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Research Article

Computation of the Complexity of Networks under Generalized Operations

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The connected and acyclic components contained in a network are identified by the computation of its complexity, where complexity of a network refers to the total number of spanning trees present within. The article in hand deals with the enumeration of the complexity of various networks' operations such as sum ($K_{2,n}+W_3$, $K_{2,n}+nK_1$, K_n+S_n), product ($K_{2,n}\boxtimes K_2$, $K_{2,n}\ltimes K_2$, $K_n\times K_2$, $K_n\boxtimes K_2$), difference ($K_{2,n}\ominus K_2$), and the conjunction of S_n with K_2. All our computations have been concluded by implementation of the methods of linear algebra and matrix theory. Our derivations will also be highlighted with the assistance of 3D plots at the end of this article.

1. Introduction

Only simple network $\mathbb{G}=(V(\mathbb{G}),E(\mathbb{G}))$ shall be dealt with throughout the paper. One of the most useful algebraic invariants is the complexity, i.e., number of spanning trees in a network admitting roots in combinatorics, algebraic graph theory, and networking. It is prominently linked with network engineering and particular branches of computer sciences that deal in the security designs specifically. Realistically, concreteness and precision in a network are based on the number of spanning trees it possesses. This indicates that complexity is an identifier for the quality of a network. Certain applications of complexity in different fields of mathematics and physics can be observed in [1–4]. For instance, we are living in an era of networking. The tools similar to complexity ensure the robustness and accuracy in a network so that one can obtain interruption free signals, since the complexity is an identifier of the number of connected and acyclic pathways present in a network, where every such pathway contains all junctions or vertices present in a network. So, this invariant helps in the enhancement of robustness of wireless sensor networks (WSNs) and other similar mobile networks by relating the total number of spanning trees present within. Another application of complexity can be observed in the security design of a sensitive area of a building. Say there are several secured chambers, and there are legitimate passages only to reach to those chambers. One legitimate passage can be identified by a unique pathway. That is, no cyclic pathway is allowed from one chamber to another. A programming-based software application will ensure if a visitor follows a legitimate passage or not through acyclic pathway mechanism, whereas such unique acyclic pathway is termed as complexity of the network.

1.1. Definitions and Preliminaries. The following lemma is a direct derivation of Temperley's equation mentioned previously.

Lemma 1 (see [5]). *Let $\mathbb{G}$ be ϱ order network; then,*

$$\tau(\mathbb{G})=\frac{1}{\varrho^2}\det\left(\varrho I-D(\overline{\mathbb{G}})+A(\overline{\mathbb{G}})\right), \tag{1}$$

where $\overline{\mathbb{G}}\cong\mathbb{G}$.

The above expression is more useful as it represents the complexity of $\mathbb{G}$ as the determinant of a particular matrix, rather than involving its eigenvalues. The eigenvalues based process is relatively difficult and complex.

The solution of the following iterative expression defines the first kind of Chebyshev polynomials.

$$\mathbb{T}_{\varrho+1}(x) - 2x\mathbb{T}_{\varrho}(x) + \mathbb{T}_{\varrho-1}(x) = 0; \mathbb{T}_0(x) = 1, \mathbb{T}_1(x) = x. \tag{2}$$

The standard solution of (2) gives

$$\mathbb{T}_{\varrho}(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2-1}\right)^{\varrho} + \left(x - \sqrt{x^2-1}\right)^{\varrho}\right]; \varrho \geq 1. \tag{3}$$

The solution of the following iterative expression defines the second kind of Chebyshev polynomials.

$$\mathbb{U}_{\varrho+1}(x) - 2x\mathbb{U}_{\varrho}(x) + \mathbb{U}_{\varrho-1}(x) = 0; \mathbb{U}_0(x) = 1, \mathbb{U}_1(x) = x. \tag{4}$$

The standard solution of (4) gives

$$\mathbb{U}_m(z) = \frac{1}{2\sqrt{z^2-1}}\left[\left(z + \sqrt{z^2-1}\right)^{\varrho+1} - \left(z - \sqrt{z^2-1}\right)^{\varrho+1}\right]; \varrho \geq 1. \tag{5}$$

Identity (4) is valid $\forall z \in \mathbb{C}$ excluding $z = \pm 1$ [6]. The determinants are closely related to both 1^{st} and 2^{nd} kind Chebyshev Polynomials.where H_1 and H_2 are non-singular matrices.

Lemma 2 (see [7, 8]).

(i) $\forall \phi \geq 3$, $det[\mathbb{A}_m(\phi)] = 2[\mathbb{T}_m(\phi/2) - 1]$, *where*

$$\mathbb{A}_m(\phi) = \begin{pmatrix} \phi & -1 & 0 & 0 & \dots & 0 & 0 & 0 & -1 \\ -1 & \phi & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & \phi & -1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & \phi & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \phi & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & \phi & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & \phi & -1 \\ -1 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & \phi \end{pmatrix}. \tag{6}$$

(i) $\forall \phi \geq 4$, $m \geq 3$, $det[\mathbb{B}_m(\phi)] = 2(\phi + m - 3)/\phi - 3[\mathbb{T}_m(\phi - 1/2) - 1]$, *where*

$$\mathbb{B}_m(\phi) = \begin{pmatrix} \phi & 0 & 1 & 1 & \dots & 1 & 1 & 1 & 0 \\ 0 & \phi & 0 & 1 & \dots & 1 & 1 & 1 & 1 \\ 1 & 0 & \phi & 0 & \dots & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & \phi & \dots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & \phi & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \dots & 0 & \phi & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 & \phi & 0 \\ 0 & 1 & 1 & 1 & \dots & 1 & 1 & 0 & \phi \end{pmatrix}. \tag{7}$$

(i) $\forall m, \phi$, $det[\mathbb{C}_m(\phi)] = (\phi - 1)\mathbb{U}_{m-1}(\phi + 1/2)$, *where*

$$\mathbb{C}_m(\phi) = \begin{pmatrix} \phi & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & \phi+1 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & \phi+1 & -1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & \phi+1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \phi+1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & \phi+1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & \phi+1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & \phi \end{pmatrix}. \tag{8}$$

(i) $\forall \phi \geq 2$, $m \geq 3$, $det[\mathbb{D}_m(\phi)] = (m + \phi - 2)U_{m-1}(\phi/2)$, *where*

$$\mathbb{D}_m(\phi) = \begin{pmatrix} \phi & 0 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \\ 0 & \phi+1 & 0 & 1 & \dots & 1 & 1 & 1 & 1 \\ 1 & 0 & \phi+1 & 0 & \dots & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & \phi+1 & \dots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & \phi+1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \dots & 0 & \phi+1 & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 & \phi+1 & 0 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 0 & \phi \end{pmatrix}. \tag{9}$$

Lemma 3 (see [9]). $\forall \phi$ *and* m, $det[\mathbb{W}_m(\phi)] = (\phi + m - 1)(\phi - 1)^{m-1}$, *where*$\mathbb{W}_m(\phi)$*is an*$m \times m$*circulant matrix given as*

$$\mathbb{W}_m(\phi) = \begin{pmatrix} \phi & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \\ 1 & \phi & 1 & 1 & \dots & 1 & 1 & 1 & 1 \\ 1 & 1 & \phi & 1 & \dots & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \phi & \dots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & \phi & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & \phi & 1 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & \phi & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & \phi \end{pmatrix}. \quad (10)$$

Lemma 4 (see [10]). *Let H_1, H_2, H_3, andH_4be the block matrices of orders$\theta \times \theta$, $\theta \times \vartheta$, $\vartheta \times \theta$, and$\vartheta \times \vartheta$, respectively. Then,*

$$\det\begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} = \det\left(H_4 - H_3 H_1^{-1} H_2\right) \times \det\left(H_1\right) = \det\left(H_1 - H_2 H_4^{-1} H_3\right) \times \det\left(H_4\right), \quad (11)$$

Lemma 5 (see [11]). *For $\phi \geq 5$, let us consider a circulant matrix given as*

$$E_\phi = \begin{pmatrix} \zeta & \eta & 1 & 0 & \dots & 0 & 0 & 1 & \eta \\ \eta & \zeta & \eta & 1 & \dots & 0 & 0 & 0 & 1 \\ 1 & \eta & \zeta & \eta & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & \eta & \zeta & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \zeta & \eta & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & \eta & \zeta & \eta & 1 \\ 1 & 0 & 0 & 0 & \dots & 1 & \eta & \zeta & \eta \\ \eta & 1 & 0 & 0 & \dots & 0 & 1 & \eta & \zeta \end{pmatrix}_{\phi \times \phi}, \quad (12)$$

$$\tau\left(E_\phi\right) = \begin{cases} \left[\zeta^2 + 4\left(\zeta - \eta^2 + 1\right)\right] \prod_{i=1}^{\frac{\phi}{2}-1} \left[\zeta + 2\eta \cos\left(\frac{2\pi i}{\phi}\right) + 2\cos\left(\frac{4\pi i}{\phi}\right)\right]^2 : & \text{for even } \phi; \\ \left[\zeta + 2(\eta + 1)\right] \prod_{i=1}^{\lfloor \frac{\phi}{2} \rfloor} \left[\zeta + 2\eta \cos\left(\frac{2\pi i}{\phi}\right) + 2\cos\left(\frac{4\pi i}{\phi}\right)\right]^2 : & \text{for odd } \phi. \end{cases} \quad (13)$$

We shall also provide a few definitions [12, 13] in Section 3 before a certain result, where necessary. Throughout the article, $\overline{G}$ represents the complement of the network G.

1.2. Main Contributions. In the present article, we will mainly compute the closed formulae for the complexity of various generalized operations on graphs such as sum ($K_{2,n} + W_3$, $K_{2,n} + nK_1$, $K_n + S_n$), product ($K_{2,n} \boxtimes K_2$, $K_{2,n} \ltimes K_2$, $K_n \times K_2$, $K_n \boxtimes K_2$), difference ($K_{2,n} \ominus K_2$), and the conjunction of S_n with K_2. Furthermore, all our computations have been concluded by implementation of the methods of linear algebra and matrix theory.

1.3. Main Structure. The main structure of this article is as follows:

1. Section 1 comprises the introduction and preliminaries of our main work.
2. Section 2 contains the salient work related to our derivations.
3. Section 3 consists of the main derivations we have obtained in the form of the complexities of various networks' operations.
4. Section 4 contains a brief summary and graphical illustrations of our work.
5. Section 5 gives the conclusion and also tells about the future work related to this paper.

2. Related Work

If we talk about the closed formulae for the complexity of an infinite family of networks, we shall not be able to locate any such generalized result. Although it is still possible to derive the new closed formulae of the complexity of classes of networks having order m, where m is sufficiently large, it is useful to obtain this invariant for the networks of finite order for the values as we increase the order of a network. If we look into the historical development of this concept, the calculation of the complexity of the complete network as $\tau(K_\beta) = \beta^{\beta-2}$ is the foremost concept that appeared in [14]. The second prominent result in this regard is the complexity of the complete bipartite network which is again derived by Cayley [14] as $\tau(K_{\mu,\nu}) = \mu^{\nu-1}\nu^{\mu-1}$. In [15], the closed formula for the complexity of Mobius ladder has been obtained as $\tau(M_\chi) = \chi/2[(2+\sqrt{3})^\chi + (2-\sqrt{\chi})^\chi + 2]$ for $\chi \geq 2$ in [15].

The determination of the total spanning trees of a network has recently reappeared as an active topic. Kirchhoff's matrix tree theorem [16] is a prominent result in this regard. It represents the complexity of a network as the determinant of a random cofactor of its Kirchhoff's matrix, where, say for a network $\mathbb{G}$, $K(\mathbb{G})$ = degree matrix of $\mathbb{G}$ – adjacency matrix of $\mathbb{G}$ indicates its Kirchhoff's matrix.

A combinatorial method for computing the complexity of a network is with the use of contraction-deletion theorem. As an iterative process for an edge $uv \in E(\mathbb{G})$, the complexity of $\mathbb{G}$ is the sum of $\tau(\mathbb{G}|uv)$ and $\tau(\mathbb{G} - uv)$. Here, $\mathbb{G}|uv$ is the network derived as the result of contraction of uv in $\mathbb{G}$ repeatedly until the end points u and v coincide [17].

In [18], the self-adapted task scheduling strategies in the wireless sensor networks have been designed and analyzed. Wang et al. [19] discussed the ant colony optimization-based location-aware routing for wireless sensor networks. In [20], a pedestrian detection method has been designed and examined based on the genetic algorithm for optimizing XGBoost training parameters. For wireless sensor networks, Wan and Xiong designed and assessed an energy-efficient sleep scheduling mechanism with similarity measure [21]. Lu et al. in [22] explored a finger vein-based personal authentication mechanism for Internet-related security. Furthermore, in [23, 24], some latest work on the enumeration of the complexity of networks can be observed.

3. Main Results

In networking, the characteristic of developing new structures from the existing ones through network operations and studying their various properties always remains active. The present section addresses our main derivations consisting of the closed formulae of the complexity of various networks obtained as the result of network operations.

Theorem 1. *For all n, the complexity of the network $K_{2,n} + W_3$ is given by*

$$\tau\left(K_{2,n} + W_3\right) = 6^{n-1}(n+4)(n+6)^4. \quad (14)$$

Proof. Consider the network $K_{2,n} + W_3$ with $|V(K_{2,n} + W_3)| = n+6$ and $|E(K_{2,n} + W_3)| = 6n+14$ (see the general formation in Figure 1).

Applying Lemma 1, we have

$$\begin{aligned}\tau\left(K_{2,n} + W_3\right) &= \frac{1}{(n+6)^2}\det[(n+6)I - \overline{D} + \overline{A}] \\ &= \frac{1}{(n+6)^2}\det\begin{pmatrix} n+6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & n+6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & n+6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n+6 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n+5 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & n+5 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 7 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 7 & \cdots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 7 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 7 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 7 \end{pmatrix}_{(n+6)\times(n+6)}.\end{aligned} \quad (15)$$

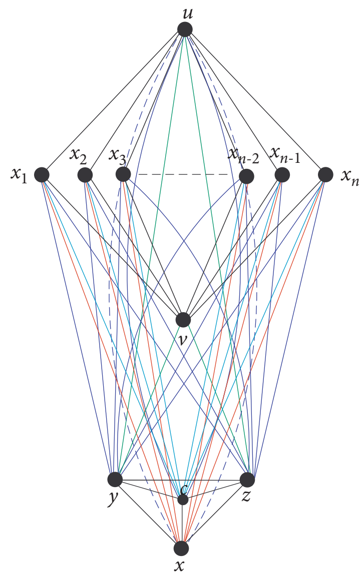

Figure 1: The network $K_{2,n} + W_3$.

Now on the above determinant, we perform the following operations simultaneously:

(i) Adding all columns to C_1.

(ii) From C1, we take the number n+5 as common.

(iii) Subtracting C_1 from all columns.

(iv) Expanding along R_1.

This yields

$$
= \det \begin{pmatrix}
n+5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & \dots & -1 & -1 & -1 & -1 \\
-1 & n+5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & \dots & -1 & -1 & -1 & -1 \\
-1 & -1 & n+5 & -1 & -1 & -1 & -1 & -1 & -1 & \dots & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & n+4 & 0 & -1 & -1 & -1 & -1 & \dots & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & 0 & n+4 & -1 & -1 & -1 & -1 & \dots & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & 6 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 6 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 6 & 0 & \dots & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 6 & \dots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & \dots & 6 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & 6 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 6 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 6
\end{pmatrix}_{(n+5)\times(n+5)},
$$

$$
\Rightarrow \tau\left(K_{2,n} + W_3\right) = \det \begin{pmatrix} P_{5\times 5} & Q_{5\times n} \\ R_{n\times 5} & S_{n\times n} \end{pmatrix}_{(n+5)\times(n+5)}. \tag{16}
$$

By using Lemma 4, we get

$$\tau\left(K_{2,n}+W_3\right)=\det(S).\det\left(P-QS^{-1}R\right)$$

$$=6^n\left(\frac{1}{6}\right)^5(-1)^5\det\begin{pmatrix} -5n-30 & n+6 & n+6 & n+6 & n+6 \\ n+6 & -5n-30 & n+6 & n+6 & n+6 \\ n+6 & n+6 & -5n-30 & n+6 & n+6 \\ n+6 & n+6 & n+6 & -5n-24 & n \\ n+6 & n+6 & n+6 & n & -5n-24 \end{pmatrix}. \tag{17}$$

Evaluating and simplifying, we obtain $\tau(K_{2,n}+W_3)=6^{n-1}(n+4)(n+6)^4$. □

Theorem 2. *For all n , the complexity of the strong product $K_{2,n}\boxtimes K_2$ is given by*

$$\tau\left(K_{2,n}\boxtimes K_2\right)=(24)^n\left(2\left(n^3+2n^2+n\right)\right). \tag{18}$$

Proof. Consider the network $K_{2,n}$) K_2 with $|V(K_{2,n}) K_2)|=2n+4$ and $|E(K_{2,n}) K_2)|=9n+2$ (see the general formation in Figure 2).

Applying Lemma 1, we have

$$\tau\left(K_{2,n}\boxtimes K_2\right)=\frac{1}{(2n+4)^2}\det\left[(2n+4)I-\overline{D}+\overline{A}\right]$$

$$=\frac{1}{(n+6)^2}\det\begin{pmatrix} n+6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & n+6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & n+6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n+6 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n+5 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & n+5 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 7 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 7 & \cdots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 7 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 7 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 7 \end{pmatrix}_{(n+6)\times(n+6)}$$

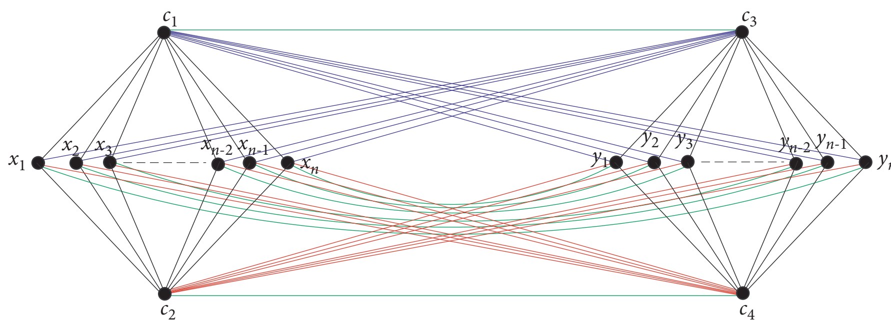

FIGURE 2: The strong product $K_{2,n} \boxtimes K_2$.

$$= \det \begin{pmatrix} n+5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & \ldots & -1 & -1 & -1 & -1 \\ -1 & n+5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & \ldots & -1 & -1 & -1 & -1 \\ -1 & -1 & n+5 & -1 & -1 & -1 & -1 & -1 & -1 & \ldots & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & n+4 & 0 & -1 & -1 & -1 & -1 & \ldots & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & n+4 & -1 & -1 & -1 & -1 & \ldots & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 6 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 6 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 6 & 0 & \ldots & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 6 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & \ldots & 6 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & \ldots & 0 & 6 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 6 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 6 \end{pmatrix}_{(n+5)\times(n+5)},$$

$$\Rightarrow \tau\left(K_{2,n} \boxtimes K_2\right) = \det \begin{pmatrix} P_{3\times 3} & Q_{3\times 2n} \\ R_{2n\times 3} & S_{2n\times 2n} \end{pmatrix}_{(2n+3)\times(2n+3)}. \quad (19)$$

By using Lemma 4, we get

$$\tau\left(K_{2,n} \boxtimes K_2\right) = \det(S).\det\left(P - QS^{-1}R\right)$$

$$= 5^n \left(\frac{24}{5}\right)^n \det \begin{pmatrix} \frac{3n+2}{2} & \frac{-n}{2} & \frac{-n-2}{2} \\ \frac{-n}{2} & \frac{3n+2}{2} & \frac{-n}{2} \\ \frac{-n-2}{2} & \frac{-n}{2} & \frac{3n+2}{2} \end{pmatrix}. \quad (20)$$

Evaluating the above determinant, we obtain finally $\Rightarrow \tau(K_{2,n} \boxtimes K_2) = (24)^n (2(n^3 + 2n^2 + n))$. □

Theorem 3. *For all n , the complexity of the homomorphic product $K_{2,n} \ltimes K_2 \cong K_{2,n} \times K_2$ is given by*

$$\tau\left(K_{2,n} \ltimes K_2\right) = 8^{n-1}\left(n^3 + 6n^2 + 8n\right). \quad (21)$$

Proof. Consider the network $K_{2,n} \ltimes K_2$ with $|V(K_{2,n} \ltimes K_2)| = 2n+4$ and $|E(K_{2,n} \ltimes K_2)| = 5n+2$ (see the general formation in Figure 3).

Applying Lemma 1, we have

$$\tau\left(K_{2,n}\ltimes K_2\right) = \frac{1}{(2n+4)^2}\det\left[(2n+4)I - \overline{D} + \overline{A}\right]$$

$$= \det\begin{pmatrix}
n+5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\
-1 & n+5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\
-1 & -1 & n+5 & -1 & -1 & -1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & n+4 & 0 & -1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & 0 & n+4 & -1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & 6 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 6 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 6 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 6 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & \cdots & 6 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 6 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 6 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 6
\end{pmatrix}_{(n+5)\times(n+5)}$$

$$= 6^n\left(\frac{1}{6}\right)^5(-1)^5\det\begin{pmatrix}
-5n-30 & n+6 & n+6 & n+6 & n+6 \\
n+6 & -5n-30 & n+6 & n+6 & n+6 \\
n+6 & n+6 & -5n-30 & n+6 & n+6 \\
n+6 & n+6 & n+6 & -5n-24 & n \\
n+6 & n+6 & n+6 & n & -5n-24
\end{pmatrix},$$

$$\Rightarrow\tau\left(K_{2,n} < \mathrm{imes}K_2\right) = \det\begin{pmatrix} P_{3\times 3} & Q_{3\times 2n} \\ R_{2n\times 3} & S_{2n\times 2n}\end{pmatrix}_{(2n+3)\times(2n+3)}. \quad (22)$$

By using Lemma 4, we get

$$\tau\left(K_{2,n}\ltimes K_2\right) = \det(S).\det\left(P - QS^{-1}R\right)$$

$$= 8^n\left(\frac{1}{8}\right)^3\det\begin{pmatrix} 5n+8 & -n & -n-8 \\ -n & 5n+8 & -3n \\ -n-8 & -3n & 5n+8\end{pmatrix}. \quad (23)$$

Evaluating the above determinant and simplifying, we obtain finally $\Rightarrow\tau(K_{2,n}\ltimes K_2) = 8^{n-1}(n^3 + 6n^2 + 8n)$. □

Theorem 4. *For all* n *, the complexity of the mirror network* $K_{2,n} + nK_1$ *is given as*

$$\tau\left(K_{2,n} + nK_1\right) = 4n(n+1)(n+2)^{2n-2}. \quad (24)$$

Proof. Consider the network $K_{2,n} + nK_1$ with $|V(K_{2,n} + nK_1)| = 2n + 2$ and $|E(K_{2,n} + nK_1)| = n^2 + 4n$ (see Figure 4).

Applying Lemma 1, we have

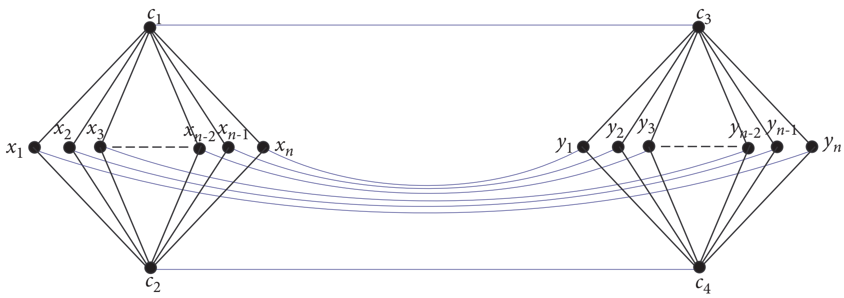

Figure 3: The homomorphic product $K_{2,n} \ltimes K_2$.

$$
\begin{aligned}
\tau\left(K_{2,n} + nK_1\right) &= \frac{1}{(2n+2)^2} \det\left[(2n+2)I - \overline{D} + \overline{A}\right] \\
&= \frac{1}{(2n+2)^2} \\
&\det\begin{pmatrix}
2n+1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
1 & 2n+1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & n+3 & 1 & 1 & \dots & 1 & 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 1 & n+3 & 1 & \dots & 1 & 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & n+3 & \dots & 1 & 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & 1 & 1 & \dots & n+3 & 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & \dots & 1 & n+3 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & \dots & 1 & 1 & n+3 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & n+3 & 1 & 1 & \dots & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & n+3 & 1 & \dots & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 1 & n+3 & \dots & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 1 & 1 & \dots & n+3 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 1 & 1 & \dots & 1 & n+3 & 1 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 1 & 1 & \dots & 1 & 1 & n+3
\end{pmatrix}
\end{aligned}
$$

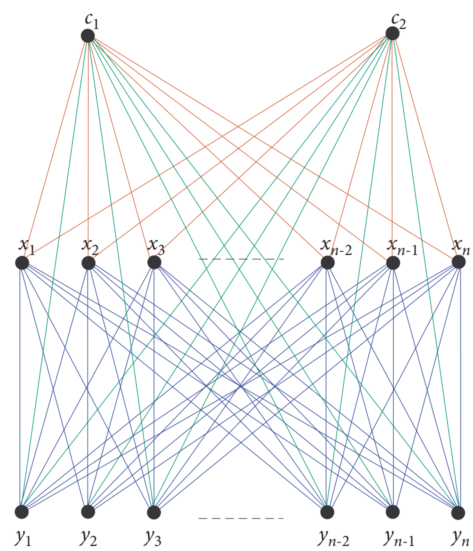

FIGURE 4: The network $K_{2,n} + nK_1$.

$$= \det\begin{pmatrix} 2n & -1 & -1 & -1 & \dots & -1 & -1 & -1 & -1 & -1 & -1 & \dots & -1 & -1 & -1 \\ -1 & n+2 & 0 & 0 & \dots & 0 & 0 & 0 & -1 & -1 & -1 & \dots & -1 & -1 & -1 \\ -1 & 0 & n+2 & 0 & \dots & 0 & 0 & 0 & -1 & -1 & -1 & \dots & -1 & -1 & -1 \\ -1 & 0 & 0 & n+2 & \dots & 0 & 0 & 0 & -1 & -1 & -1 & \dots & -1 & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \dots & n+2 & 0 & 0 & -1 & -1 & -1 & \dots & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & \dots & 0 & n+2 & 0 & -1 & -1 & -1 & \dots & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & \dots & 0 & 0 & n+2 & -1 & -1 & -1 & \dots & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & \dots & -1 & -1 & -1 & n+2 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & \dots & -1 & -1 & -1 & 0 & n+2 & 0 & \dots & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & \dots & -1 & -1 & -1 & 0 & 0 & n+2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \dots & -1 & -1 & -1 & 0 & 0 & 0 & \dots & n+2 & 0 & 0 \\ -1 & -1 & -1 & -1 & \dots & -1 & -1 & -1 & 0 & 0 & 0 & \dots & 0 & n+2 & 0 \\ -1 & -1 & -1 & -1 & \dots & -1 & -1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & n+2 \end{pmatrix}_{(2n+1)\times(2n+1)},$$

$$\Rightarrow \tau\left(K_{2,n} + nK_1\right) = \det\begin{pmatrix} P_{1\times 1} & Q_{1\times 2n} \\ R_{2n\times 1} & S_{2n\times 2n} \end{pmatrix}_{(2n+1)\times(2n+1)}. \quad (25)$$

By using Lemma 4, we have

$$\tau\left(K_{2,n} + nK_1\right) = \det(S).\det\left(P - QS^{-1}R\right)$$
$$= 3^n \times \det\begin{bmatrix} \frac{2n+3}{3} & \frac{-n}{3} \\ \frac{-n}{3} & \frac{2n+3}{3} \end{bmatrix}. \quad (26)$$

Simplification finally gives $\tau(K_{2,n} + nK_1) = 4n(n+1)(n+2)^{2n-2}$. □

Theorem 5. *For all n, the complexity of the cartesian product $K_n \times K_2$ is given as*

$$\tau(K_n \times K_2) = n^{n-2}(n+2)^{n-1}. \quad (27)$$

Proof. Consider thenetwork $K_n \times K_2$ with $|V(K_n \times K_2)| = 2n$ and $|E(K_n \times K_2)| = n^2$ (see Figure 5).

Applying Lemma 1, we have

$$\tau(K_n \times K_2) = \frac{1}{(2n)^2}\det[(2n)I - \overline{D} + \overline{A}]$$
$$= \frac{1}{(2n)^2}\det\begin{pmatrix}
n+1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 & 1 & 1 \\
0 & n+1 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 1 & \dots & 1 & 1 & 1 \\
0 & 0 & n+1 & \dots & 0 & 0 & 0 & 1 & 1 & 0 & \dots & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & n+1 & 0 & 0 & 1 & 1 & 1 & \dots & 0 & 1 & 1 \\
0 & 0 & 0 & \dots & 0 & n+1 & 0 & 1 & 1 & 1 & \dots & 1 & 0 & 1 \\
0 & 0 & 0 & \dots & 0 & 0 & n+1 & 1 & 1 & 1 & \dots & 1 & 1 & 0 \\
0 & 1 & 1 & \dots & 1 & 1 & 1 & n+1 & 0 & 0 & \dots & 0 & 0 & 0 \\
1 & 0 & 1 & \dots & 1 & 1 & 1 & 0 & n+1 & 0 & \dots & 0 & 0 & 0 \\
1 & 1 & 0 & \dots & 1 & 1 & 1 & 0 & 0 & n+1 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \dots & 0 & 1 & 1 & 0 & 0 & 0 & \dots & n+1 & 0 & 0 \\
1 & 1 & 1 & \dots & 1 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & n+1 & 0 \\
1 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & n+1
\end{pmatrix}_{2n\times 2n}, \quad (28)$$
$$\Rightarrow \tau(K_n \times K_2) = \frac{1}{(2n)^2}\det\begin{pmatrix} P_{n\times n} & Q_{n\times n} \\ R_{n\times n} & S_{n\times n} \end{pmatrix}_{2n\times 2n}.$$

By using Lemma 4, we have

$$\tau(K_n \times K_2) = \frac{1}{(2n)^2}\det(S).\det\left(P - QS^{-1}R\right)$$

$$= \frac{1}{(2n)^2} \times \det\begin{pmatrix} \frac{n^2+n+2}{2-n} & 1 & 1 \\ 1 & \frac{n^2+n+2}{2-n} & 1 \\ 1 & 1 & \frac{n^2+n+2}{2-n} \end{pmatrix}. \tag{29}$$

Simplifying, we get $\tau(K_n \times K_2) = n^{n-2}(n+2)^{n-1}$. □

Corollary 1. *For all n, the complexity of the symmetric difference $K_n \ominus K_2$ is given as*

$$\tau(K_n \ominus K_2) = n^{n-2}(n+2)^{n-1}. \tag{30}$$

Proof. Since $K_n \times K_2 \cong K_n \ominus K_2$ (see Figure 6), $\tau(K_n \ominus K_2) = n^{n-2}(n+2)^{n-1} = \tau(K_n \times K_2)$. □

Theorem 6. *For all n, the complexity of the strong product $K_n \boxtimes K_2 \cong K_n + K_n$ is given as*

$$\tau(K_n \boxtimes K_2) = (2n)^{2n-2}. \tag{31}$$

Proof. Consider the network $K_n \boxtimes K_2$ with $|V(K_n \boxtimes K_2)| = 2n$ and $|E(K_n \boxtimes K_2)| = n(2n-1)$ (see Figure 7).

Applying Lemma 1, we have

$$\tau(K_n \boxtimes K_2) = \frac{1}{(2n)^2}\det((2n)I - \overline{D} + \overline{A})$$

$$= \frac{1}{(2n)^2}\det\begin{pmatrix}
2n & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 2n & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 2n & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 2n & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 2n & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 2n & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 2n & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 2n & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 2n & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 2n & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 2n & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 2n
\end{pmatrix}_{2n\times 2n}, \tag{32}$$

$$\Rightarrow \tau(K_n \boxtimes K_2) = \frac{1}{(2n)^2}\det\begin{pmatrix} P_{n\times n} & O_{n\times n} \\ O_{n\times n} & S_{n\times n} \end{pmatrix}_{2n\times 2n},$$

$$\Rightarrow \tau(K_n \boxtimes K_2) = \frac{1}{(2n)^2}(2n)^{2n}.$$

Simplifying, we get $\tau(K_n \boxtimes K_2) = (2n)^{2n-2}$. □

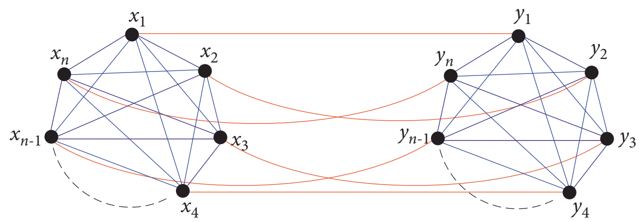

FIGURE 5: The Cartesian product $K_n \times K_2$.

Theorem 7. *For all n, the complexity of the symmetric difference $K_{2,n} \ominus K_2$ is given by*

$$\tau\left(K_{2,n} \ominus K_2\right) = (n+2)^{2n+2}. \quad (33)$$

Proof. Consider the network $K_{2,n} \ominus K_2$ with $|V(K_{2,n} \ominus K_2)| = 2n+4$ and $|E(K_{2,n} \ominus K_2)| = n^2 + 4n + 4$ (see the general formation in Figure 8).

Applying Lemma 1, we have

$$\begin{aligned}
\tau\left(K_{2,n} \ominus K_2\right) &= \frac{1}{(2n+4)^2} \det\left((2n+4)I - \overline{D} + \overline{A}\right) \\
&= \frac{1}{(2n)^2} \det \begin{pmatrix}
2n & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 2n & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 2n & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 2n & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 2n & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 2n & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 2n & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 2n & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 2n & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 2n & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 2n & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 2n
\end{pmatrix}_{2n\times 2n}, \\
&\Rightarrow \tau\left(K_n \boxtimes K_2\right) = \frac{1}{(2n)^2} \det \begin{pmatrix} P_{n\times n} & O_{n\times n} \\ O_{n\times n} & S_{n\times n} \end{pmatrix}_{2n\times 2n}, \\
&\Rightarrow \tau\left(K_{2,n} \ominus K_2\right) = \det \begin{pmatrix} P_{3\times 3} & Q_{3\times 2n} \\ R_{2n\times 3} & S_{2n\times 2n} \end{pmatrix}_{(2n+3)\times(2n+3)}.
\end{aligned} \quad (34)$$

By using Lemma 4, we get

$$\begin{aligned}
\tau\left(K_{2,n} \ominus K_2\right) &= \det(P).\det\left(S - RP^{-1}Q\right) \\
&= \frac{(n+2)^{2n+1}}{4\left(n^2+3n+2\right)} \times \frac{n+2}{4n+4} \det \begin{pmatrix} 3n+4 & -n-2 & -n-2 \\ -n-2 & 3n+4 & -n \\ -n-2 & -n & 3n+4 \end{pmatrix}.
\end{aligned} \quad (35)$$

Evaluating the determinant and simplifying, we obtain finally $\Rightarrow \tau(K_{2,n} \ominus K_2) = (n+2)^{2n+2}$. □

Theorem 8. *For all n, the complexity of the network $K_n + S_n$ is given by*

$$\tau\left(K_n + S_n\right) = (2n+1)^n (n+1)^{n-1}. \quad (36)$$

Proof. Consider the network $K_n + S_n$ with $|V(K_n + S_n)| = 2n + 1$ and $|E(K_n + S_n)| = 3n/2(n + 1)$. Applying Lemma 1, we have

$$\begin{aligned}
\tau(K_n + S_n) &= \frac{1}{(2n+1)^2}\det[(2n+1)I - \overline{D} + \overline{A}] \\
&= \frac{1}{(2n+1)^2} \\
&\det\begin{pmatrix}
2n+1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & n+2 & 1 & 1 & \ldots & 1 & 1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & n+2 & 1 & \ldots & 1 & 1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 1 & n+2 & \ldots & 1 & 1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 1 & 1 & 1 & \ldots & n+2 & 1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & \ldots & 1 & n+2 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & \ldots & 1 & 1 & n+2 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 2n+1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 2n+1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 2n+1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 2n+1 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 2n+1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 2n+1
\end{pmatrix} \\
&= \det\begin{pmatrix}
n+1 & 0 & 0 & \ldots & 0 & 0 & 0 & -1 & -1 & -1 & \ldots & -1 & -1 & -1 \\
0 & n+1 & 0 & \ldots & 0 & 0 & 0 & -1 & -1 & -1 & \ldots & -1 & -1 & -1 \\
0 & 0 & n+1 & \ldots & 0 & 0 & 0 & -1 & -1 & -1 & \ldots & -1 & -1 & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & n+1 & 0 & 0 & -1 & -1 & -1 & \ldots & -1 & -1 & -1 \\
0 & 0 & 0 & \ldots & 0 & n+1 & 0 & -1 & -1 & -1 & \ldots & -1 & -1 & -1 \\
0 & 0 & 0 & \ldots & 0 & 0 & n+1 & -1 & -1 & -1 & \ldots & -1 & -1 & -1 \\
-1 & -1 & -1 & \ldots & -1 & -1 & -1 & 2n & 0 & 0 & \ldots & 0 & 0 & 0 \\
-1 & -1 & -1 & \ldots & -1 & -1 & -1 & 0 & 2n & 0 & \ldots & 0 & 0 & 0 \\
-1 & -1 & -1 & \ldots & -1 & -1 & -1 & 0 & 0 & 2n & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-1 & -1 & -1 & \ldots & -1 & -1 & -1 & 0 & 0 & 0 & \ldots & 2n & 0 & 0 \\
-1 & -1 & -1 & \ldots & -1 & -1 & -1 & 0 & 0 & 0 & \ldots & 0 & 2n & 0 \\
-1 & -1 & -1 & \ldots & -1 & -1 & -1 & 0 & 0 & 0 & \ldots & 0 & 0 & 2n
\end{pmatrix}_{2n\times 2n}, \\
&\Rightarrow \tau(K_n + S_n) = \det\begin{pmatrix} P_{n\times n} & Q_{n\times n} \\ R_{n\times n} & S_{n\times n} \end{pmatrix}_{2n\times 2n}.
\end{aligned} \tag{37}$$

By using Lemma 4, we obtain

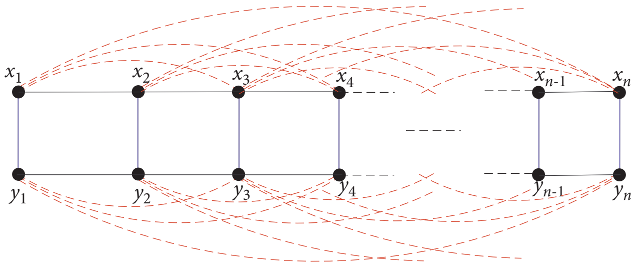

FIGURE 6: The symmetric difference $K_n \ominus K_2$.

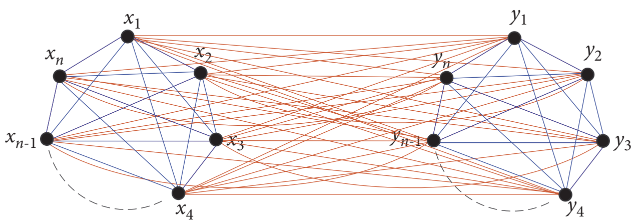

FIGURE 7: The strong product $K_n \boxtimes K_2$.

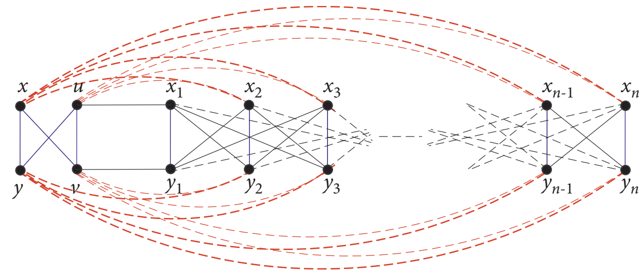

FIGURE 8: The symmetric difference $K_{2,n} \ominus K_2$.

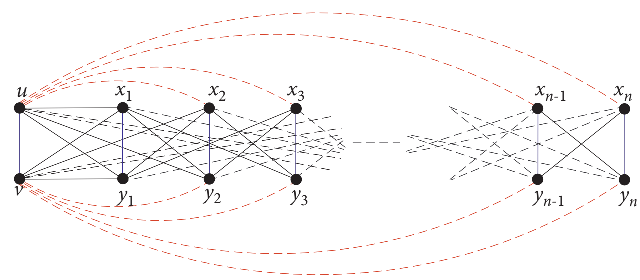

FIGURE 9: The disjunction network $S_n \wedge K_2$.

$$\tau(K_n + S_n) = \det(S).\det\left(P - QS^{-1}R\right)$$

$$= (n+1)^n \times \left(-\frac{2n+1}{n+1}\right)^n \det \begin{pmatrix} -n & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & \\ 1 & -n & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\ 1 & 1 & -n & 1 & \ldots & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -n & \ldots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & -n & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \ldots & 1 & -n & 1 & 1 \\ 1 & 1 & 1 & 1 & \ldots & 1 & 1 & -n & 1 \\ 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & -n \end{pmatrix}_{n\times n}. \quad (38)$$

Using Lemma 3 and simplifying, we get $\Rightarrow \tau(K_n + S_n) = (2n+1)^n (n+1)^{n-1}$. □

Theorem 9. *For all n, the complexity of the conjunction $S_n \wedge K_2$ is given as*

$$\tau(S_n \wedge K_2) = 4(n+1)^2 (n+2)^{2n-2}. \quad (39)$$

Proof. Consider the network $S_n \wedge K_2$ with $|V(S_n \wedge K_2)| = 2n + 2$ and $|E(S_n \wedge K_2)| = n^2 + 4n + 1$ (see Figure 9).

Applying Lemma 1, we have

$$
\begin{aligned}
\tau(S_n \wedge K_2) &= \frac{1}{(2n+2)^2} \det\left[(2n+2)I - \overline{D} + \overline{A}\right] \\
&= \frac{1}{(2n+2)^2} \\
&\det\begin{pmatrix}
2n+2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 2n+2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & n+3 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & n+3 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & n+3 & \cdots & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & 1 & 1 & \cdots & n+3 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & \cdots & 1 & n+3 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & n+3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & n+3 & 1 & 1 & \cdots & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & n+3 & 1 & \cdots & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & n+3 & \cdots & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 & \cdots & n+3 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 & \cdots & 1 & n+3 & 1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & n+3
\end{pmatrix} \\
&= \det\begin{pmatrix}
2n+1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 \\
-1 & n+2 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 \\
-1 & 0 & n+2 & 0 & \cdots & 0 & 0 & 0 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 \\
-1 & 0 & 0 & n+2 & \cdots & 0 & 0 & 0 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & 0 & \cdots & n+2 & 0 & 0 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 \\
-1 & 0 & 0 & 0 & \cdots & 0 & n+2 & 0 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 \\
-1 & 0 & 0 & 0 & \cdots & 0 & 0 & n+2 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & n+2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & 0 & n+2 & 0 & \cdots & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & 0 & 0 & n+2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & 0 & 0 & 0 & \cdots & n+2 & 0 & 0 \\
-1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & 0 & 0 & 0 & \cdots & 0 & n+2 & 0 \\
-1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & n+2
\end{pmatrix}_{(2n+1)\times(2n+1)}, \\
&\Rightarrow \tau(S_n \wedge K_2) = \det\begin{pmatrix} P_{1\times 1} & Q_{1\times 2n} \\ R_{2n\times 1} & S_{2n\times 2n} \end{pmatrix}_{(2n+1)\times(2n+1)}.
\end{aligned} \tag{40}
$$

By using Lemma 4, we have

Table 1: Synopsis of the results.

Network	Parameters	Complexity	Planar ∨ non-planar
$K_{2,n}+W_3$	$\forall n \in \mathbb{N}$	$6^{n-1}(n+4)(n+6)^4$	Non-planar
$K_{2,n}\boxtimes K_2$	$\forall n \in \mathbb{N}$	$(24)^n(2(n^3+2n^2+n))$	Non-planar
$K_{2,n}\ltimes K_2$	$\forall n \in \mathbb{N}$	$8^{n-1}(n^3+6n^2+8n)$	Non-planar
$K_{2,n}+nK_1$	$\forall n \in \mathbb{N}$	$4n(n+1)(n+2)^{2n-2}$	Non-planar
$K_n \times K_2$	$\forall n \in \mathbb{N}$	$n^{n-2}(n+2)^{n-1}$	Non-planar
$K_n\boxtimes K_2$	$\forall n \in \mathbb{N}$	$(2n)^{2n-2}$	Non-planar
$K_{2,n}\ominus K_2$	$\forall n \in \mathbb{N}$	$(n+2)^{2n+2}$	Non-planar
K_n+S_n	$\forall n \in \mathbb{N}$	$(2n+1)^n(n+1)^{n-1}$	Non-planar
$S_n\wedge K_2$	$\forall n \in \mathbb{N}$	$4(n+1)^2(n+2)^{2n-2}$	Non-planar

$$
\begin{aligned}
\tau(S_n\wedge K_2) &= \det(S).\det\left(P - QS^{-1}R\right)\\
&= \frac{(-n)^n}{n+2}\\
&\quad\times \det\begin{pmatrix}
\frac{n^2+3n+4}{-n} & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \\
1 & \frac{n^2+3n+4}{-n} & 1 & 1 & \cdots & 1 & 1 & 1 & 1\\
1 & 1 & \frac{n^2+3n+4}{-n} & 1 & \cdots & 1 & 1 & 1 & 1\\
1 & 1 & 1 & \frac{n^2+3n+4}{-n} & \cdots & 1 & 1 & 1 & 1\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
1 & 1 & 1 & 1 & \cdots & \frac{n^2+3n+4}{-n} & 1 & 1 & 1\\
1 & 1 & 1 & 1 & \cdots & 1 & \frac{n^2+3n+4}{-n} & 1 & 1\\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & \frac{n^2+3n+4}{-n} & 1\\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \frac{n^2+3n+4}{-n}
\end{pmatrix}_{n\times n}\\
&\quad\times \det([2n+1]-[n]).
\end{aligned}
\tag{41}
$$

Lemma 3 and simplification finally give

$$\tau(S_n\wedge K_2) = 4(n+1)^2(n+2)^{2n-2}. \tag{42}$$

□

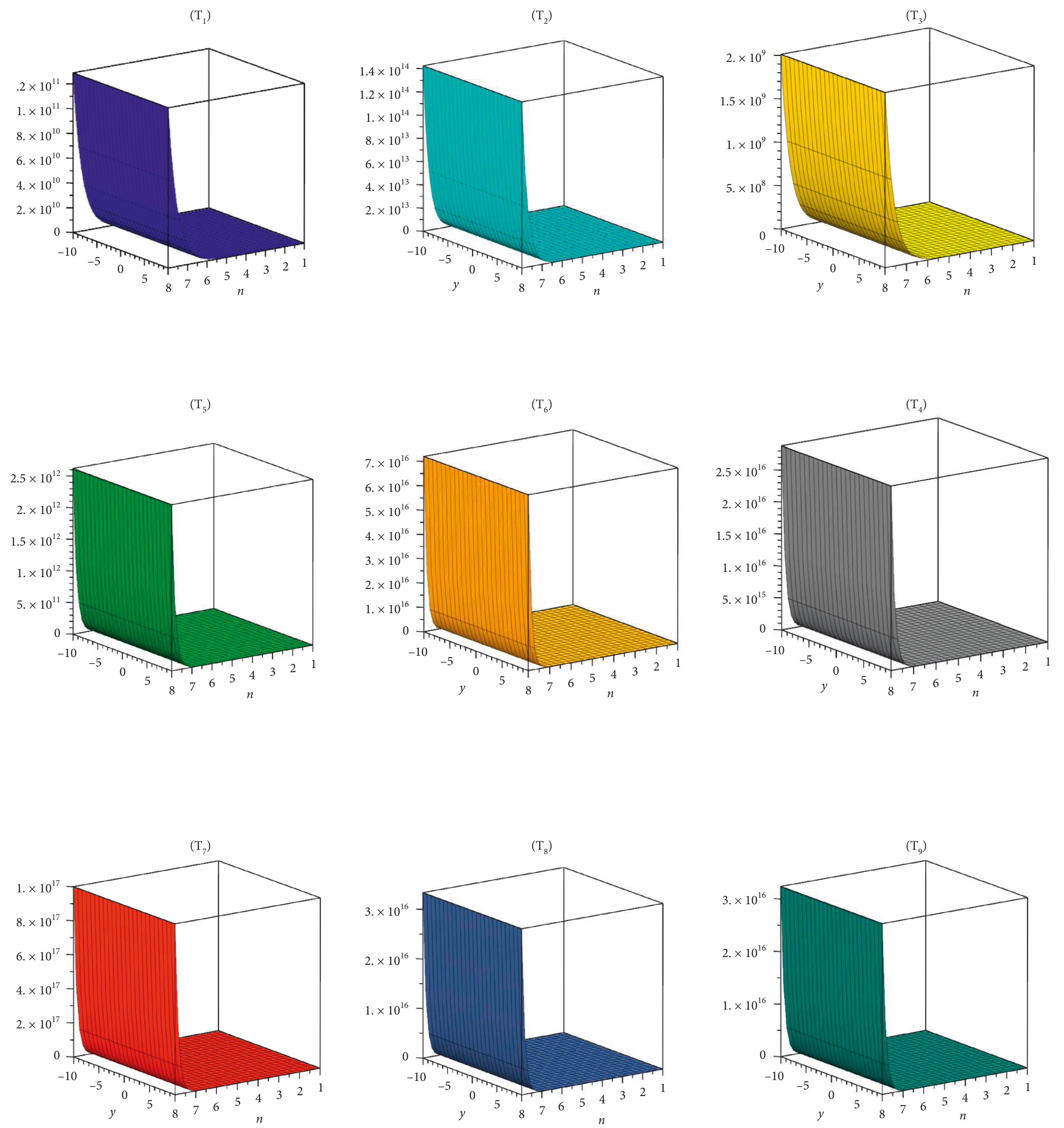

FIGURE 10: Trends of the enumerated complexities of $K_{2,n}+W_3 \longrightarrow (T_1)$, $K_{2,n}\boxtimes K_2 \longrightarrow (T_2)$, $K_{2,n}\ltimes K_2 \longrightarrow (T_3)$, $K_{2,n}+nK_1 \longrightarrow (T_4)$, $K_n\times K_2 \longrightarrow (T_5)$, $K_n\boxtimes K_2 \longrightarrow (T_6)$, $K_{2,n}\ominus K_2 \longrightarrow (T_7)$, $K_n+S_n \longrightarrow (T_8)$, and $S_n\wedge K_2 \longrightarrow (T_9)$.

4. Synopsis and the Diagrammatic Comparison of the Complexities of the Networks Obtained

This section consists of a briefing and graphical plots and juxtaposition of the values of complexities of the networks enumerated in this note.

Table 1 indicates a synopsis of our results in the shape of complexities of various networks and also categorically recognizes it being planar or not.

Figure 10 shows the discrete graphical shapes of the values of the complexity of networks obtained here, whereas Figure 11 addresses the relative comparison of the complexities of these networks, revealing the red one to be the dominating layer.

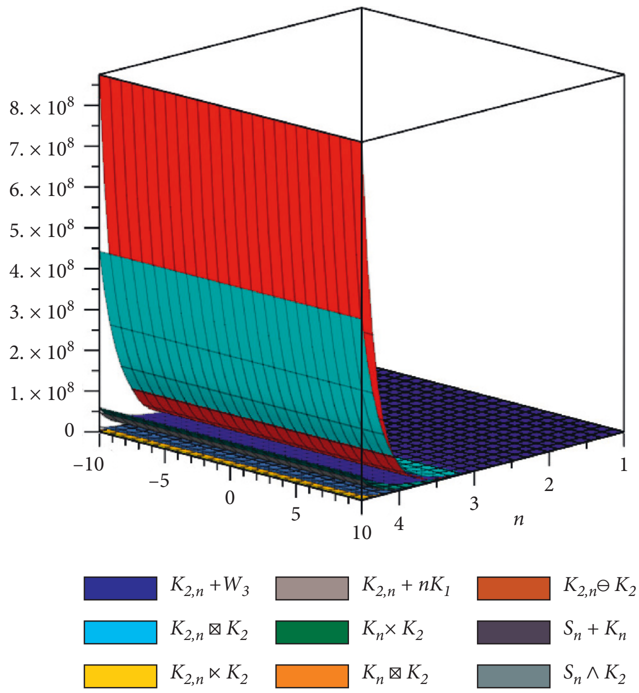

Figure 11: Comparison of the trends of the enumerated complexities.

5. Conclusion

One of the meaningful algebraic invariants in networking nowadays is complexity. This invariant provides us the information of the total number of acyclic networks present within the base network, which ultimately ensures the reliability and accuracy in the network. We have enumerated here the complexity of various operations on networks such as $K_{2,n}+W_3$, $K_{2,n}\boxtimes K_2$, $K_{2,n}\ltimes K_2$, $K_{2,n}+nK_1$, $K_n\times K_2$, $K_n\boxtimes K_2$, $K_{2,n}\ominus K_2$, K_n+S_n, and $S_n\wedge K_2$. The adopted methods are mainly algebraic and feature Chebyshev polynomials and the matrix theory in the calculations. As future work, we encourage the researchers to obtain the complexities of further generalized operations on networks such as corona product, zig zag product, homomorphic product, join, shadow, conjunction, and disjunction of various classes of networks.

Data Availability

The whole data are included within this article. However, the reader may contact the corresponding author for more details on the data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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