ldots, \mathrm{cn}_{\bar{a}}) = \left(\frac{2}{\bar{a}(\bar{a}+1)} \left(\sum_{\substack{i,j=1 \\ j\neq i}}^{\bar{a}} \left(\bar{a}\ddot{Y}_i \mathrm{cn}_i \right)^{\ddot{U}} \otimes_{SS} \left(\bar{a}\ddot{Y}_j \mathrm{cn}_j \right)^{\tilde{O}} \right) \right)^{1/(\ddot{U}+\text{Ө})}$$

$$= \left(\frac{2}{\bar{a}(\bar{a}+1)} \left(\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}} \left(\bar{a}\ddot{Y}_i cn'_j\right)^{\ddot{U}} \otimes_{SS} \left(\bar{a}\ddot{Y}_j cn'_j\right)^{\tilde{O}} \right) \right)^{1/(\ddot{U}+\Theta)}$$

$$= \mathrm{CSSPHM}^{\ddot{U},\Theta}\left(cn'_1, cn'_2, \ldots, cn'_{\bar{a}}\right). \tag{43}$$

Theorem 5 (boundedness). *Let* $cn_i = \langle [I_i^L, I_i^U], f_i \rangle$ $(i = 1, 2, \ldots, \bar{a})$ *be a faction of CNs,* $cn^- = \langle [\min_i I_i^L, \min_i I_i^U], \max_i f_i \rangle$ *and* $cn^+ = \langle [\max_i I_i^L, \max_i I_i^U], \min_i f_i \rangle$. *Then,*

$$cn^- \le \mathrm{CSSPHM}^{\ddot{U},\Theta}\left(cn_1, cn_2, \ldots, cn_{\bar{a}}\right) \le cn^+. \tag{44}$$

Proof. By the judgement technique in Definition 2, we have $cn_i \ge cn'_j$, and then based on Theorems 3 and 4, we have

$$\begin{aligned} &\mathrm{CSSPHM}^{\ddot{U},\Theta}\left(cn_1, cn_2, \ldots, cn_{\bar{a}}\right) \\ &\quad \ge \mathrm{CSSPHM}^{\ddot{U},\Theta}\left(cn_1^-, cn_2^-, \ldots, cn_{\bar{a}}^-\right) = cn^-. \end{aligned} \tag{45}$$

Similarly, we can have

$$\begin{aligned} &\mathrm{CSSPHM}^{\ddot{U},\Theta}\left(cn_1, cn_2, \ldots, cn_{\bar{a}}\right) \\ &\quad \le \mathrm{CSSPHM}^{\ddot{U},\Theta}\left(cn_1^+, cn_2^+, \ldots, cn_{\bar{a}}^+\right) = cn^+. \end{aligned} \tag{46}$$

So, we have

$$\mathrm{CSSPHEM}^{\ddot{U},\Theta}\left(cn_1^-, cn_2^-, \ldots, cn_{\bar{a}}^-\right) \le \mathrm{CSSPHM}^{\ddot{U},\Theta}\left(cn_1, cn_2, \ldots, cn_{\bar{a}}\right) \le \mathrm{CSSPHM}^{\ddot{U},\Theta}\left(cn_1^+, cn_2^+, \ldots, cn_{\bar{a}}^+\right). \tag{47}$$

□

Hence,

$$cn^- \le \mathrm{CSrSrPHOM}^{\ddot{U},\Theta}\left(cn_1, cn_2, \ldots, cn_{\dagger}\right) \le cn^+. \tag{48}$$

By apportioning different values to the constraints Ж, $\ddot{U}$ and Θ, numerous different circumstances of the CSSPHM operator can be attained, and are specified below as follows:

Case 1. If Ж $= 0$, then the CSSPHM operator relapses to the CPHM operator, which can be stated as follows:

$$\mathrm{CSSPHM}^{\ddot{U},\Theta}\left(cn_1, cn_2, \ldots, cn_{\bar{a}}\right) = \left(\frac{2}{\bar{a}(\bar{a}+1)} \sum_{\substack{i,j=1\\ j=i}}^{\bar{a}} \left(\left(\bar{a}\ddot{Y}_i cn_i\right)^{\ddot{U}} \otimes \left(\bar{a}\ddot{Y}_j cn_j\right)^{\tilde{O}} \right) \right)^{1/(\ddot{U}+\Theta)}$$

$$= \left\langle \left[\begin{aligned} &\left(1 - \prod_{\substack{i,j=1\\ j=i}}^{\bar{a}} \left(1 - \left(1 - \left(1 - I_i^L\right)^{\bar{a}\ddot{Y}_i}\right)^{\ddot{U}} \left(1 - \left(1 - I_j^L\right)^{\bar{a}\ddot{Y}_j}\right)^{\tilde{O}} \right)^{2/\bar{a}(\bar{a}+1)} \right)^{1/(\ddot{U}+\Theta)}, \\ &\left(1 - \prod_{\substack{i,j=1\\ j=i}}^{\bar{a}} \left(1 - \left(1 - \left(1 - I_i^U\right)^{\bar{a}\ddot{Y}_i}\right)^{\ddot{U}} \left(1 - \left(1 - I_j^U\right)^{\bar{a}\ddot{Y}_j}\right)^{\tilde{O}} \right)^{2/\bar{a}(\bar{a}+1)} \right)^{1/(\ddot{U}+\Theta)} \end{aligned} \right], \right.$$

$$\left. 1 - \left(1 - \prod_{\substack{i,j=1\\ j=i}}^{\bar{a}} \left(1 - \left(1 - \left(f_i\right)^{\bar{a}\ddot{Y}_i}\right)^{\ddot{U}} \left(1 - \left(f_j\right)^{\bar{a}\ddot{Y}_j}\right)^{\tilde{O}} \right)^{2/\bar{a}(\bar{a}+1)} \right)^{1/(\ddot{U}+\Theta)} \right\rangle. \tag{49}$$

Case 2. If $\tilde{O} \longrightarrow 0$, then the CSSPHM operator reverts to the cubic descending PA operator and is quantified as follows:

$$\begin{aligned}
&\text{CSSPHM}^{\ddot{U},\text{Ө}}\left(cn_1, cn_2, \ldots, cn_{\bar{a}}\right)\\
&= \lim_{\tilde{O}\longrightarrow 0}\left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(\left(\bar{a}\ddot{Y}_i cn_i\right)^{\ddot{U}} \otimes_{SS}\left(\bar{a}\ddot{Y}_j cn_j\right)^{\tilde{O}}\right)\right)^{1/(\ddot{U}+\text{Ө})}\\
&= \left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\sum_{i=1}^{\bar{a}}\left(\left(\bar{a}+1-i\right)\left(\bar{a}\ddot{Y}_i cn_i\right)^{\ddot{U}}\right)\right)^{1/\ddot{U}}\\
&= \left\langle \left[\begin{array}{l}
\left(\frac{1}{\ddot{U}}\left(1-\left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\sum_{i=1}^{\bar{a}}\left(\left(\bar{a}+1-i\right)\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i\left(1-I_i^L\right)^{\text{Ж}}-\left(\bar{a}\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\ddot{U}-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)-\left(\left(2\bar{a}-i\right)\left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\right)-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}}-1\right)\right)^{1/\text{Ж}},\\
\left(\frac{1}{\ddot{U}}\left(1-\left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\sum_{i=1}^{\bar{a}}\left(\left(\bar{a}+1-i\right)\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i\left(1-I_i^U\right)^{\text{Ж}}-\left(\bar{a}\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\ddot{U}-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)-\left(\left(2\bar{a}-i\right)\left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\right)-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}}-1\right)\right)^{1/\text{Ж}}
\end{array}\right],\right.\\
&\left. 1-\left(\frac{1}{\ddot{U}}\left(1-\left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\sum_{i=1}^{\bar{a}}\left(\left(\bar{a}+1-i\right)\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i f_i^{\text{Ж}}-\left(\bar{a}\ddot{Y}_i-1\right)\right)\frac{1}{\text{Ж}}\right)^{\text{Ж}}-\left(\ddot{U}-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)-\left(\left(2\bar{a}-i\right)\left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\right)-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}}-1\right)\right)^{1/\text{Ж}}\right\rangle.
\end{aligned} \tag{50}$$

Case 3. If $\ddot{U} \longrightarrow 0$, then the CSSPHM operator reverts to the cubic ascending PA operator and is quantified as follows:

$$\begin{aligned}
&\text{CSSPHM}^{0,\tilde{O}}\left(cn_1, cn_2, \ldots, cn_{\bar{a}}\right)\\
&= \lim_{\ddot{U}\longrightarrow 0}\left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(\left(\bar{a}\ddot{Y}_i cn_i\right)^{\ddot{U}} \otimes_{SS}\left(\bar{a}\ddot{Y}_j cn_j\right)^{\tilde{O}}\right)\right)^{1/(\ddot{U}+\text{Ө})}\\
&= \left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\sum_{i=1}^{\bar{a}}\left(i\left(\bar{a}\ddot{Y}_i cn_i\right)^{\tilde{O}}\right)\right)^{1/\tilde{O}}\\
&= \left\langle \left[\begin{array}{l}
\left(\frac{1}{\tilde{O}}\left(1-\left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\sum_{i=1}^{\bar{a}}\left(i\left(1-\left(\tilde{O}\left(1-\left(\bar{a}\ddot{Y}_i\left(1-I_i^L\right)^{\text{Ж}}-\left(\bar{a}\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\tilde{O}-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)-\left(\left(\bar{a}+i-1\right)\left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\right)-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\tilde{O}}-1\right)\right)^{1/\text{Ж}},\\
\left(\frac{1}{\tilde{O}}\left(1-\left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\sum_{i=1}^{\bar{a}}\left(i\left(1-\left(\tilde{O}\left(1-\left(\bar{a}\ddot{Y}_i\left(1-I_i^U\right)^{\text{Ж}}-\left(\bar{a}\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\tilde{O}-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)-\left(\left(\bar{a}+i-1\right)\left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\right)-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\tilde{O}}-1\right)\right)^{1/\text{Ж}}
\end{array}\right],\right.\\
&\left. 1-\left(\frac{1}{\tilde{O}}\left(1-\left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\sum_{i=1}^{\bar{a}}\left(i\left(1-\left(\tilde{O}\left(1-\left(\bar{a}\ddot{Y}_i f_i^{\text{Ж}}-\left(\bar{a}\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\tilde{O}-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)-\left(\left(\bar{a}+i-1\right)\left(\frac{2}{\bar{a}\left(\bar{a}+1\right)}\right)-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\tilde{O}}-1\right)\right)^{1/\text{Ж}}\right\rangle.
\end{aligned} \tag{51}$$

Case 4. If $\tilde{O} \longrightarrow 0$ and $\sup(cn_i, cn_j) = h\,(h \in [0,1])$ (for all $i \neq j$), then the CSSPHM operator reverts to the cubic linear descending WA operator and is stated as follows:

$$\begin{aligned}
&\mathrm{CSSPHM}^{\ddot{U},0}(cn_1, cn_2, \ldots, cn_{\bar{a}}) \\
&= \lim_{\tilde{O} \longrightarrow 0}\left(\frac{2}{\bar{a}(\bar{a}+1)} \sum_{\substack{i,j=1 \\ j=1}}^{\bar{a}} \left(\bar{a}\ddot{Y}_i cn_i\right)^{\ddot{U}} \otimes_{SS} \left(\bar{a}\ddot{Y}_j cn_j\right)^{\tilde{O}}\right)^{1/\ddot{U}+\Theta} \\
&= \left(\frac{2}{\bar{a}(\bar{a}+1)} \sum_{i=1}^{\bar{a}} (\bar{a}+1-i) cn_i^{\ddot{U}}\right)^{1/\ddot{U}} \\
&= \left\langle \begin{bmatrix} \left(\frac{1}{\ddot{U}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{i=1}^{\bar{a}}\left((\bar{a}+1-i)\left(1-\left(\ddot{U}\left(I_i^L\right)^{\text{Ж}}-(\ddot{U}-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)-\left((2\bar{a}-i)\left(\frac{2}{\bar{a}(\bar{a}+1)}\right)-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}}-1\right)\right)^{1/\text{Ж}}, \\
\left(\frac{1}{\ddot{U}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{i=1}^{\bar{a}}\left((\bar{a}+1-i)\left(1-\left(\ddot{U}\left(I_i^U\right)^{\text{Ж}}-(\ddot{U}-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)-\left((2\bar{a}-i)\left(\frac{2}{\bar{a}(\bar{a}+1)}\right)-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}}-1\right)\right)^{1/\text{Ж}} \end{bmatrix}, \right. \\
&\qquad \left. 1-\left(\frac{1}{\ddot{U}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{i=1}^{\dagger}\left((\bar{a}+1-i)\left(1-\left(\ddot{U}(1-f_i)^{\text{Ж}}-(\ddot{U}-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)-\left((2\bar{a}-i)\left(\frac{2}{\bar{a}(\bar{a}+1)}\right)-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}}-1\right)\right)^{1/\text{Ж}} \right\rangle.
\end{aligned} \tag{52}$$

Certainly, the significant degree of CNs $cn_i^{\ddot{U}}\,(i = 1, 2, \ldots, \bar{a})$ is $(\bar{a}, \bar{a}-1, \ldots, 1)$.

Case 5. If $\ddot{U} \longrightarrow 0$ and $\sup(cn_i, cn_j) = h\,(h \in [0,1])$ (for all $i \neq j$), then the CSSPHOM operator reverts to the cubic linear ascending WA operator and is stated as follows:

$$\mathrm{CSSPHM}^{0,\tilde{\mathrm{O}}}(\mathrm{cn}_1, \mathrm{cn}_2, \ldots, \mathrm{cn}_{\bar{\mathrm{a}}})$$

$$= \lim_{\ddot{\mathrm{U}} \longrightarrow 0} \left(\frac{2}{\bar{\mathrm{a}}(\bar{\mathrm{a}}+1)} \sum_{\substack{i,j=1 \\ j=1}}^{\bar{\mathrm{a}}} (\bar{\mathrm{a}}\ddot{\mathrm{Y}}_i \mathrm{cn}_i)^{\ddot{\mathrm{U}}} \otimes_{SS} (\bar{\mathrm{a}}\ddot{\mathrm{Y}}_j \mathrm{cn}_j)^{\tilde{\mathrm{O}}} \right)^{1/(\ddot{\mathrm{U}}+\Theta)}$$

$$= \left(\frac{2}{\bar{\mathrm{a}}(\bar{\mathrm{a}}+1)} \sum_{i=1}^{\bar{\mathrm{a}}} \left(i \mathrm{cn}_i^{\tilde{\mathrm{O}}} \right) \right)^{1/\tilde{\mathrm{O}}}$$

$$= \left\langle \begin{array}{c} \left[\begin{array}{c} \left(\frac{1}{\tilde{\mathrm{O}}} \left(1 - \left(\frac{2}{\bar{\mathrm{a}}(\bar{\mathrm{a}}+1)} \sum_{i=1}^{\bar{\mathrm{a}}} \left((i) \left(1 - \left(\tilde{\mathrm{O}} \left(I_i^L \right)^{\aleph} - (\tilde{\mathrm{O}}-1) \right)^{1/Ж} \right)^{Ж} \right) - \left((\bar{\mathrm{a}}+i-1) \left(\frac{2}{\bar{\mathrm{a}}(\bar{\mathrm{a}}+1)} \right) - 1 \right) \right)^{1/Ж} \right)^{Ж} - \left(\frac{1}{\tilde{\mathrm{O}}} - 1 \right) \right)^{1/Ж}, \\ \left(\frac{1}{\tilde{\mathrm{O}}} \left(1 - \left(\frac{2}{\bar{\mathrm{a}}(\bar{\mathrm{a}}+1)} \sum_{i=1}^{\bar{\mathrm{a}}} \left((i) \left(1 - \left(\tilde{\mathrm{O}} \left(I_i^U \right)^{Ж} - (\tilde{\mathrm{O}}-1) \right)^{1/Ж} \right)^{Ж} \right) - \left((\bar{\mathrm{a}}+i-1) \left(\frac{2}{\bar{\mathrm{a}}(\bar{\mathrm{a}}+1)} \right) - 1 \right) \right)^{1/Ж} \right)^{Ж} - \left(\frac{1}{\tilde{\mathrm{O}}} - 1 \right) \right)^{1/Ж} \end{array} \right], \\ 1 - \left(\frac{1}{\tilde{\mathrm{O}}} \left(1 - \left(\frac{2}{\bar{\mathrm{a}}(\bar{\mathrm{a}}+1)} \sum_{i=1}^{\bar{\mathrm{a}}} \left((i) \left(1 - \left(\tilde{\mathrm{O}}(1-f_i)^{Ж} - (\tilde{\mathrm{O}}-1) \right)^{1/Ж} \right)^{Ж} \right) - \left((\bar{\mathrm{a}}+i-1) \left(\frac{2}{\bar{\mathrm{a}}(\bar{\mathrm{a}}+1)} \right) - 1 \right) \right)^{1/Ж} \right)^{Ж} - \left(\frac{1}{\tilde{\mathrm{O}}} - 1 \right) \right)^{1/Ж} \end{array} \right\rangle. \tag{53}$$

Case 6. If $\ddot{\mathrm{U}} = \Theta = 1/2$ and $\sup(\mathrm{cn}_i, \mathrm{cn}_j) = h(h \in [0,1])$ (for all $i \neq j$), then the CSSPHM operator reverts to the cubic basic HM operator and is stated as follows:

$$\mathrm{CSSPHM}^{(1/2),(1/2)}(\mathrm{cn}_1, \mathrm{cn}_2, \ldots, \mathrm{cn}_{\bar{\mathrm{a}}})$$

$$= \frac{2}{\bar{\mathrm{a}}(\bar{\mathrm{a}}+1)} \sum_{\substack{i,j=1 \\ j=1}}^{\bar{\mathrm{a}}} \left((\mathrm{cn}_i)^{(1/2)} \otimes (\mathrm{cn}_j)^{(1/2)} \right)$$

$$= \left\langle \begin{array}{c} \left[1 - \left(\frac{2}{\bar{\mathrm{a}}(\bar{\mathrm{a}}+1)} \sum_{\substack{i,j=1 \\ j=i}}^{\bar{\mathrm{a}}} \left(1 - \left(\frac{1}{2} \left(I_i^L \right)^{Ж} + \frac{1}{2} \left(I_j^L \right)^{Ж} \right)^{1/Ж} \right)^{Ж} \right)^{1/Ж}, 1 - \left(\frac{2}{\bar{\mathrm{a}}(\bar{\mathrm{a}}+1)} \sum_{\substack{i,j=1 \\ j=i}}^{\bar{\mathrm{a}}} \left(1 - \left(\frac{1}{2} \left(I_i^U \right)^{Ж} + \frac{1}{2} \left(I_j^U \right)^{Ж} \right)^{1/Ж} \right)^{Ж} \right)^{1/Ж} \right], \\ \left(\frac{2}{\bar{\mathrm{a}}(\bar{\mathrm{a}}+1)} \sum_{\substack{i,j=1 \\ j=i}}^{\bar{\mathrm{a}}} \left(1 - \left(\frac{1}{2} (1-f_i)^{Ж} + \frac{1}{2} (1-fs_j)^{Ж} \right)^{1/Ж} \right)^{Ж} \right)^{1/Ж} \end{array} \right\rangle. \tag{54}$$

Case 7. If $\ddot{\mathrm{U}} = \Theta = 1$, and $\sup(\mathrm{cn}_i, \mathrm{cn}_j) = h(h \in [0,1])$ (for all $i \neq j$), then the CSSPHM operator reverts to the cubic linear HM operator, and is stated as follows:

$$\mathrm{CSSPHM}^{1,1}\left(\mathrm{cn}_1,\mathrm{cn}_2,\ldots,\mathrm{cn}_{\bar{a}}\right)=\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=1}}^{\dagger}\left(\mathrm{cn}_i\otimes\mathrm{cn}_j\right)\right)^{1/2}$$

$$=\left\langle\begin{bmatrix}\left(\frac{1}{2}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\dagger}\left(1-\left(\left(\left(I_i^L\right)^{Ж}+\left(I_j^L\right)^{Ж}-1\right)^{1/Ж}\right)^{Ж}\right)\right)^{1/Ж}\right)^{Ж}+\frac{1}{2}\right)^{1/Ж}, \\ \left(\frac{1}{2}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\dagger}\left(1-\left(\left(\left(I_i^U\right)^{Ж}+\left(I_j^U\right)^{Ж}-1\right)^{1/Ж}\right)^{Ж}\right)\right)^{1/Ж}\right)^{Ж}+\frac{1}{2}\right)^{1/Ж}\end{bmatrix}, 1-\left(\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\dagger}\left(1-\left(\left(1-f_i\right)^{Ж}+\left(1-f_j\right)^{Ж}-1\right)^{1/Ж}\right)^{Ж}\right)^{1/Ж}\right)^{Ж}+\frac{1}{2}\right)^{1/Ж}\right\rangle. \tag{55}$$

□

Definition 9. Let $\mathrm{cn}_i=\langle[I_i^L,I_i^U],f_i\rangle\,(i=1,2,\ldots,\bar{a})$ be a faction of CNs, then the cubic SS power geometric HM (CSSPGHM) operator is explicated as follows:

$$\mathrm{CSSPGHM}^{Ü,Ѳ}\left(\mathrm{cn}_1,\mathrm{cn}_2,\ldots,\mathrm{cn}_{\bar{a}}\right)=\frac{1}{Ü+Ѳ}\left(\prod_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(Ü\mathrm{cn}_{\mathrm{i}}^{\left(\bar{a}\left(1+\mathrm{T}\left(\mathrm{cn}_{\mathrm{i}}\right)\right)/\sum_{z=1}^{\bar{a}}\left(1+\mathrm{T}\left(\mathrm{cn}_z\right)\right)\right)}\oplus_{\mathrm{SS}}Ѳ\mathrm{cn}_{\mathrm{j}}^{\left(\bar{a}\left(1+\mathrm{T}\left(\mathrm{cn}_{\mathrm{j}}\right)\right)/\sum_{z=1}^{\bar{a}}\left(1+\mathrm{T}\left(\mathrm{cn}_z\right)\right)\right)}\right)\right)^{2/\bar{a}(\bar{a}+1)}, \tag{56}$$

where $Ü,Ѳ\geq 0,\ \mathrm{T}(\mathrm{cn_i})=\sum_{\substack{j=1\\ j\neq i}}^{\bar{a}}\sup(\mathrm{cn_i},\mathrm{cn_j}),\ \sup(\mathrm{cn_i},\mathrm{cn}_j)=1-\overline{\overline{\overline{\mathrm{DNE}}}}(\mathrm{cn}_i,\mathrm{cn}_j)$, and $\overline{\overline{\overline{\mathrm{DNE}}}}(\mathrm{cn}_i,\mathrm{cn}_j)$ can be totalized by (6).

Let $\ddot{\mathrm{Y}}_{\mathrm{i}}=(1+T(\mathrm{cn}_i))/\sum_{z=1}^{\bar{a}}(1+T(cn_z))$, then the definition of CSSPGHM operator is correspondent to the subsequent formula:

$$\mathrm{CSSPGHM}^{Ü,Ѳ}\left(\mathrm{cn}_1,\mathrm{cn}_2,\ldots,\mathrm{cn}_{\bar{a}}\right)=\frac{1}{Ü+Ѳ}\left(\prod_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(Ü\mathrm{cn}_{\mathrm{i}}^{\bar{a}\ddot{\mathrm{Y}}_{\mathrm{i}}}\oplus_{\mathrm{SS}}Ѳ\mathrm{cn}_{\mathrm{j}}^{\bar{a}\ddot{\mathrm{Y}}_{\mathrm{j}}}\right)\right)^{2/\bar{a}(\bar{a}+1)}. \tag{57}$$

Theorem 6. *Let $Ü\geq 0$, $Ѳ\geq 0$ and $Ü,Ѳ$ take no more than one value of 0 at a time, $cn_i=\langle[I_i^L,I_i^U],f_i\rangle$ be a faction of CNs. Then, exploiting the CSSPGHM operator, their merged values are CNs, and*

$$\text{CSSPGHM}^{\text{Ü},\text{Ѳ}}(\text{cn}_1,\text{cn}_2,\ldots,\text{cn}_{\bar{\text{a}}})$$

$$=\left\langle\left[\begin{array}{l}1-\left(\frac{1}{\text{Ü}+\text{Ѳ}}\left(1-\left(\frac{2}{\bar{\text{a}}(\bar{\text{a}}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{\text{a}}}\left(1-\left(\text{Ü}\left(1-\left(\bar{\text{a}}\text{Ÿ}_i\left(\text{I}_i^{\text{L}}\right)^{\text{Ж}}-\left(\bar{\text{a}}\text{Ÿ}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(\bar{\text{a}}\text{Ÿ}_j\left(\text{I}_j^{\text{L}}\right)^{\text{Ж}}-\left(\bar{\text{a}}\text{Ÿ}_j-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\text{Ü}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\text{Ü}+\text{Ѳ}}-1\right)\right)^{1/\text{Ж}},\\ 1-\left(\frac{1}{\text{Ü}+\text{Ѳ}}\left(1-\left(\frac{2}{\bar{\text{a}}(\bar{\text{a}}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{\text{a}}}\left(1-\left(\text{Ü}\left(1-\left(\bar{\text{a}}\text{Ÿ}_i\left(\text{I}_i^{\text{U}}\right)^{\text{Ж}}-\left(\bar{\text{a}}\text{Ÿ}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(\bar{\text{a}}\text{Ÿ}_j\left(\text{I}_j^{\text{U}}\right)^{\text{Ж}}-\left(\bar{\text{a}}\text{Ÿ}_j-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\text{Ü}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\text{Ü}+\text{Ѳ}}-1\right)\right)^{1/\text{Ж}}\end{array}\right],\right.$$

$$\left.\left(\frac{1}{\text{Ü}+\text{Ѳ}}\left(1-\left(\frac{2}{\bar{\text{a}}(\bar{\text{a}}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{\text{a}}}\left(1-\left(\text{Ü}\left(1-\left(\bar{\text{a}}\text{Ÿ}_i(1-\text{f}_i)^{\text{Ж}}-\left(\bar{\text{a}}\text{Ÿ}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ѳ}\left(1-\left(\bar{\text{a}}\text{Ÿ}_j(1-\text{f}_j)^{\text{Ж}}-\left(\bar{\text{a}}\text{Ÿ}_j-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\text{Ü}-\text{Ѳ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\text{Ü}+\text{Ѳ}}-1\right)\right)^{1/\text{Ж}}\right\rangle. \tag{58}$$

Proof. Proof of Theorem 6 is similar to that of Theorem 2. Therefore, omitted here.

Now, we deliberate some privileged properties of the projected CSSPGHM operator. □

Theorem 7 (idempotency). *Let $\text{Ü}\geq 0$, $\text{Ѳ}\geq 0$, and $\text{Ü},\text{Ѳ}$ grab no more than one value of 0 at a time, $cn_i=\langle[I_i^L,I_i^U],f_i\rangle$ $(i=1,2,\ldots,\bar{a})$ be a faction of CNs, and $cn_i=\langle[I_i^L,I_i^U],f_i\rangle=cn=\langle[I^L,I^U],f\rangle$ $(i=1,2,\ldots,\bar{a})$. Then,*

$$\text{CSSPGHM}^{\text{Ü},\text{Ѳ}}(\text{cn}_1,\text{cn}_2,\ldots,\text{cn}_{\bar{\text{a}}})=\text{cn}=\left\langle\left[I^L,I^U\right],f\right\rangle. \tag{59}$$

Theorem 8 (commutativity). *Let $(cn_1',cn_2',\ldots,cn_{\bar{a}}')$ be any permutation of $(cn_1,cn_2,\ldots,cn_{\bar{a}})$. Then,*

$$\text{CSSPGHM}^{\text{Ü},\text{Ѳ}}\left(\text{cn}_1',\text{cn}_2',\ldots,\text{cn}_{\bar{\text{a}}}'\right)=\text{CSSPGHM}^{\text{Ü},\text{Ѳ}}(\text{cn}_1,\text{cn}_2,\ldots,\text{cn}_{\bar{\text{a}}}). \tag{60}$$

Theorem 9 (boundedness). *Let $cn_i=\langle[I_i^L,I_i^U],f_i\rangle$ $(i=1,2,\ldots,\bar{a})$ be a faction of CNs, $cn^-=\langle[\min_i I_i^L,\min_i I_i^U],\max_i f_i\rangle$ and $\Theta^+=\langle[\max_i I_i^L,\max_i I_i^U],\min_i f_i\rangle$. Then,*

$$\text{cn}^-\leq\text{CSSPGHM}^{\text{Ü},\text{Ѳ}}(\text{cn}_1,\text{cn}_2,\ldots,\text{cn}_{\bar{\text{a}}})\leq\text{cn}^+. \tag{61}$$

The proofs of these properties are the same as for CSSPHM operators. Hence, these are executed here.

By specifying distinctive values of the parameters Ж, Ü and Ѳ, numerous specific AOs are obtained from the CSSPGHM operator, and are stated as given below:

Case 1. If Ж = 0, then the CSSPGHM operator reverts to the CPGHM operator and is stated as follows:

$$\text{CSSPGHM}^{\text{Ü},\text{Ѳ}}(\text{cn}_1,\text{cn}_2,\ldots,\text{cn}_{\bar{\text{a}}})=\frac{1}{\text{Ü}+\text{Ѳ}}\left(\prod_{\substack{i,j=1\\ j=i}}^{\bar{\text{a}}}\left(\text{Ü}\text{cn}_{\text{i}}^{\bar{\text{a}}\text{Ÿ}_{\text{i}}}\oplus\text{Ѳ}\text{cn}_{\text{j}}^{\bar{\text{a}}\text{Ÿ}_{\text{j}}}\right)\right)^{2/\bar{\text{a}}(\bar{\text{a}}+1)}$$

$$=\left\langle\left[\begin{array}{l}1-\left(1-\prod_{\substack{i,j=1\\ j=i}}^{\bar{\text{a}}}\left(1-\left(1-\left(I_i^L\right)^{\bar{\text{a}}\text{Ÿ}_{\text{i}}}\right)^{\text{Ü}}\left(1-\left(I_j^L\right)^{\bar{\text{a}}\text{Ÿ}_{\text{j}}}\right)^{\text{Õ}}\right)^{2/\bar{\text{a}}(\bar{\text{a}}+1)}\right)^{1/(\text{Ü}+\text{Ѳ})},\\ 1-\left(1-\prod_{\substack{i,j=1\\ j=i}}^{\bar{\text{a}}}\left(1-\left(1-\left(I_i^U\right)^{\bar{\text{a}}\text{Ÿ}_{\text{i}}}\right)^{\text{Ü}}\left(1-\left(I_j^U\right)^{\bar{\text{a}}\text{Ÿ}_{\text{j}}}\right)^{\text{Õ}}\right)^{2/\bar{\text{a}}(\bar{\text{a}}+1)}\right)^{1/(\text{Ü}+\text{Ѳ})}\end{array}\right],\right. \tag{62}$$

$$\left.\left(1-\prod_{\substack{i,j=1\\ j=i}}^{\bar{\text{a}}}\left(1-\left(1-(1-f_i)^{\bar{\text{a}}\text{Ÿ}_{\text{i}}}\right)^{\text{Ü}}\left(1-(1-f_j)^{\bar{\text{a}}\text{Ÿ}_{\text{j}}}\right)^{\text{Õ}}\right)^{2/\bar{\text{a}}(\bar{\text{a}}+1)}\right)^{1/(\text{Ü}+\text{Ѳ})}\right\rangle.$$

Case 2. If $\tilde{O} \longrightarrow 0$, then the CSSPGHM operator reverts to the cubic descending PG operator and is stated as follows:

$$
\begin{aligned}
&\mathrm{CSSPGHM}^{\ddot{U},0}(\mathrm{cn}_1, \mathrm{cn}_2, \ldots, \mathrm{cn}_{\bar{a}})\\
&= \lim_{\tilde{O}\longrightarrow 0} \frac{1}{\ddot{U}+\Theta}\left(\prod_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(\ddot{U}\mathrm{cn}_i^{\bar{a}\ddot{Y}_i}\oplus\Theta\mathrm{cn}_j^{\bar{a}\ddot{Y}_j}\right)\right)^{2/\bar{a}(\bar{a}+1)}\\
&= \frac{1}{\ddot{U}}\left(\prod_{i=1}^{\bar{a}}\left(\ddot{U}\mathrm{cn}_i^{\bar{a}\ddot{Y}_i}\right)^{(\bar{a}+1-i)}\right)^{2/\bar{a}(\bar{a}+1)}\\
&= \left\langle \left[1-\left(\frac{1}{\ddot{U}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{i=1}^{\bar{a}}\left((\bar{a}+1-i)\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i(I_i^L)^{\text{Ж}}-(\bar{a}\ddot{Y}_i-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-(\ddot{U}-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)-\left((2\bar{a}-i)\left(\frac{2}{\bar{a}(\bar{a}+1)}\right)-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}}-1\right)\right)^{1/\text{Ж}}, \right.\right.\\
&\left.\left. 1-\left(\frac{1}{\ddot{U}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{i=1}^{\bar{a}}\left((\bar{a}+1-i)\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i(I_i^U)^{\text{Ж}}-(\bar{a}\ddot{Y}_i-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-(\ddot{U}-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)-\left((2\bar{a}-i)\left(\frac{2}{\bar{a}(\bar{a}+1)}\right)-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}}-1\right)\right)^{1/\text{Ж}}\right], \right.\\
&\left. \left(\frac{1}{\ddot{U}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{i=1}^{\bar{a}}\left((\bar{a}+1-i)\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\ddot{Y}_i(1-f_i)^{\text{Ж}}-(\bar{a}\ddot{Y}_i-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-(\ddot{U}-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)-\left((2\bar{a}-i)\left(\frac{2}{\bar{a}(\bar{a}+1)}\right)-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}}-1\right)\right)^{1/\text{Ж}} \right\rangle.
\end{aligned}
\tag{63}
$$

Case 3. If $\ddot{U} \longrightarrow 0$, then the CSSPGHM operator reverts to the cubic ascending PG operator and is stated as follows:

$$
\begin{aligned}
&\mathrm{CSSPGHM}^{0,\tilde{O}}(\mathrm{cn}_1, \mathrm{cn}_2, \ldots, \mathrm{cn}_{\bar{a}})\\
&= \lim_{G\longrightarrow 0} \frac{1}{\tilde{O}}\left(\prod_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(\ddot{U}\mathrm{cn}_i^{\bar{a}\ddot{Y}_i}\oplus\Theta\mathrm{cn}_j^{\bar{a}\ddot{Y}_j}\right)\right)^{2/\bar{a}(\bar{a}+1)}\\
&= \frac{1}{\tilde{O}}\left(\prod_{i=1}^{\bar{a}}\left(\tilde{O}\mathrm{cn}_i^{\bar{a}\ddot{Y}_i}\right)^{i}\right)^{2/\bar{a}(\bar{a}+1)}\\
&= \left\langle \left[1-\left(\frac{1}{\tilde{O}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{i=1}^{\bar{a}}\left(i\left(1-\left(\tilde{O}\left(1-\left(\bar{a}\ddot{Y}_i(I_i^L)^{\text{Ж}}-(\bar{a}\ddot{Y}_i-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-(\tilde{O}-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)-\left((\bar{a}+i-1)\left(\frac{2}{\bar{a}(\bar{a}+1)}\right)-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\tilde{O}}-1\right)\right)^{1/\text{Ж}}, \right.\right.\\
&\left.\left. 1-\left(\frac{1}{\tilde{O}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{i=1}^{\bar{a}}\left(i\left(1-\left(\tilde{O}\left(1-\left(\bar{a}\ddot{Y}_i(I_i^U)^{\text{Ж}}-(\bar{a}\ddot{Y}_i-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-(\tilde{O}-1)\right)^{\frac{1}{\text{Ж}}}\right)^{\text{Ж}}\right)-\left((\bar{a}+i-1)\left(\frac{2}{\bar{a}(\bar{a}+1)}\right)-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\tilde{O}}-1\right)\right)^{1/\text{Ж}}\right], \right.\\
&\left. \left(\frac{1}{\tilde{O}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{i=1}^{\bar{a}}\left(i\left(1-\left(\tilde{O}\left(1-\left(\bar{a}\ddot{Y}_i(1-f_i)^{\text{Ж}}-(\bar{a}\ddot{Y}_i-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-(\tilde{O}-1)\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)-\left((\bar{a}+i-1)\left(\frac{2}{\bar{a}(\bar{a}+1)}\right)-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\tilde{O}}-1\right)\right)^{1/\text{Ж}} \right\rangle.
\end{aligned}
\tag{64}
$$

Case 4. If $\tilde{O} \longrightarrow 0$ and $\sup(cn_i, cn_j) = h(h \in [0,1])(\forall i \neq j)$, then the CSSPGHM operator reverts to the cubic linear descending weighted geometric operator and is stated as follows:

$$\begin{aligned}
&\mathrm{CSSPGHM}^{\ddot{U},0}(cn_1, cn_2, \ldots, cn_{\bar{a}})\\
&= \lim_{\tilde{O}\longrightarrow 0} \frac{1}{\ddot{U}}\left(\prod_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(\ddot{U}\mathrm{cn}_i^{\bar{a}\ddot{Y}_i} \oplus \Theta \mathrm{cn}_j^{\bar{a}\ddot{Y}_j}\right)\right)^{2/\bar{a}(\bar{a}+1)}\\
&= \frac{1}{\ddot{U}}\left(\prod_{i=1}^{\bar{a}}(\ddot{U}\mathrm{cn}_i)^{(\bar{a}+1-i)}\right)^{2/\bar{a}(\bar{a}+1)}\\
&= \left\langle \begin{bmatrix} 1-\left(\frac{1}{\ddot{U}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{i=1}^{\bar{a}}\left((\bar{a}+1-i)\left(1-\left(\ddot{U}\left(1-\mathrm{I}_i^L\right)^{Ж}-(\ddot{U}-1)\right)^{1/Ж}\right)^{Ж}\right)-\left((2\bar{a}-i)\left(\frac{2}{\bar{a}(\bar{a}+1)}\right)-1\right)\right)^{1/Ж}\right)^{Ж}-\left(\frac{1}{\ddot{U}}-1\right)\right)^{1/Ж},\\ 1-\left(\frac{1}{\ddot{U}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{i=1}^{\bar{a}}\left((\bar{a}+1-i)\left(1-\left(\ddot{U}\left(1-\mathrm{I}_i^U\right)^{Ж}-(\ddot{U}-1)\right)^{1/Ж}\right)^{Ж}\right)-\left((2\bar{a}-i)\left(\frac{2}{\bar{a}(\bar{a}+1)}\right)-1\right)\right)^{1/Ж}\right)^{Ж}-\left(\frac{1}{\ddot{U}}-1\right)\right)^{1/Ж} \end{bmatrix}, \right.\\
&\qquad \left. \left(\frac{1}{\ddot{U}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{i=1}^{\bar{a}}\left((\bar{a}+1-i)\left(1-(\ddot{U}\mathrm{f}_i-(\ddot{U}-1))^{1/Ж}\right)^{Ж}\right)-\left((2\bar{a}-i)\left(\frac{2}{\bar{a}(\bar{a}+1)}\right)-1\right)\right)^{1/Ж}\right)^{Ж}-\left(\frac{1}{\ddot{U}}-1\right)\right)^{1/Ж}\right\rangle.
\end{aligned} \tag{65}$$

Certainly, the significant degree of CNs $\mathrm{cn}_i\,(i = 1, 2, \ldots, \bar{a})$ is $(\bar{a}, \bar{a}-1, \ldots, 1)$.

Case 5. If $\tilde{O} \longrightarrow 0$ and $\sup(\mathrm{cn}_i, \mathrm{cn}_j) = h(h \in [0,1])$ $(\forall i \neq j)$, then the CSSPGHM operator reverts to the cubic linear ascending weighted geometric operator and is stated as follows:

$$\begin{aligned}
&\mathrm{CSSPGHM}^{0,\tilde{O}}(\mathrm{cn}_1, \mathrm{cn}_2, \ldots, \mathrm{cn}_{\bar{a}})\\
&= \lim_{\tilde{O}\longrightarrow 0} \frac{1}{\ddot{U}+\Theta}\left(\prod_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(\ddot{U}\mathrm{cn}_i^{\bar{a}\ddot{Y}_i} \oplus \Theta \mathrm{cn}_j^{\bar{a}\ddot{Y}_j}\right)\right)^{2/\bar{a}(\bar{a}+1)}\\
&= \frac{1}{\tilde{O}}\left(\prod_{i=1}^{\bar{a}}(\tilde{O}\mathrm{cn}_i)^{i}\right)^{2/\bar{a}(\bar{a}+1)},\\
&= \left\langle \begin{bmatrix} 1-\left(\frac{1}{\tilde{O}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{i=1}^{\bar{a}}\left((i)\left(1-\left(\tilde{O}\left(1-I_i^L\right)^{Ж}-(\tilde{O}-1)\right)^{1/Ж}\right)^{Ж}\right)-\left((\bar{a}+i-1)\left(\frac{2}{\bar{a}(\bar{a}+1)}\right)-1\right)\right)^{1/Ж}\right)^{Ж}-\left(\frac{1}{\tilde{O}}-1\right)\right)^{1/Ж},\\ 1-\left(\frac{1}{\tilde{O}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{i=1}^{\bar{a}}\left((i)\left(1-\left(\tilde{O}\left(1-I_i^U\right)^{Ж}-(\tilde{O}-1)\right)^{1/Ж}\right)^{Ж}\right)-\left((\bar{a}+i-1)\left(\frac{2}{\bar{a}(\bar{a}+1)}\right)-1\right)\right)^{1/Ж}\right)^{Ж}-\left(\frac{1}{\tilde{O}}-1\right)\right)^{1/Ж} \end{bmatrix}\right.\\
&\qquad \left. \left(\frac{1}{\tilde{O}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{i=1}^{\dagger}\left((i)\left(1-(\tilde{O}f_i^{Ж}-(\tilde{O}-1))^{1/Ж}\right)^{Ж}\right)-\left((\bar{a}+i-1)\left(\frac{2}{\bar{a}(\bar{a}+1)}\right)-1\right)\right)^{1/Ж}\right)^{Ж}-\left(\frac{1}{\tilde{O}}-1\right)\right)^{1/Ж}\right\rangle.
\end{aligned} \tag{66}$$

Case 6. If $\text{Ü} = \text{ϴ} = 1/2$ and $\sup(\text{cn}_i, \text{cn}_j) = h(h \in [0,1])(\forall i \neq j)$, then the CSSPGHM operator reverts to the cubic GHM operator and is stated as follows:

$$\text{CSSPGHM}^{(1/2),(1/2)}(\text{cn}_1, \text{cn}_2, \ldots, \text{cn}_{\bar{a}}) = \frac{1}{\text{Ü}+\text{ϴ}}\left(\prod_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(\text{Ücn}_i^{\bar{a}\text{Ÿ}_i} \oplus \text{ϴcn}_j^{\bar{a}\text{Ÿ}_j}\right)\right)^{2/\bar{a}(\bar{a}+1)}$$

$$= \left(\frac{1}{2}\prod_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(\text{cn}_i \oplus \text{cn}_j\right)\right)^{2/\bar{a}(\bar{a}+1)}$$

$$= \left\langle \begin{bmatrix} \left(\dfrac{2}{\bar{a}(\bar{a}+1)}\displaystyle\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\frac{1}{2}\left(1-I_i^L\right)^{Ж}+\frac{1}{2}\left(1-I_j^L\right)^{Ж}\right)^{1/Ж}\right)^{Ж}\right)^{1/Ж}, \\ \left(\dfrac{2}{\bar{a}(\bar{a}+1)}\displaystyle\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\frac{1}{2}\left(1-I_i^U\right)^{Ж}+\frac{1}{2}\left(1-I_j^U\right)^{Ж}\right)^{1/Ж}\right)^{Ж}\right)^{1/Ж} \end{bmatrix}, 1-\left(\frac{2}{†(†+1)}\sum_{\substack{i,j=1\\ j=i}}^{†}\left(1-\left(\frac{1}{2}f_i^{Ж}+\frac{1}{2}f_j^{Ж}\right)^{1/Ж}\right)^{Ж}\right)^{1/Ж} \right\rangle. \tag{67}$$

Case 7. If $\text{Ü} = \text{ϴ} = 1$ and $\sup(cn_i, cn_j) = h(h \in [0,1])(\forall i \neq j)$, then the CSSPGHM operator reverts to the cubic basic GHM operator and is stated as follows:

$$\text{CSSPGHM}^{1,1}(\text{cn}_1, \text{cn}_2, \ldots, \text{cn}_{\bar{a}})$$

$$= \frac{1}{\text{Ü}+\text{ϴ}}\left(\prod_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(\text{Ücn}_i^{\bar{a}\text{Ÿ}_i} \oplus \text{ϴcn}_j^{\bar{a}\text{Ÿ}_j}\right)\right)^{2/\bar{a}(\bar{a}+1)}$$

$$= \frac{1}{2}\left(\prod_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(\text{cn}_i \oplus \text{cn}_j\right)\right)^{2/†(†+1)}$$

$$= \left\langle \begin{bmatrix} 1-\left(\left(1-\left(\dfrac{2}{\bar{a}(\bar{a}+1)}\displaystyle\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\left(1-I_i^L\right)^{Ж}+\left(1-_j^L\right)^{Ж}-1\right)^{1/Ж}\right)^{Ж}\right)^{1/Ж}\right)^{Ж}+\dfrac{1}{2}\right)^{1/Ж}, \\ 1-\left(\left(1-\left(\dfrac{2}{\bar{a}(\bar{a}+1)}\displaystyle\sum_{\substack{i,j=1\\ j=i}}^{†}\left(1-\left(\left(1-I_i^U\right)^{Ж}+\left(1-I_j^U\right)^{Ж}-1\right)^{1/Ж}\right)^{Ж}\right)^{1/Ж}\right)^{Ж}+\dfrac{1}{2}\right)^{1/Ж} \end{bmatrix}, \left(\frac{1}{2}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{†}\left(1-\left(f_i^{Ж}+f_j^{Ж}-1\right)^{1/Ж}\right)^{Ж}\right)^{1/Ж}\right)^{Ж}+\frac{1}{2}\right)^{1/Ж} \right\rangle. \tag{68}$$

4.2. CSSWPHM and CSSWPGHM Operators. In this portion, we instigate the CSSWPHM operator and the CSSWPGHM operator by taking the significance of the attributes.

Definition 10. Let $\ddot{U},\Theta\geq 0, \mathrm{cn}_i=\langle[I_i^L,I_i^U],f_i\rangle$ $(i=1,2,\ldots,\bar{a})$ be a faction of CNs and $\varsigma=(\varsigma_1,\varsigma_2,\ldots,\varsigma_{\bar{a}})^T$ be the importance degree of cn_i $(i=1,2,\ldots,\bar{a})$, where $\varsigma_i\geq 0$ and $\sum_{i=1}^{\bar{a}}\varsigma_i=1$. The CSSWPHM operator is rationalized as follows:

$$\mathrm{CSSWPHM}^{\ddot{U},\Theta}(\mathrm{cn}_1,\mathrm{cn}_2,\ldots,\mathrm{cn}_{\bar{a}})=\left(\frac{2}{\bar{a}(\bar{a}+1)}\left(\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(\frac{\bar{a}\varsigma_i(1+T(\mathrm{cn}_i))}{\sum_{z=1}^{\bar{a}}\varsigma_z(1+T(\mathrm{cn}_z))}\mathrm{cn}_i\right)^{\ddot{U}}\otimes_{SS}\left(\frac{\bar{a}\varsigma_j\left(1+T\left(\mathrm{cn}_j\right)\right)}{\sum_{z=1}^{\bar{a}}\varsigma_z(1+T(\mathrm{cn}_z))}\mathrm{cn}_j\right)^{\tilde{O}}\right)\right)^{1/(\ddot{U}+\Theta)}, \tag{69}$$

where $\ddot{U},\Theta\geq 0, T(\mathrm{cn}_i)=\sum_{\substack{j=1\\ j\neq i}}^{\dagger}\sup(\mathrm{cn}_i,\mathrm{cn}_j), \sup(\mathrm{cn}_i,\mathrm{cn}_j)=1-\overline{\overline{\mathrm{DNE}}}(\mathrm{cn}_i,\mathrm{cn}_j)$, and $\overline{\overline{\mathrm{DNE}}}(\mathrm{cn}_i,\mathrm{cn}_j)$ can be reckoned by (6).

Let $\ddot{Y}_i=(1+T(\mathrm{cn}_i))/\sum_{z=1}^{\bar{a}}(1+T(\mathrm{cn}_z))$, then (69) be correspondent to the following shape:

$$\mathrm{CSSWPHM}^{\ddot{U},\Theta}(\mathrm{cn}_1,\mathrm{cn}_2,\ldots,\mathrm{cn}_{\bar{a}})=\left(\frac{2}{\bar{a}(\bar{a}+1)}\left(\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(\bar{a}\varsigma_i\ddot{Y}_i\mathrm{cn}_i\right)^{\ddot{U}}\otimes_{SS}\left(\bar{a}\ddot{Y}_j\varsigma_j\mathrm{cn}_j\right)^{\tilde{O}}\right)\right)^{1/(\ddot{U}+\Theta)}. \tag{70}$$

Theorem 10. *Let $\ddot{U}\geq 0$, $\Theta\geq 0$ and $\ddot{U},\Theta$ take no more than one value of 0 at a time, $cn_i=\langle[I_i^L,I_i^U],f_i\rangle$ be a faction of CNs. Then, exploiting the CSSWPHM operator, their merged values are CNs, and*

$$\begin{aligned}
&\mathrm{CSSWPHM}^{\ddot{U},\Theta}(\mathrm{cn}_1,\mathrm{cn}_2,\ldots,\mathrm{cn}_{\bar{a}})\\
&=\left\langle\left[\begin{array}{l}
\left(\left(\frac{1}{\ddot{U}+\Theta}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\varsigma_i\ddot{Y}_i\left(1-I_i^L\right)^{\text{Ж}}-\left(\bar{a}\varsigma_i\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\Theta\left(1-\left(\bar{a}\varsigma_j\ddot{Y}_j\left(1-I_j^L\right)^{\text{Ж}}-\left(\bar{a}\varsigma_j\ddot{Y}_j-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\Theta+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}+\Theta}-1\right)\right)^{1/\text{Ж}},\\
\left(\left(\frac{1}{\ddot{U}+\Theta}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\varsigma_i\ddot{Y}_i\left(1-I_i^U\right)^{\text{Ж}}-\left(\bar{a}\varsigma_i\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\Theta\left(1-\left(\bar{a}\varsigma_j\ddot{Y}_j\left(1-I_j^U\right)^{\text{Ж}}-\left(\bar{a}\varsigma_j\ddot{Y}_j-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\Theta+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}+\Theta}-1\right)\right)^{1/\text{Ж}}
\end{array}\right],\right.\\
&\left.1-\left(\frac{1}{\ddot{U}+\Theta}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\varsigma_i\ddot{Y}_if_i^{\text{Ж}}-\left(\bar{a}\varsigma_i\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\Theta\left(1-\left(\bar{a}\varsigma_j\ddot{Y}_jf_j^{\text{Ж}}-\left(\bar{a}\varsigma_j\ddot{Y}_j-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\Theta+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}+\Theta}-1\right)\right)^{1/\text{Ж}}\right\rangle.
\end{aligned} \tag{71}$$

Definition 11. Let $\ddot{U},\Theta\geq 0, \mathrm{cn}_i=\langle[I_i^L,I_i^U],f_i\rangle$ $(i=1,2,\ldots,\bar{a})$ be a faction of CNs and $\varsigma=(\varsigma_1,\varsigma_2,\ldots,\varsigma_{\bar{a}})^T$ be the significant degree of cn_i $(i=1,2,\ldots,\bar{a})$, where $\varsigma_i\geq 0$ and $\sum_{i=1}^{\bar{a}}\varsigma_i=1$. The CSSWPGHM operator is rationalized as follows:

$$\mathrm{CSSWPGHM}^{\ddot{U},\Theta}(\mathrm{cn}_1,\mathrm{cn}_2,\ldots,\mathrm{cn}_{\bar{a}})=\frac{1}{\ddot{U}+\Theta}\left(\prod_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(\ddot{U}\mathrm{cn}_i^{\bar{a}\varsigma_i(1+T(\mathrm{cn}_i))/\sum_{z=1}^{\bar{a}}\varsigma_z(1+T(\mathrm{cn}_z))}\oplus_{SrSr}\Theta\mathrm{cn}_j^{\bar{a}\varsigma_j\left(1+T\left(\mathrm{cn}_j\right)\right)/\sum_{z=1}^{\bar{a}}\varsigma_z(1+T(\mathrm{cn}_z))}\right)\right)^{2/\bar{a}(\bar{a}+1)}, \tag{72}$$

Table 1: Cubic decision making matrix.

Alternative/attributes	$\overline{\overline{Cei}}_1$	$\overline{\overline{Cei}}_2$	$\overline{\overline{Cei}}_3$	$\overline{\overline{Cei}}_4$	$\overline{\overline{Cei}}_5$
$\overline{\overline{tr}}_1$	$\langle[0.3, 0.4], 0.2\rangle$	$\langle[0.4, 0.6], 0.2\rangle$	$\langle[0.3, 0.7], 0.9\rangle$	$\langle[0.2, 0.3], 0.8\rangle$	$\langle[0.4, 0.5], 0.7\rangle$
$\overline{\overline{tr}}_2$	$\langle[0.1, 0.2], 0.7\rangle$	$\langle[0.3, 0.7], 0.2\rangle$	$\langle[0.2, 0.4], 0.3\rangle$	$\langle[0.4, 0.5], 0.4\rangle$	$\langle[0.3, 0.8], 0.6\rangle$
$\overline{\overline{tr}}_3$	$\langle[0.5, 0.8], 0.3\rangle$	$\langle[0.4, 0.6], 0.8\rangle$	$\langle[0.2, 0.3], 0.4\rangle$	$\langle[0.5, 0.8], 0.5\rangle$	$\langle[0.3, 0.6], 0.8\rangle$
$\overline{\overline{tr}}_4$	$\langle[0.2, 0.4], 0.3\rangle$	$\langle[0.3, 0.8], 0.2\rangle$	$\langle[0.3, 0.5], 0.5\rangle$	$\langle[0.4, 0.5], 0.2\rangle$	$\langle[0.3, 0.4], 0.6\rangle$
$\overline{\overline{tr}}_5$	$\langle[0.8, 0.9], 0.2\rangle$	$\langle[0.4, 0.6], 0.3\rangle$	$\langle[0.2, 0.4], 0.8\rangle$	$\langle[0.3, 0.7], 0.2\rangle$	$\langle[0.2, 0.3], 0.5\rangle$

where $\ddot{U}, \text{Ɵ} \geq 0, T(cn_i) = \sum_{j=1, j\neq i}^{\bar{a}} \sup(cn_i, cn_j), \sup(cn_i, cn_j) = 1 - \overline{\overline{DNE}}(cn_i, cn_j)$, and $\overline{\overline{DNE}}(cn_i, cn_j)$ can be processed by (6).

Let $\ddot{Y}_i = (1 + T(cn_i))/\sum_{z=1}^{\bar{a}}(1 + T(cn_z))$, then (72) is correspondent to the following type:

$$\mathrm{CSSWPGHM}^{\ddot{U},\text{Ɵ}}(cn_1, cn_2, \ldots, cn_{\dagger}) = \frac{1}{\ddot{U} + \text{Ɵ}}\left(\prod_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(\ddot{U}cn_i^{\bar{a}\varsigma_i\ddot{Y}_i}\oplus_{SS}\text{Ɵ}cn_j^{\bar{a}\varsigma_j\ddot{Y}_j}\right)\right)^{2/\bar{a}(\bar{a}+1)}. \tag{73}$$

Theorem 11. *Let $\ddot{U} \geq 0$, $\text{Ɵ} \geq 0$ and $\ddot{U}, \text{Ɵ}$ take no more than one value of 0 at a time, $cn_i = \langle[I_i^L, I_i^U], f_i\rangle\ (i = 1, 2, \ldots, \bar{a})$ be a faction of CNs. Then, employing the CSSWPGHM operator, their fused values are CNs, and*

$$\begin{aligned}
&\mathrm{CSSWPGHM}^{\ddot{U},\text{Ɵ}}(cn_1, cn_2, \ldots, cn_{\bar{a}})\\
&= \Bigg\langle\Bigg[1-\left(\frac{1}{\ddot{U}+\text{Ɵ}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\varsigma_i\ddot{Y}_i\left(I_i^L\right)^{\text{Ж}}-\left(\bar{a}\varsigma_i\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ɵ}\left(1-\left(\bar{a}\varsigma_j\ddot{Y}_j\left(I_j^L\right)^{\text{Ж}}-\left(\bar{a}\varsigma_j\ddot{Y}_j-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ɵ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}+\text{Ɵ}}-1\right)\right)^{1/\text{Ж}},\\
&\quad 1-\left(\frac{1}{\ddot{U}+\text{Ɵ}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\varsigma_i\ddot{Y}_i\left(I_i^U\right)^{\text{Ж}}-\left(\bar{a}\varsigma_i\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ɵ}\left(1-\left(\bar{a}\varsigma_j\ddot{Y}_j\left(I_j^U\right)^{\text{Ж}}-\left(\bar{a}\varsigma_j\ddot{Y}_j-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ɵ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}+\text{Ɵ}}-1\right)\right)^{1/\text{Ж}}\Bigg],\\
&\quad \left(\frac{1}{\ddot{U}+\text{Ɵ}}\left(1-\left(\frac{2}{\bar{a}(\bar{a}+1)}\sum_{\substack{i,j=1\\ j=i}}^{\bar{a}}\left(1-\left(\ddot{U}\left(1-\left(\bar{a}\varsigma_i\ddot{Y}_i(1-f_i)^{\text{Ж}}-\left(\bar{a}\varsigma_i\ddot{Y}_i-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}+\text{Ɵ}\left(1-\left(\bar{a}\varsigma_j\ddot{Y}_j\left(1-f_j\right)^{\text{Ж}}-\left(\bar{a}_j\ddot{Y}_j-1\right)\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\ddot{U}-\text{Ɵ}+1\right)^{1/\text{Ж}}\right)^{\text{Ж}}\right)^{1/\text{Ж}}\right)^{\text{Ж}}-\left(\frac{1}{\ddot{U}+\text{Ɵ}}-1\right)\right)^{1/\text{Ж}}\Bigg\rangle.
\end{aligned} \tag{74}$$

The proofs of Theorems 10 and 11 are the same as Theorems 2 and 6. Therefore, here we omit their proofs.

5. An Application of SSPHM Operator to MADM

In this portion, we pertain to the aforementioned CSSPHM AOs to discover creative approaches for MADM under cubic environments. Let $\overline{\overline{Tr}} = \left\{\overline{\overline{tr}}_1, \overline{\overline{tr}}_2, \ldots, \overline{\overline{tr}}_g\right\}$ be the group of distant alternatives, the group of attributes is verbalized by $\overline{\overline{\mathrm{Cei}}} = \left\{\overline{\overline{\mathrm{Cei}}}_1, \overline{\overline{\mathrm{Cei}}}_2, \ldots, \overline{\overline{\mathrm{Cei}}}_h\right\}$, the importance degree of the attributes is epitomised by $\overline{\overline{\mathrm{Wit}}} = (\overline{\overline{\mathrm{wit}}}_1, \overline{\overline{\mathrm{wit}}}_2, \ldots, \overline{\overline{\mathrm{wit}}}_h)^T$ such that $\overline{\overline{\mathrm{wit}}}_e \in [0, 1], \sum_{e=1}^{h}\overline{\overline{\mathrm{wit}}}_e = 1$. In the procedure of decision making, the evaluation information about the alternative $\overline{\overline{tr}}_u\ (u = 1, 2, \ldots, g)$ concerning the attribute $\overline{\overline{\mathrm{Cei}}}_w\ (w = 1, 2, \ldots, h)$ is expressed by a cubic decision matrix $\overline{\overline{\mathrm{CD}}} = (cn_{de})_{g\times h}$, where $cn_{de} = \langle[\Psi^L, \Psi^U], \Upsilon\rangle$ is a CN.

Afterwards, take a chance on actual decision scenarios where the weight vector of attributes has already been determined. As a result, we launch MADM methods based on the suggested CSSPHA operators.

5.1. MADM with Known Weight Vectors of Attributes. In this section, to pact with real decision circumstances in which the importance degrees of attributes are known in advance, we apply the CSSWPHM operator and CSSWPGHM operator to launch the following approach to solve MADM problems under cubic environments. To do so, instantly follow the steps given:

Step 1. Locate support $\mathrm{Sup}(\mathrm{cn}_{de}, \mathrm{cn}_{dx})$ by utilizing the following formula:

$$\mathrm{Sup}(\mathrm{cn}_{de}, \mathrm{cn}_{ex}) = 1 - \overline{\overline{\mathrm{DNE}}}(\mathrm{cn}_{de}, \mathrm{cn}_{ex}), \ (d = 1, 2, \ldots, g, e = 1, 2, \ldots, h, e \neq x), \tag{75}$$

where $\overline{\overline{\mathrm{DNE}}}(\mathrm{cn}_{de}, \mathrm{cn}_{ex})$ is the distance measure and is intended by utilizing (6).

Step 2. Discover the weighted support degree $T(\mathrm{cn}_{de})$ that CN cn_{de} collects from other CNs $\overline{\overline{\mathrm{cn}}}_{dx}$ $(x = 1, 2, \ldots, h; e \neq x)$,

$$T(\mathrm{cn}_{de}) = \sum_{x=1, x\neq e}^{h} \mathrm{sup}(\mathrm{cn}_{de}, \mathrm{cn}_{dx}). \tag{76}$$

Step 3. Establish a weighting vector Ξ_{de} $(d = 1, 2, \ldots, g, e = 1, 2, \ldots, h)$ associated with cn_{de} ,

$$\Xi_{de} = \frac{g\overline{\overline{\mathrm{wit}}}_e (1 + T(\mathrm{cn}_{de}))}{\sum_{e=1}^{h} \overline{\overline{\mathrm{wit}}}_e (1 + T(\mathrm{cn}_{de}))}. \tag{77}$$

Step 4. Employ CSSWPHOM or CSSWPGHOM operators to aggregate the evaluation values cn_{de} $(d = 1, 2, \ldots, g, e = 1, 2, \ldots, h)$ into an overall evaluation value cn_d $(d = 1, 2, \ldots, g)$ matching the alternatives $\overline{\overline{tr}}_d$ $(d = 1, 2, \ldots, g)$;

$$\mathrm{cn}_d = \mathrm{CSSWPHOM}_{\zeta, \overline{\overline{\mathrm{wit}}}}(d_1, \mathrm{cn}_{d2}, \ldots, \mathrm{cn}_{ds}), \tag{78}$$

or

$$\mathrm{cn}_d = \mathrm{CSSWPGHOM}_{\zeta, \overline{\overline{\mathrm{We}}}}(\mathrm{cn}_{d1}, \mathrm{cn}_{d2}, \ldots, \mathrm{cn}_{ds}). \tag{79}$$

Step 5. Locate the scores $\overline{\overline{\mathrm{SoF}}}(\mathrm{cn}_d)$ for the overall CNs of the alternatives $\overline{\overline{tr}}_d$ $(d = 1, 2, \ldots, g)$ by manipulating Definition 2.

Step 6. Rank all alternatives $\overline{\overline{tr}}_d$ $(d = 1, 2, \ldots, g)$ and select the best one (s) with the ranking order cn_d $(d = 1, 2, \ldots, g)$.

5.2. Illustrative Example. In this subpart, a numerical example adapted from [26] about enterprise resource planning in order to verify the unassailability and compensations of the initiated approach.

Let us say a corporation decides to use an ERP system (enterprise resource planning). The specialist's panel chose $\overline{\overline{tr}}_d$ $(g = 1, 2, 3, 4, 5)$ five prospective investors after gathering all relevant information on ERP dealers and systems. Some external decision-making specialists are among the organization's members. The group decides on five attributes $\overline{\overline{\mathrm{Cei}}}_e$ $(e = 1, 2, 3, 4, 5)$. To assess the alternatives, (1) function and technology $\overline{\overline{\mathrm{Cei}}}_1$, (2) strategic fitness $\overline{\overline{\mathrm{Cei}}}_2$, (3) the ability of the vendor $\overline{\overline{\mathrm{Cei}}}_3$, (4) reputation of the vendor $\overline{\overline{\mathrm{Cei}}}_4$, and (5) growth analysis of the vendor $\overline{\overline{\mathrm{Cei}}}_5$, with weight vectors of the attributes are $(0.2, 0.15, 0.15, 0.25, 0.25)^T$. CFN's will be used by the expert's committee to create the initial decision matrix given in Table 1. To solve this decision making, the following steps to be followed:

Step 1. Analyse the support $\mathrm{Sup}(\mathrm{cn}_{de}^{d}, \mathrm{cn}_{dx}^{d})$ by the following formula (61), and we have

$$
\begin{aligned}
S_{12}^{1} &= S_{21}^{1} = 0.9000,\\
S_{13}^{1} &= S_{31}^{1} = 0.6667,\\
S_{14}^{1} &= S_{41}^{1} = 0.7333,\\
S_{15}^{1} &= S_{51}^{1} = 0.7667,\\
S_{23}^{1} &= S_{32}^{1} = 0.7000,\\
S_{24}^{1} &= S_{42}^{1} = 0.6333,\\
S_{25}^{1} &= S_{52}^{1} = 0.8000,\\
S_{34}^{1} &= S_{43}^{1} = 0.8333,\\
S_{35}^{1} &= S_{53}^{1} = 0.8667,\\
S_{45}^{1} &= S_{54}^{1} = 0.8333,\\
S_{12}^{2} &= S_{21}^{2} = 0.6000,\\
S_{13}^{2} &= S_{31}^{2} = 0.7667,\\
S_{14}^{2} &= S_{41}^{2} = 0.7000,\\
S_{15}^{2} &= S_{51}^{2} = 0.7000,\\
S_{23}^{2} &= S_{32}^{2} = 0.8333,\\
S_{24}^{2} &= S_{42}^{2} = 0.8333,\\
S_{25}^{2} &= S_{52}^{2} = 0.8333,\\
S_{34}^{2} &= S_{43}^{2} = 0.8667,\\
S_{35}^{2} &= S_{53}^{2} = 0.7333,\\
S_{45}^{2} &= S_{54}^{2} = 0.8000,\\
S_{12}^{3} &= S_{21}^{3} = 0.7333,\\
S_{13}^{3} &= S_{31}^{3} = 0.7000,\\
S_{14}^{3} &= S_{41}^{3} = 0.9333,\\
S_{15}^{3} &= S_{51}^{3} = 0.7000,\\
S_{23}^{3} &= S_{32}^{3} = 0.7000,\\
S_{24}^{3} &= S_{42}^{3} = 0.8333,\\
S_{25}^{3} &= S_{52}^{3} = 0.9667,\\
S_{34}^{3} &= S_{43}^{3} = 0.7333,\\
S_{35}^{3} &= S_{53}^{3} = 0.7333,\\
S_{45}^{3} &= S_{54}^{3} = 0.8000,\\
S_{12}^{4} &= S_{21}^{4} = 0.8000,\\
S_{13}^{4} &= S_{31}^{4} = 0.8667,\\
S_{14}^{4} &= S_{41}^{4} = 0.8667,\\
S_{15}^{4} &= S_{51}^{4} = 0.8667,\\
S_{23}^{4} &= S_{32}^{4} = 0.8000,\\
S_{24}^{4} &= S_{42}^{4} = 0.8667,\\
S_{25}^{4} &= S_{52}^{4} = 0.7333,\\
S_{34}^{4} &= S_{43}^{4} = 0.8667,\\
S_{35}^{4} &= S_{53}^{4} = 0.9333,\\
S_{45}^{4} &= S_{54}^{4} = 0.8000,\\
S_{12}^{5} &= S_{21}^{5} = 0.7333,\\
S_{13}^{5} &= S_{31}^{5} = 0.4333,\\
S_{14}^{5} &= S_{41}^{5} = 0.7667,\\
S_{15}^{5} &= S_{51}^{5} = 0.5,\\
S_{23}^{5} &= S_{32}^{5} = 0.7000,\\
S_{24}^{5} &= S_{42}^{5} = 0.9000,\\
S_{25}^{5} &= S_{52}^{5} = 0.7667,\\
S_{34}^{5} &= S_{43}^{5} = 0.6667,\\
S_{35}^{5} &= S_{53}^{5} = 0.8667,\\
S_{45}^{5} &= S_{54}^{5} = 0.7333.
\end{aligned} \tag{80}
$$

Step 2. Exploiting Equation (62), to discover the weighted support $T(\mathrm{cn}_{de})$ that CN cn_{de} collects from other CNs $\overline{\overline{\mathrm{cn}}}_{dx}\,(d, x = 1, 2, \ldots, 5; e \neq x)$.,. For simplicity, we indicate $T(\mathrm{cn}_{de})$ by T_{de}, we have

$$
\begin{aligned}
T_{11} &= 3.0667,\\
T_{12} &= 3.0333,\\
T_{13} &= 3.0000,\\
T_{14} &= 3.0000,\\
T_{15} &= 3.2333,\\
T_{21} &= 2.7667,\\
T_{22} &= 3.1000,\\
T_{23} &= 3.2000,\\
T_{24} &= 3.2000,\\
T_{25} &= 3.0667,\\
T_{31} &= 3.0667,\\
T_{32} &= 3.2000,\\
T_{33} &= 2.8333,\\
T_{34} &= 3.2000,\\
T_{35} &= 3.1667,\\
T_{41} &= 3.4000,\\
T_{42} &= 3.2000,\\
T_{43} &= 3.4667,
\end{aligned}
$$

$$
\begin{aligned}
T_{44} &= 3.4000,\\
T_{45} &= 3.3333,\\
T_{51} &= 2.4333,\\
T_{52} &= 3.1000,\\
T_{53} &= 2.6667,\\
T_{54} &= 3.0667,\\
T_{55} &= 2.8667.
\end{aligned}
\tag{81}
$$

Step 3. Manipulating Equation (63), to locate the weight $\Xi_{de}\,(d = 1, \ldots., 5, e = 1, \ldots, 5)$, we have

$$
\begin{aligned}
\Xi_{11} &= 0.9975,\\
\Xi_{12} &= 0.7420,\\
\Xi_{13} &= 0.7359,\\
\Xi_{14} &= 1.2265,\\
\Xi_{15} &= 0.0173,\\
\Xi_{21} &= 0.9266,\\
\Xi_{22} &= 0.7565,\\
\Xi_{23} &= 0.7749,\\
\Xi_{24} &= 1.2915,\\
\Xi_{25} &= 0.0167,\\
\Xi_{31} &= 0.9895,\\
\Xi_{32} &= 0.7664,\\
\Xi_{33} &= 0.6995,\\
\Xi_{34} &= 1.2774,\\
\Xi_{35} &= 1.2672,\\
\Xi_{41} &= 1.0084,\\
\Xi_{42} &= 0.7219,\\
\Xi_{43} &= 0.7678,\\
\Xi_{44} &= 1.2605,\\
\Xi_{45} &= 1.2414,\\
\Xi_{51} &= 0.8953,\\
\Xi_{52} &= 0.8018,\\
\Xi_{53} &= 0.7171,\\
\Xi_{54} &= 1.3255,\\
\Xi_{55} &= 1.2603.
\end{aligned}
\tag{82}
$$

Step 4Manipulating Equation (64) or (65), to locate the overall assessment values of each alternative, we have $(\ddot{U} = 2, \Theta = 2, Ж = -2)$

$$
\begin{aligned}
\overline{\overline{tr_1}} &= \langle [0.2063, 0.3921], 0.5230\rangle,\\
\overline{\overline{tr_2}} &= \langle [0.1599, 0.3776], 0.4751\rangle,\\
\overline{\overline{tr_3}} &= \langle [0.3693, 0.6628], 0.5562\rangle,\\
\overline{\overline{tr_4}} &= \langle [0.2892, 0.5329], 0.3297\rangle,\\
\overline{\overline{tr_5}} &= \langle [0.4041, 0.6646], 0.3415\rangle\\
\text{Or}\ \overline{\overline{tr_1}} &= \langle [0.3866, 0.5985], 0.6196\rangle,\\
\overline{\overline{tr_2}} &= \langle [0.2869, 0.5297], 0.6201\rangle,\\
\overline{\overline{tr_3}} &= \langle [0.3697, 0.6298], 0.6091\rangle,\\
\overline{\overline{tr_4}} &= \langle [0.2973, 0.5170], 0.3474\rangle,\\
\overline{\overline{tr_5}} &= \langle [0.3292, 0.5763], 0.4192\rangle.
\end{aligned}
\tag{83}
$$

Step 5. Operating Equation (6) given in Definition 2, to locate the score values of overall $\overline{\overline{tr_d}}\,(d = 1, \ldots, 5)$ CNs, to rank the alternatives, we have

$$
\begin{aligned}
\overline{\overline{\mathrm{SoF}}}\left(\overline{\overline{tr_1}}\right) &= 0.0251,\\
\overline{\overline{\mathrm{SoF}}}\left(\overline{\overline{tr_2}}\right) &= 0.0208,\\
\overline{\overline{\mathrm{SoF}}}\left(\overline{\overline{tr_3}}\right) &= 0.1586,\\
\overline{\overline{\mathrm{SoF}}}\left(\overline{\overline{tr_4}}\right) &= 0.1641,\\
\overline{\overline{\mathrm{SoF}}}\left(\overline{\overline{tr_5}}\right) &= 0.2424.\\
\text{Or}\ \overline{\overline{\mathrm{SoF}}}\left(\overline{\overline{tr_1}}\right) &= 0.1218,\\
\overline{\overline{\mathrm{SoF}}}\left(\overline{\overline{tr_2}}\right) &= 0.0655,\\
\overline{\overline{\mathrm{SoF}}}\left(\overline{\overline{tr_3}}\right) &= 0.1301,\\
\overline{\overline{\mathrm{SoF}}}\left(\overline{\overline{tr_4}}\right) &= 0.1556,\\
\overline{\overline{\mathrm{SoF}}}\left(\overline{\overline{tr_5}}\right) &= 0.1621.
\end{aligned}
\tag{84}
$$

Step 6. According to the score values, ranking order of the alternatives $\overline{\overline{tr_d}}\,(d = 1, 2, \ldots, 5)$ is as follows:

$$
\begin{aligned}
&\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_1}} > \overline{\overline{tr_2}}\\
&\text{or}\ \overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_1}} > \overline{\overline{tr_2}}.
\end{aligned}
\tag{85}
$$

Therefore, according to the ranking order, the best alternative is $\overline{\overline{tr_5}}$, while the worst one is $\overline{\overline{tr_2}}$.

5.3. Impact of the Parameter Ж on Final Ranking Orders Applying CSrSrWPHM and CSrSrWPGHM Operators. In this subportion, the impact of the parameter Ж on last ranking orders employing CSSWPHM and CSSWPGHM operators is explored, and the value of parameters $\ddot{U} = \Theta = 2$ are permanent. For distinct values of the parameter Ж, the score values and ranking orders while operating CSSWPHM and CSSWPGHM operators are given in Table 2. One can observe from Table 2 that for different values of the parameter Ж the ranking orders are different. That is,

Table 2: Effect of the parameter Ж on final ranking order.

Parameter	CSSWPHM operator	CSSWPGHM operator	Ranking order
Ж = −3	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = 0.0558, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = 0.0461, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = 0.1662, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.1694, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = 0.2681.$	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = 0.0847, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = 0.0226, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = 0.1203, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.1526, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = 0.1453.$	$\overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_1}}} > \overline{\overline{\underline{tr_2}}}$ or $\overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_1}}} > \overline{\overline{\underline{tr_2}}}$
Ж = −4	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = 0.0882, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = 0.0741, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = 0.1788, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.1782, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = 0.3019.$	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = 0.0505, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = -0.0161, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = 0.1052, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.1454, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = 0.1196.$	$\overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_1}}} > \overline{\overline{\underline{tr_2}}}$ or $\overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_1}}} > \overline{\overline{\underline{tr_2}}}$
Ж = −7	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = 0.1557, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = 0.1420, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = 0.2193, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.2158, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = 0.3889.$	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = -0.0154, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = -0.0914, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = 0.0475, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.1166, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = 0.0394.$	$\overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_1}}} > \overline{\overline{\underline{tr_2}}}$ or $\overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_1}}} > \overline{\overline{\underline{tr_2}}}$
Ж = −10	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = 0.1922, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = 0.1885, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = 0.2490, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.2460, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = 0.4301.$	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = -0.0502, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = -0.1291, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = 0.0007, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.0909, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = -0.0079.$	$\overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_1}}} > \overline{\overline{\underline{tr_2}}}$ or $\overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_1}}} > \overline{\overline{\underline{tr_2}}}$
Ж = −20	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = 0.2426, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = 0.2612, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = 0.2939, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.2865, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = 0.4717.$	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = -0.0964, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = -0.1709, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = -0.0581, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.0455, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = -0.0615.$	$\overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_1}}} > \overline{\overline{\underline{tr_2}}}$ or $\overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_1}}} > \overline{\overline{\underline{tr_2}}}$
Ж = −35	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = 0.2677, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = 0.2954, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = 0.3128, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.3070, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = 0.4860.$	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = -0.1148, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = -0.1856, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = -0.0788, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.0238, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = -0.0806.$	$\overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_1}}} > \overline{\overline{\underline{tr_2}}}$ or $\overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_1}}} > \overline{\overline{\underline{tr_2}}}$
Ж = −50	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = 0.2780, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = 0.3079, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = 0.3196, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.3155, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = 0.4908.$	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = -0.1212, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = -0.1907, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = -0.0859, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.0159, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = -0.0872.$	$\overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_2}}} > \overline{\overline{\underline{tr_1}}}$ or $\overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_1}}} > \overline{\overline{\underline{tr_2}}}$
Ж = −100	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = 0.2895, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = 0.3213, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = 0.3268, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.3248, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = 0.4958.$	$\overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_1}}}) = -0.1277, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_2}}}) = -0.1958, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_3}}}) = -0.0934, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_4}}}) = 0.0075, \overline{\overline{\underline{\mathrm{SoF}}}}(\overline{\overline{\underline{tr_5}}}) = -0.0940.$	$\overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_2}}} > \overline{\overline{\underline{tr_1}}}$ or $\overline{\overline{\underline{tr_4}}} > \overline{\overline{\underline{tr_3}}} > \overline{\overline{\underline{tr_5}}} > \overline{\overline{\underline{tr_1}}} > \overline{\overline{\underline{tr_2}}}$

Table 3: Effect of the parameter Ü, Ѳ on final ranking order.

Parameter	CSSWPHM operator	CSSWPGHM operator	Ranking order
Ü = 2, Ѳ = 3	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.0170, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.0176, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1532, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1639, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.2316.$	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.1285, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.0802, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1281, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1544, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.1603.$	$\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}} > \overline{\overline{tr_1}}$ or $\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_1}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}}$
Ü = 3, Ѳ = 5	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.0041, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.0108, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1473, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1614, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.2185.$	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.1458, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.1010, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1316, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1565, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.1665.$	$\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}} > \overline{\overline{tr_1}}$ or $\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_1}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}}$
Ü = 5, Ѳ = 3	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.0096, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.0104, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1556, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1603, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.2337.$	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.1502, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.0867, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1418, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1622, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.1810.$	$\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}} > \overline{\overline{tr_1}}$ or $\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_1}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}}$
Ü = 7, Ѳ = 15	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = -0.0166, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.0011, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1379, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1580, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.1972.$	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.1743, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.1366, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1350, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1586, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.1737.$	$\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}} > \overline{\overline{tr_1}}$ or $\overline{\overline{tr_4}} > \overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_2}} > \overline{\overline{tr_3}}$
Ü = 15, Ѳ = 7	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = -0.0073, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = -0.0009, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1500, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1570, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.2216.$	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.1813, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.1169, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1514, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1672, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.1973.$	$\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}} > \overline{\overline{tr_1}}$ or $\overline{\overline{tr_4}} > \overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_2}} > \overline{\overline{tr_3}}$
Ü = 12, Ѳ = 25	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = -0.0237, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = -0.0033, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1346, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1559, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.1904.$	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.1863, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.1493, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1384, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1606, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.1789.$	$\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}} > \overline{\overline{tr_1}}$ or $\overline{\overline{tr_4}} > \overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_2}} > \overline{\overline{tr_3}}$
Ü = 25, Ѳ = 12	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = -0.0144, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = -0.0056, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1462, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1551, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.2142.$	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.1931, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.1308, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1545, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1689, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.2023.$	$\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}} > \overline{\overline{tr_1}}$ or $\overline{\overline{tr_5}} > \overline{\overline{tr_1}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}}$
Ü = 30, Ѳ = 50	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = -0.0307, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = -0.0097, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1311, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1526, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.1846.$	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.2015, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.1623, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1455, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1648, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.1893.$	$\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}} > \overline{\overline{tr_1}}$ or $\overline{\overline{tr_1}} > \overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_2}} > \overline{\overline{tr_3}}$

Table 4: Comparison with existing approaches.

Approaches	Score values	Ranking orders
CWA [12]	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.1098, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.1447, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1850, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1692, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.2549$	$\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}} > \overline{\overline{tr_1}}$
CWG [12]	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.0277, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.0747, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1225, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1309, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.1418$	$\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}} > \overline{\overline{tr_1}}$
CEWA [26]	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.0962, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.1348, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1762, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1635, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.2378$	$\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}} > \overline{\overline{tr_1}}$
CFWHMDA [26]	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.0029, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.0735, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.0077, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.574, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.0718$	$\overline{\overline{tr_2}} > \overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_1}}$
CFWGHMDA [26]	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.1210, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.1635, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.2144, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1574, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.2784$	$\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_2}} > \overline{\overline{tr_1}}$
Proposed AOs	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.0251, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.0208, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1586, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1641, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.2424$	$\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_1}} > \overline{\overline{tr_2}}$
Proposed AOs	$\overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_1}}) = 0.1218, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_2}}) = 0.0655, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_3}}) = 0.1301, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_4}}) = 0.1556, \overline{\overline{\mathrm{SoF}}}(\overline{\overline{tr_5}}) = 0.1621$	$\overline{\overline{tr_5}} > \overline{\overline{tr_4}} > \overline{\overline{tr_3}} > \overline{\overline{tr_1}} > \overline{\overline{tr_2}}$

employing the CSSWPHM operator and CSSWPGHM operator the best alternative is either $\overline{\overline{tr}}_5$ or $\overline{\overline{tr}}_4$ for different values of Ж while the worst alternative is either $\overline{\overline{tr}}_2$ or $\overline{\overline{tr}}_1$. One can also see from Table 2 that when the values of the parameter Ж decrease, the score values of the alternative $\overline{\overline{tr}}_g (g = 1, \ldots, 5)$ increases while exploiting the CSSWPHM operator. Similarly, from Table 2, when the values of the parameter Ж decrease, the score values of the alternatives $\overline{\overline{tr}}_g (g = 1, \ldots, 5)$ decreases.

5.4. Effect of the Parameters Ü, ϴ on Final Ranking Order. In this subsegment, the effect of the parameters Ü, ϴ on last ranking orders operating CSSWPHM and CSSWPGHM operators are inspected, and the value of parameter Ж $= -2$ are permanent. For distinct values of the parameters A, B, the score values and ranking orders while operating CSSWPHM and CSSWPGHM operators are quantified in Table 3. From Table 3, one can observe that when the values of the parameters Ü, ϴ are different, the ranking orders obtained are different. That is, to employ the CSSWPHM operator and CSSWPGHM operator, the best alternative is either $\overline{\overline{tr}}_5$ or $\overline{\overline{tr}}_4$ or $\overline{\overline{tr}}_1$ for different values of the parameters Ü, ϴ, while the worst alternative is either $\overline{\overline{tr}}_2$ or $\overline{\overline{tr}}_1$ or $\overline{\overline{tr}}_3$. One can also see from Table 3 that when the values of the parameters Ü, ϴ increase, the score values of the alternatives $\overline{\overline{tr}}_g (g = 1, \ldots, 5)$ decrease, while exploiting the CSSWPHM operator. Similarly, from Table 3, when the values of the parameter Ü, ϴ decrease, the score values of the alternatives $\overline{\overline{tr}}_g (g = 1, \ldots, 5)$ increase, while exploiting the CSSWPHM operator. The basic reason for this is because the above AOs are more adjustable since they are constituted of generic parameters, limit the influence of inconvenient information, and take into account the relationship between input information. As a result, the MADM model developed on these aggregation operators is more adaptable. As a consequence, the decision-maker may modify the values of these parameters to the specific requirements of the scenario.

5.5. Comparison with Existing Approaches. In this section, the developed MADM model, which is predicated on this newly established novel AGOs, to various current approaches, namely, Mahmood et al. [12], Ayub et al. [24], and Fahmi et al. [26] MADM models. The comparison between these approaches and the proposed approach is given in Table 4. From Table 4, we can observe that the best alternative obtained from the existing approaches and the proposed approach is the same except for utilizing the CWHMDA operator, while the worst alternative obtained from the existing approaches and the proposed approach is different. The MADM model that will be implemented is based on the recently launched aggregation operators. To put it another way, these aggregation operators are suggested for CNs using Schweizer–Sklar operational, which are generic parameters that make the decision-making procedure more adaptable. In the meanwhile, existing MADM models are drawn on aggregation operators that are launched using algebraic operational laws or Dombi operational laws. The aggregation operator proposed by Mahmood et al. [12] is a simple weighted aggregation operator, which does not have the capacity of taking interrelationships or removing the effect of awkward data from the final ranking results. While, the aggregation operators developed by Ayub et al. [24], are drawn on Dombi operational laws, which have the capacity of tinterrelationships among input arguments and also consist of a generic parameter. The aggregation operators developed by Fahmi et al. [26] are simple weighted averaging operators based on Einstein operational laws for CNs. These aggregation operators do not have the characteristic of taking interrelationship among input arguments or removing the effect of awkward data from the final ranking results. Up to now, the existing aggregation operators for CNs have only the capacity of considering interrelationships among input arguments and cannot remove the effect of awkward data from the final ranking results. Whereas, anticipated aggregation operators will be able to remove the effect of awkward data while also taking into account the interrelationship between the input data at the same time. The predicted AOs also have the benefit of including generic parameters, which sorts the decision-making procedure more supple. As a result, while solving MADM models using cubic information, the initiated AOs are more practical and comparable in their application.

6. Conclusion

The evaluation of the ERP (enterprise resource planning) system is one of the numerous applications of MADM. The goal of this article is to introduce a cubic set-based decision support as a practical way to explain ambiguity, reluctance, and uncertainty. This article makes a four-fold contribution. Firstly, inimitable Schweizer–Sklar operational rules for CNs are developed, and some of their key characteristics are examined. Secondly, using these inimitable Schweizer–Sklar operational laws, some CSSPHM operators are discussed, including the cubic Schweizer–Sklar power Heronian mean operator, the cubic Schweizer–Sklar power geometric Heronian mean operator, the cubic Schweizer–Sklar power weighted Heronian mean operator, and the cubic Schweizer–Sklar power weighted geometric Heronian mean operator, as well as their vital properties. We can see that some of the existing AOs are special instances of these freshly launched AOs by supplying particular values to the generic parameters. These AOs offer benefits over current AOs. While the current aggregate operators for CSS can only consider interrelationships among input data, the initiated AOs can remove the effect of awkward data, examine the interrelationship among the input data, and also have a general parameter at the same time. Finally, a MADM model is expected based on these AOs. The suggested technique is supported by numerical examples from enterprise resource planning. We also investigate the impact of the decision's outcome using the recently released cubic fuzzy Schweizer–Sklar power Heronian mean aggregation operators. Then, we compare our work to that of others and also discussed the advantages of the proposed aggregation operators.

In future, we will define Schweizer–Sklar operational laws for trapezoidal intuitionistic fuzzy numbers, dual hesitant fuzzy soft sets [41], Neutrosophic cubic sets [30], complex intuitionistic fuzzy sets [42], and extend several aggregation operators such as power average [27–29], robust aggregation operators [43], Choquet Integral for Spherical Fuzzy Sets [44], initiated for these structures and initiate some MADM models and apply these models to solve MADM problem under the said structure. We will also apply the anticipated approach to some new applications, such as detecting hate speech in social media [42], public transportation, technologies selection [45], traffic control, the digital twin model, and so on, or extend the anticipated model to some more extended form of CSS.

Data Availability

Data sharing does not apply to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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