Hindawi
Complexity
Volume 2022, Article ID 6927111, 15 pages
https://doi.org/10.1155/2022/6927111

Research Article

Two Complex Graph Operations and their Exact Formulations on Topological Properties

Shehla Hameed,[1] **Muhammad Kamran Jamil**,[1] **Muhammad Waheed**,[1] **Muhammad Azeem**,[1] **and Senesie Swaray**[2]

[1]*Department of Mathematics, Riphah Institute of Computing and Applied Sciences, Riphah International University, Lahore, Pakistan*
[2]*Tree Crops Unit, Sierra Leone Agricultural Research Institute, Freetown, Sierra Leone*

Correspondence should be addressed to Senesie Swaray; senesieswaray74@gmail.com

Received 3 March 2022; Revised 28 April 2022; Accepted 16 May 2022; Published 7 June 2022

Academic Editor: Shahzad Sarfraz

Graph operations are utilized for developing complicated graph structures from basic graphs, and these basic graphs can help to understand the properties of complex networks. While on the other side, the topological descriptor is known as a numeric value that is associated with the graph of a network. It has enormous practical applications in chemistry and other fields of science. This particular work in this draft is the extended work and investigated the first, second, first multiplicative, first reformulated Zagreb indices, and the forgotten index of subdivision double corona and subdivision double neighborhood corona products.

1. Introduction

A topological index is a number associated with a graph of some network. With the help of this number, we can describe some properties of the network. Especially in organic chemistry, topological indices are used to predict some physical, chemical, or biological properties of organic compounds. Topological indices are key topics in the study of quantitative structural properties of a chemical network [1–5].

For a graph $\lambda = (V_\lambda, E_\lambda)$, the vertex and edge sets are denoted by V_λ and E_λ. The number of elements in V_λ and E_λ is called the order (n) and the size (m), respectively, of λ. A graph of order n and size m is denoted by $\lambda(n, m)$. The set of vertices adjacent to the vertex $v \in V_\lambda$ is called the neighborhood set of v and the number of elements in the neighborhood set the degree of v in λ is denoted by $d_\lambda(v)$ [6–10].

The study of the topological index started in 1947, and after that, hundreds of topological indices have been presented depending on the nature of applications for different chemical compounds. In 1972, the researchers in [11] introduced the first and second Zagreb indices. For a graph λ, these topological indices are defined as

$$M_1(\lambda) = \sum_{v\in V_\lambda} d_\lambda(v)^2 = \sum_{uv\in E_\lambda} [d_\lambda(u) + d_\lambda(v)],$$
$$M_2(\lambda) = \sum_{uv\in E_\lambda} d_\lambda(u) d_\lambda(v). \tag{1}$$

The researchers of [12, 13], introduced multiplicative variants of ordinary Zagreb indices. These topological indices are used to study molecular chirality, complexity, heterosystems, and Ze-isomerism. The first and second multiplicative Zagreb indices are defined as

$$\prod_1(\lambda) = \prod_{v\in V_\lambda} d_\lambda(v)^2,$$
$$\prod_2(\lambda) = \prod_{uv\in E_\lambda} d_\lambda(u) d_\lambda(v). \tag{2}$$

In 2015, researchers in [14, 15] suggested a forgotten topological index that is comparable to the first Zagreb index in its applications. The forgotten topological index is also known as F-index, and it is defined as

$$F(\lambda) = \sum_{v \in V_\lambda} d_\lambda(v)^3 = \sum_{uv \in E_\lambda} \left[d_\lambda(u)^2 + d_\lambda(v)^2\right]. \quad (3)$$

In 2004, Milicevic et al. [16] proposed reformulated Zagreb indices using edge-degrees rather than vertex-degrees. Mathematically, it is expressed as

$$EM_1(\lambda) = \sum_{e \in E_\lambda} d(e)^2 \text{where } d(e) = d(u) + d(v) - 2. \quad (4)$$

For $d(e) = d(u) + d(v)$, the above expression is known as the first hyper Zagreb index $HM_1(\lambda)$.

Complex network structures or large molecular structures can be constructed by applying some graph operations on simple graphs. Furthermore, these simple graphs can help to describe some properties of these structures. For example, the Cartesian product provides a significant model for connecting computers [17, 18].

For graphs $\lambda_1(n_1, m_1)$ and $\lambda_2(n_2, m_2)$, the corona product $\lambda_1 \circ \lambda_2$ is obtained by taking one copy of λ_1, n_1 copies of λ_2, and joining j^{th} vertex of λ_1 to every vertex J^{th} copy of λ_2 [19]. A special graph obtained by attaching a vertex in the each edge of λ is called the subdivision graph of λ and is symbolized by λ^s [20].

Let $\lambda(n, m)$, $\lambda_1(n_1, m_1)$, and $\lambda_2(n_2, m_2)$ be three graphs. The operation known as subdivision double corona product of λ, λ_1, and λ_2 and symbolized by $\lambda^s \circ (\lambda_1, \lambda_2)$, and is attained by making single copy of λ^s, n copies of λ_1, m copies of λ_2, and after that, by attaching the i^{th} old vertex, $u(o)$, of λ^s to each vertex of the i^{th} copy of λ_1 and j^{th} new vertex $u(N)$ of λ^s to each vertex of the j^{th} copy of λ_2 [21]. An illustration of subdivision double corona product is shown in Figure 1.

For the above three graphs, the subdivision double neighborhood corona product, $\lambda^s \cdot (\lambda_1, \lambda_2)$, is the graph attained by making single copy of λ^s, n copies of λ_1, m copies of λ_2 and after that, by attaching the neighborhood vertices of the i^{th} old vertex $u(o)$ of λ^s to every vertex of the i^{th} duplicate of λ_1 and joining the neighborhood vertices of the j^{th} new vertex $u(N)$ of λ^s to every vertex of the j^{th} duplicate of λ_2 [21]. Figure 2 explains the notation of $(\lambda^s \cdot (\lambda_1, \lambda_2))$.

In [22], the authors investigated the first and second Zagreb indices of the Cartesian, composition, join, disjunction, and symmetric difference graph operations. The author in [23] computed the forgotten topological index of different corona products of graphs and the author in [24] gave the exact expressions of Zagreb indices of the generalized hierarchical product of graphs. For more discussion and results, we refer to [25, 26]. There are some new and recent topics related to this study is found, one can see [27–31].

The Laplacian spectrum of double neighborhood corona graphs are found in the literature of [21], and the main results are presented.

Theorem 1. *Let λ be a t-regular graph on n vertices, m edges, λ_1 and λ_2 be any two graphs on n_1 and n_2 vertices, respectively. Then, the Laplacian spectrum of $\lambda^s \circ (\lambda_1 . \lambda_2)$ comprises*

(i) $\eta^4 - (n_1 + n_2 + t + 4)\eta^4 + [(n_1 + 1)(n_2 + 3) + 2(t + 1) + n_2 t + \eta_j(\lambda)]\eta^2 - [t(n_2 + 1) + 2 (n_1 + \eta_j(\lambda) + 1)]\eta + \eta_j(\lambda) = 0$, *for* $1 \le j \le n$;

(ii) $n_2 + 3 \pm \sqrt{(n_2 + 3)^2 - 8/2}$ *repeated* $m - n$ *times each;*

(iii) $\eta_j(\lambda_1) + 1$ *repeated n times, for* $2 \le j \le n_1$;

(iv) $\eta_j(\lambda_2) + 1$ *repeated m times, for* $2 \le j \le n_2$.

Theorem 2 (see [21]). *Let λ be a t-regular graph on n vertices, m edges, λ_1 and λ_2 be any two graphs on n_1 and n_2 vertices, respectively. Then, the Laplacian spectrum of $\lambda^s \cdot (\lambda_1, \lambda_2)$ comprises*

(i) *all the roots of the equation*

$$\begin{aligned}&\eta - (n_1 + n_2 + t + 4) + \Big[(n_2 + 4)(n_1 + t) + 1 - \big(2r - \eta_j(\lambda)\big)(n_1 + n_2) \\ &- \sqrt{2t - \eta_j(\lambda)}\,\Big]\eta^2 - \Big[n_1 + n - 2 + t + 2 - \big(2t - \eta_j(\lambda)\big)n_1(n_1 + t + 1) \\ &+ n_2(n_2 + 3)\big) + 2\Big(\big(n_1 + t\big)(n_2 + 2) - \sqrt{2t\eta_j(\lambda)}\,\Big)\Big] + (n_1 + t)(n_2 + 2) - \\ &\sqrt{2t - \eta_j(\lambda)} + \big(2t - \eta_j(\lambda)\big)(n_1 n_2 - n_1 - n_2) = 0, \quad \text{for } 1 \ge j \ge n.\end{aligned} \quad (5)$$

(ii) $n_2 + 2$ *repeated* $m - n$ *times;*

(iii) *1 repeated* $m - n$ *times;*

(iv) $\eta_i(\lambda_1) + 1$ *repeated n times, for* $2 \le i \le n_1$;

(v) $\eta_i(\lambda_2) + 1$ *repeated n times, for* $2 \le i \le n_2$.

In this paper, we extend the work and investigated some degree-based topological indices of these graph operations.

2. Main Results

The current section contained the main results which include the formulation of some degree-based topological indices such as first and second Zagreb, first multiplicative Zagreb, first reformulated Zagreb, the forgotten indices of subdivision double corona product, and subdivision double neighborhood corona product of graphs.

Following is the famous relationship between arithmetic and geometric means.

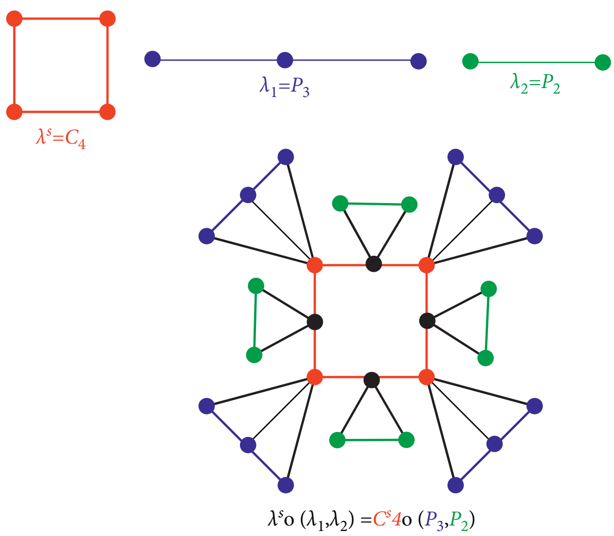

Figure 1: Subdivision double corona product $(C_4^s \circ (P_3, P_2))$.

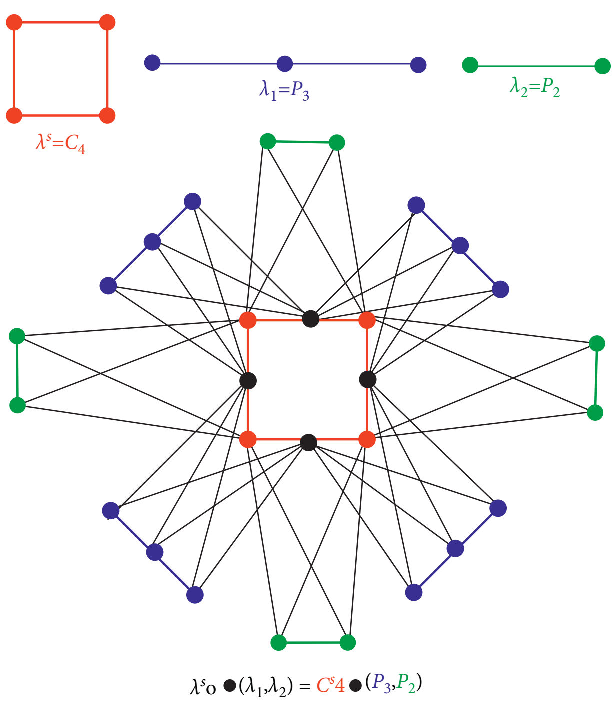

Figure 2: Subdivision double neighborhood corona product $(C_4^s \cdot (P_3, P_2))$.

Lemma 1 (AM-GM Inequality). *Let $a_1, a_2, \ldots, a_n$ be non-negative numbers. Then,*

$$\frac{a_1 + a_2 + \cdots + a_n}{2} \geq (a_1 a_2 \ldots a_n)^{1/n}, \tag{6}$$

and the equality holds if and only if $a_1 = a_2 = \ldots = a_n$ are equal.

Next to lemmas are the direct results from the definitions of the subdivision double corona product and subdivision neighborhood corona product of graphs.

Lemma 2. *Let λ, λ_1, and λ_2 are three graphs having order n, n_1, and n_2, respectively. Then, the degree behavior of the vertices in subdivision double corona product is given as*

$$d_{(\lambda^s \circ (\lambda_1, \lambda_2))}(v) = \begin{cases} d_\lambda(v) + n_1 & \text{if } v \in V_\lambda; \\ d_{\lambda_1}(v) + 1 & \text{if } v \in V_{\lambda_1}; \\ d_{\lambda_2}(v) + 1 & \text{if } v \in V_{\lambda_2}; \\ n_2 + 2 & \text{if } v \in V_{\lambda^s}. \end{cases} \tag{7}$$

Lemma 3. *Let λ, λ_1, and λ_2 are graphs having order n, n_1, and n_2, respectively, then the degrees of the vertices in subdivision double neighborhood corona product is*

$$d_{(\lambda^s \bullet (\lambda_1, \lambda_2))}(v) = \begin{cases} d_\lambda(v) + 2n_2 & \text{if } v \in V_\lambda; \\ d_{\lambda_1}(v) + 2 & \text{if } v \in V_{\lambda_1}; \\ d_{\lambda_2}(v) + 2 & \text{if } v \in V_{\lambda_2}; \\ 2(1 + n_1) & \text{if } v \in V_{\lambda^s}. \end{cases} \tag{8}$$

Following is our first main result, which gives the first Zagreb index of the subdivision double corona product in terms of the first Zagreb indices of basic graphs, their orders, and sizes.

Theorem 3. *Let $\lambda(n, m)$, $\lambda_1(n_1, m_1)$, and $\lambda_2(n_2, m_2)$ be the simple connected graphs. Then, the first Zagreb index of the subdivision double corona product, $(\lambda^s \circ (\lambda_1, \lambda_2))$, is given as*

$$\begin{aligned} M_1(\lambda^s \circ (\lambda_1, \lambda_2)) = {} & M_1(\lambda) + 4mn_1 + nn_1^2 \\ & + nM_1(\lambda_1) + 4m_1 n + nn_1 + mM_1(\lambda_2) \\ & + 4m_2 m + mn_2 + m(2 + n_2)^2. \end{aligned} \tag{9}$$

Proof. From the concept of topological descriptor named the first Zagreb index, we have got

$$M_1(\lambda^s \circ (\lambda_1, \lambda_2)) = \sum_{v \in V(\lambda^s \circ (\lambda_1, \lambda_2))} d_{\lambda^s \circ (\lambda_1, \lambda_2)}(v)^2. \tag{10}$$

Now, we apply Lemma 2,

$$\begin{aligned} M_1(\lambda^s \circ (\lambda_1, \lambda_2)) = {} & \sum_{v \in V_\lambda} (d_\lambda(v) + n_1)^2 \\ & + n \sum_{v \in V_{\lambda_1}} (d_{\lambda_1}(v) + 1)^2 + m \sum_{v \in V_{\lambda_2}} (d_{\lambda_2}(v) + 1)^2 + \sum_{v \in V_{\lambda^s}} (2 + n_2)^2 \end{aligned}$$

$$= \sum_{v \in V_\lambda} d_\lambda(v)^2 + 2n_1 \sum_{v \in V_\lambda} d_\lambda(v) + n_1^2 \sum_{v \in V_\lambda} 1 + n\left[\sum_{v \in V_{\lambda_1}} d_{\lambda_1}(v)^2 + 2\sum_{v \in V_{\lambda_1}} d_{\lambda_1}(v) + \sum_{v \in V_{\lambda_1}} 1\right]$$

$$+ m\left[\sum_{v \in V_{\lambda_2}} d_{\lambda_2}(v)^2 + 2\sum_{v \in V_{\lambda_2}} d_{\lambda_2}(v) + \sum_{v \in V_{\lambda_2}} 1\right] + (2+n_2)^2 \sum_{v \in V_{\lambda^s}} 1$$

$$= M_1(\lambda) + 4mn_1 + nn_1^2 + nM_1(\lambda_1) + 4m_1 n + nn_1 + mM_1(\lambda_2) + 4m_2 m + mn_2 + m(2+n_2)^2. \tag{11}$$

Hence, the required expression.

The next results put a bound on the first multiplicative Zagreb index for the subdivision double corona product. □

Theorem 4. *Let $\lambda(n,m)$, $\lambda_1(n_1,m_1)$, and $\lambda_2(n_2,m_2)$ be the simple connected graphs. Then, the first multiplicative Zagreb index of the subdivision double corona product $(\lambda^s \circ(\lambda_1,\lambda_2))$ is given as*

$$\prod_1 (\lambda^s \circ(\lambda_1,\lambda_2)) \leq \left[\frac{M_1(\lambda) + 4n_1 m + nn_1^2}{n}\right]^n \cdot \left[\frac{M_1(\lambda_1) + 4m_1 + n_1}{n_1}\right]^{nn_1} \cdot \left[\frac{M_1(\lambda_2) + 4m_2 + n_2}{n_2}\right]^{mn_2} \cdot (2+n_2)^{2m}. \tag{12}$$

Proof. From using the concept of the first multiplicative Zagreb index and Lemma 2, we have got

$$\prod_1 (\lambda^s \circ(\lambda_1,\lambda_2)) = \prod_{v \in V(\lambda^s \circ(\lambda_1,\lambda_2))} d_{(\lambda^s \circ(\lambda_1,\lambda_2))}(v)^2$$

$$= \prod_{v \in V_\lambda} (d_\lambda(v) + n_1)^2 \times \left[\prod_{v \in V_{\lambda_1}} \left(d_{\lambda_1}(v) + 1\right)^2\right]^n \times \left[\prod_{v \in V_{\lambda_2}} \left(d_{\lambda_2}(v) + 1\right)^2\right]^m \times \prod_{v \in V_{\lambda^s}} (2+n_2)^2$$

$$= \prod_{v \in V_\lambda} \left(d_\lambda(v)^2 + 2d_\lambda(v)(n_1) + n_1^2\right) \times \left[\prod_{v \in V_{\lambda_1}} \left(d_{\lambda_1}(v)^2 + 2d_{\lambda_1}(v) + 1\right)\right]^n$$

$$\times \left[\prod_{v \in V_{\lambda_2}} \left(d_{\lambda_2}(v)^2 + 2d_{\lambda_2}(v) + 1\right)\right]^m \times \prod_{v \in V_{\lambda^s}} (2+n_2)^2$$

$$\leq \left[\frac{\sum_{v \in V_\lambda}\left(d_\lambda(v)^2 + 2d_\lambda(v)n_1 + n_1^2\right)}{n}\right]^n \times \left[\frac{\sum_{v \in V_{\lambda_1}}\left(d_{\lambda_1}(v)^2 + 2d_{\lambda_1}(v) + 1\right)}{n_1}\right]^{nn_1}$$

$$\times \left[\frac{\sum_{v \in V_{\lambda_2}}\left(d_{\lambda_2}(v)^2 + 2d_{\lambda_2}(v) + 1\right)}{n_2}\right]^{mn_2} \times \left[\frac{\sum_{v \in V_{\lambda^s}} (2+n_2)^2}{m}\right]^m$$

$$\leq \left[\frac{\sum_{v \in V_\lambda} d_\lambda(v)^2 + 2n_1 \sum_{v \in V_\lambda} d_\lambda(v) + n_1^2 \sum_{v \in V_\lambda} 1}{n}\right]^n$$

$$\times \left[\frac{\sum_{v \in V_{\lambda_1}} d_{\lambda_1}(v)^2 + 2\sum_{v \in V_{\lambda_1}} d_{\lambda_1}(v) + \sum_{v \in V_{\lambda_1}} 1}{n_1}\right]^{nn_1}$$

$$\times\left[\frac{\sum_{v\in V_{\lambda_2}} d_{\lambda_2}(v)^2 + 2\sum_{v\in V_{\lambda_2}} d_{\lambda_2}(v) + \sum_{v\in V_{\lambda_2}} 1}{n_2}\right]^{mn_2} \times\left[\frac{(2+n_2)^2\sum_{v\in V_{\lambda^s}} 1}{m}\right]^{m}$$
$$=\left[\frac{M_1(\lambda)+4n_1m+nn_1^2}{n}\right]^{n}\times\left[\frac{M_1(\lambda_1)+4m_1+n_1}{n_1}\right]^{nn_1}$$
$$\times\left[\frac{M_1(\lambda_2)+4m_2+n_2}{n_2}\right]^{mn_2}\times(2+n_2)^{2m}. \tag{13}$$

The inequality is due to the Lemma 1. Equality in the last expression holds if and only if λ, λ_1, and λ_2 are regular graphs. □

Theorem 5. *Let $\lambda(n,m)$, $\lambda_1(n_1,m_1)$, and $\lambda_2(n_2,m_2)$ are three simple graphs and $\lambda^s(n\prime,m\prime)$ be the subdivided graph of λ. Then, the second Zagreb index of subdivision double corona product $(\lambda^s\circ(\lambda_1,\lambda_2))$ is given as*

$$\begin{aligned}
&M_2(\lambda^s\circ(\lambda_1,\lambda_2))\\
&= nM_1(\lambda_1)+mM_1(\lambda_2)+nM_2(\lambda_1)+mM_2(\lambda_2)+4m(m_1+m_2)\\
&\quad+(2n_1+n_1n_2)m\prime+mm_2+n_1(3m+2m_1n+nn_1)+mn_2(2+n_2+2m_2)+\\
&\quad\sum_{uv\in E_{\lambda^s}}(2d_\lambda(v)+n_2d_\lambda(v)).
\end{aligned} \tag{14}$$

Proof. From using the concept of the second Zagreb index and Lemma 2, we have got

$$\begin{aligned}
M_2(\lambda^s\circ(\lambda_1,\lambda_2)) &= \sum_{uv\in E_{\lambda^s\circ(\lambda_1,\lambda_2)}} d_{\lambda^s\circ(\lambda_1,\lambda_2)}u\cdot d_{\lambda^s\circ(\lambda_1,\lambda_2)}v.\\
&= \sum_{uv\in E_{\lambda^s}}(d_\lambda+n_1)(2+n_2)+n\sum_{uv\in E_{\lambda_1}}\left(d_{\lambda_1}(u)+1\right)\left(d_{\lambda_1}(v)+1\right)\\
&\quad+m\sum_{uv\in E_{\lambda_2}}\left(d_{\lambda_2}(u)+1\right)\left(d_{\lambda_2}(v)+1\right)+\sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_1}}}(d_\lambda(u)+n_1)\left(d_{\lambda_1}(v)+1\right)\\
&\quad+\sum_{\substack{u(N)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}}(2+n_2)\left(d_{\lambda_2}(v)+1\right)\\
&= \sum_{uv\in E_{\lambda^s}}(2d_\lambda(u)+n_2d_\lambda(u)+2n_1+n_1n_2)+n\sum_{uv\in E_{\lambda_1}}\left(d_{\lambda_1}(u)d_{\lambda_1}(v)+d_{\lambda_1}(u)\right.\\
&\quad\left.+d_{\lambda_1}(v)+1\right)+m\sum_{uv\in E_{\lambda_2}}\left(d_{\lambda_2}(u)d_{\lambda_2}(v)+d_{\lambda_2}(u)+d_{\lambda_2}(v)+1\right)\\
&\quad+\sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_1}}}\left(d_\lambda(u)d_{\lambda_1}(v)+d_\lambda(u)+n_1d_{\lambda_1}(v)+n_1\right)\\
&\quad+\sum_{\substack{u(N)\in V_{\lambda^s}\\ v\in V\lambda_2}}\left(2d_{\lambda_2}(v)+2+n_2d_{\lambda_2}(v)+n_2\right)
\end{aligned}$$

$$= \sum_{uv\in E_{\lambda^s}} (2d_\lambda(u) + n_2 d_\lambda(u)) + \sum_{uv\in E_{\lambda^s}} (2n_1 + n_1 n_2) + n\left[\sum_{uv\in E_{\lambda_1}} d_{\lambda_1}(u) d_{\lambda_1}(v) + \sum_{uv\in E_{\lambda_1}} \left(d_{\lambda_1}(u) + d_{\lambda_1}(v)\right) + \sum_{uv\in E_{\lambda_1}} 1\right]$$

$$+ m\left[\sum_{uv\in E_{\lambda_2}} d_{\lambda_2}(u) d_{\lambda_2}(v) + \sum_{uv\in E_{\lambda_2}} \left(d_{\lambda_2}(u) + d_{\lambda_2}(v)\right) + \sum_{uv\in E_{\lambda_2}} 1\right] +$$

$$+ \sum_{\substack{u(o)\in V_{\lambda^s} \\ v\in V_{\lambda_1}}} d_\lambda(u) d_{\lambda_1}(v) + \sum_{\substack{u(o)\in V_{\lambda^s} \\ v\in V_{\lambda_1}}} \left(d_\lambda(u) + n_1 d_{\lambda_1}(v)\right)$$

$$+ n_1 \sum_{u(o)\in V_{\lambda^s} v\in V_{\lambda_1}} 1 + \sum_{\substack{u(N)\in V_{\lambda^s} \\ v\in V_{\lambda_2}}} \left(2d_{\lambda_2}(v) + n_2 d_{\lambda_2}(v)\right) + \sum_{\substack{u(N)\in V_{\lambda^s} \\ v\in V_{\lambda_2}}} (2 + n_2)$$

$$\sum_{uv\in E_{\lambda^s}} (2d_\lambda(u) + n_2 d_\lambda(u)) + (2n_1 + n_1 n_2)\left|E_{\lambda^s}\right| + nM_2(\lambda_1) + nM_1(\lambda_1)$$

$$+ n\left|E_{\lambda_1}\right| + mM_2(\lambda_2) + mM_1(\lambda_2) + m\left|E_{\lambda_2}\right| + \sum_{u(o)\in V_{\lambda^s}} d_\lambda(u) \sum_{v\in V_{\lambda_1}} d_{\lambda_1}(v)$$

$$+ \sum_{u(o)\in V_{\lambda^s}} d_\lambda(u) \sum_{v\in V_{\lambda_1}} 1 + n_1 \sum_{u(o)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_1}} d_{\lambda_1}(v) + n_1 \sum_{u(o)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_1}} 1$$

$$+ 2 \sum_{u(N)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_2}} d_{\lambda_2}(v) + n_2 \sum_{u(N)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_2}} d_{\lambda_2}(v) + (2 + n_2) \sum_{u(N)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_2}} 1$$

$$= \sum_{uv\in E_{\lambda^s}} (2d_\lambda(u) + n_2 d_\lambda(u)) + (2n_1 + n_1 n_2)m\prime + nM_2(\lambda_1) + nM_1(\lambda_1) + nm_1$$

$$+ mM_2(\lambda_2) + mM_1(\lambda_2) + mm_2 + 4mm_1 + 2mn_1 + 2m_1 nn_1 + nn_1^2 + 4mm_2$$

$$+ 2mm_2 n_2 + mn_2(2 + n_2). \tag{15}$$

After some simplification, we can get the required result.

The next result is about the first reformulated Zagreb index of the subdivision double corona product of graphs. □

Theorem 6. *Let $\lambda(n, m)$, $\lambda_1(n_1, m_1)$, and $\lambda_2(n_2, m_2)$ be the simple graphs and $\lambda^s(n\prime, m\prime)$ is the subdivision of the graph λ. Then, the first reformulated Zagreb index of the subdivision double corona product $(\lambda^s \circ (\lambda_1, \lambda_2))$ is given as*

$$\begin{aligned} EM_1(\lambda^s \circ (\lambda_1, \lambda_2)) = {} & n_1 M_1(\lambda) + nM_1(\lambda_1) + mM_1(\lambda_2) + nHM_1(\lambda_1) + mHM_1(\lambda_2) + \\ & (n_1 + n_2)^2 m' + 8mm_1 + nn_1(n_1 - 1)^2 + 2(n_1 - 1)(2mn_1 + 2nm_1) + (1 + n_2)^2 mn_2 + \\ & 4mm_2(1 + n_2) + \sum_{uv\in E_{\lambda^s}} \left(d_\lambda(u)^2 + 2(n_1 + n_2) d_\lambda(u)\right). \end{aligned} \tag{16}$$

Proof. Using the Lemma 2 and the concept of the first reformulated Zagreb index, we have got

$$
\begin{aligned}
EM_1(\lambda^{s}\circ(\lambda_1,\lambda_2)) &= \sum_{uv\in E_{\lambda^{s}\circ(\lambda_1,\lambda_2)}} \left(d_{\lambda^{s}\circ(\lambda_1,\lambda_2)}(u) + d_{\lambda^{s}\circ(\lambda_1,\lambda_2)}(v) - 2\right)^2 \\
&= \sum_{uv\in E_{\lambda^s}} (d_{\lambda}(u) + n_1 + 2 + n_2 - 2)^2 + n\sum_{uv\in E_{\lambda_1}} \left(d_{\lambda_1}(u) + 1 + d_{\lambda_1}(v) + 1 - 2\right)^2 \\
&\quad + m\sum_{uv\in E_{\lambda_2}} \left(d_{\lambda_2}(u) + 1 + d_{\lambda_2}(v) + 1 - 2\right)^2 + \sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_1}}} \left(d_{\lambda}(u) + n_1 + d_{\lambda_1}(v) + 1 - 2\right)^2 \\
&\quad + \sum_{\substack{u(N)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}} \left(2 + n_2 + d_{\lambda_2}(v) + 1 - 2\right)^2 = \sum_{uv\in E_{\lambda^s}} [d_{\lambda}(u) + (n_1 + n_2)]^2 + n\sum_{uv\in E_{\lambda_1}} \left(d_{\lambda_1}(u) + d_{\lambda_1}(v)\right)^2 \\
&\quad + m\sum_{uv\in E_{\lambda_2}} \left(d_{\lambda_2}(u) + d_{\lambda_2}(v)\right)^2 + \sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_1}}} \left[\left(d_{\lambda}(u) + d_{\lambda_1}(v)\right) + (n_1 - 1)\right]^2 + \sum_{\substack{u(N)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}} \left[d_{\lambda_2}(v) + (1 + n_2)\right]^2 \\
&= \sum_{uv\in E_{\lambda^s}} \left[d_{\lambda}(u)^2 + (n_1 + n_2)^2 + 2d_{\lambda}(u)(n_1 + n_2)\right] + n\sum_{uv\in E_{\lambda_1}} \left(d_{\lambda_1}(u) + d_{\lambda_1}(v)\right)^2 \\
&\quad + m\sum_{uv\in E_{\lambda_2}} \left(d_{\lambda_2}(u) + d_{\lambda_2}(v)\right)^2 + \sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_1}}} \Big[\left(d_{\lambda}(u) + d_{\lambda_1}(v)\right)^2 + (n_1 - 1)^2 \\
&\quad + 2\left(d_{\lambda}(u) + d_{\lambda_1}(v)\right)(n_1 - 1)\Big] + \sum_{\substack{u(N)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}} \left[d_{\lambda_2}(v)^2 + (1 + n_2)^2 + 2d_{\lambda_2}(v)(1 + n_2)\right] \\
&= \sum_{uv\in E_{\lambda^s}} d_{\lambda}(u)^2 + 2(n_1 + n_2)\sum_{uv\in E_{\lambda^s}} d_{\lambda}(u) + \sum_{uv\in E_{\lambda^s}} (n_1 + n_2)^2 + nHM_1(\lambda_1) \\
&\quad + mHM_1(\lambda_2) + \sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_1}}} \left(d_{\lambda}(u) + d_{\lambda_1}(v)\right)^2 + (n_1 - 1)^2 \sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_1}}} 1 + 2(n_1 - 1) \\
&\qquad \sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_1}}} \left(d_{\lambda}(u) + d_{\lambda_1}(v)\right) + \sum_{\substack{u(N)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}} d_{\lambda_2}(v)^2 + (1 + n_2)^2 \sum_{\substack{u(N)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}} 1 + 2(1 + n_2) \\
&\qquad \sum_{\substack{u(N)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}} d_{\lambda_2}(v) = \sum_{uv\in E_{\lambda^s}} \left(d_{\lambda}(u)^2 + 2(n_1 + n_2)d_{\lambda}(u)\right) + (n_1 + n_2)^2 m\prime + nHM_1(\lambda_1) \\
&\quad + mHM_1(\lambda_2) + \sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_1}}} d_{\lambda}(u)^2 + \sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_1}}} d_{\lambda_1}(v)^2 + 2\sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_1}}} d_{\lambda}(u)d_{\lambda_1}(v) \\
&\quad + nn_1(n_1 - 1)^2 + 2(n_1 - 1)\sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_1}}} \left(d_{\lambda}(u) + d_{\lambda_1}(v)\right) + \sum_{\substack{u(N)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}} d_{\lambda_2}(v)^2 \\
&\quad + mn_2(1 + n_2)^2 + 2(1 + n_2)\sum_{u(N)\in V_{\lambda^s}\, v\in V_{\lambda_2}} d_{\lambda_2}(v) \\
&= \sum_{uv\in E_{\lambda^s}} \left(d_{\lambda}(u)^2 + 2(n_1 + n_2)d_{\lambda}(u)\right) + (n_1 + n_2)^2 m\prime + nHM_1(\lambda_1) + mHM_1(\lambda_2) \\
&\quad + \sum_{u(o)\in V_{\lambda^s}} d_{\lambda}(u)^2 \sum_{v\in V_{\lambda_1}} 1 + \sum_{u(o)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_1}} d_{\lambda_1}(v)^2 + 2\sum_{u(o)\in E_{\lambda^s}} d_{\lambda}(u) \sum_{v\in V_{\lambda_1}} d_{\lambda_1}(v)
\end{aligned}
$$

$$+ nn_1(n_1-1)^2 + 2(n_1-1)\left[\sum_{u(o)\in V_{\lambda^s}} d_\lambda(u) \sum_{v\in V_{\lambda_1}} 1 + \sum_{u(o)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_1}} d_{\lambda_1}(v)\right]$$
$$+ \sum_{u(N)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_2}} d_{\lambda_2}(v)^2 + mn_2(1+n_2)^2 + 2(1+n_2)\left[\sum_{u(N)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_2}} d_{\lambda_2}(v)\right]$$
$$= \sum_{uv\in E_{\lambda^s}} \left(d_\lambda(u)^2 + 2(n_1+n_2)d_\lambda(u)\right) + (n_1+n_2)^2 m\prime + nHM_1(\lambda_1) + mHM_1(\lambda_2)$$
$$+ n_1M_1(\lambda) + nM_1(\lambda_1) + 2\cdot 2m\cdot 2m_1 + (n_1-1)^2\cdot nn_1 + 2(n_1-1)(2mn_1+2m_1n)$$
$$+ mM_1(\lambda_2) + (1+n_2)^2 mn_2 + 2(1+n_2).2m_2m$$
$$= \sum_{uv\in E_{\lambda^s}} \left(d_\lambda(u)^2 + 2(n_1+n_2)d_\lambda(u)\right) + (n_1+n_2)^2 m\prime + nHM_1(\lambda_1)$$
$$+ mHM_1(\lambda_2) + n_1M_1(\lambda) + nM_1(\lambda_1) + 8mm_1 + nn_1(n_1-1)^2$$
$$+ 4(n_1-1)(mn_1+nm_1) + mM_1(\lambda_2) + mn_2(1+n_2)^2 + 4mm_2(1+n_2). \tag{17}$$

Hence, the proof is done. □

Theorem 7. *For graphs $\lambda(n,m)$, $\lambda_1(n_1,m_1)$, and $\lambda_2(n_2,m_2)$, then, the first Zagreb index of the subdivision double neighborhood corona product $(\lambda^s\cdot(\lambda_1,\lambda_2))$ is given as*

$$M_1(\lambda^s\cdot(\lambda_1,\lambda_2)) = M_1(\lambda) + nM_1(\lambda_1) + mM_1(\lambda_2) + 8mn_2 + 4n_2^2n + 8nm_1 + 4nn_1 + 8mm_2 + 4mn_2 + m(2+2n_1)^2. \tag{18}$$

Proof. From the concept of the first Zagreb index we have

$$M_1(\lambda^s\cdot(\lambda_1,\lambda_2)) = \sum_{v\in V_{(\lambda^s\cdot(\lambda_1,\lambda_2))}} d_{(\lambda^s\cdot(\lambda_1,\lambda_2))}(v)^2. \tag{19}$$

Now, we apply Lemma 3,

$$= \sum_{v\in V_\lambda} (d_\lambda(v)+2n_2)^2 + n\sum_{v\in V_{\lambda_1}} \left(d_{\lambda_1}(v)+2\right)^2 + m\sum_{v\in V\lambda_2}\left(d_{\lambda_2}(v)+2\right)^2 + \sum_{v\in V_{\lambda^s}} (2+2n_1)^2$$
$$= \sum_{v\in V_\lambda}\left(d_\lambda(v)^2 + 2d_\lambda(v)(2n_2) + (2n_2)^2\right) + n\sum_{v\in V_{\lambda_1}}\left(d_{\lambda_1}(v)^2 + (2)2d_{\lambda_1}(v) + (2)^2\right)$$
$$+ m\sum_{v\in V_{\lambda_2}}\left(d_{\lambda_2}(v)^2 + 2d_{\lambda_2}(v)(2) + (2)^2\right) + \sum_{v\in V_{\lambda^s}} (2+2n_1)^2$$
$$= \sum_{v\in V_\lambda} d_\lambda(v)^2 + 4n_2\sum_{v\in V_\lambda} d_\lambda(v) + 4n_2^2\sum_{v\in V_\lambda} 1 + n\left[\sum_{v\in V_{\lambda_1}} d_{\lambda_1}(v)^2 + 4\sum_{v\in V_{\lambda_1}} d_{\lambda_1}(v) + 4\sum_{v\in V_{\lambda_1}} 1\right]$$
$$+ m\left[\sum_{v\in V_{\lambda_2}} d_{\lambda_2}(v)^2 + 4\sum_{v\in V_{\lambda_2}} d_{\lambda_2}(v) + 4\sum_{v\in V_{\lambda_2}} 1\right] + (2+2n_1)^2\sum_{v\in V_{\lambda^s}} 1$$
$$= M_1(\lambda) + 8mn_2 + 4n_2^2n + nM_1(\lambda_1) + 8nm_1 + 4nn_1 + mM_1(\lambda_2) + 8mm_2$$
$$+ 4mn_2 + m(2+2n_1)^2. \tag{20}$$

Hence, the proof is done. □

Theorem 8. *Let $\lambda(n,m)$, $\lambda^s(n\prime,m\prime)$, $\lambda_1(n_1,m_1)$, and $\lambda_2(n_2,m_2)$ be the simple connected graphs. Then, the second Zagreb index of subdivision double neighborhood corona product $(\lambda^s\cdot(\lambda_1,\lambda_2))$ is given as*

$$M_2(\lambda^s\cdot(\lambda_1,\lambda_2)) = 2\sum_{uv\in E_{\lambda^s}}(d_\lambda(u)+n_1d_\lambda(u)) + 4n_2m' + 4n_1n_2m' + nM_2(\lambda_1) + 2nM_1(\lambda_1) + 4nm_1 + mM_2(\lambda_2) + 2mM_1(\lambda_2) + 12mm_2 + 8mn_2 + 8nn_2m_2 + 8nn_2^2 + 8mm_1 + 8mm_1n_1 + 8mn_1 + 8mn_1^2. \tag{21}$$

Proof. From the concept of the second Zagreb index, we have

$$M_2(\lambda^s\cdot(\lambda_1,\lambda_2)) = \sum_{uv\in E_{\lambda^s\cdot(\lambda_1,\lambda_2)}} d_{\lambda^s\cdot(\lambda_1,\lambda_2)}u.d_{\lambda^s\cdot(\lambda_1,\lambda_2)}v. \tag{22}$$

Now, we apply Lemma 3,

$$\begin{aligned}
&= \sum_{uv\in E_{\lambda^s}}(d_\lambda(u)+2n_2)(2+2n_1) + n\sum_{uv\in E_{\lambda_1}}\big(d_{\lambda_1}(u)+2\big)\big(d_{\lambda_1}(v)+2\big)\\
&\quad + m\sum_{uv\in E_{\lambda_2}}\big(d_{\lambda_2}(u)+2\big)\big(d_{\lambda_2}(v)+2\big) + 2\sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}}(d_\lambda(u)+2n_2)\big(d_{\lambda_2}(v)+2\big)\\
&\quad + 2\sum_{\substack{u(N)\in V_{\lambda^s}\\ v\in V_{\lambda_1}}}(2+2n_1)\big(d_{\lambda_1}(v)+2\big) = \sum_{uv\in E_{\lambda^s}}\big(2d_\lambda(u)+2n_1d_\lambda(u)+4n_2+4n_1n_2\big)\\
&\quad + n\sum_{uv\in E_{\lambda_1}}\big(d_{\lambda_1}(u)d_{\lambda_1}(v)+2d_{\lambda_1}(u)+2d_{\lambda_1}(v)+4\big)\\
&\quad + m\sum_{uv\in E_{\lambda_2}}\big(d_{\lambda_2}(u)d_{\lambda_2}(v)+2d_{\lambda_2}(u)+2d_{\lambda_2}(v)+4\big) + 2\sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}}\Big[\big(d_\lambda(u)d_{\lambda_2}(v)\big)+2d_\lambda(u)+2n_2d_{\lambda_2}(v)+4n_2\Big]\\
&\quad + 2\sum_{\substack{u(N)\in V_{\lambda^s}\\ v\in V_{\lambda_1}}}\big(2d_{\lambda_1}(v)+4+2n_1d_{\lambda_1}(v)+4n_1\big)\\
&= 2\sum_{uv\in E_{\lambda^s}}(d_\lambda(u)+n_1d_\lambda(u)) + 4n_2\sum_{uv\in E_{\lambda^s}}1 + 4n_1n_2\sum_{uv\in E_{\lambda^s}}1 + n\left[\sum_{uv\in E_{\lambda_1}}d_{\lambda_1}(u)d_{\lambda_1}(v) + 2\sum_{uv\text{I}E_{\lambda_1}}\big(d_{\lambda_1}(u)+d_{\lambda_1}(v)\big) + 4\sum_{uv\in E_{\lambda_1}}1\right]\\
&\quad + m\left[\sum_{uv\in E_{\lambda_2}}d_{\lambda_2}(u)d_{\lambda_2}(v) + 2\sum_{uv\in E_{\lambda_2}}\big(d_{\lambda_2}(u)+d_{\lambda_2}(v)\big) + 4\sum_{uv\in E_{\lambda_2}}1\right]\\
&\quad + 2\left[\sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}}d_\lambda(u)d_{\lambda_2}(v) + \sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}}\big(2d_\lambda(u)+2n_2d_{\lambda_2}(v)\big) + 4n_2\sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}}1\right]\\
&\quad + 2\left[\sum_{u(N)\in V_{\lambda^s}v\in V_{\lambda_1}}\big(2d_{\lambda_1}(v)+2n_1d_{\lambda_1}(v)\big) + 4\sum_{u(N)\in V_{\lambda^s}v\in V_{\lambda_1}}1 + 4n_1\sum_{u(N)\in V_{\lambda^s}v\in V_{\lambda_1}}1\right]
\end{aligned}$$

$$= 2\sum_{uv\in E_{\lambda^s}} (d_\lambda(u) + n_1 d_\lambda(u)) + 4n_2 m\prime + 4n_1 n_2 m\prime + n[M_2(\lambda_1) + 2M_1(\lambda_1) + 4m_1]$$

$$+ m[M_2(\lambda_2) + 2M_1(\lambda_2) + 4m_2] + 2\left[\sum_{u(o)\in V_{\lambda^s}} d_\lambda(u) \sum_{v\in V_{\lambda_2}} d_{\lambda_2}(v) + 2\sum_{u(o)\in V_{\lambda^s}} d_\lambda(u)\right.$$

$$\left.\sum_{v\in V_{\lambda_2}} 1 + 2n_2 \sum_{u(o)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_2}} d_{\lambda_2}(v) + 4n_2 \sum_{u(o)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_2}} 1\right] + 2\left[2\sum_{u(N)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_1}} d_{\lambda_1}(v)\right.$$

$$\left.+2n_1 \sum_{u(N)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_1}} d_{\lambda_1}(v) + 4\sum_{u(N)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_1}} 1 + 4n_1 \sum_{u(N)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_1}} 1\right]$$

$$= 2\sum_{uv\in E_{\lambda^s}} (d_\lambda(u) + n_1 d_\lambda(u)) + 4n_2 m\prime + 4n_1 n_2 m\prime + n[M_2(\lambda_1) + 2M_1(\lambda_1) + 4m_1]$$

$$+ m[M_2(\lambda_2) + 2M_1(\lambda_2) + 4m_2] + 2[2m \cdot 2m_2 + 2\cdot 2m \cdot n_2 + 2n_2 n \cdot 2m_2 + 4n_2 n \cdot n_2]$$

$$+ 2\left[2m \cdot 2m_1 + 2n_1 m \cdot 2m_1 + 4mn_1 + 4mn_1^2\right]$$

$$= 2\sum_{uv\in E_{\lambda^s}} (d_\lambda(u) + n_1 d_\lambda(u)) + 4n_2 m\prime + 4n_1 n_2 m\prime + n[M_2(\lambda_1) + 2M_1(\lambda_1) + 4m_1]$$

$$+ m[M_2(\lambda_2) + 2M_1(\lambda_2) + 4m_2] + 2\left[4mm_2 + 4mn_2 + 4nn_2 m_2 + 4nn_2^2\right]$$

$$+ 2\left[4mm_1 + 4mm_1 n_1 + 4mn_1 + 4mn_1^2\right]$$

$$= 2\sum_{uv\in E_{\lambda^s}} (d_\lambda(u) + n_1 d_\lambda(u)) + 4n_2 m\prime + 4n_1 n_2 m\prime + nM_2(\lambda_1) + 2nM_1(\lambda_1) + 4nm_1$$

$$+ mM_2(\lambda_2) + 2mM_1(\lambda_2) + 12mm_2 + 8mn_2 + 8nn_2 m_2 + 8nn_2^2 + 8mm_1$$

$$+ 8mm_1 n_1 + 8mn_1 + 8mn_1^2. \tag{23}$$

Hence, the proof is done. ☐

Theorem 9. *Let $\lambda(n,m)$, $\lambda_1(n_1,m_1)$, and $\lambda_2(n_2,m_2)$ be the simple connected graphs. Then, the forgotten topological index of subdivision double neighborhood corona product $\lambda^s \cdot (\lambda_1,\lambda_2)$ is given as*

$$F(\lambda^s \cdot (\lambda_1,\lambda_2)) = F(\lambda) + 24mn_2^2 + 6n_2 M_1(\lambda) + 8nn_2^3 + nF(\lambda_1) + 24nm_1 + 6nM_1(\lambda_1) + 8nn_1 + mF(\lambda_2) + 24mm_2 + 6mM_1(\lambda_2) + 8mn_2 + m(2+2n_1)^3. \tag{24}$$

Proof. From the concept of the forgotten index, we have

$$F(\lambda^s \cdot (\lambda_1,\lambda_2)) = \sum_{v\in V_{(\lambda^s\bullet(\lambda_1,\lambda_2))}} d_{(\lambda^s\cdot(\lambda_1,\lambda_2))}(v)^3. \tag{25}$$

Now, we apply Lemma 3,

$$= \sum_{v\in V_\lambda} (d_\lambda(v) + 2n_2)^3 + n\sum_{v\in V_{\lambda_1}} \left(d_{\lambda_1}(v) + 2\right)^3 + m\sum_{v\in V_{\lambda_2}} \left(d_{\lambda_2}(v) + 2\right)^3$$

$$+ \sum_{v\in V_{\lambda^s}} (2 + 2n_1)^3$$

$$
\begin{aligned}
&= \sum_{v\in V_\lambda}\left(d_\lambda(v)^3 + 3d_\lambda(v)(2n_2)^2 + 3d_\lambda(v)^2(2n_2) + (2n_2)^3\right)\\
&\quad + n\sum_{v\in V_{\lambda_1}}\left(d_{\lambda_1}(v)^3 + 3d_{\lambda_1}(v)(2)^2 + 3d_{\lambda_1}(v)^2\cdot(2) + 2^3\right)\\
&\quad + m\sum_{v\in V_{\lambda_2}}\left(d_{\lambda_2}(v)^3 + 3d_{\lambda_2}(v)(2)^2 + 3d_{\lambda_2}(v)^2\cdot 2 + 2^3\right) + \sum_{v\in V_{\lambda^s}}(2+2n_1)^3\\
&= \sum_{v\in V_\lambda} d_\lambda(v)^3 + 12n_2^2\sum_{v\in V_\lambda} d_\lambda(v) + 6n_2\sum_{v\in V_\lambda} d_\lambda(v)^2 + 8n_2^3\sum_{v\in V_\lambda}1 + n\sum_{v\in V_{\lambda_1}} d_{\lambda_1}(v)^3\\
&\quad + 12n\sum_{v\in V_{\lambda_1}} d_{\lambda_1}(v) + 6n\sum_{v\in V_{\lambda_1}} d_{\lambda_1}(v)^2 + 8n\sum_{v\in V_{\lambda_1}}1 + m\sum_{v\in V_{\lambda_2}} d_{\lambda_2}(v)^3 + 12m\\
&\quad \sum_{v\in V_{\lambda_2}} d_{\lambda_2}(v) + 6m\sum_{v\in V_{\lambda_2}} d_{\lambda_2}(v)^2 + 8m\sum_{v\in V_{\lambda_2}}1 + (2+2n_1)^3\sum_{v\in V_{\lambda^s}}1\\
&= F(\lambda) + 24mn_2^2 + 6n_2M_1(\lambda) + 8nn_2^3 + nF(\lambda_1) + 24nm_1 + 6nM_1(\lambda_1)\\
&\quad + 8nn_1 + mF(\lambda_2) + 24mm_2 + 6mM_1(\lambda_2) + 8mn_2 + m(2+2n_1)^3.
\end{aligned}
\tag{26}
$$

Hence, the proof is done. □

Theorem 10. *Let $\lambda(n,m)$, $\lambda_1(n_1,m_1)$, and $\lambda_2(n_2,m_2)$ be the simple connected graphs. Then, the first multiplicative Zagreb index of subdivision double neighborhood corona product $(\lambda^s\cdot(\lambda_1,\lambda_2))$ is given as*

$$
\begin{aligned}
\prod_1(\lambda^s\cdot(\lambda_1,\lambda_2)) &\le \left[\frac{M_1(\lambda)+8n_2m+4nn_2^2}{n}\right]^n \times \left[\frac{M_1(\lambda_1)+8m_1+4n_1}{n_1}\right]^{nn_1}\\
&\quad \times \left[\frac{M_1(\lambda_2)+8m_2+4n_2}{n_2}\right]^{mn_2} \times (2+2n_1)^{2m}.
\end{aligned}
\tag{27}
$$

Proof. From the concept of the first multiplicative Zagreb index, we have

$$
\prod_1(\lambda^s\cdot(\lambda_1,\lambda_2)) = \prod_{v\in V_{(\lambda^s\cdot(\lambda_1,\lambda_2))}} d_{(\lambda^s\cdot(\lambda_1,\lambda_2))}(v)^2.
\tag{28}
$$

Now, we apply Lemma 3,

$$
\begin{aligned}
&= \prod_{v\in V_\lambda}(d_\lambda(v)+2n_2)^2 \times \left[\prod_{v\in V_{\lambda_1}}\left(d_{\lambda_1}(v)+2\right)^2\right]^n \times \left[\prod_{v\in V_{\lambda_2}}\left(d_{\lambda_2}(v)+2\right)^2\right]^m \times \prod_{v\in V_{\lambda^s}}(2+2n_1)^2\\
&= \prod_{v\in V_\lambda}\left(d_\lambda(v)^2 + 2d_\lambda(v)(2n_2) + (2n_2)^2\right) \times \left[\prod_{v\in V_{\lambda_1}}\left(d_{\lambda_1}(v)^2 + 2d_{\lambda_1}(v)\times 2 + 2^2\right)\right]^n\\
&\quad \times \left[\prod_{v\in V_{\lambda_2}}\left(d_{\lambda_2}(v)^2 + 2d_{\lambda_2}(v)\times 2 + 2^2\right)\right]^m \times \prod_{v\in V_{\lambda^s}}(2+2n_1)^2.
\end{aligned}
\tag{29}
$$

By Lemma 1,

$$\begin{aligned}
&\leq \left[\frac{\sum_{v\in V_\lambda}\left(d_\lambda(v)^2+4n_2d_\lambda(v)+4n_2^2\right)}{n}\right]^n \times \left[\frac{\sum_{v\in V\lambda_1}\left(d_{\lambda_1}(v)^2+4d_{\lambda_1}(v)+4\right)}{n_1}\right]\\
&{}^{nn_1}\times\left[\frac{\sum_{v\in V_{\lambda_2}}\left(d_{\lambda_2}(v)^2+4d_{\lambda_2}(v)+4\right)}{n_2}\right]^{mn_2}\times\left[\frac{\sum_{v\in V_{\lambda^s}}(2+2n_1)^2}{m}\right]^m\\
&\leq \left[\frac{\sum_{v\in V_\lambda}d_\lambda(v)^2+4n_2\sum_{v\in V_\lambda}d_\lambda(v)+4n_2^2\sum_{v\in V_\lambda}1}{n}\right]^n\times\left[\frac{\sum_{v\in V\lambda_1}d_{\lambda_1}(v)^2+4\sum_{v\in V_{\lambda_1}}d_{\lambda_1}(v)+4\sum_{v\in V_{\lambda_1}}1}{n_1}\right]^{nn_1}\\
&\times\left[\frac{\sum_{v\in V_{\lambda_2}}d_{\lambda_2}(v)^2+4\sum_{v\in V_{\lambda_2}}d_{\lambda_2}(v)+4\sum_{v\in V_{\lambda_2}}1}{n_2}\right]^{mn_2}\times\left[\frac{(2+2n_1)^2\sum_{v\in V_{\lambda^s}}1}{m}\right]^m\\
&=\left[\frac{M_1(\lambda)+8n_2m+4nn_2^2}{n}\right]^n\times\left[\frac{M_1(\lambda_1)+8m_1+4n_1}{n_1}\right]^{nn_1}\\
&\times\left[\frac{M_1(\lambda_2)+8m_2+4n_2}{n_2}\right]^{mn_2}\times(2+2n_1)^{2m}.
\end{aligned} \tag{30}$$

Hence, equality holds in 13 iff λ, λ_1, and λ_2 are regular graphs.

Hence, the proof is done. □

Theorem 11. *Let $\lambda(n,m)$, $\lambda^s(n\prime,m\prime)$, $\lambda_1(n_1,m_1)$, and $\lambda_2(n_2,m_2)$ be the simple connected graphs. Then, the first reformulated Zagreb index of subdivision double neighborhood corona product $(\lambda^s\cdot(\lambda_1,\lambda_2))$ is given as*

$$\begin{aligned}
EM_1(\lambda^s\cdot(\lambda_1,\lambda_2)) = &\sum_{uv\in E_{\lambda^s}}\left(d_\lambda(u)^2+4(n_1+n_2)d_\lambda(u)\right)+4(n_1+n_2)^2m\prime\\
&+nHM_1(\lambda_1)+4nm_1+4nM_1(\lambda_1)+mHM_1(\lambda_2)+4mm_2+4mM_1(\lambda_2)\\
&+2n_2M_1(\lambda)+2nM_1(\lambda_2)+6mm_2+8nn_2^3+16mn_2^2+16nn_2m_2+2mM_1(\lambda_1)\\
&+2mn_1(2+2n_1)^2+8mm_1(2+2n_1).
\end{aligned} \tag{31}$$

Proof. From the concept of the first reformulated Zagreb, we have

$$EM_1(\lambda^s\cdot(\lambda_1,\lambda_2)) = \sum_{uv\in E_{\lambda^s\cdot(\lambda_1,\lambda_2)}}\left(d_{\lambda^s\cdot(\lambda_1,\lambda_2)}(u)+d_{\lambda^s\cdot(\lambda_1,\lambda_2)}(v)-2\right)^2. \tag{32}$$

Now, we apply Lemma 3,

$$\begin{aligned}
&=\sum_{uv\in E_{\lambda^s}}(d_\lambda(u)+2n_2+2+2n_1-2)^2+n\sum_{uv\in E_{\lambda_1}}\left(d_{\lambda_1}(u)+2+d_{\lambda_1}(v)+2-2\right)^2\\
&+m\sum_{uv\in E_{\lambda_2}}\left(d_{\lambda_2}(u)+2+d_{\lambda_2}(v)+2-2\right)^2+2\sum_{u(o)\in V_{\lambda^s}v\in V_{\lambda_2}}\left(d_\lambda(u)+2n_2+d_{\lambda_2}(v)+2-2\right)^2
\end{aligned}$$

$$+ 2 \sum_{\substack{u(N)\in V_{\lambda^s} \\ v\in V_{\lambda_1}}} \left(2 + 2n_1 + d_{\lambda_1}(v) + 2 - 2\right)^2 = \sum_{uv\in E_{\lambda^s}} \left(d_\lambda(u) + 2(n_1 + n_2)\right)^2 + n \sum_{uv\in E_{\lambda_1}} \left(d_{\lambda_1}(u) + d_{\lambda_1}(v) + 2\right)^2$$

$$+ m \sum_{uv\in E_{\lambda_2}} \left(d_{\lambda_2}(u) + d_{\lambda_2}(v) + 2\right)^2 + 2 \sum_{u(o)\in V_{\lambda^s} v\in V_{\lambda_2}} \left(d_\lambda(u) + d_{\lambda_2}(v) + 2n_2\right)^2 + 2 \sum_{u(N)\in V_{\lambda^s} v\in V_{\lambda_1}} \left(d_{\lambda_1}(v) + 2 + 2n_1\right)^2$$

$$= \sum_{uv\in E_{\lambda^s}} \left[d_\lambda(u)^2 + \left(2(n_1 + n_2)\right)^2 + 2d_\lambda(u)\left(2(n_1 + n_2)\right)\right] + n \sum_{uv\in E_{\lambda_1}} \left[\left(d_{\lambda_1}(u) + d_{\lambda_1}v\right)^2 + 4 + 4\left(d_{\lambda_1}(u) + d_{\lambda_1}(v)\right)\right]$$

$$+ m \sum_{uv\in E_{\lambda_2}} \left[\left(d_{\lambda_2}(u) + d_{\lambda_2}(v)\right)^2 + 4 + 4\left(d_{\lambda_2}(u) + d_{\lambda_2}(v)\right)\right] + 2 \sum_{\substack{u(o)\in V_{\lambda^s} \\ v\in V_{\lambda_2}}} \left[\left(d_\lambda(u) + d_{\lambda_2}(v)\right)^2 + (2n_2)^2 + 2\left(d_\lambda(u) + d_{\lambda_2}(v)\right)2n_2\right]$$

$$+ 2 \sum_{u(N)\in V_{\lambda^s} v\in V_{\lambda_1}} \left[d_{\lambda_1}(v)^2 + (2 + 2n_1)^2 + 2\left(d_{\lambda_1}(v)(2 + 2n_1)\right)\right]$$

$$= \sum_{uv\in E_{\lambda^s}} d_\lambda(u)^2 + 4(n_1 + n_2)^2 \sum_{uv\in E_{\lambda^s}} 1 + 4(n_1 + n_2) \sum_{uv\in E_{\lambda^s}} d_\lambda(u)$$

$$+ n\left[\sum_{uv\in E_{\lambda_1}} \left(d_{\lambda_1}(u) + d_{\lambda_1}(v)\right)^2 + 4 \sum_{uv\in E_{\lambda_1}} 1 + 4 \sum_{uv\in E_{\lambda_1}} \left(d_{\lambda_1}(u) + d_{\lambda_1}(v)\right)\right]$$

$$+ m\left[\sum_{uv\in E_{\lambda_2}} \left(d_{\lambda_2}(u) + d_{\lambda_2}(v)\right)^2 + 4 \sum_{uv\in E_{\lambda_2}} 1 + 4 \sum_{uv\in E_{\lambda_2}} \left(d_{\lambda_2}(u) + d_{\lambda_2}(v)\right)\right]$$

$$+ 2\left[\sum_{\substack{u(o)\in V_{\lambda^s} \\ v\in V_{\lambda_2}}} \left(d_\lambda(u) + d_{\lambda_2}(v)\right)^2 + 4n_2^2 \sum_{\substack{u(o)\in V_{\lambda^s} \\ v\in V_{\lambda_2}}} 1 + 4n_2 \sum_{\substack{u(o)\in V_{\lambda^s} \\ v\in V_{\lambda_2}}} \left(d_\lambda(u) + d_{\lambda_2}(v)\right)\right]$$

$$+ 2\left[\sum_{\substack{u(N)\in V_{\lambda^s} \\ v\in V_{\lambda_1}}} d_{\lambda_1}(v)^2 + (2 + 2n_1)^2 \sum_{\substack{u(N)\in V_{\lambda^s} \\ v\in V_{\lambda_1}}} 1 + 2(2 + 2n_1) \sum_{\substack{u(N)\in V_{\lambda^s} \\ v\in V_{\lambda_1}}} d_{\lambda_1}(v)\right],$$

$$= \sum_{uv\in E_{\lambda^s}} d_\lambda(u)^2 + 4(n_1 + n_2)^2 m\prime + 4(n_1 + n_2) \sum_{uv\in E_{\lambda^s}} d_\lambda(u) + nHM_1(\lambda_1) + 4nm_1 + 4nM_1(\lambda_1)$$

$$+ mHM_1(\lambda_2) + 4mm_2 + 4mM_1(\lambda_2)$$

$$+ 2\left[\sum_{\substack{u(o)\in V_{\lambda^s} \\ v\in V_{\lambda_2}}} \left(d_\lambda(u)^2 + d_{\lambda_2}(v)^2 + 2d_\lambda(u)d_{\lambda_2}(v)\right) + 4n_2^2 \sum_{\substack{u(o)\in V_{\lambda^s} \\ v\in V_{\lambda_2}}} 1 + 4n_2\left(\sum_{\substack{u(o)\in V_{\lambda^s} \\ v\in V_{\lambda_2}}} d_\lambda(u) + \sum_{\substack{u(o)\in V_{\lambda^s} \\ v\in V_{\lambda_2}}} d_{\lambda_2}(v)\right)\right]$$

$$+ 2\left[\sum_{\substack{u(N)\in V_{\lambda^s} \\ v\in V_{\lambda_1}}} d_{\lambda_1}(v)^2 + (2 + 2n_1)^2 \sum_{\substack{u(N)\in V_{\lambda^s} \\ v\in V_{\lambda_1}}} 1 + 2(2 + 2n_1) \sum_{\substack{u(N)\in V_{\lambda^s} \\ v\in V_{\lambda_1}}} d_{\lambda_1}(v)\right]$$

$$
\begin{aligned}
&= \sum_{uv\in E_{\lambda^s}} d_\lambda(u)^2 + 4(n_1+n_2)^2 m\prime + 4(n_1+n_2)\sum_{uv\in E_{\lambda^s}} d_\lambda(u) + nHM_1(\lambda_1)\\
&\quad + 2\left[\sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}}\left(d_\lambda(u)^2 + d_{\lambda_2}(v)^2 + 2d_\lambda(u)d_{\lambda_2}(v)\right) + 4n_2^2\sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}} 1 + 4n_2\left(\sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}} d_\lambda(u) + \sum_{\substack{u(o)\in V_{\lambda^s}\\ v\in V_{\lambda_2}}} d_{\lambda_2}(v)\right)\right]\\
&\quad + 2\left[\sum_{u(N)\in V_{\lambda^s} v\in V_{\lambda_1}} d_{\lambda_1}(v)^2 + (2+2n_1)^2\sum_{u(N)\in V_{\lambda^s} v\in V_{\lambda_1}} 1 + 2(2+2n_1)\sum_{u(N)\in V_{\lambda^s} v\in V_{\lambda_1}} d_{\lambda_1}(v)\right]\\
&= \sum_{uv\in E_{\lambda^s}} d_\lambda(u)^2 + 4(n_1+n_2)^2 m\prime + 4(n_1+n_2)\sum_{uv\in E_{\lambda^s}} d_\lambda(u) + nHM_1(\lambda_1)\\
&\quad + 4nm_1 + 4nM_1(\lambda_1) + mHM_1(\lambda_2) + 4mm_2 + 4mM_1(\lambda_2)\\
&\quad + 2\left[\sum_{u(o)\in V_{\lambda^s}} d_\lambda(u)^2 \sum_{v\in V_{\lambda_2}} 1 + \sum_{u(o)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_2}} d_\lambda(v)^2 + 2\sum_{u(o)\in V_{\lambda^s}} d_\lambda(u)\sum_{v\in V_{\lambda_2}} d_\lambda(v) + 4n_2^2 \sum_{u(o)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_2}} 1\right.\\
&\quad \left. +4n_2\left(\sum_{u(o)\in V_{\lambda^s}} d_\lambda(u)\sum_{v\in V_{\lambda_2}} 1 + \sum_{u(o)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_2}} d_{\lambda_2}(v)\right)\right]\\
&\quad + 2\left[\sum_{u(N)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_1}} d_{\lambda_1}(v)^2 + (2+2n_1)^2 \sum_{u(N)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_1}} 1 + 2(2+2n_1)\sum_{u(N)\in V_{\lambda^s}} 1 \sum_{v\in V_{\lambda_1}} d_{\lambda_1}(v)\right]\\
&= \sum_{uv\in E_{\lambda^s}} d_\lambda(u)^2 + 4(n_1+n_2)^2 m\prime + 4(n_1+n_2)\sum_{uv\in E_{\lambda^s}} d_\lambda(u) + nHM_1(\lambda_1)\\
&\quad + 4nm_1 + 4nM_1(\lambda_1) + mHM_1(\lambda_2) + 4mm_2 + 4mM_1(\lambda_2)\\
&\quad + 2\left[n_2M_1(\lambda) + nM_1(\lambda_2) + 2\cdot 2m\cdot 2m_2 + 4nn_2n_2^2 + 4n_2(2m\cdot n_2 + n\cdot 2m_2)\right]\\
&\quad + 2\left[mM_1(\lambda_1) + (2+2n_1)^2 m\cdot n_1 + 2(2+2n_1)m\cdot 2m_1\right]\\
&= \sum_{uv\in E_{\lambda^s}}\left(d_\lambda(u)^2 + 4(n_1+n_2)d_\lambda(u)\right) + 4(n_1+n_2)^2 m\prime + nHM_1(\lambda_1) + 4nm_1\\
&\quad + 4nM_1(\lambda_1) + mHM_1(\lambda_2) + 4mm_2 + 4mM_1(\lambda_2) + 2n_2M_1(\lambda) + 2nM_1(\lambda_2)\\
&\quad + 16mm_2 + 8nn_2^3 + 16mn_2^2 + 16nn_2m_2 + 2mM_1(\lambda_1) + 2mn_1(2+2n_1)^2 + 8mm_1(2+2n_1).
\end{aligned} \tag{33}
$$

Hence, the proof is done. □

3. Conclusion

The first and second Zagreb indices, the first multiplicative Zagreb index, the first reformed Zagreb index, and the forgotten topological index were explored in this work, and their exact expressions were investigated. Other degree and distance-based topological indices of these complicated network operations could be calculated in the future work.

Data Availability

All the data supporting the results are included in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] C. Hansch and L. Leo, “Exploring QSAR fundamentals and applicability in chemistry and biology,” Amer. Chem. Soc, Washington DC, USA, 1996.

[2] M. F. Nadeem, M. Imran, H. M. A. Siddiqui, M. Azeem, A. Khalil, and Y. Ali, “Yasir, “Topological aspects of metal-organic structure with the help of underlying networks” Arabian,” *Journal of Chemistry*, vol. 14, no. 6, 2021.

[3] A. Shabbir, M. F. Nadeem, S. Mukhtar, and A. Raza, “On edge version of some degree-based topological indices of HAC5C7

[p,q] and VC5C7[p,q] nanotubes," *Polycyclic Aromatic Compounds*, vol. 42, no. 3, pp. 849–865, 2020.

[4] M. F. Nadeem, M. Azeem, and H. M. A. Siddiqui, "Comparative study of Zagreb indices for capped, semi-capped, and uncapped carbon nanotubes," *Polycyclic Aromatic Compounds*, vol. 1-18, pp. 1–18, 2021.

[5] M. F. Nadeem, M. Azeem, and I. Farman, "Comparative study of topological indices for capped and uncapped carbon nanotubes," *Polycyclic Aromatic Compounds*, vol. 1-18, pp. 1–18, 2021.

[6] M. F. Nadeem, S. Zafar, and Z. Zahid, "On topological properties of the line graphs of subdivision graphs of certain nanostructures," *Applied Mathematics and Computation*, vol. 273, pp. 125–130, 2016.

[7] A. Ahmad and S. C. López, "Distance-based topological polynomials associated with zero-divisor graphs," *Mathematical Problems in Engineering*, vol. 2021, Article ID 4959559, 8 pages, 2021.

[8] A. Ahmad, A. Ali, M. A. Asim, and M. F. Nadeem, "Polynomials of degree-based indices of metal-organic networks," *Combinatorial Chemistry & High Throughput Screening*, vol. 23, 2020.

[9] A. Ahmad, R. Hasni, K. Elahi, and M. A. Asim, "Polynomials of degree-based indices for swapped networks modeled by optical transpose interconnection system," *IEEE Access*, Institute of Electrical and Electronics Engineers, vol. 8, , Article ID 214293, 2020.

[10] N. A. Koam and A. Ahmad, "Polynomials of degree-based indices for three-dimensional mesh network," *Computers, Materials & Continua*, vol. 65, no. 2, pp. 1271–1282, 2020.

[11] I. Gutman and N. Tirnajsti, "Graph theory and molecular orbitals. III. Total pi-electron energy bydrocarbon," *Chemical Physics Letters*, vol. 17, pp. 535–538, 1972.

[12] R. Todeschini and V. Consonni, "New local vertex invariants and molecular descriptors based on functions of the vertex degrees," *MATCH Commun. Math. Comput. Chem.*vol. 64, pp. 359–372, 2010.

[13] M. Eliasi, A. Iranmanesh, and I. Gutman, "Multiplicative versions of first Zagreb index," *MATCH Commun. Math. Comput. Chem.*vol. 68, no. 1, pp. 217–230, 2012.

[14] B. Furtula and I. Gutman, "A forgotten topological index," *Journal of Mathematical Chemistry*, vol. 53, no. 4, pp. 1184–1190, 2015.

[15] B. Furtula, I. Gutman, Z. K. Vukicevic, G. Lekishvili, and G. Popivoda, "On an old degree-based topologicalindex," *Sci. Math.*vol. 148, pp. 19–31, 2015.

[16] A. Mili, evi, S. Nikolic and N. Tirnajsti, "On reformulated Zagreb indices č c c," *Molecular Diversity*, vol. 8, pp. 393–399, 2004.

[17] J. A. Bondy and U. S. R. Murty, *Graph Theory in Graduate Texts in Mathematics*, Vol. 29, Springer, , Berlin, Germany, 2008.

[18] F. Harary, *Graph Theory, Reading*, Addison-Wesley, Boston, Massachusetts, MA, USA, 1994.

[19] G. P. Singh, A. Borah, and S. Ray, "A review paper on corona product of graphs," *Advances and Applications in Mathematical Sciences*, vol. 19, no. 10, pp. 1047–1054, 2020.

[20] P. Sarkar, N. De, and A. Pal, "The forgotten topological index of double corona of graphs related to the different subdivision graphs," *Mathematical Advances in Pure and Applied Sciences*, vol. 2, no. 1, pp. 2636–8021, 2019.

[21] S. Barik and G. Sahoo, "On the Laplacian spectra of some variants of corona," *Linear Algebra Appl*, vol. 512, 2016.

[22] M. H. Khalifeh, H. Yousefi-Azari, and A. R. Ashrafi, "The first and second Zagreb indices of some graph operations," *Discrete Applied Mathematics*, vol. 157, no. 4, pp. 804–811, 2009.

[23] N. De, "Computing f-index of different corona products of graphs," *Bulletin of Mathematical Sciences and Applications*, vol. 19, pp. 24–30, 2017.

[24] M. Arezoomand and B. Taeri, "Zagreb indices of the generalized hierarchical product of graphs," *MATCH Commun. Math. Comput. Chem.*vol. 69, no. 1, pp. 131–140, 2013.

[25] A. R. Ashraf, A. R. Ashrafi, and H. Shabani, "The modified Wiener index of some graph operations," *ARS Mathematica Contemporanea*, vol. 11, no. 2, pp. 277–284, 2015.

[26] N. De, "The vertex Zagreb indices of some graph operations," *Carpathian Math. Publ.*vol. 8, no. 2, pp. 215–223, 2016.

[27] A. Ahmad, "On the degree based topological indices of benzene ring embedded in P-type-surface in 2D network," *Hacettepe Journal of Mathematics and Statistics*, vol. 4, no. 46, pp. 9–18, 2017.

[28] H. Alshehri, A. Ahmad, Y. Alqahtani, and M. Azeem, "Vertex metric-based dimension of generalized perimantanes diamondoid structure," *IEEE Access*, vol. 10, Article ID 43320, 2022.

[29] A. N. A. Koam, A. Ahmad, M. Azeem, and M. F. Nadeem, "Bounds on the partition dimension of one pentagonal carbon nanocone structure," *Arabian Journal of Chemistry*, vol. 15, no. 7, Article ID 103923, 2022.

[30] H. Raza, M. Waheed, M. K. Jamil, and M. Azeem, "Structures devised by the generalizations of two graph operations and their topological descriptors," *Main Group Metal Chemistry*, vol. 45, no. 1, pp. 44–56, 2022.

[31] A. N. A. H. Ahmad, A. Ahmad, and M. Azeem, "Computation of edge- and vertex-degree-based topological indices for tetrahedral sheets of clay minerals," *Main Group Metal Chemistry*, vol. 45, no. 1, pp. 26–34, 2022.