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Research Article

A Flexible Extension of Reduced Kies Distribution: Properties, Inference, and Applications in Biology

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The extended reduced Kies distribution (ExRKD), which is an asymmetric flexible extension of the reduced Kies distribution, is the subject of this research. Some of its most basic mathematical properties are deduced from its formal definitions. We computed the ExRKD parameters using eight well-known methods. A full simulation analysis was done that allows the study of these estimators' asymptotic behavior. The efficiency and applicability of the ExRKD are investigated via the modeling of COVID-19 and milk data sets, which demonstrates that the ExRKD delivers a better match to the data sets when compared to competing models.

1. Introduction

In order to emulate real-world applications, several authors have proposed ways for adding shape characteristics to standard distributions. In particular, statisticians and data modelers are highly interested in the new distributions that are formed by the extra parameter of a good generator(s). Deriving distributions with limit support is becoming an increasingly popular topic of study among researchers. The necessity for modeling and analyzing bounded data may be seen in a variety of real-world domains, including medicine, politics, and psychology. There are a variety of natural and artificial phenomena that are quantified in terms of indices, percentages, proportions, rates, and ratios. All of these measurements are circumscribed by a certain interval, which is typically the unit interval. According to the literature, a number of distributions with support on the unit interval have been presented, which are derived from transformations of the cumulative distribution function (CDF). Many writers have developed and examined unit distributions such as unit Teissier distribution [1], exponentiated Top-p–Leone distribution [2], log-Lindley distribution [3], transformed gamma distribution [4], Cosine-sine distribution [5], power log-Lindley distribution, power logarithmic distribution and unit Weibull distribution [6, 7, 8], respectivly.

In 2013, Kumar and Dharmaja [9] proposed the reduced Kies distribution (RKD) with CDF and probability density function (PDF) defined, respectively, as follows:

$$F(x) = 1 - e^{-(x/1-x)^{\alpha}}, \; 0 < x < 1,$$
$$f(x) = \frac{\alpha e^{-(x/1-x)^{\alpha}} (x/1-x)^{\alpha}}{(1-x)x}. \tag{1}$$

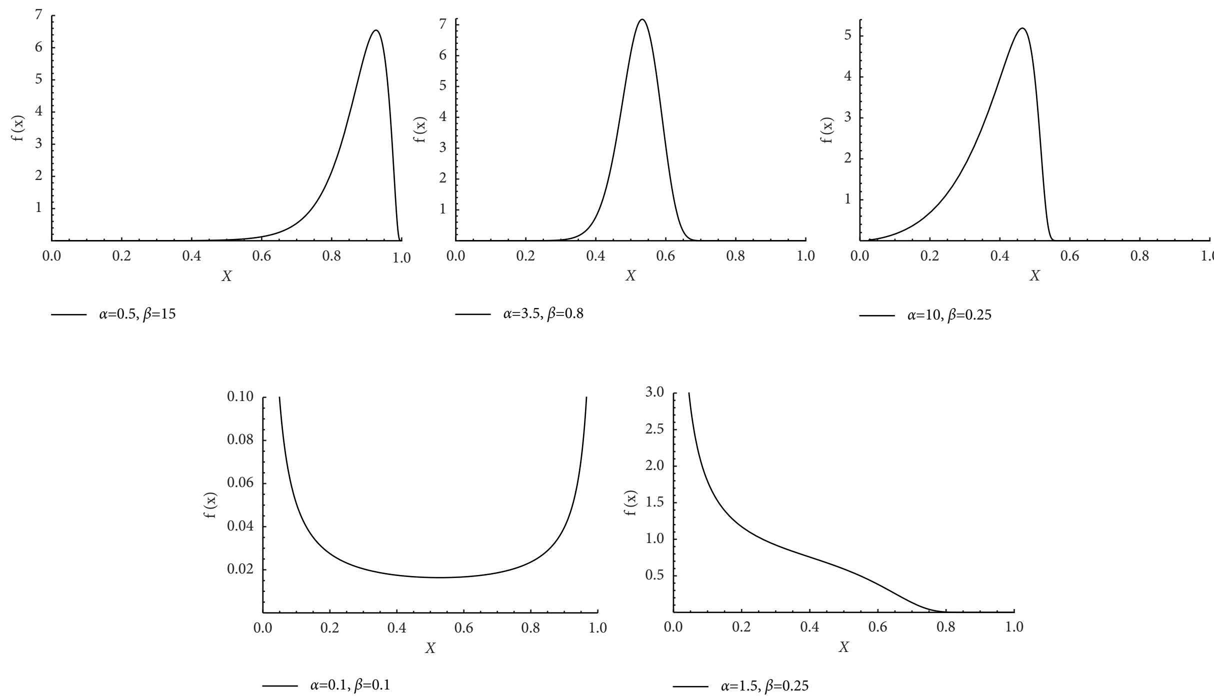

Figure 1: Plots of the PDF of the ExRKD with various parametric values.

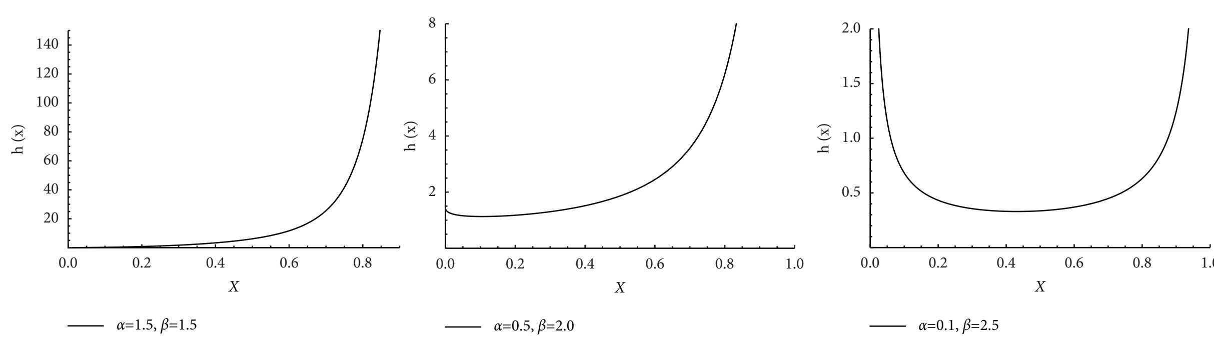

Figure 2: Plots of the HF of the ExRKD with various parametric values.

Our motivation for developing a new flexible version of the RKD, which we refer to as the extended reduced Kies distribution (ExRKD), is to provide greater flexibility in fitting real-world data sets. Furthermore, the article's main objectives are as follows: The first objective is dedicated to the investigation of a novel version of the RKD based on the weighted-G (WD-G) family [10], which is referred to as the ExRKD. The essential distributional characteristics of the ExRKD are thoroughly studied. The ExRKD is endowed with a variety of favorable characteristics. According to the following section, it is capable of handling a variety of PDF and hazard function(HF) shapes. The second goal is to investigate the estimate of the ExRKD parameters using conventional techniques. As a result, eight different estimating methodologies are used for this objective. On the basis of empirical simulation findings, we investigate and rate the performance of various estimators in order to establish a guideline for selecting the most appropriate estimation technique for estimating the ExRKD parameters.

The rest of the paper is divided into six sections, which are as follows: In Section 2, we define the ExRKD and plot its PDF and HF. Section 3 delves into the fundamental characteristics of ExRKD. In Section 4, the parameters of the model are determined using traditional estimation methods. The findings of the simulation are presented in Section 5. Section 6 is dedicated to the analysis of real-world data. Finally, in Section 7, some concluding observations are presented.

2. Formulation of ExRK Distribution

In this section, we define the CDF and the PDF of ExRK distribution, respectively, by using the WD-G family of [10] and CDF in equation (1) as follows:

Table 1: Simulation values of BIAS, MSE, and MRE for $\alpha = 0.25, \beta = 0.75$.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE
25	BIAS	$\hat{\alpha}$	$0.081622^{\{4\}}$	$0.065258^{\{3\}}$	$0.122917^{\{8\}}$	$0.053236^{\{1\}}$	$0.090325^{\{7\}}$	$0.084864^{\{6\}}$	$0.062003^{\{2\}}$	$0.082455^{\{5\}}$
		$\hat{\beta}$	$0.16451^{\{1\}}$	$0.187985^{\{4\}}$	$0.215351^{\{6\}}$	$0.167454^{\{2\}}$	$0.216096^{\{7\}}$	$0.217825^{\{8\}}$	$0.185377^{\{3\}}$	$0.195407^{\{5\}}$
	MSE	$\hat{\alpha}$	$0.009973^{\{2\}}$	$0.014156^{\{3\}}$	$0.070766^{\{8\}}$	$0.007123^{\{1\}}$	$0.030151^{\{7\}}$	$0.015345^{\{5\}}$	$0.01441^{\{4\}}$	$0.028916^{\{6\}}$
		$\hat{\beta}$	$0.04479^{\{1\}}$	$0.057887^{\{4\}}$	$0.074164^{\{7\}}$	$0.046973^{\{2\}}$	$0.073642^{\{6\}}$	$0.078046^{\{8\}}$	$0.055738^{\{3\}}$	$0.060312^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.326489^{\{4\}}$	$0.261031^{\{3\}}$	$0.491669^{\{8\}}$	$0.212945^{\{1\}}$	$0.3613^{\{7\}}$	$0.339455^{\{6\}}$	$0.248012^{\{2\}}$	$0.329821^{\{5\}}$
		$\hat{\beta}$	$0.219347^{\{1\}}$	$0.250647^{\{4\}}$	$0.287134^{\{6\}}$	$0.223272^{\{2\}}$	$0.288128^{\{7\}}$	$0.290433^{\{8\}}$	$0.247169^{\{3\}}$	$0.260543^{\{5\}}$
	$\sum$ ranks		$13^{\{2\}}$	$21^{\{4\}}$	$43^{\{8\}}$	$9^{\{1\}}$	$41^{\{6.5\}}$	$41^{\{6.5\}}$	$17^{\{3\}}$	$31^{\{5\}}$
50	BIAS	$\hat{\alpha}$	$0.053746^{\{7\}}$	$0.039116^{\{3\}}$	$0.058017^{\{8\}}$	$0.032691^{\{1\}}$	$0.050696^{\{5\}}$	$0.05304^{\{6\}}$	$0.038016^{\{2\}}$	$0.046704^{\{4\}}$
		$\hat{\beta}$	$0.10746^{\{1\}}$	$0.126549^{\{3\}}$	$0.148825^{\{7\}}$	$0.110671^{\{2\}}$	$0.144277^{\{6\}}$	$0.159685^{\{8\}}$	$0.126768^{\{4\}}$	$0.133584^{\{5\}}$
	MSE	$\hat{\alpha}$	$0.005371^{\{5\}}$	$0.003024^{\{3\}}$	$0.010234^{\{8\}}$	$0.002024^{\{1\}}$	$0.006508^{\{7\}}$	$0.004869^{\{4\}}$	$0.002743^{\{2\}}$	$0.005481^{\{6\}}$
		$\hat{\beta}$	$0.019585^{\{1\}}$	$0.027016^{\{4\}}$	$0.035223^{\{7\}}$	$0.021115^{\{2\}}$	$0.033684^{\{6\}}$	$0.040851^{\{8\}}$	$0.024993^{\{3\}}$	$0.028816^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.214985^{\{7\}}$	$0.156463^{\{3\}}$	$0.232068^{\{8\}}$	$0.130764^{\{1\}}$	$0.202786^{\{5\}}$	$0.212161^{\{6\}}$	$0.152064^{\{2\}}$	$0.186816^{\{4\}}$
		$\hat{\beta}$	$0.14328^{\{1\}}$	$0.168731^{\{3\}}$	$0.198433^{\{7\}}$	$0.147561^{\{2\}}$	$0.19237^{\{6\}}$	$0.212914^{\{8\}}$	$0.169024^{\{4\}}$	$0.178112^{\{5\}}$
	$\sum$ ranks		$22^{\{4\}}$	$19^{\{3\}}$	$45^{\{8\}}$	$9^{\{1\}}$	$35^{\{6\}}$	$40^{\{7\}}$	$17^{\{2\}}$	$29^{\{5\}}$
75	BIAS	$\hat{\alpha}$	$0.050815^{\{8\}}$	$0.028764^{\{2\}}$	$0.04524^{\{7\}}$	$0.024581^{\{1\}}$	$0.041811^{\{5\}}$	$0.044249^{\{6\}}$	$0.029942^{\{3\}}$	$0.035437^{\{4\}}$
		$\hat{\beta}$	$0.084779^{\{2\}}$	$0.09661^{\{3\}}$	$0.121307^{\{6\}}$	$0.082531^{\{1\}}$	$0.127078^{\{8\}}$	$0.122239^{\{7\}}$	$0.109596^{\{4\}}$	$0.110472^{\{5\}}$
	MSE	$\hat{\alpha}$	$0.002684^{\{4\}}$	$0.001601^{\{2\}}$	$0.004687^{\{8\}}$	$0.001066^{\{1\}}$	$0.003741^{\{7\}}$	$0.003194^{\{5\}}$	$0.001627^{\{3\}}$	$0.003236^{\{6\}}$
		$\hat{\beta}$	$0.01297^{\{2\}}$	$0.016376^{\{3\}}$	$0.023753^{\{6\}}$	$0.012253^{\{1\}}$	$0.025487^{\{8\}}$	$0.024522^{\{7\}}$	$0.019014^{\{4\}}$	$0.019493^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.20326^{\{8\}}$	$0.115057^{\{2\}}$	$0.180961^{\{7\}}$	$0.098323^{\{1\}}$	$0.167245^{\{5\}}$	$0.176996^{\{6\}}$	$0.119769^{\{3\}}$	$0.141747^{\{4\}}$
		$\hat{\beta}$	$0.113039^{\{2\}}$	$0.128814^{\{3\}}$	$0.161743^{\{6\}}$	$0.110041^{\{1\}}$	$0.169437^{\{8\}}$	$0.162985^{\{7\}}$	$0.146128^{\{4\}}$	$0.147296^{\{5\}}$
	$\sum$ ranks		$26^{\{4\}}$	$15^{\{2\}}$	$40^{\{7\}}$	$6^{\{1\}}$	$41^{\{8\}}$	$38^{\{6\}}$	$21^{\{3\}}$	$29^{\{5\}}$
100	BIAS	$\hat{\alpha}$	$0.045262^{\{8\}}$	$0.024581^{\{2\}}$	$0.034583^{\{5\}}$	$0.019686^{\{1\}}$	$0.036614^{\{6\}}$	$0.039413^{\{7\}}$	$0.026108^{\{3\}}$	$0.028806^{\{3\}}$
		$\hat{\beta}$	$0.071516^{\{2\}}$	$0.080407^{\{3\}}$	$0.103425^{\{6\}}$	$0.07047^{\{1\}}$	$0.108323^{\{8\}}$	$0.107061^{\{7\}}$	$0.090832^{\{4\}}$	$0.094763^{\{5\}}$
	MSE	$\hat{\alpha}$	$0.000995^{\{2\}}$	$0.00122^{\{4\}}$	$0.002207^{\{6\}}$	$0.000703^{\{1\}}$	$0.002658^{\{8\}}$	$0.002532^{\{7\}}$	$0.001175^{\{3\}}$	$0.001517^{\{5\}}$
		$\hat{\beta}$	$0.009947^{\{2\}}$	$0.011883^{\{3\}}$	$0.017578^{\{6\}}$	$0.009252^{\{1\}}$	$0.018299^{\{7\}}$	$0.018385^{\{8\}}$	$0.013158^{\{4\}}$	$0.014533^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.181047^{\{8\}}$	$0.098322^{\{2\}}$	$0.138331^{\{5\}}$	$0.078745^{\{1\}}$	$0.146455^{\{6\}}$	$0.157654^{\{7\}}$	$0.104431^{\{3\}}$	$0.115222^{\{4\}}$
		$\hat{\beta}$	$0.095354^{\{2\}}$	$0.107209^{\{3\}}$	$0.1379^{\{6\}}$	$0.09396^{\{1\}}$	$0.144431^{\{8\}}$	$0.142748^{\{7\}}$	$0.12111^{\{4\}}$	$0.12635^{\{5\}}$
	$\sum$ ranks		$24^{\{4\}}$	$17^{\{2\}}$	$34^{\{6\}}$	$6^{\{1\}}$	$43^{\{7.5\}}$	$43^{\{7.5\}}$	$21^{\{3\}}$	$28^{\{5\}}$
125	BIAS	$\hat{\alpha}$	$0.042024^{\{8\}}$	$0.021384^{\{2\}}$	$0.03378^{\{7\}}$	$0.01728^{\{1\}}$	$0.029932^{\{5\}}$	$0.033378^{\{6\}}$	$0.02344^{\{3\}}$	$0.025707^{\{4\}}$
		$\hat{\beta}$	$0.058643^{\{2\}}$	$0.070858^{\{3\}}$	$0.097561^{\{7\}}$	$0.057506^{\{1\}}$	$0.092583^{\{6\}}$	$0.098095^{\{8\}}$	$0.083185^{\{4\}}$	$0.084214^{\{5\}}$
	MSE	$\hat{\alpha}$	$0.000518^{\{1\}}$	$0.000934^{\{3\}}$	$0.002101^{\{8\}}$	$0.000548^{\{2\}}$	$0.00163^{\{6\}}$	$0.00184^{\{7\}}$	$0.000936^{\{4\}}$	$0.001198^{\{5\}}$
		$\hat{\beta}$	$0.007231^{\{2\}}$	$0.009414^{\{3\}}$	$0.015016^{\{7\}}$	$0.006427^{\{1\}}$	$0.013678^{\{6\}}$	$0.015056^{\{8\}}$	$0.010814^{\{4\}}$	$0.011244^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.168094^{\{8\}}$	$0.085536^{\{2\}}$	$0.135121^{\{7\}}$	$0.069119^{\{1\}}$	$0.119729^{\{5\}}$	$0.133511^{\{6\}}$	$0.093762^{\{3\}}$	$0.102828^{\{4\}}$
		$\hat{\beta}$	$0.07819^{\{2\}}$	$0.094477^{\{3\}}$	$0.130082^{\{7\}}$	$0.076675^{\{1\}}$	$0.123444^{\{6\}}$	$0.130793^{\{8\}}$	$0.110913^{\{4\}}$	$0.112286^{\{5\}}$
	$\sum$ ranks		$23^{\{4\}}$	$16^{\{2\}}$	$43^{\{7.5\}}$	$7^{\{1\}}$	$34^{\{6\}}$	$43^{\{7.5\}}$	$22^{\{3\}}$	$28^{\{5\}}$
150	BIAS	$\hat{\alpha}$	$0.033995^{\{8\}}$	$0.018661^{\{2\}}$	$0.027968^{\{6\}}$	$0.015277^{\{1\}}$	$0.027097^{\{5\}}$	$0.031338^{\{7\}}$	$0.020924^{\{3\}}$	$0.023931^{\{4\}}$
		$\hat{\beta}$	$0.054205^{\{1\}}$	$0.061158^{\{3\}}$	$0.087696^{\{8\}}$	$0.056196^{\{2\}}$	$0.087088^{\{6\}}$	$0.087545^{\{7\}}$	$0.075761^{\{4\}}$	$0.079985^{\{5\}}$
	MSE	$\hat{\alpha}$	$0.000394^{\{1\}}$	$0.000717^{\{3\}}$	$0.001377^{\{7\}}$	$0.000438^{\{2\}}$	$0.001307^{\{6\}}$	$0.001591^{\{8\}}$	$0.000727^{\{4\}}$	$0.001063^{\{5\}}$
		$\hat{\beta}$	$0.006265^{\{2\}}$	$0.007217^{\{4\}}$	$0.011789^{\{6\}}$	$0.006085^{\{1\}}$	$0.011989^{\{7\}}$	$0.012343^{\{8\}}$	$0.009136^{\{4\}}$	$0.010003^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.155981^{\{8\}}$	$0.074646^{\{2\}}$	$0.111872^{\{6\}}$	$0.061109^{\{1\}}$	$0.10839^{\{5\}}$	$0.125352^{\{7\}}$	$0.083695^{\{3\}}$	$0.095722^{\{4\}}$
		$\hat{\beta}$	$0.072274^{\{1\}}$	$0.081543^{\{3\}}$	$0.116927^{\{8\}}$	$0.074928^{\{2\}}$	$0.116118^{\{6\}}$	$0.116727^{\{7\}}$	$0.101014^{\{4\}}$	$0.106646^{\{5\}}$
	$\sum$ ranks		$21^{\{3\}}$	$16^{\{2\}}$	$41^{\{7\}}$	$9^{\{1\}}$	$35^{\{6\}}$	$44^{\{8\}}$	$22^{\{4\}}$	$28^{\{5\}}$

$$F(x) = \frac{\log\left(\left(1 - e^{-(x/1-x)^{\alpha}}\right)^{\beta} + 1\right)}{\log(2)},$$
$$f(x) = \frac{\alpha\beta (x/1-x)^{\alpha-1}\left(1 - e^{-(x/1-x)^{\alpha}}\right)^{\beta}}{(x-1)^2 \log(2)\left(e^{(x/1-x)^{\alpha}} - 1\right)\left(\left(1 - e^{-(x/1-x)^{\alpha}}\right)^{\beta} + 1\right)}. \quad (2)$$

Its hazard function is defined as follows:

$$h(x) = \frac{f(x)}{S(x)} = \frac{\alpha\beta (1/x-1)^{-\alpha}\left(1 - e^{-(x/1-x)^{\alpha}}\right)^{\beta}}{(x-1)x\left(e^{(x/1-x)^{\alpha}} - 1\right)\left(\left(1 - e^{-(x/1-x)^{\alpha}}\right)^{\beta} + 1\right)\left(\log(2) - \log\left(\left(1 - e^{-(x/1-x)^{\alpha}}\right)^{\beta} + 1\right)\right)}, \quad (3)$$

Table 2: Simulation values of BIAS, MSE, and MRE for $\alpha = 0.75, \beta = 1.5$.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE
25	BIAS	$\hat{\alpha}$	$0.117882^{\{3\}}$	$0.112061^{\{2\}}$	$0.165051^{\{8\}}$	$0.107679^{\{1\}}$	$0.145588^{\{7\}}$	$0.123216^{\{5\}}$	$0.120017^{\{4\}}$	$0.129074^{\{6\}}$
		$\hat{\beta}$	$0.259965^{\{2\}}$	$0.271556^{\{3\}}$	$0.299962^{\{8\}}$	$0.244616^{\{1\}}$	$0.278657^{\{5\}}$	$0.275586^{\{4\}}$	$0.296926^{\{7\}}$	$0.284412^{\{6\}}$
	MSE	$\hat{\alpha}$	$0.028782^{\{5\}}$	$0.022124^{\{2\}}$	$0.118446^{\{8\}}$	$0.019317^{\{1\}}$	$0.052388^{\{7\}}$	$0.027362^{\{4\}}$	$0.026159^{\{3\}}$	$0.034504^{\{6\}}$
		$\hat{\beta}$	$0.108514^{\{2\}}$	$0.119873^{\{3\}}$	$0.148218^{\{7\}}$	$0.099379^{\{1\}}$	$0.12962^{\{5\}}$	$0.121339^{\{4\}}$	$0.150547^{\{8\}}$	$0.137926^{\{6\}}$
	MRE	$\hat{\alpha}$	$0.157176^{\{3\}}$	$0.149415^{\{2\}}$	$0.220068^{\{8\}}$	$0.143572^{\{1\}}$	$0.194117^{\{7\}}$	$0.164288^{\{5\}}$	$0.160023^{\{4\}}$	$0.172099^{\{6\}}$
		$\hat{\beta}$	$0.17331^{\{2\}}$	$0.181037^{\{3\}}$	$0.199975^{\{8\}}$	$0.163077^{\{1\}}$	$0.185771^{\{5\}}$	$0.183724^{\{4\}}$	$0.197951^{\{7\}}$	$0.189608^{\{6\}}$
	$\sum$ ranks		$17^{\{3\}}$	$15^{\{2\}}$	$47^{\{8\}}$	$6^{\{1\}}$	$36^{\{6.5\}}$	$26^{\{4\}}$	$33^{\{5\}}$	$36^{\{6.5\}}$
50	BIAS	$\hat{\alpha}$	$0.07402^{\{1\}}$	$0.077653^{\{3\}}$	$0.104456^{\{8\}}$	$0.077545^{\{2\}}$	$0.093938^{\{7\}}$	$0.080405^{\{5\}}$	$0.077729^{\{4\}}$	$0.081263^{\{6\}}$
		$\hat{\beta}$	$0.181696^{\{2\}}$	$0.185623^{\{3\}}$	$0.209063^{\{8\}}$	$0.176866^{\{1\}}$	$0.202077^{\{7\}}$	$0.193209^{\{5\}}$	$0.201481^{\{6\}}$	$0.187236^{\{4\}}$
	MSE	$\hat{\alpha}$	$0.009703^{\{2\}}$	$0.010413^{\{3\}}$	$0.022908^{\{8\}}$	$0.009263^{\{1\}}$	$0.015877^{\{7\}}$	$0.010442^{\{4\}}$	$0.010491^{\{5\}}$	$0.011499^{\{6\}}$
		$\hat{\beta}$	$0.05272^{\{2\}}$	$0.05428^{\{3\}}$	$0.070501^{\{8\}}$	$0.049971^{\{1\}}$	$0.065014^{\{6\}}$	$0.059947^{\{5\}}$	$0.066791^{\{7\}}$	$0.058^{\{4\}}$
	MRE	$\hat{\alpha}$	$0.098694^{\{1\}}$	$0.103538^{\{3\}}$	$0.139274^{\{8\}}$	$0.103394^{\{2\}}$	$0.12525^{\{7\}}$	$0.107207^{\{5\}}$	$0.103639^{\{4\}}$	$0.10835^{\{6\}}$
		$\hat{\beta}$	$0.121131^{\{2\}}$	$0.123749^{\{3\}}$	$0.139376^{\{8\}}$	$0.117911^{\{1\}}$	$0.134718^{\{7\}}$	$0.128806^{\{5\}}$	$0.134321^{\{6\}}$	$0.124824^{\{4\}}$
	$\sum$ ranks		$10^{\{2\}}$	$18^{\{3\}}$	$48^{\{8\}}$	$8^{\{1\}}$	$41^{\{7\}}$	$29^{\{4\}}$	$32^{\{6\}}$	$30^{\{5\}}$
75	BIAS	$\hat{\alpha}$	$0.061778^{\{2\}}$	$0.062994^{\{4\}}$	$0.076351^{\{8\}}$	$0.060783^{\{1\}}$	$0.074807^{\{7\}}$	$0.068885^{\{6\}}$	$0.062589^{\{3\}}$	$0.066892^{\{5\}}$
		$\hat{\beta}$	$0.145697^{\{2\}}$	$0.154409^{\{3\}}$	$0.170298^{\{8\}}$	$0.142167^{\{1\}}$	$0.160494^{\{6\}}$	$0.158044^{\{5\}}$	$0.163073^{\{7\}}$	$0.155157^{\{4\}}$
	MSE	$\hat{\alpha}$	$0.006557^{\{4\}}$	$0.006518^{\{2\}}$	$0.010007^{\{8\}}$	$0.005551^{\{1\}}$	$0.00954^{\{7\}}$	$0.007545^{\{5\}}$	$0.006529^{\{3\}}$	$0.007606^{\{6\}}$
		$\hat{\beta}$	$0.033801^{\{2\}}$	$0.037605^{\{3\}}$	$0.046341^{\{8\}}$	$0.031609^{\{1\}}$	$0.042229^{\{6\}}$	$0.039689^{\{5\}}$	$0.042514^{\{7\}}$	$0.038408^{\{4\}}$
	MRE	$\hat{\alpha}$	$0.08237^{\{2\}}$	$0.083992^{\{4\}}$	$0.101801^{\{8\}}$	$0.081043^{\{1\}}$	$0.099742^{\{7\}}$	$0.091846^{\{6\}}$	$0.083452^{\{3\}}$	$0.089189^{\{5\}}$
		$\hat{\beta}$	$0.097131^{\{2\}}$	$0.102939^{\{3\}}$	$0.113532^{\{8\}}$	$0.094778^{\{1\}}$	$0.106996^{\{6\}}$	$0.105363^{\{5\}}$	$0.108715^{\{7\}}$	$0.103438^{\{4\}}$
	$\sum$ ranks		$14^{\{2\}}$	$19^{\{3\}}$	$48^{\{8\}}$	$6^{\{1\}}$	$39^{\{7\}}$	$32^{\{6\}}$	$30^{\{5\}}$	$28^{\{4\}}$
100	BIAS	$\hat{\alpha}$	$0.050862^{\{1\}}$	$0.05658^{\{4\}}$	$0.064863^{\{7\}}$	$0.052378^{\{2\}}$	$0.065046^{\{8\}}$	$0.056738^{\{5\}}$	$0.054369^{\{3\}}$	$0.057854^{\{6\}}$
		$\hat{\beta}$	$0.125442^{\{2\}}$	$0.132155^{\{4\}}$	$0.143719^{\{7\}}$	$0.121916^{\{1\}}$	$0.14539^{\{8\}}$	$0.137344^{\{5\}}$	$0.138746^{\{6\}}$	$0.132091^{\{3\}}$
	MSE	$\hat{\alpha}$	$0.00431^{\{2\}}$	$0.005302^{\{5\}}$	$0.006966^{\{7\}}$	$0.004103^{\{1\}}$	$0.007192^{\{8\}}$	$0.005069^{\{4\}}$	$0.004964^{\{3\}}$	$0.005772^{\{6\}}$
		$\hat{\beta}$	$0.024675^{\{2\}}$	$0.02777^{\{3\}}$	$0.033143^{\{8\}}$	$0.02385^{\{1\}}$	$0.032864^{\{7\}}$	$0.029737^{\{5\}}$	$0.029878^{\{6\}}$	$0.02827^{\{4\}}$
	MRE	$\hat{\alpha}$	$0.067816^{\{1\}}$	$0.07544^{\{4\}}$	$0.086484^{\{7\}}$	$0.069837^{\{2\}}$	$0.086727^{\{8\}}$	$0.075651^{\{5\}}$	$0.072491^{\{3\}}$	$0.077139^{\{6\}}$
		$\hat{\beta}$	$0.083628^{\{2\}}$	$0.088104^{\{4\}}$	$0.095813^{\{7\}}$	$0.081277^{\{1\}}$	$0.096926^{\{8\}}$	$0.091563^{\{5\}}$	$0.092498^{\{6\}}$	$0.088061^{\{3\}}$
	$\sum$ ranks		$10^{\{2\}}$	$24^{\{3\}}$	$43^{\{7\}}$	$8^{\{1\}}$	$47^{\{8\}}$	$29^{\{6\}}$	$27^{\{4\}}$	$28^{\{5\}}$
125	BIAS	$\hat{\alpha}$	$0.046076^{\{1\}}$	$0.049459^{\{4\}}$	$0.059037^{\{8\}}$	$0.046894^{\{2\}}$	$0.056816^{\{7\}}$	$0.0516^{\{6\}}$	$0.048696^{\{3\}}$	$0.049737^{\{5\}}$
		$\hat{\beta}$	$0.111973^{\{2\}}$	$0.116232^{\{3\}}$	$0.129633^{\{8\}}$	$0.109042^{\{1\}}$	$0.127908^{\{7\}}$	$0.123999^{\{6\}}$	$0.123779^{\{5\}}$	$0.117006^{\{4\}}$
	MSE	$\hat{\alpha}$	$0.003409^{\{2\}}$	$0.003916^{\{4\}}$	$0.005662^{\{8\}}$	$0.003354^{\{1\}}$	$0.005285^{\{7\}}$	$0.004201^{\{6\}}$	$0.003831^{\{3\}}$	$0.004026^{\{5\}}$
		$\hat{\beta}$	$0.019726^{\{2\}}$	$0.022038^{\{4\}}$	$0.026632^{\{8\}}$	$0.019442^{\{1\}}$	$0.026065^{\{7\}}$	$0.023897^{\{5\}}$	$0.024712^{\{6\}}$	$0.0218^{\{3\}}$
	MRE	$\hat{\alpha}$	$0.061434^{\{1\}}$	$0.065945^{\{4\}}$	$0.078717^{\{8\}}$	$0.062525^{\{2\}}$	$0.075755^{\{7\}}$	$0.0688^{\{6\}}$	$0.064928^{\{3\}}$	$0.066316^{\{5\}}$
		$\hat{\beta}$	$0.074648^{\{2\}}$	$0.077488^{\{3\}}$	$0.086422^{\{8\}}$	$0.072695^{\{1\}}$	$0.085272^{\{7\}}$	$0.082666^{\{6\}}$	$0.082519^{\{5\}}$	$0.078004^{\{4\}}$
	$\sum$ ranks		$10^{\{2\}}$	$22^{\{3\}}$	$48^{\{8\}}$	$8^{\{1\}}$	$42^{\{7\}}$	$35^{\{6\}}$	$25^{\{4\}}$	$26^{\{5\}}$
150	BIAS	$\hat{\alpha}$	$0.040845^{\{1\}}$	$0.044847^{\{3\}}$	$0.054291^{\{8\}}$	$0.041118^{\{2\}}$	$0.052331^{\{7\}}$	$0.045308^{\{4\}}$	$0.045597^{\{5\}}$	$0.046057^{\{6\}}$
		$\hat{\beta}$	$0.102559^{\{2\}}$	$0.10974^{\{3\}}$	$0.116942^{\{8\}}$	$0.09909^{\{1\}}$	$0.114989^{\{6\}}$	$0.111479^{\{4\}}$	$0.115777^{\{7\}}$	$0.111507^{\{5\}}$
	MSE	$\hat{\alpha}$	$0.002734^{\{2\}}$	$0.003241^{\{3\}}$	$0.004805^{\{8\}}$	$0.002593^{\{1\}}$	$0.004415^{\{7\}}$	$0.003332^{\{5\}}$	$0.003284^{\{4\}}$	$0.003443^{\{6\}}$
		$\hat{\beta}$	$0.01668^{\{2\}}$	$0.019294^{\{3\}}$	$0.021686^{\{8\}}$	$0.015809^{\{1\}}$	$0.02074^{\{6\}}$	$0.019633^{\{4\}}$	$0.021631^{\{7\}}$	$0.019823^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.05446^{\{1\}}$	$0.059797^{\{3\}}$	$0.072388^{\{8\}}$	$0.054825^{\{2\}}$	$0.069775^{\{7\}}$	$0.06041^{\{4\}}$	$0.060797^{\{5\}}$	$0.061409^{\{6\}}$
		$\hat{\beta}$	$0.068372^{\{2\}}$	$0.07316^{\{3\}}$	$0.077961^{\{8\}}$	$0.06606^{\{1\}}$	$0.076659^{\{6\}}$	$0.074319^{\{4\}}$	$0.077185^{\{7\}}$	$0.074338^{\{5\}}$
	$\sum$ ranks		$10^{\{2\}}$	$18^{\{3\}}$	$48^{\{8\}}$	$8^{\{1\}}$	$39^{\{7\}}$	$25^{\{4\}}$	$35^{\{6\}}$	$33^{\{5\}}$

where $S(x)$ is its survival function (SF). Figures 1 and 2 show the possible plots of the PDF and the HF of the ExRKD, respectively.

3. Statistical Properties

3.1. Quantile Function. To get the quantile function (QF) of the ExRKD, one must first determine the inverse function of the CDF of the ExRKD, which is used to have randomly generated data sets from our proposed model and it is defined as follows:

$$Q = 1 - \frac{1}{\left(-\log\left(1 - \left(2^{p} - 1\right)^{1/\beta}\right)\right)^{1/\alpha} + 1}, \quad 0 < p < 1. \tag{4}$$

3.2. Linear Representation. In this subsection, we derive linear representations of the CDF and PDF of the ExRKD that are helpful in many calculations.

For $-1 < x \leq 1$, we define the following expansion:

$$\log(1 + x) = \sum_{w=1}^{\infty} (-1)^{w+1} \frac{x^{w}}{w}. \tag{5}$$

By using last relation to CDF of ExRKD in equation (3), we have

$$F(x) = \sum_{w=1}^{\infty} \Phi_{w} G_{w}(x), \tag{6}$$

and its corresponding PDF is defined as follows:

Table 3: Simulation values of BIAS, MSE, and MRE for $\alpha = 1.5, \beta = 0.75$.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE
25	BIAS	$\hat{\alpha}$	$0.403938^{\{5\}}$	$0.394719^{\{3\}}$	$0.599086^{\{8\}}$	$0.353052^{\{1\}}$	$0.522094^{\{7\}}$	$0.360686^{\{2\}}$	$0.398035^{\{4\}}$	$0.501736^{\{6\}}$
		$\hat{\beta}$	$0.171923^{\{1\}}$	$0.184513^{\{4\}}$	$0.208659^{\{7\}}$	$0.178267^{\{2\}}$	$0.208677^{\{8\}}$	$0.178345^{\{3\}}$	$0.189428^{\{5\}}$	$0.205418^{\{6\}}$
	MSE	$\hat{\alpha}$	$0.477018^{\{3\}}$	$0.493378^{\{4\}}$	$1.376029^{\{8\}}$	$0.357452^{\{2\}}$	$0.929049^{\{6\}}$	$0.354195^{\{1\}}$	$0.674488^{\{5\}}$	$1.012718^{\{7\}}$
		$\hat{\beta}$	$0.046371^{\{1\}}$	$0.052691^{\{4\}}$	$0.069169^{\{7\}}$	$0.051124^{\{3\}}$	$0.069188^{\{8\}}$	$0.050753^{\{2\}}$	$0.05782^{\{5\}}$	$0.065421^{\{6\}}$
	MRE	$\hat{\alpha}$	$0.269292^{\{5\}}$	$0.263146^{\{3\}}$	$0.399391^{\{8\}}$	$0.235368^{\{1\}}$	$0.348063^{\{7\}}$	$0.240457^{\{2\}}$	$0.265356^{\{4\}}$	$0.334491^{\{6\}}$
		$\hat{\beta}$	$0.229231^{\{1\}}$	$0.246018^{\{4\}}$	$0.278212^{\{7\}}$	$0.237689^{\{2\}}$	$0.278236^{\{8\}}$	$0.237793^{\{3\}}$	$0.252571^{\{5\}}$	$0.27389^{\{6\}}$
	$\sum$ ranks		$16^{\{3\}}$	$22^{\{4\}}$	$45^{\{8\}}$	$11^{\{1\}}$	$44^{\{7\}}$	$13^{\{2\}}$	$28^{\{5\}}$	$37^{\{6\}}$
50	BIAS	$\hat{\alpha}$	$0.221619^{\{2\}}$	$0.23885^{\{5\}}$	$0.34319^{\{8\}}$	$0.209574^{\{1\}}$	$0.328961^{\{7\}}$	$0.228378^{\{3\}}$	$0.237229^{\{4\}}$	$0.285926^{\{6\}}$
		$\hat{\beta}$	$0.118107^{\{2\}}$	$0.132868^{\{4\}}$	$0.154687^{\{8\}}$	$0.116901^{\{1\}}$	$0.151413^{\{7\}}$	$0.131583^{\{3\}}$	$0.134993^{\{5\}}$	$0.14188^{\{6\}}$
	MSE	$\hat{\alpha}$	$0.102835^{\{4\}}$	$0.105831^{\{5\}}$	$0.334914^{\{8\}}$	$0.075844^{\{1\}}$	$0.289094^{\{7\}}$	$0.088987^{\{2\}}$	$0.10164^{\{3\}}$	$0.212808^{\{6\}}$
		$\hat{\beta}$	$0.021414^{\{2\}}$	$0.028344^{\{4\}}$	$0.036168^{\{7\}}$	$0.02138^{\{1\}}$	$0.036571^{\{8\}}$	$0.026762^{\{3\}}$	$0.028403^{\{5\}}$	$0.031995^{\{6\}}$
	MRE	$\hat{\alpha}$	$0.147746^{\{2\}}$	$0.159233^{\{5\}}$	$0.228793^{\{8\}}$	$0.139716^{\{1\}}$	$0.219307^{\{7\}}$	$0.152252^{\{3\}}$	$0.158153^{\{4\}}$	$0.190618^{\{6\}}$
		$\hat{\beta}$	$0.157476^{\{2\}}$	$0.177158^{\{4\}}$	$0.206249^{\{8\}}$	$0.155867^{\{1\}}$	$0.201884^{\{7\}}$	$0.175444^{\{3\}}$	$0.17999^{\{5\}}$	$0.189173^{\{6\}}$
	$\sum$ ranks		$14^{\{2\}}$	$27^{\{5\}}$	$47^{\{8\}}$	$6^{\{1\}}$	$43^{\{7\}}$	$17^{\{3\}}$	$26^{\{4\}}$	$36^{\{6\}}$
75	BIAS	$\hat{\alpha}$	$0.169662^{\{2\}}$	$0.193867^{\{5\}}$	$0.270825^{\{8\}}$	$0.165032^{\{1\}}$	$0.25135^{\{7\}}$	$0.178703^{\{4\}}$	$0.177601^{\{3\}}$	$0.209395^{\{6\}}$
		$\hat{\beta}$	$0.093936^{\{1\}}$	$0.106846^{\{5\}}$	$0.128505^{\{8\}}$	$0.094145^{\{2\}}$	$0.120559^{\{7\}}$	$0.10356^{\{3\}}$	$0.105625^{\{4\}}$	$0.108721^{\{6\}}$
	MSE	$\hat{\alpha}$	$0.053903^{\{2\}}$	$0.066424^{\{5\}}$	$0.144329^{\{8\}}$	$0.044168^{\{1\}}$	$0.131125^{\{7\}}$	$0.054543^{\{3\}}$	$0.056073^{\{4\}}$	$0.093976^{\{6\}}$
		$\hat{\beta}$	$0.013993^{\{2\}}$	$0.017994^{\{5\}}$	$0.025477^{\{8\}}$	$0.013988^{\{1\}}$	$0.02418^{\{7\}}$	$0.01686^{\{3\}}$	$0.01753^{\{4\}}$	$0.018504^{\{6\}}$
	MRE	$\hat{\alpha}$	$0.113108^{\{2\}}$	$0.129245^{\{5\}}$	$0.18055^{\{8\}}$	$0.110022^{\{1\}}$	$0.167567^{\{7\}}$	$0.119135^{\{4\}}$	$0.118401^{\{3\}}$	$0.139597^{\{6\}}$
		$\hat{\beta}$	$0.125248^{\{1\}}$	$0.142461^{\{5\}}$	$0.17134^{\{8\}}$	$0.125526^{\{2\}}$	$0.160746^{\{7\}}$	$0.138081^{\{3\}}$	$0.140833^{\{4\}}$	$0.144962^{\{6\}}$
	$\sum$ *Ranks*		$10^{\{2\}}$	$30^{\{5\}}$	$48^{\{8\}}$	$8^{\{1\}}$	$42^{\{7\}}$	$20^{\{3\}}$	$22^{\{4\}}$	$36^{\{6\}}$
100	BIAS	$\hat{\alpha}$	$0.143207^{\{2\}}$	$0.170465^{\{6\}}$	$0.207384^{\{7\}}$	$0.13941^{\{1\}}$	$0.216216^{\{8\}}$	$0.156581^{\{3\}}$	$0.16659^{\{5\}}$	$0.164009^{\{4\}}$
		$\hat{\beta}$	$0.084284^{\{2\}}$	$0.092792^{\{4\}}$	$0.104823^{\{7\}}$	$0.080592^{\{1\}}$	$0.107247^{\{8\}}$	$0.092391^{\{3\}}$	$0.096125^{\{6\}}$	$0.093657^{\{5\}}$
	MSE	$\hat{\alpha}$	$0.037562^{\{2\}}$	$0.053^{\{6\}}$	$0.094128^{\{7\}}$	$0.028895^{\{1\}}$	$0.127587^{\{8\}}$	$0.039867^{\{3\}}$	$0.046498^{\{4\}}$	$0.052543^{\{5\}}$
		$\hat{\beta}$	$0.011233^{\{2\}}$	$0.013858^{\{4\}}$	$0.017662^{\{7\}}$	$0.010207^{\{1\}}$	$0.018116^{\{8\}}$	$0.013373^{\{3\}}$	$0.0143^{\{6\}}$	$0.014001^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.095471^{\{2\}}$	$0.113644^{\{6\}}$	$0.138256^{\{7\}}$	$0.09294^{\{1\}}$	$0.144144^{\{8\}}$	$0.104387^{\{3\}}$	$0.11106^{\{5\}}$	$0.10934^{\{4\}}$
		$\hat{\beta}$	$0.112378^{\{2\}}$	$0.123722^{\{4\}}$	$0.139764^{\{7\}}$	$0.107456^{\{1\}}$	$0.142995^{\{8\}}$	$0.123189^{\{3\}}$	$0.128166^{\{6\}}$	$0.124876^{\{5\}}$
	$\sum$ *Ranks*		$12^{\{2\}}$	$30^{\{5\}}$	$42^{\{7\}}$	$6^{\{1\}}$	$48^{\{8\}}$	$18^{\{3\}}$	$32^{\{6\}}$	$28^{\{4\}}$
125	BIAS	$\hat{\alpha}$	$0.125323^{\{1\}}$	$0.140446^{\{4\}}$	$0.196285^{\{8\}}$	$0.12934^{\{2\}}$	$0.186987^{\{7\}}$	$0.143845^{\{5\}}$	$0.138527^{\{3\}}$	$0.145408^{\{6\}}$
		$\hat{\beta}$	$0.071757^{\{1\}}$	$0.081591^{\{3\}}$	$0.096374^{\{8\}}$	$0.072325^{\{2\}}$	$0.094955^{\{7\}}$	$0.083968^{\{6\}}$	$0.082654^{\{4\}}$	$0.082885^{\{5\}}$
	MSE	$\hat{\alpha}$	$0.028017^{\{2\}}$	$0.03325^{\{4\}}$	$0.076361^{\{8\}}$	$0.026108^{\{1\}}$	$0.071163^{\{7\}}$	$0.034206^{\{5\}}$	$0.032985^{\{3\}}$	$0.037533^{\{6\}}$
		$\hat{\beta}$	$0.008159^{\{1\}}$	$0.010109^{\{3\}}$	$0.014827^{\{8\}}$	$0.008263^{\{2\}}$	$0.014461^{\{7\}}$	$0.011302^{\{6\}}$	$0.010758^{\{4\}}$	$0.010818^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.083548^{\{1\}}$	$0.093631^{\{4\}}$	$0.130857^{\{8\}}$	$0.086227^{\{2\}}$	$0.124658^{\{7\}}$	$0.095896^{\{5\}}$	$0.092351^{\{3\}}$	$0.096939^{\{6\}}$
		$\hat{\beta}$	$0.095675^{\{1\}}$	$0.108788^{\{3\}}$	$0.128498^{\{8\}}$	$0.096433^{\{2\}}$	$0.126606^{\{7\}}$	$0.111957^{\{6\}}$	$0.110205^{\{4\}}$	$0.110513^{\{5\}}$
	$\sum$ *Ranks*		$7^{\{1\}}$	$21^{\{3.5\}}$	$48^{\{8\}}$	$11^{\{2\}}$	$42^{\{7\}}$	$33^{\{5.5\}}$	$21^{\{3.5\}}$	$33^{\{5.5\}}$
150	BIAS	$\hat{\alpha}$	$0.115159^{\{1\}}$	$0.134124^{\{5\}}$	$0.172374^{\{8\}}$	$0.115194^{\{2\}}$	$0.163079^{\{7\}}$	$0.13224^{\{4\}}$	$0.120341^{\{3\}}$	$0.135241^{\{6\}}$
		$\hat{\beta}$	$0.065425^{\{1\}}$	$0.073621^{\{3\}}$	$0.086416^{\{8\}}$	$0.065783^{\{2\}}$	$0.083708^{\{7\}}$	$0.075534^{\{4\}}$	$0.075598^{\{5\}}$	$0.076739^{\{6\}}$
	MSE	$\hat{\alpha}$	$0.023219^{\{2\}}$	$0.031351^{\{6\}}$	$0.054339^{\{8\}}$	$0.020216^{\{1\}}$	$0.048258^{\{7\}}$	$0.028359^{\{4\}}$	$0.023266^{\{3\}}$	$0.030862^{\{5\}}$
		$\hat{\beta}$	$0.006828^{\{1\}}$	$0.008545^{\{3\}}$	$0.011775^{\{8\}}$	$0.006833^{\{2\}}$	$0.011559^{\{7\}}$	$0.00896^{\{4\}}$	$0.00901^{\{5\}}$	$0.009189^{\{6\}}$
	MRE	$\hat{\alpha}$	$0.076773^{\{1\}}$	$0.089416^{\{5\}}$	$0.114916^{\{8\}}$	$0.076796^{\{2\}}$	$0.10872^{\{7\}}$	$0.08816^{\{4\}}$	$0.080227^{\{3\}}$	$0.09016^{\{6\}}$
		$\hat{\beta}$	$0.087233^{\{1\}}$	$0.098162^{\{3\}}$	$0.115221^{\{8\}}$	$0.087711^{\{2\}}$	$0.11161^{\{7\}}$	$0.100712^{\{4\}}$	$0.100798^{\{5\}}$	$0.102319^{\{6\}}$
	$\sum$ *Ranks*		$7^{\{1\}}$	$25^{\{5\}}$	$48^{\{8\}}$	$11^{\{2\}}$	$42^{\{7\}}$	$24^{\{3.5\}}$	$24^{\{3.5\}}$	$35^{\{6\}}$

$$f(x) = \sum_{w=1}^{\infty} \Phi_w g_w(x), \quad (7)$$

where $\Phi_w = (-1)^{w+1}/w \log(2)$ and $G_w(x) = (1-e^{-(x/1-x)^{\alpha}})^{\beta w}$ follow the exponentiated reduced Kies distribution (ERKD) [14] with parameters α and βw.

3.3. Moments. The ExRKD's k^{th} moments have the following form:

$$\begin{aligned} \mu_k' = E\left(X^k\right) &= \int_0^{\infty} x^k f(x)\mathrm{d}x \\ &= \sum_{w=1}^{\infty} \Phi_w \int_0^{\infty} x^k g_w(x)\mathrm{d}x = \sum_{w=1}^{\infty} \Phi_w B_k, \end{aligned} \quad (8)$$

where B_k is the k^{th} moments of ERKD, see [14] which is defined. With the values of $k = 1, 2, 3$, and 4, we get the first four moments around the origin of the ExRKD, which are then used to derive the coefficients of skewness and kurtosis, respectively.

Table 4: Simulation values of BIAS, MSE, and MRE for $\alpha = 2.5, \beta = 2.5$.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE
25	BIAS	$\hat{\alpha}$	$0.304119^{\{4\}}$	$0.296276^{\{1\}}$	$0.385635^{\{8\}}$	$0.301092^{\{3\}}$	$0.357767^{\{7\}}$	$0.299733^{\{2\}}$	$0.329087^{\{5\}}$	$0.331094^{\{6\}}$
		$\hat{\beta}$	$0.411856^{\{6\}}$	$0.401001^{\{3\}}$	$0.461255^{\{7\}}$	$0.365757^{\{1\}}$	$0.410637^{\{5\}}$	$0.370429^{\{2\}}$	$0.461765^{\{8\}}$	$0.406516^{\{4\}}$
	MSE	$\hat{\alpha}$	$0.173413^{\{4\}}$	$0.150549^{\{3\}}$	$0.286677^{\{8\}}$	$0.135627^{\{1\}}$	$0.233482^{\{7\}}$	$0.141902^{\{2\}}$	$0.195735^{\{5\}}$	$0.216509^{\{6\}}$
		$\hat{\beta}$	$0.281335^{\{4\}}$	$0.26855^{\{3\}}$	$0.363038^{\{7\}}$	$0.214857^{\{1\}}$	$0.299694^{\{6\}}$	$0.228874^{\{2\}}$	$0.38591^{\{8\}}$	$0.289853^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.121648^{\{4\}}$	$0.118511^{\{1\}}$	$0.154254^{\{8\}}$	$0.120437^{\{3\}}$	$0.143107^{\{7\}}$	$0.119893^{\{2\}}$	$0.131635^{\{5\}}$	$0.132438^{\{6\}}$
		$\hat{\beta}$	$0.164742^{\{6\}}$	$0.1604^{\{3\}}$	$0.184502^{\{7\}}$	$0.146303^{\{1\}}$	$0.164255^{\{5\}}$	$0.148172^{\{2\}}$	$0.184706^{\{8\}}$	$0.162606^{\{4\}}$
	$\sum$ *Ranks*		$28^{\{4\}}$	$14^{\{3\}}$	$45^{\{8\}}$	$10^{\{1\}}$	$37^{\{6\}}$	$12^{\{2\}}$	$39^{\{7\}}$	$31^{\{5\}}$
50	BIAS	$\hat{\alpha}$	$0.20402^{\{1\}}$	$0.211866^{\{3\}}$	$0.254903^{\{8\}}$	$0.209012^{\{2\}}$	$0.239911^{\{7\}}$	$0.215345^{\{4\}}$	$0.218514^{\{5\}}$	$0.220951^{\{6\}}$
		$\hat{\beta}$	$0.277733^{\{3\}}$	$0.277755^{\{4\}}$	$0.298225^{\{7\}}$	$0.258464^{\{2\}}$	$0.291377^{\{6\}}$	$0.25659^{\{1\}}$	$0.312824^{\{8\}}$	$0.27948^{\{5\}}$
	MSE	$\hat{\alpha}$	$0.069994^{\{2\}}$	$0.075141^{\{4\}}$	$0.110322^{\{8\}}$	$0.067262^{\{1\}}$	$0.092775^{\{7\}}$	$0.071149^{\{3\}}$	$0.076677^{\{5\}}$	$0.082901^{\{6\}}$
		$\hat{\beta}$	$0.124769^{\{3\}}$	$0.127553^{\{4\}}$	$0.151987^{\{7\}}$	$0.105374^{\{1\}}$	$0.141063^{\{6\}}$	$0.105421^{\{2\}}$	$0.163508^{\{8\}}$	$0.127786^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.081608^{\{1\}}$	$0.084746^{\{3\}}$	$0.101961^{\{8\}}$	$0.083605^{\{2\}}$	$0.095964^{\{7\}}$	$0.086138^{\{4\}}$	$0.087406^{\{5\}}$	$0.08838^{\{6\}}$
		$\hat{\beta}$	$0.111093^{\{3\}}$	$0.111102^{\{4\}}$	$0.11929^{\{7\}}$	$0.103385^{\{2\}}$	$0.116551^{\{6\}}$	$0.102636^{\{1\}}$	$0.12513^{\{8\}}$	$0.111792^{\{5\}}$
	$\sum$ *Ranks*		$13^{\{2\}}$	$22^{\{4\}}$	$45^{\{8\}}$	$10^{\{1\}}$	$39^{\{6.5\}}$	$15^{\{3\}}$	$39^{\{6.5\}}$	$33^{\{5\}}$
75	BIAS	$\hat{\alpha}$	$0.166547^{\{1\}}$	$0.170835^{\{4\}}$	$0.196685^{\{7\}}$	$0.169062^{\{2\}}$	$0.20057^{\{8\}}$	$0.169818^{\{3\}}$	$0.175179^{\{5\}}$	$0.182211^{\{6\}}$
		$\hat{\beta}$	$0.22025^{\{3\}}$	$0.225596^{\{4\}}$	$0.24252^{\{7\}}$	$0.207421^{\{1\}}$	$0.241819^{\{6\}}$	$0.212417^{\{2\}}$	$0.249624^{\{8\}}$	$0.23013^{\{5\}}$
	MSE	$\hat{\alpha}$	$0.045789^{\{3\}}$	$0.04581^{\{4\}}$	$0.06405^{\{7\}}$	$0.042931^{\{1\}}$	$0.066407^{\{8\}}$	$0.044833^{\{2\}}$	$0.048791^{\{5\}}$	$0.053745^{\{6\}}$
		$\hat{\beta}$	$0.077378^{\{3\}}$	$0.081369^{\{4\}}$	$0.093009^{\{6\}}$	$0.069242^{\{1\}}$	$0.094038^{\{7\}}$	$0.070275^{\{2\}}$	$0.101808^{\{8\}}$	$0.084736^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.066619^{\{1\}}$	$0.068334^{\{4\}}$	$0.078674^{\{7\}}$	$0.067625^{\{2\}}$	$0.080228^{\{8\}}$	$0.067927^{\{3\}}$	$0.070072^{\{5\}}$	$0.072885^{\{6\}}$
		$\hat{\beta}$	$0.0881^{\{3\}}$	$0.090238^{\{4\}}$	$0.097008^{\{7\}}$	$0.082968^{\{1\}}$	$0.096728^{\{6\}}$	$0.084967^{\{2\}}$	$0.09985^{\{8\}}$	$0.092052^{\{5\}}$
	$\sum$ *Ranks*		$14^{\{2.5\}}$	$24^{\{4\}}$	$41^{\{7\}}$	$8^{\{1\}}$	$43^{\{8\}}$	$14^{\{2.5\}}$	$39^{\{6\}}$	$33^{\{5\}}$
100	BIAS	$\hat{\alpha}$	$0.141906^{\{2\}}$	$0.146114^{\{3\}}$	$0.169241^{\{7\}}$	$0.140648^{\{1\}}$	$0.173418^{\{8\}}$	$0.153637^{\{6\}}$	$0.152786^{\{4\}}$	$0.153514^{\{5\}}$
		$\hat{\beta}$	$0.190564^{\{3\}}$	$0.195179^{\{4\}}$	$0.210297^{\{7\}}$	$0.180092^{\{1\}}$	$0.196464^{\{5\}}$	$0.18454^{\{2\}}$	$0.220167^{\{8\}}$	$0.201465^{\{6\}}$
	MSE	$\hat{\alpha}$	$0.032574^{\{2\}}$	$0.033924^{\{3\}}$	$0.046013^{\{7\}}$	$0.030339^{\{1\}}$	$0.047511^{\{8\}}$	$0.036331^{\{4\}}$	$0.03827^{\{5\}}$	$0.038545^{\{6\}}$
		$\hat{\beta}$	$0.057766^{\{3\}}$	$0.062308^{\{5\}}$	$0.071568^{\{7\}}$	$0.051122^{\{1\}}$	$0.062295^{\{4\}}$	$0.052888^{\{2\}}$	$0.077332^{\{8\}}$	$0.063063^{\{6\}}$
	MRE	$\hat{\alpha}$	$0.056763^{\{2\}}$	$0.058446^{\{3\}}$	$0.067697^{\{7\}}$	$0.056259^{\{1\}}$	$0.069367^{\{8\}}$	$0.061455^{\{6\}}$	$0.061114^{\{4\}}$	$0.061406^{\{5\}}$
		$\hat{\beta}$	$0.076225^{\{3\}}$	$0.078072^{\{4\}}$	$0.084119^{\{7\}}$	$0.072037^{\{1\}}$	$0.078586^{\{5\}}$	$0.073816^{\{2\}}$	$0.088067^{\{8\}}$	$0.080586^{\{6\}}$
	$\sum$ *Ranks*		$15^{\{2\}}$	$22^{\{3.5\}}$	$42^{\{8\}}$	$6^{\{1\}}$	$38^{\{7\}}$	$22^{\{3.5\}}$	$37^{\{6\}}$	$34^{\{5\}}$
125	BIAS	$\hat{\alpha}$	$0.118476^{\{1\}}$	$0.134094^{\{5\}}$	$0.155452^{\{8\}}$	$0.125452^{\{2\}}$	$0.147565^{\{7\}}$	$0.130606^{\{3\}}$	$0.132839^{\{4\}}$	$0.134527^{\{6\}}$
		$\hat{\beta}$	$0.167993^{\{2\}}$	$0.176812^{\{5\}}$	$0.180364^{\{6\}}$	$0.165227^{\{1\}}$	$0.180729^{\{7\}}$	$0.170747^{\{3\}}$	$0.194579^{\{8\}}$	$0.171912^{\{4\}}$
	MSE	$\hat{\alpha}$	$0.023258^{\{1\}}$	$0.029345^{\{6\}}$	$0.038982^{\{8\}}$	$0.024233^{\{2\}}$	$0.034698^{\{7\}}$	$0.026704^{\{3\}}$	$0.028139^{\{4\}}$	$0.028436^{\{5\}}$
		$\hat{\beta}$	$0.046381^{\{3\}}$	$0.050439^{\{5\}}$	$0.051135^{\{6\}}$	$0.042368^{\{1\}}$	$0.052449^{\{7\}}$	$0.044546^{\{2\}}$	$0.062435^{\{8\}}$	$0.047364^{\{4\}}$
	MRE	$\hat{\alpha}$	$0.04739^{\{1\}}$	$0.053638^{\{5\}}$	$0.062181^{\{8\}}$	$0.050181^{\{2\}}$	$0.059026^{\{7\}}$	$0.052243^{\{3\}}$	$0.053136^{\{4\}}$	$0.053811^{\{6\}}$
		$\hat{\beta}$	$0.067197^{\{2\}}$	$0.070725^{\{5\}}$	$0.072145^{\{6\}}$	$0.066091^{\{1\}}$	$0.072292^{\{7\}}$	$0.068299^{\{3\}}$	$0.077832^{\{8\}}$	$0.068765^{\{4\}}$
	$\sum$ *Ranks*		$10^{\{2\}}$	$31^{\{5\}}$	$42^{\{7.5\}}$	$9^{\{1\}}$	$42^{\{7.5\}}$	$17^{\{3\}}$	$36^{\{6\}}$	$29^{\{4\}}$
150	BIAS	$\hat{\alpha}$	$0.112162^{\{1\}}$	$0.121655^{\{4\}}$	$0.139967^{\{8\}}$	$0.114556^{\{2\}}$	$0.137315^{\{7\}}$	$0.120293^{\{3\}}$	$0.122264^{\{5\}}$	$0.122876^{\{6\}}$
		$\hat{\beta}$	$0.159623^{\{3\}}$	$0.163072^{\{5\}}$	$0.168161^{\{6\}}$	$0.152283^{\{2\}}$	$0.168542^{\{7\}}$	$0.150036^{\{1\}}$	$0.180895^{\{8\}}$	$0.161729^{\{4\}}$
	MSE	$\hat{\alpha}$	$0.0203^{\{2\}}$	$0.022597^{\{3\}}$	$0.031855^{\{8\}}$	$0.020019^{\{1\}}$	$0.029988^{\{7\}}$	$0.022608^{\{4\}}$	$0.023563^{\{5\}}$	$0.023642^{\{6\}}$
		$\hat{\beta}$	$0.039966^{\{3\}}$	$0.041972^{\{4\}}$	$0.045767^{\{6\}}$	$0.037165^{\{2\}}$	$0.046095^{\{7\}}$	$0.035307^{\{1\}}$	$0.051321^{\{8\}}$	$0.04252^{\{5\}}$
	MRE	$\hat{\alpha}$	$0.044865^{\{1\}}$	$0.048662^{\{4\}}$	$0.055987^{\{8\}}$	$0.045823^{\{2\}}$	$0.054926^{\{7\}}$	$0.048117^{\{3\}}$	$0.048906^{\{5\}}$	$0.049151^{\{6\}}$
		$\hat{\beta}$	$0.063849^{\{3\}}$	$0.065229^{\{5\}}$	$0.067264^{\{6\}}$	$0.060913^{\{2\}}$	$0.067417^{\{7\}}$	$0.060014^{\{1\}}$	$0.072358^{\{8\}}$	$0.064692^{\{4\}}$
	$\sum$ *Ranks*		$13^{\{2.5\}}$	$25^{\{4\}}$	$42^{\{7.5\}}$	$11^{\{1\}}$	$42^{\{7.5\}}$	$13^{\{2.5\}}$	$39^{\{6\}}$	$31^{\{5\}}$

The e^{th} central moment of X, say μ_n, is determined as follows:

$$\mu_e = E(x-\mu)^e = \sum_{w=0}^{\infty} (-1)^w \binom{e}{w} \mu_1^{\prime w} \mu_{e-w}^{\prime}. \tag{9}$$

The cumulants (H_c) of X can be obtained as follows:

$$H_c = \mu_c^{\prime} - \sum_{w=0}^{c-1} \binom{c-1}{w-1} w_r \mu_{c-r}^{\prime}. \tag{10}$$

The moment generating function of the ExRKD takes the following form:

$$M(t) = \sum_{m=0}^{\infty} \sum_{w=1}^{\infty} \Phi_w \frac{t^m}{m!} B_m, \tag{11}$$

and its characteristic function is generated by substituting *it* for t in the last equation.

Table 5: Simulation values of BIAS, MSE, and MRE for $\alpha = 5, \beta = 7$.

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE
25	BIAS	$\hat{\alpha}$	0.424125{3}	0.418816{1}	0.502114{8}	0.453139{4}	0.471347{7}	0.420383{2}	0.457444{5}	0.470156{6}
		$\hat{\beta}$	1.213283{3}	1.26109{4}	1.52221{7}	1.134502{1}	1.331473{5}	1.142832{2}	1.552684{8}	1.34303{6}
	MSE	$\hat{\alpha}$	0.313475{4}	0.295721{2}	0.401758{8}	0.306792{3}	0.351378{6}	0.281471{1}	0.338885{5}	0.369043{7}
		$\hat{\beta}$	2.888571{3}	2.973171{4}	4.691322{7}	2.170101{2}	3.443173{5}	2.132579{1}	5.549997{8}	3.520322{6}
	MRE	$\hat{\alpha}$	0.084825{3}	0.083763{1}	0.100423{8}	0.090628{4}	0.094269{7}	0.084077{2}	0.091489{5}	0.094031{6}
		$\hat{\beta}$	0.173326{3}	0.180156{4}	0.217459{7}	0.162072{1}	0.19021{5}	0.163262{2}	0.221812{8}	0.191861{6}
	∑ *Ranks*		19{4}	16{3}	45{8}	15{2}	35{5}	10{1}	39{7}	37{6}
50	BIAS	$\hat{\alpha}$	0.289599{2}	0.293456{3}	0.350989{7}	0.307804{5}	0.359051{8}	0.287945{1}	0.321541{6}	0.30213{4}
		$\hat{\beta}$	0.784693{2}	0.821394{4}	0.952284{7}	0.773842{1}	0.92615{6}	0.786926{3}	1.047538{8}	0.833305{5}
	MSE	$\hat{\alpha}$	0.137515{3}	0.136897{2}	0.200054{7}	0.145307{4}	0.201669{8}	0.125261{1}	0.16455{6}	0.149501{5}
		$\hat{\beta}$	1.051162{3}	1.156676{4}	1.844135{7}	0.959457{1}	1.446362{6}	0.960695{2}	2.032209{8}	1.257739{5}
	MRE	$\hat{\alpha}$	0.05792{2}	0.058691{3}	0.070198{7}	0.061561{5}	0.07181{8}	0.057589{1}	0.064308{6}	0.060426{4}
		$\hat{\beta}$	0.112099{2}	0.117342{4}	0.136041{7}	0.110549{1}	0.132307{6}	0.112418{3}	0.149648{8}	0.119044{5}
	∑ *Ranks*		14{2}	20{4}	42{7}	17{3}	42{7}	11{1}	42{7}	28{5}
75	BIAS	$\hat{\alpha}$	0.238391{2}	0.248294{3}	0.276135{8}	0.248778{4}	0.273071{7}	0.228686{1}	0.272073{6}	0.250092{5}
		$\hat{\beta}$	0.688096{5}	0.679673{4}	0.768152{6}	0.631936{1}	0.783999{7}	0.63501{2}	0.855302{8}	0.677334{3}
	MSE	$\hat{\alpha}$	0.089056{2}	0.098175{5}	0.123015{8}	0.096679{3}	0.122167{7}	0.084542{1}	0.117451{6}	0.097492{4}
		$\hat{\beta}$	0.769224{5}	0.747234{3}	1.006987{6}	0.63311{2}	1.039185{7}	0.624892{1}	1.238218{8}	0.750067{4}
	MRE	$\hat{\alpha}$	0.047678{2}	0.049659{3}	0.055227{8}	0.049756{4}	0.054614{7}	0.045737{1}	0.054415{6}	0.050018{5}
		$\hat{\beta}$	0.098299{5}	0.097096{4}	0.109736{6}	0.090277{1}	0.112{7}	0.090716{2}	0.122186{8}	0.096762{3}
	∑ *Ranks*		21{3}	22{4}	42{7}	15{2}	42{7}	8{1}	42{7}	24{5}
100	BIAS	$\hat{\alpha}$	0.207156{2}	0.217747{5}	0.244345{8}	0.206672{1}	0.233912{7}	0.209156{3}	0.232231{6}	0.212134{4}
		$\hat{\beta}$	0.618257{5}	0.594219{4}	0.692723{7}	0.549872{1}	0.644486{6}	0.579368{2}	0.738799{8}	0.593609{3}
	MSE	$\hat{\alpha}$	0.066763{3}	0.07673{5}	0.094387{8}	0.064204{1}	0.08983{7}	0.066149{2}	0.086771{6}	0.071822{4}
		$\hat{\beta}$	0.623719{5}	0.576807{3}	0.809345{7}	0.479949{1}	0.711039{6}	0.510618{2}	0.906671{8}	0.585592{4}
	MRE	$\hat{\alpha}$	0.041431{2}	0.043549{5}	0.048869{8}	0.041334{1}	0.046782{7}	0.041831{3}	0.046446{6}	0.042427{4}
		$\hat{\beta}$	0.088322{5}	0.084888{4}	0.09896{7}	0.078553{1}	0.092069{6}	0.082767{2}	0.105543{8}	0.084801{3}
	∑ *Ranks*		22{3.5}	26{5}	45{8}	6{1}	39{6}	14{2}	42{7}	22{3.5}
125	BIAS	$\hat{\alpha}$	0.175227{1}	0.183615{4}	0.212005{8}	0.181934{3}	0.205874{6}	0.177247{2}	0.208815{7}	0.189139{5}
		$\hat{\beta}$	0.483692{1}	0.531625{4}	0.596861{7}	0.518691{3}	0.580656{6}	0.511083{2}	0.652667{8}	0.55218{5}
	MSE	$\hat{\alpha}$	0.050023{1}	0.054502{4}	0.072149{8}	0.051608{3}	0.069579{7}	0.05096{2}	0.067353{6}	0.05644{5}
		$\hat{\beta}$	0.394679{1}	0.445153{4}	0.570226{7}	0.416821{3}	0.521715{6}	0.414464{2}	0.664917{8}	0.483635{5}
	MRE	$\hat{\alpha}$	0.035045{1}	0.036723{4}	0.042401{8}	0.036387{3}	0.041175{6}	0.035449{2}	0.041763{7}	0.037828{5}
		$\hat{\beta}$	0.069099{1}	0.075946{4}	0.085266{7}	0.074099{3}	0.082951{6}	0.073012{2}	0.093238{8}	0.078883{5}
	∑ *Ranks*		6{1}	24{4}	45{8}	18{3}	37{6}	12{2}	44{7}	30{5}
150	BIAS	$\hat{\alpha}$	0.16481{1}	0.177895{5}	0.197563{8}	0.170441{2}	0.195316{7}	0.174931{4}	0.183598{6}	0.170571{3}
		$\hat{\beta}$	0.464907{3}	0.483013{5}	0.550978{7}	0.442317{1}	0.520645{6}	0.455147{2}	0.559226{8}	0.48077{4}
	MSE	$\hat{\alpha}$	0.042474{1}	0.048229{5}	0.062796{8}	0.044627{2}	0.059323{7}	0.046728{4}	0.052764{6}	0.045857{3}
		$\hat{\beta}$	0.352247{3}	0.374937{4}	0.495016{7}	0.313611{1}	0.429923{6}	0.321892{2}	0.517236{8}	0.375403{5}
	MRE	$\hat{\alpha}$	0.032962{1}	0.035579{5}	0.039513{8}	0.034088{2}	0.039063{7}	0.034986{4}	0.03672{6}	0.034114{3}
		$\hat{\beta}$	0.066415{3}	0.069002{5}	0.078711{7}	0.063188{1}	0.074378{6}	0.065021{2}	0.079889{8}	0.068681{4}
	∑ *Ranks*		12{2}	29{5}	45{8}	9{1}	39{6}	18{3}	42{7}	22{4}

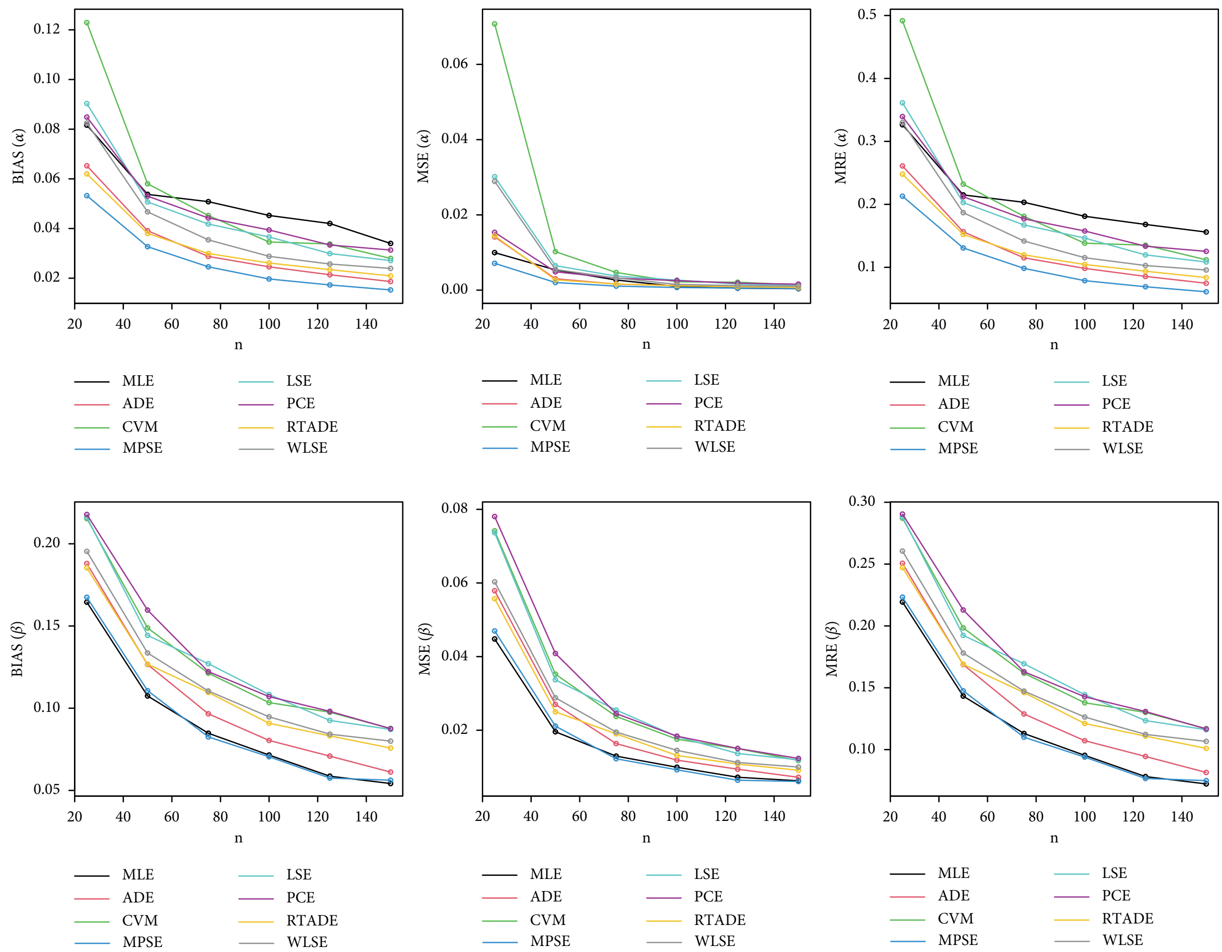

Figure 3: Graphical representation of BIAS, MSE, and MRE values in Table 1.

3.4. Order Statistics. The PDF and CDF of the ith order statistic for the ExRKD are defined, respectively, as follows:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x) = \alpha\beta n!\log^{-i}(2)\left(\frac{x}{1-x}\right)^{\alpha-1} \times \frac{\left(1-e^{-(x/1-x)^{\alpha}}\right)^{\beta}\log^{i-1}\left(\left(1-e^{-(x/1-x)^{\alpha}}\right)^{\beta}+1\right)\left(1-\log\left(\left(1-e^{-(x/1-x)^{\alpha}}\right)^{\beta}+1\right)/\log(2)\right)^{n-i}}{(x-1)^{2}\,\Gamma(i)\left(e^{(x/1-x)^{\alpha}}-1\right)\Gamma(-i+n+1)\left(\left(1-e^{-(x/1-x)^{\alpha}}\right)^{\beta}+1\right)},$$

$$F_{i:n}(x) = \sum_{r=i}^{n}\binom{n}{r}(F(x))^{r}(1-F(x))^{n-r} = \frac{\log^{-i}(2)\Gamma(n+1)\log^{i}\left(\left(1-e^{-(x/1-x)^{\alpha}}\right)^{\beta}+1\right)\left(1-\log\left(\left(1-e^{-(x/1-x)^{\alpha}}\right)^{\beta}+1\right)/\log(2)\right)^{n-i}T}{\Gamma(-i+n+1)}, \quad (12)$$

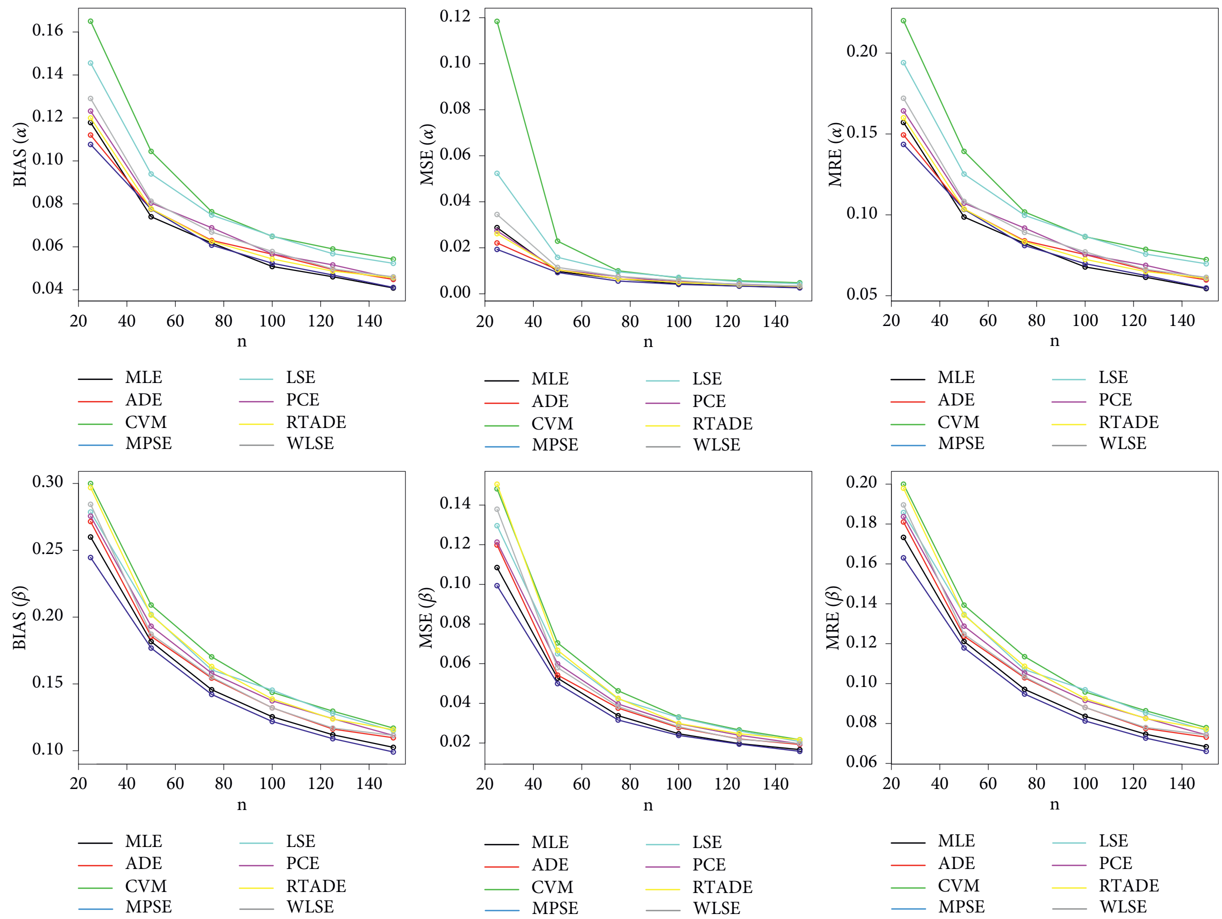

Figure 4: Graphical representation of BIAS, MSE, and MRE values in Table 2.

where $T = {}_2\widetilde{F}_1\,(1, i-n; i+1; \log(2)/\log(1/2\,((1- e^{-(x/1-x)^{\alpha}})^{\beta} +1)) + 1)$ is a regularized hypergeometric function.

4. Estimation Methods

This section discusses how to estimate the ExRK model parameters using several classical estimation approaches by maximization or minimization of the considered function. For more details, see [15–19].

For $x_1, \ldots, x_n$ defined as a random sample from ExRK distribution, maximum likelihood estimation (MLE) obtains estimators of ExRK distribution by maximizing the log-likelihood function specified in the following equation:

$$l = \beta \sum_{i=1}^{n} \log\left(1 - e^{-(x_i/1-x_i)^{\alpha}}\right) - \sum_{i=1}^{n} \log\left(\left(1 - e^{-(x_i/1-x_i)^{\alpha}}\right)^{\beta} + 1\right) - \sum_{i=1}^{n} \log\left(e^{(x_i/1-x_i)^{\alpha}} - 1\right) + (\alpha - 1) \sum_{i=1}^{n} \log\left(\frac{x_i}{1-x_i}\right) - \sum_{i=1}^{n} \log\left((x_i - 1)^2\right) + n \log\left(\frac{\alpha\beta}{\log(2)}\right). \quad (13)$$

For $x_1 \leq \cdots \leq x_n$ defined as a sorted random sample from ExRK distribution, Anderson–Darling estimation (ADE) obtains estimators of ExRK distribution by minimizing the function specified in the following equation:

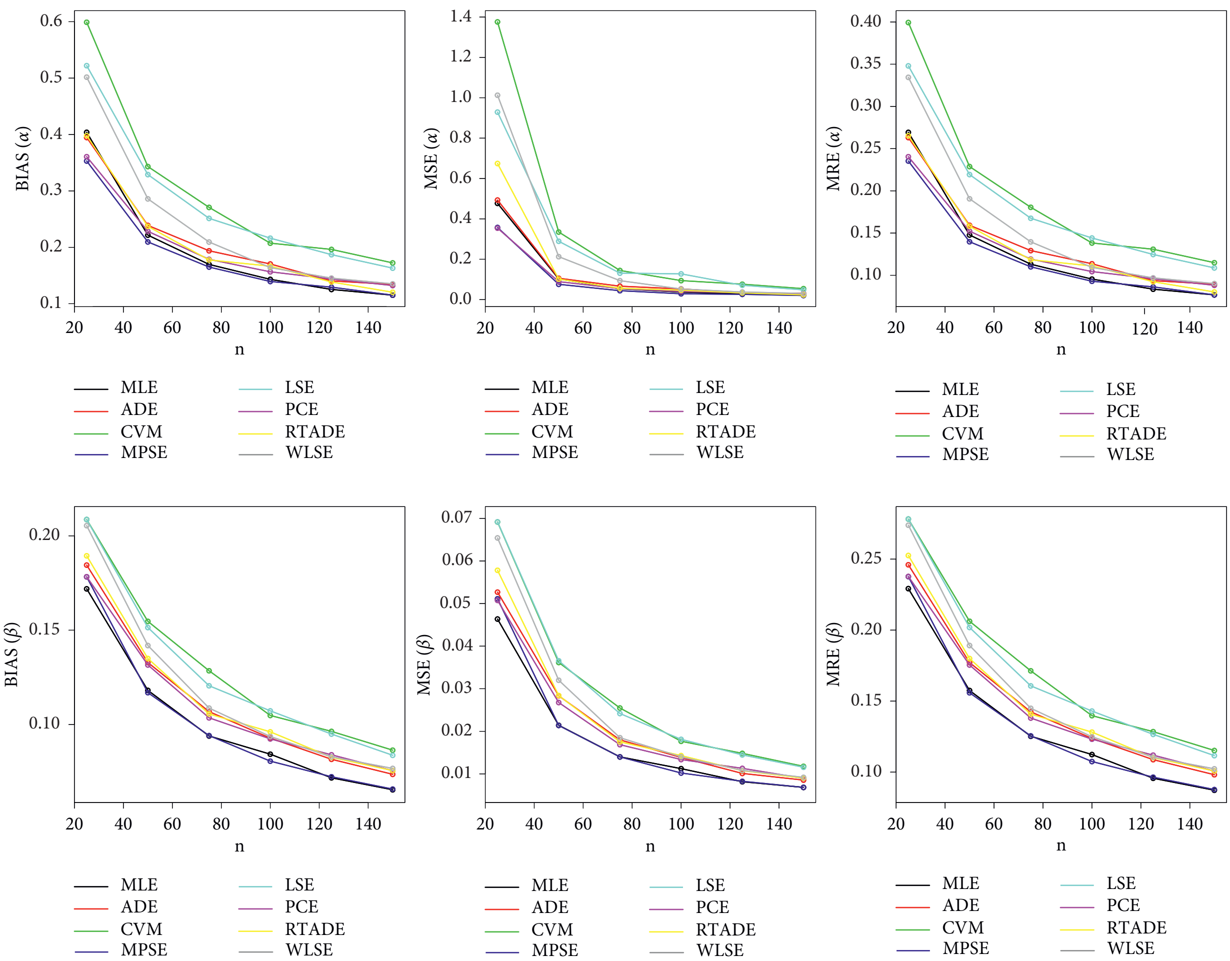

Figure 5: Graphical representation of BIAS, MSE, and MRE values in Table 3.

$$A = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(x_i) + \log S(x_i)\right]$$
$$= -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(\frac{\log\left(\left(1-e^{-\left(x_i/1-x_i\right)^{\alpha}}\right)^{\beta}+1\right)}{\log(2)}\right) + \log\left(1 - \frac{\log\left(\left(1-e^{-\left(x_i/1-x_i\right)^{\alpha}}\right)^{\beta}+1\right)}{\log(2)}\right)\right]. \tag{14}$$

For $x_1 \le \cdots \le x_n$, defined as a sorted random sample from ExRK distribution, Cramér–von Mises estimation (CVME) obtains estimators of ExRK distribution by minimizing the function specified in the following equation:

$$C = -\frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_i) - \frac{2i-1}{2n}\right]^2$$
$$= -\frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{\log\left(\left(1-e^{-\left(x_i/1-x_i\right)^{\alpha}}\right)^{\beta}+1\right)}{\log(2)} - \frac{2i-1}{2n}\right]^2. \tag{15}$$

For $x_1 \le \cdots \le x_n$, defined as a sorted random sample from ExRK distribution, maximum product of spacings estimation (MPSE) obtains estimators of ExRK distribution by maximizing the function specified in the following equation:

$$MP = \frac{1}{n+1}\sum_{i=1}^{n+1}\log H_i, \tag{16}$$

where

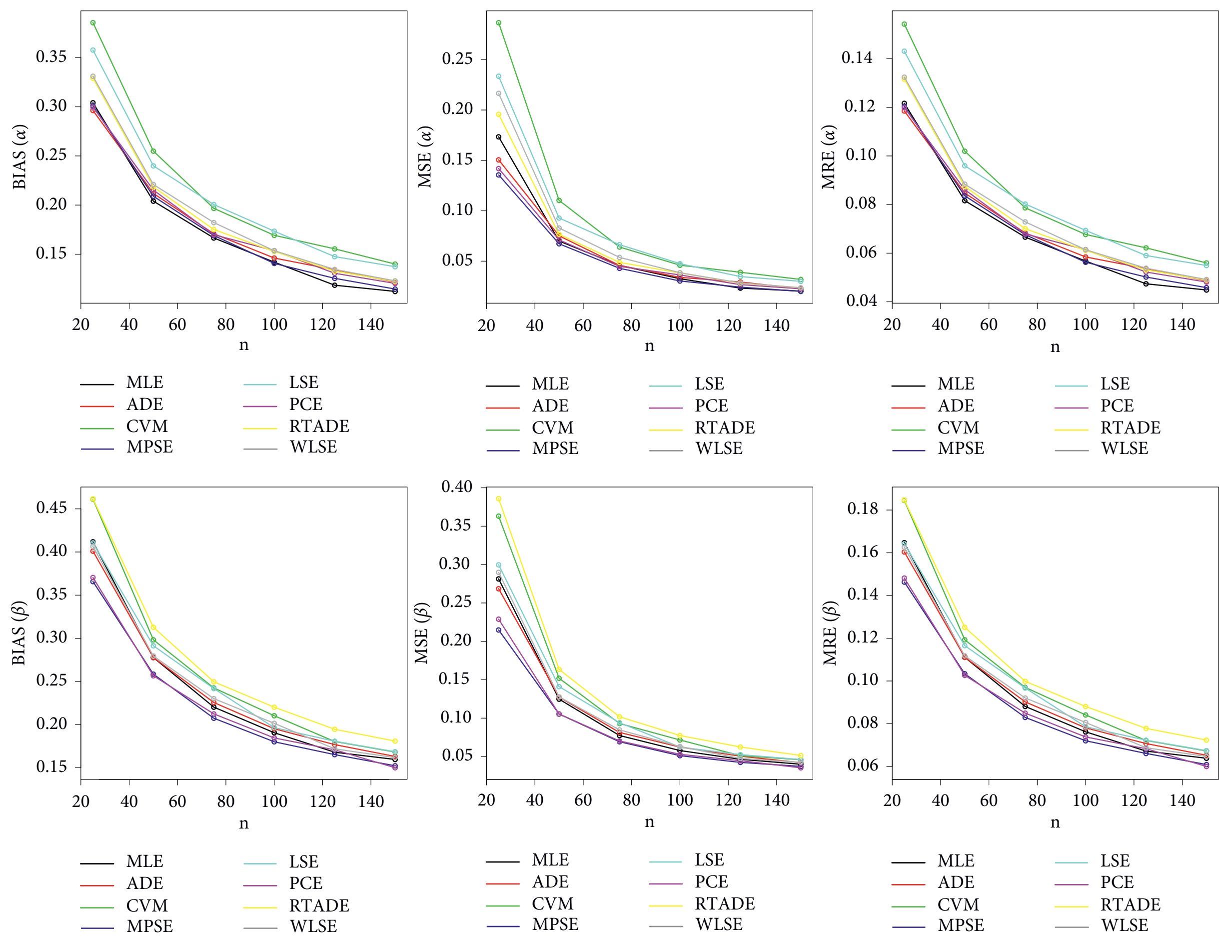

Figure 6: Graphical representation of BIAS, MSE, and MRE values in Table 4.

$$H_i = F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right) = \frac{\log\left(\left(1 - e^{-\left(x_i/1-x_i\right)^{\alpha}}\right)^{\beta} + 1\right)}{\log(2)} - \frac{\log\left(\left(1 - e^{-\left(x_{i-1}/1-x_{i-1}\right)^{\alpha}}\right)^{\beta} + 1\right)}{\log(2)}. \tag{17}$$

For $x_1 \leq \cdots \leq x_n$, defined as a sorted random sample from ExRK distribution, least-squares estimation (LSE) obtains estimators of ExRK distribution by minimizing the function specified in the following equation:

$$V = \sum_{i=1}^{n}\left[F\left(x_i\right) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\left[\frac{\log\left(\left(1 - e^{-\left(x_i/1-x_i\right)^{\alpha}}\right)^{\beta} + 1\right)}{\log(2)} - \frac{i}{n+1}\right]^2. \tag{18}$$

For $x_1 \leq \cdots \leq x_n$, defined as a sorted random sample from ExRK distribution, percentile estimation (PCE) obtains estimators of ExRK distribution by minimizing the function specified in the following equation:

$$PC = \sum_{i=1}^{n}\left[x_i - Q\left(k_i\right)\right]^2 = \sum_{i=1}^{n}\left[x_i - 1 + \frac{1}{\left(-\log\left(1 - \left(2^{p} - 1\right)^{1/\beta}\right)\right)^{1/\alpha} + 1}\right]^2, k_i = \frac{i}{n+1}. \tag{19}$$

For $x_1 \leq \cdots \leq x_n$, defined as a sorted random sample from ExRK distribution, right-tail Anderson–Darling estimation (RTADE) obtains estimators of ExRK distribution by minimizing the function specified in the following equation:

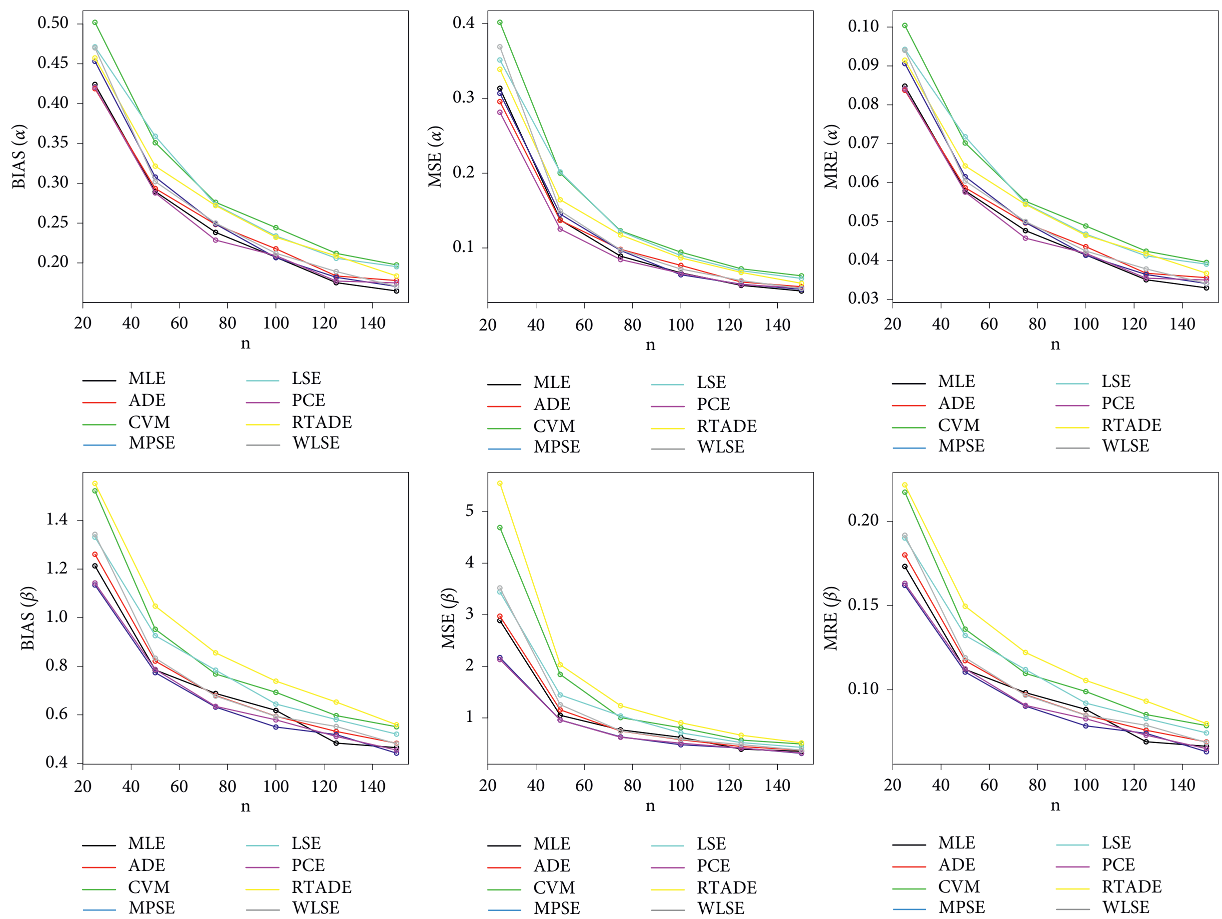

Figure 7: Graphical representation of BIAS, MSE, and MRE values in Table 5.

$$R = \frac{n}{2} - 2\sum_{i=1}^{n} F(x_i) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S(x_{n+1-i})$$
$$= \frac{n}{2} - 2\sum_{i=1}^{n}\left(\frac{\log\left(\left(1-e^{-(x_i/1-x_i)^{\alpha}}\right)^{\beta}+1\right)}{\log(2)}\right)$$
$$-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(\frac{\log\left(\left(1-e^{-(x_{n+1-i}/1-x_{n+1-i})^{\alpha}}\right)^{\beta}+1\right)}{\log(2)}\right). \tag{20}$$

For $x_1 \leq \cdots \leq x_n$, defined as a sorted random sample from ExRK distribution, weighted least-squares estimation (WLSE) obtains estimators of ExRK distribution by minimizing the function specified in the following equation:

$$W = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_i) - \frac{i}{n+1}\right]^2$$
$$= \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}$$
$$\bullet\left[\frac{\log\left(\left(1-e^{-(x_i/1-x_i)^{\alpha}}\right)^{\beta}+1\right)}{\log(2)} - \frac{i}{n+1}\right]^2. \tag{21}$$

Table 6: Partial and overall ranks of all the methods of estimation of proposed distribution by various values of model parameters.

Parameter	n	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE
$\alpha = 0.25, \beta = 0.75$	25	2.0	4.0	8.0	1.0	6.5	6.5	3.0	5.0
	50	4.0	3.0	8.0	1.0	6.0	7.0	2.0	5.0
	75	4.0	2.0	7.0	1.0	8.0	6.0	3.0	5.0
	100	4.0	2.0	6.0	1.0	7.5	7.5	3.0	5.0
	125	4.0	2.0	7.5	1.0	6.0	7.5	3.0	5.0
	150	3.0	2.0	7.0	1.0	6.0	8.0	4.0	5.0
$\alpha = 0.75, \beta = 1.5$	25	3.0	2.0	8.0	1.0	6.5	4.0	5.0	6.5
	50	2.0	3.0	8.0	1.0	7.0	4.0	6.0	5.0
	75	2.0	3.0	8.0	1.0	7.0	6.0	5.0	4.0
	100	2.0	3.0	7.0	1.0	8.0	6.0	4.0	5.0
	125	2.0	3.0	8.0	1.0	7.0	6.0	4.0	5.0
	150	2.0	3.0	8.0	1.0	7.0	4.0	6.0	5.0
$\alpha = 1.5, \beta = 0.75$	25	3.0	4.0	8.0	1.0	7.0	2.0	5.0	6.0
	50	2.0	5.0	8.0	1.0	7.0	3.0	4.0	6.0
	75	2.0	5.0	8.0	1.0	7.0	3.0	4.0	6.0
	100	2.0	5.0	7.0	1.0	8.0	3.0	6.0	4.0
	125	1.0	3.5	8.0	2.0	7.0	5.5	3.5	5.5
	150	1.0	5.0	8.0	2.0	7.0	3.5	3.5	6.0
$\alpha = 2.5, \beta = 2.5$	25	4.0	3.0	8.0	1.0	6.0	2.0	7.0	5.0
	50	2.0	4.0	8.0	1.0	6.5	3.0	6.5	5.0
	75	2.5	4.0	7.0	1.0	8.0	2.5	6.0	5.0
	100	2.0	3.5	8.0	1.0	7.0	3.5	6.0	5.0
	125	2.0	5.0	7.5	1.0	7.5	3.0	6.0	4.0
	150	2.5	4.0	7.5	1.0	7.5	2.5	6.0	5.0
$\alpha = 5, \beta = 7$	25	4.0	3.0	8.0	2.0	5.0	1.0	7.0	6.0
	50	2.0	4.0	7.0	3.0	7.0	1.0	7.0	5.0
	75	3.0	4.0	7.0	2.0	7.0	1.0	7.0	5.0
	100	3.5	5.0	8.0	1.0	6.0	2.0	7.0	3.5
	125	1.0	4.0	8.0	3.0	6.0	2.0	7.0	5.0
	150	2.0	5.0	8.0	1.0	6.0	3.0	7.0	4.0
$\sum$ ranks		75.5	108.0	229.5	38.0	205.0	119.0	153.5	151.5
Overall rank		2	3	8	1	7	4	6	5

Table 7: List of the compared distributions.

Distribution	Abbreviation	Author(s)
Reduced Kies distribution	RKD	Kumar and Dharmaja [14]
Exponentiated Topp–Leone distribution	ETLD	Pourdarvish et al. [2]
Topp–Leone distribution	TLD	Topp and Leone [13]
Log-Lindley distribution	LLD	Gómez-Déniz et al. [3]
Power log-Lindley distribution	PLLD	Abd El-Bar et al. [6]
Cosine-sine distribution	CSD	Abd El-Bar et al. [6]
Power logarithmic distribution	PLD	Abd El-Bar et al. [7]
Transformed gamma distribution	TGD	Grassia [4]
Log-gamma distribution	LGD	Amini et al. [11]
Log-weighted power distribution	LWPD	Chesneau [12]
Transmuted power distribution	TPD	Chesneau [12]
Beta distribution	BD	

Table 8: Numerical values for analyzing the milk real data set.

Model	$-L$	C_1	C_2	C_3	C_4	GoF_1	GoF_2	GoF_3	$GoF_3(p)$	Est. parameters (SEs)
ExRKD	−27.1801	−50.3602	−50.2448	−45.0146	−48.1932	0.619746	0.0962025	0.0733981	0.611643	$\widehat{\alpha} = 0.975641\,(0.0721869)$ $\widehat{\beta} = 1.72127\,(0.167809)$
RKD	−22.1004	−42.2008	−42.1627	−39.528	−41.1173	4.93197	1.00894	0.177934	0.00228316	$\widehat{a} = 1.09136\ (0.0688131)$
BD	−23.7772	−43.5545	−43.4391	−38.2088	−41.3874	1.38527	0.22823	0.0909919	0.338375	$\widehat{a} = 2.41252\ (0.314491)$ $\widehat{b} = 2.82967\ (0.374422)$
ETLD	−23.3428	−42.6856	−42.5702	−37.3399	−40.5185	1.54214	0.263196	0.0950351	0.288614	$\widehat{a} = 2.46832\ (0.301618)$ $\widehat{b} = 1.30496\ (0.179222)$
TLD	−21.5262	−41.0524	−41.0143	−38.3796	−39.9689	1.88132	0.284751	0.0972409	0.263762	$\widehat{a} = 2.08023\ (0.201103)$
LLD	−20.7039	−37.4077	−37.2923	−32.0621	−35.2407	2.18556	0.338619	0.103804	0.199137	$\widehat{\beta} = 2.22468\ (0.0812055)$ $\widehat{\lambda} = 1 \times 10^{-9}\ (0.0519599)$
PLLD	−20.6977	−35.3954	−35.1624	−27.3769	−32.1448	2.18706	0.338862	0.103846	0.198773	$\widehat{\alpha} = 55.9927\ (17.2231)$ $\widehat{\sigma} = 0.0397265\ (0.0100298)$ $\widehat{\lambda} = 0.00551514\ (0.128159)$
CSD	−6.56401	−11.128	−11.0899	−8.45519	−10.0445	15.0079	3.28194	0.304419	< 0.00001	$\widehat{\lambda} = 1.18423 \times 10^{-9}\ (7.99982 \times 10^{-7})$
PLD	−20.7039	−35.4077	−35.1747	−27.3892	−32.1571	2.18556	0.338618	0.103804	0.199136	$\widehat{\alpha} = 1.22468\ (0.152076)$ $\widehat{\beta} = 0.00140746\ (6.03799)$ $\widehat{\delta} = 810216\ (249258)$
TGD	−23.1814	−44.3627	−44.3246	−41.6899	−43.2792	1.98331	0.378296	0.120438	0.0897177	$\widehat{\sigma} = 2.83623\ (0.193881)$
LGD	−20.7039	−39.4077	−39.3696	−36.7349	−38.3242	2.18556	0.338618	0.103804	0.199136	$\widehat{\sigma} = 2.22468\ (0.152076)$
LWPD	−20.7039	−37.4077	−37.2923	−32.0621	−35.2407	2.18556	0.338618	0.103804	0.199136	$\widehat{\alpha} = 2.22468\ (0.372222)$ $\widehat{\lambda} = 1.0\ (0.414309)$
TPD	−20.434	−36.8681	−36.7527	−31.5224	−34.701	2.42687	0.379845	0.11484	0.118918	$\widehat{\alpha} = 1.66014\ (0.253083)$ $\widehat{\lambda} = 1.0\ (0.549492)$

Table 9: Numerical values for analyzing the COVID-19 real data set.

Model	$-L$	C_1	C_2	C_3	C_4	GoF_1	GoF_2	GoF_3	$GoF_3(p)$	Est. parameters (SEs)
ExRKD	−58.7972	−113.594	−113.404	−109.215	−111.864	0.900701	0.155322	0.106187	0.446282	$\hat{\alpha} = 0.817392$ (0.0475411) $\hat{\beta} = 8.2891$ (1.03689)
RKD	8.72551	19.451	19.5135	21.6407	20.3163	53.0981	10.4062	0.616969	< 0.00001	$\hat{a} = 0.664682$ (0.051041)
BD	−57.5743	−111.149	−110.958	−106.769	−109.418	1.05197	0.178341	0.114785	0.349424	$\hat{a} = 12.7943$ (2.2291) $\hat{b} = 4.89941$ (0.826972)
ETLD	−56.7872	−109.574	−109.384	−105.195	−107.844	1.23021	0.214243	0.12548	0.249774	$\hat{a} = 15.3167$ (2.19208) $\hat{b} = 1.86404$ (0.348079)
TLD	−51.5544	−101.109	−101.046	−98.9191	−100.244	2.04752	0.27752	0.18133	0.0260671	$\hat{a} = 10.5284$ (1.29596)
LLD	−46.7273	−89.4547	−89.2642	−85.0753	−87.7242	3.39896	0.509096	0.220321	0.00329848	$\hat{\beta} = 5.96527$ (0.376555) $\hat{\lambda} = 9.40395 \times 10^{-9}$ (0.0200914)
PLLD	−46.7271	−87.4542	−87.0671	−80.8853	−84.8585	3.39834	0.508938	0.220297	0.003303	$\hat{\alpha} = 814.282$ (266.225) $\hat{\sigma} = 0.0073251$ (0.0020101) $\hat{\lambda} = 0.001136$ (0.778915)
CSD	32.6342	67.2684	67.3309	69.4581	68.1337	76.827	13.4103	0.6956	< 0.00001	$\hat{\lambda} = 1.18423 \times 10^{-9}$ (1.01859×10^{-7})
PLD	−46.7273	−87.4547	−87.0676	−80.8857	−84.8589	3.39896	0.509096	0.220321	0.00329848	$\hat{\alpha} = 4.96527$ (0.51921) $\hat{\beta} = 0.001398$ (13.1574) $\hat{\delta} = 1.31665 \times 10^{7}$ (4.30822×10^{6})
TGD	−26.0342	−50.0683	−50.0058	−47.8787	−49.2031	9.06226	1.64558	0.273367	0.000103993	$\hat{\sigma} = 1.46942$ (0.127897)
LGD	−46.7273	−91.4547	−91.3922	−89.265	−90.5894	3.39896	0.509096	0.220321	0.00329848	$\hat{\sigma} = 5.96527$ (0.51921)
LWPD	−46.7273	−89.4547	−89.2642	−85.0753	−87.7242	3.39896	0.509096	0.220321	0.00329849	$\hat{\alpha} = 5.96527$ (1.40819) $\hat{\lambda} = 1.0$ (0.715182)
TPD	−45.3289	−86.6579	−86.4674	−82.2785	−84.9274	3.73667	0.565396	0.226964	0.00222829	$\hat{\alpha} = 4.38457$ (0.777979) $\hat{\lambda} = 1.0$ (0.759)

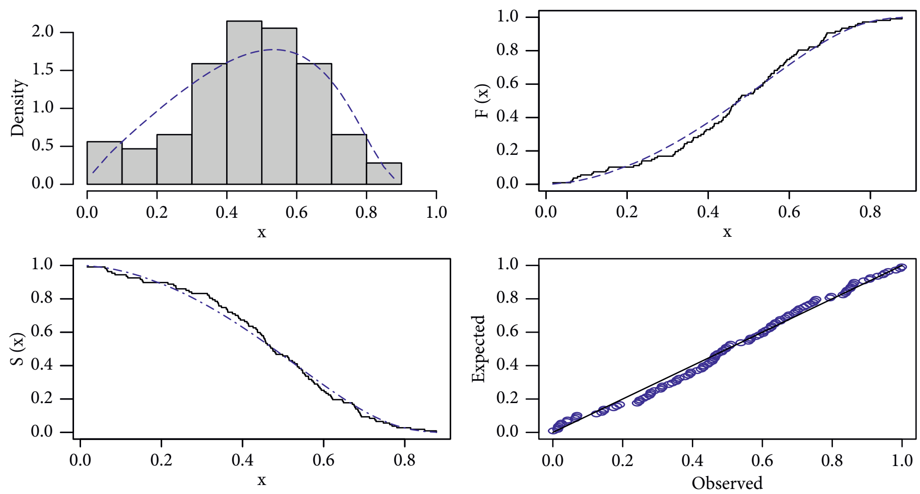

Figure 8: Histogram of the milk data set with the fitted PDF, CDF, SF, and P–P plots.

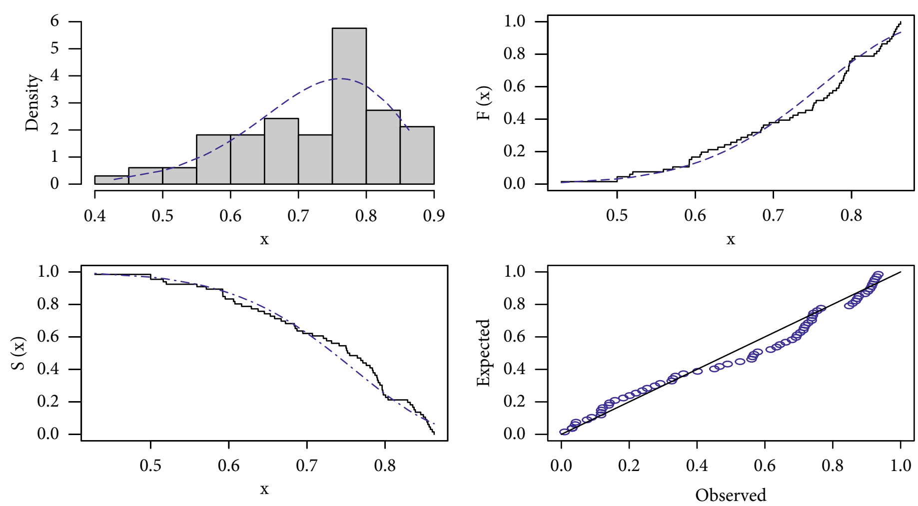

Figure 9: Histogram of the COVID-19 data set with the fitted PDF, CDF, SF, and P–P plots.

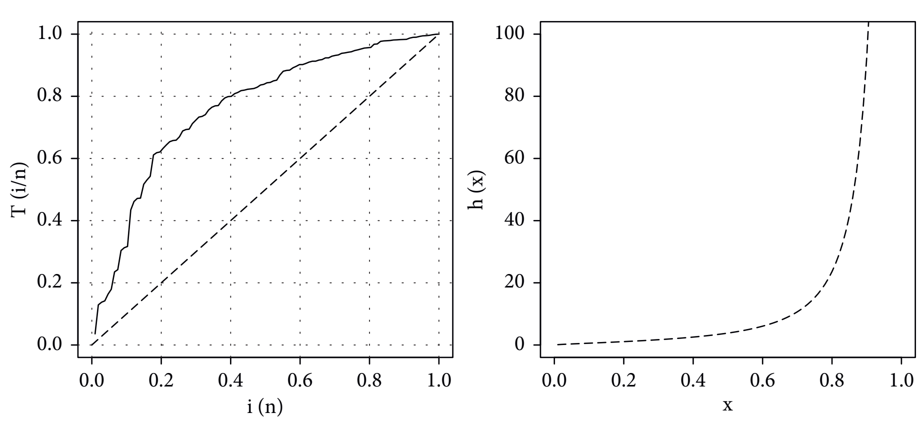

Figure 10: TTT plot and fitted HRF of ExRK model for the milk data set.

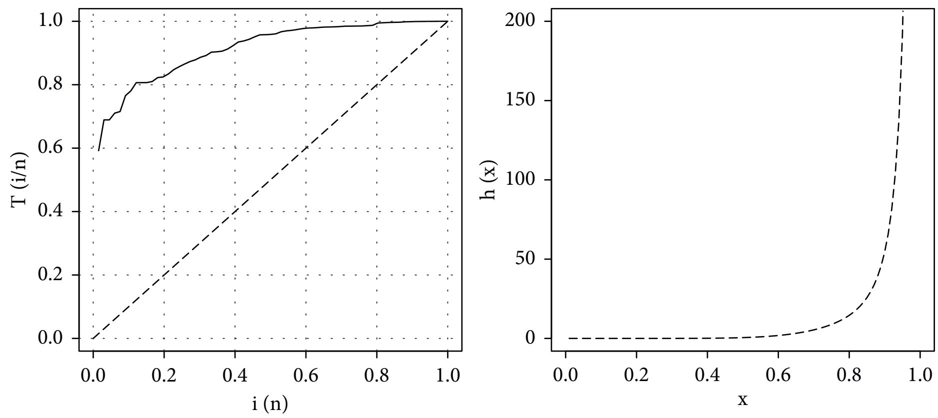

FIGURE 11: TTT plot and fitted HRF of ExRK model for the COVID-19 data set.

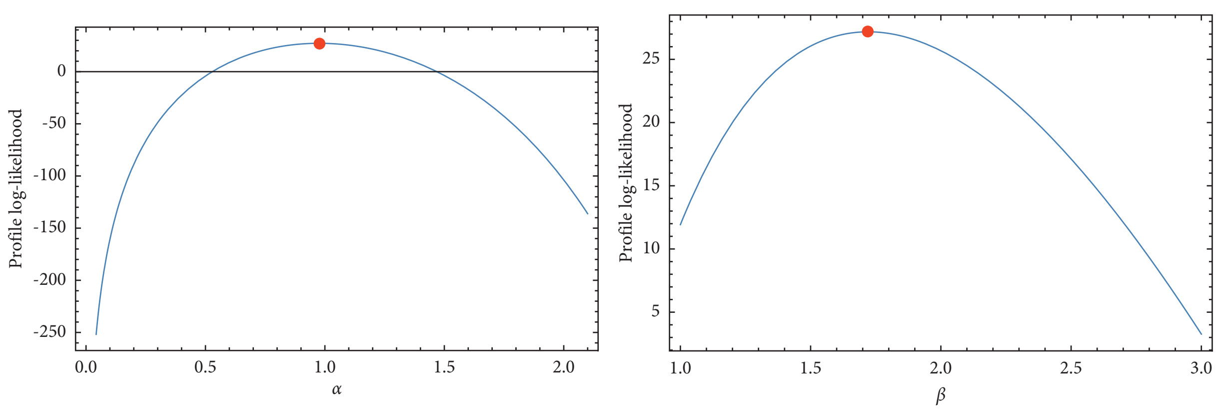

FIGURE 12: The profile of the log-likelihood functions for α and β of the milk data set.

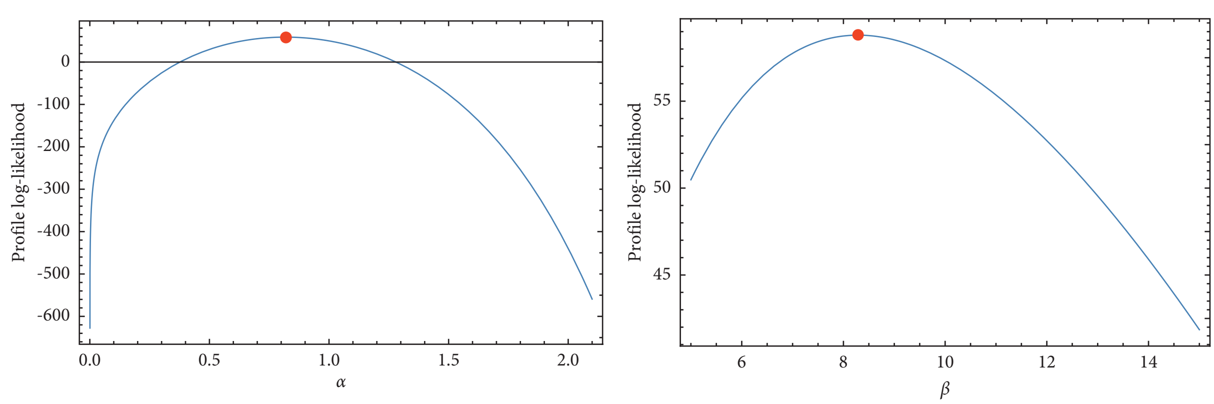

FIGURE 13: The profile of the log-likelihood functions for α and β of the COVID-19 data set.

5. Numerical Simulation

In order to find estimators of our proposed model using data sets are generated at random, all of the estimation techniques discussed in the previous section will be employed in this section. Our goal is to examine both the performance of these estimation methods and the behavior of the model estimators that we have presented in this research. In addition, we will assess the effectiveness of these strategies using various measures such as average of bias (BIAS),

$|\text{Bias}(\widehat{\omega})| = 1/H \sum_{i=1}^{H} |\widehat{\omega} - \omega|$; mean squared errors (MSE), $\text{MSE} = 1/H \sum_{i=1}^{H} (\widehat{\omega} - \omega)^2$; and mean relative errors (MRE), $\text{MRE} = 1/H \sum_{i=1}^{H} |\widehat{\omega} - \omega|/\omega$, $\omega = (\alpha, \beta)$. Using the simulation, you can figure out what the most reasonable estimation method for the model parameters would be. In our simulation, we generated one thousand ($H = 1000$) samples of size $n = 25$, 50, 75, 100, 125, and 150.

The results of simulations are presented in Tables 1–5, while the graphical representations of these tables, which correlate to the numerical results of simulations, are shown in Figures 3–7, respectively. The power value reveals how effective a technique is in comparison to all other techniques. Our estimators' partial and overall rankings are displayed in Table 6.

5.1. Simulation Outcomes. Based on the results of the simulation and the ranking tables, we reach the following conclusion:

(i) Almost all of the estimators exhibit the property of consistency in their results.

(ii) As sample size rises, the BIAS of all estimators decreases for all techniques of estimation, regardless of method.

(iii) As sample size rises, the MSE of all estimators decreases for all techniques of estimation, regardless of method.

(iv) As sample size rises, the MRE of all estimators decreases for all techniques of estimation, regardless of method.

(v) The most preferred technique for estimation is to use the maximum spacing product. If researchers have data that matches our proposed model, we recommend that they use this technique.

6. Real Data Analysis

Real-world data is used to demonstrate the distribution's adaptability in this section. The first real data set consists of 107 observations about the total milk production in the first birth of 107 cows from SINDI race which was studied by Cordeiro and dos Santos Brito [20]. The second real data set consists of 66 observations which refer to the rates of recovery from COVID-19 infections in Spain (from the 3rd of March to the 7th of May, 2020).

In order to show the flexibility of our proposed model, we compare it with well-known models. All of the compared models are specified in Table 7 for $0 < x < 1$.

In order to choose the most appropriate model for the real-world data set, we use a set of analytical criteria, including Akaike information criterion (C_1), correct Akaike information criterion (C_2), Bayesian information criterion (C_3), and Hannan–Quinn information criterion (C_4). In addition, we consider a variety of goodness-of-fit statistics in our selection, such as Anderson–Darling (GoF_1), Cramér–von Mises (GoF_2), and Kolmogorov–Smirnov (GoF_3) with its p value ($GoF_3(p)$).

As demonstrated in Tables 8 and 9, the analytical measures, as well as the MLE and corresponding standard errors (SE), are provided for the real data sets that were under consideration for evaluation. As a result, we may conclude that the ExRK model performs much better than the other comparable models. Additionally, the P–P plot and the estimated PDF, CDF, and SF plots are used to fit the proposed ExRK model to the milk and the COVID-19 data sets shown in Figures 8 and 9. Using the milk and the COVID-19 data sets, the suggested model was shown to be a good fit. TTT and estimated HRF of the ExRK model plots are shown in Figures 10 and 11 for the two real data sets, respectively. The behavior of the log-likelihood function, which is a unimodal function, with estimated parameters is shown in Figures 12 and 13 for the two real data sets.

7. Conclusion

An extension of the reduced Kies distribution, which is discussed in more depth and detailed , is built in this paper as a statistical model. Its PDF may be left- or right-skewed, symmetrical, decreasing, or even a "bathtub" form. Some of its most notable statistical properties are explored through mathematical analysis. Eight traditional estimating methods are discussed in details, and these methods are used to estimate the ExRKD parameters. Simulation findings show that the proposed estimators perform quite well. In addition, the ExRKD's practical use is discussed by examining its performance in comparison to well known models using real-world data sets.

Data Availability

The data used to support the findings of this study are included in the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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