Research Article

Bifurcation Analysis and Synchronous Patterns between Field Coupled Neurons with Time Delay

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Neurons encode and transmit signals through chemical synaptic or electrical synaptic connections in the actual nervous system. Exploring the biophysical properties of coupling channels is of great significance for further understanding the rhythm transitions of neural network electrical activity patterns and preventing neurological diseases. From the perspective of biophysics, the activation of magnetic field coupling is the result of the continuous release and propagation of intracellular and extracellular ions, which is very similar to the activation of chemical synaptic coupling through the continuous release of neurotransmitters. In this article, an induction coil is used to connect two HR neurons to stimulate the effect of magnetic field coupling. It is inevitable that time delays can affect the coupling process in the transmission of information, and it should be considered in the coupled model. Firstly, the firing characteristics and bifurcation modes of two coupled HR neurons are studied by using one parameter and two parameters bifurcation. With the increase of propagation delay and coupling gain, the chaotic state of neurons disappears and the high-period window decreases due to the influence of energy transfer between neurons. Then, the synchronization patterns of two HR neurons with different stimulation are analyzed by error diagrams and time series diagrams. It is confirmed that the synchronous pattern has certain regularity and is related not only to the neurons with large stimulation current but also to the time delay and coupling gain. The research conclusions of this article will provide the corresponding theoretical basis for medical experiments.

1. Introduction

The nervous system plays a leading role in the regulation of physiological function activities in organisms, mainly relying on the current generated between neurons to complete the transmission of information [1, 2]. The neuron model is very important to further analyze the electrical activity and synchronization of neurons, and even help to understand the potential mechanism of disease. Since the last century, great progress has been made in the building and research of neuron models, and all kinds of improved mathematical models have been established successively, such as Hodgkin–Huxley (HH) model [3–5], FitzHugh–Nagumo (FHN) model [6–8], Hindmarsh–Rose (HR) model [9, 10], and Morris–Lecar (ML) model [11]. Researchers have built various nonlinear circuits based on these theoretical models to reproduce the same characteristics of electrical activity as in biological neurons [12, 13]. In particular, these improved neuronal circuits by embedding special functional devices such as piezoelectric ceramics, phototubes, and thermistors can sense special physical signals from the outside world, resulting in various types of dimensionless functional neuronal models such as thermosensitive neurons [14], photosensitive neurons [15], and auditory neurons [16]. Moreover, any neuron can be considered as a complex charged body from a physical point of view, whose static charged ions can induce a static electric field, while the intracellular transport and transmembrane flow of ions induce a magnetic field, so that any neuron is in the electromagnetic field induced by other neurons [17–23].

Neurons transmit information through synaptic connections to realize communication between cells [24]. As
more neurons are activated and synaptic currents travel from one neuron to the other along the coupling pathway, the transfer of information and energy between the two neurons is complete. Structurally, synapses can be divided into chemical synapses and electrical synapses. Electrical synaptic coupling [25, 26] is a gap coupling with a very fast response process, and its physical process is based on voltage coupling achieved by gap resistance. Chemical synaptic coupling [27, 28] relies on the release of neurotransmitters at the synaptic terminal, which in turn affects calcium ion flow. Thus, its course of action is slower and is a field coupling from a physical point of view. Researchers have regulated the output voltage of neuronal circuits through resistor-connected voltage coupling in the traditional discussion of synchronization stability of neuronal circuits [29, 30]. In fact, the coupling process is accompanied by the consumption of Joule heat necessarily and has a cumulative effect when the resistor is embedded in the coupling channel. The power consumption of the controller or the coupling resistor will increase significantly when more nonlinear circuits are coupled in the network to obtain synchronization. This coupling will consume and accumulate more Joule heat for the nervous system, which may cause potential abnormalities. Therefore, it is important to discuss the energy dissipation mechanisms and neural energy encoding in the neural system by exploring the synchronization between neuronal circuits with different coupling modes.

Synaptic plasticity is the variety of strength of association between nerve cells, which is reflected in the change of signal transmission efficiency between synapses, the size of a single synapse, or the number of synapses. Synaptic plasticity must be considered to be more practical when analyzing the dynamic behavior of neural networks. The migration of sodium, potassium, and chloride ions in and out of the membrane induces a magnetic field when a neuron generates an action potential. The corresponding induced magnetic field can also be generated when the synaptic current passes through the coupling channel. Therefore, it is necessary to establish the corresponding field coupled model of synapses based on the above biological and physical perspectives. For nonlinear circuits and neuronal circuits, many electronic devices such as memristors [31–33], capacitors [34, 35], induction coils [36, 37], Josephson junctions [38, 39], and phototubes [40] can be used to connect the outputs of the circuits for different types of coupling and to analyze their synchronization stability.

Voltage coupling based on resistive connections achieves energy balance by consuming the energy of the nonlinear circuit actually, which in turn leads to synchronous control. Part of the energy of the coupling circuit is pumped and stored temporarily in the coupling channel when the capacitor and induction coil are embedded in the coupling channel. Changing the capacity or density of the dielectric and magnetic media in the coupling channel enables the manipulation of the coupling channel, which in turn controls the energy balance process and synchronizes the nonlinear circuit ultimately. The capacitor-based coupling is a differential coupling from the dynamics point of view, while the induction coil-based coupling is an integral coupling [41]. In practical control, a single electronic element or a combination of multiple electronic elements can be used to create the coupling channel. The hybrid synapses can be designed by combining resistors, capacitors, induction coils, and memristors [42–44] to study neuronal circuits and network synchronization stability. The dendrites of biological neurons are very dense from the actual situation, which can realize multichannel signal input, and the neuronal cells themselves also have certain flexibility. Therefore, the coupling process can be very complex when neuronal synapses are entwined.

In neural networks, the neural system has a time delay due to the limited propagation speed of neuronal impulses along axons and the lag caused by synaptic and dendrite processing. Research shows that time delay has a great impact on the firing activity of neuronal, phase synchronization degree, and conversion between different synchronization states [45–50].

The release of neurotransmitters can cause the pumping and transport of ions, which can excite the changes of electromagnetic fields inside and outside the cells. Therefore, this article proposes to use the induction coil to couple two HR neurons to stimulate magnetic field coupling considering that the chemical synaptic coupling between neurons can be described by field coupling, and add coupling delays to represent the lags of information transmission, which provides new insight into the physical mechanism of the chemical synapse. The structure of this article is as follows: in Section 2, the mathematical model of the coupling of two neurons is introduced. In Section 3, the bifurcation behaviors of coupling two identical neurons are studied. In Section 4, the synchronization patterns of two neurons with different stimulation currents are discussed. Section 5 is a discussion of open problems for neuronal networks. The conclusions are given in Section 6.

2. Model and Scheme

In 1982, Hindmash and Rose proposed the HR neuron model [9] after repeated research, which is established by three autonomous differential equations:

$$\begin{align*}
\dot{x} &= y - ax^3 + bx^2 - z + I, \\
\dot{y} &= c - dx^3 - y, \\
\dot{z} &= r [s(x - \chi_0) - z],
\end{align*}$$

(1)

where the state variables $x$, $y$, and $z$ represent the membrane potential, the recovery variable associated with fast current and the adaptive slow current, respectively. The parameters $a$ and $b$ indicate the activation and deactivation of fast ion channels. $r$ is the control of the rate of adaptive current change, which is related to the calcium ion concentration. $s$ describes the activation of slow ion channels. $\chi_0$ denotes the calcium ion quiescent potential and $I$ is the external stimulation current. Conventionally, $a, b, c, d$, and $s$ are taken as the baseline values $a = 1, b = 3, c = 1, d = 5, s = 4$ set when the HR neuron system is established. $r$ and $\chi_0$ can be fine-tuned, where $r$ is generally taken in the order of $10^{-3}$, and the values in this article are $r = 0.015, \chi_0 = -1.6$. $I$ is
often used as a bifurcation parameter to study the firing pattern of HR neurons, usually taking values within \([-10, 10]\).

HR neurons showed different firing patterns when different external stimulation currents were selected. Draw bifurcation diagram and Lyapunov exponent spectrum of HR neurons with external stimulation current \(I\), as shown in Figure 1. The third Lyapunov exponent curve is omitted for the picture clarity in Figure 1(b). It can be observed that the Lyapunov exponent spectrum is basically consistent with the bifurcation diagram. And Figure 2 shows the time series of \(x\) variable of the HR neuron for some values of \(I\). The results indicate that the external stimulation current can be changed to trigger a quiescent state \((I = 0)\), spiking firing \((I = 2)\), bursting firing \((I = 3)\), and periodical firing \((I = 6)\).

Now we couple two HR neurons via an induction coil, as shown in Figure 3. According to the Law of electromagnetic induction, the time-varying current across the coupling coil, which connects the synapses of neurons, is estimated by

\[
I_L = \frac{1}{L} \int (V_1 - V_2) \, dt, \tag{2}
\]

where \(L\) is the inductance of the coupling coil, \(V_1\) and \(V_2\) are the output voltages from HR neuronal circuit. In fact, this physical current across the coupling coil can be mapped into dimensionless current by applying scale transformation for the variables and parameters in the nonlinear circuits. It can be obtained by the following formula:

\[
I_L = D \int (x_1 - x_2) \, dt, \tag{3}
\]

where \(D\) means the coupling strength between two HR neurons, which is determined by the inductance value of coupling. Thus, the dynamics for two coupled HR neurons can be described by:

\[
\begin{align*}
\dot{x}_1 &= y_1 - ax_1^3 + bx_2^2 - z_1 + I_1 - D \int (x_1 - x_2) \, dt, \\
\dot{y}_1 &= c - dx_1^2 - y_1, \\
\dot{z}_1 &= r[s(x_1 - \chi_0) - z_1], \\
\dot{x}_2 &= y_2 - ax_2^3 + bx_1^2 - z_2 + I_2 + D \int (x_1 - x_2) \, dt, \\
\dot{y}_2 &= c - dx_2^2 - y_2, \\
\dot{z}_2 &= r[s(x_2 - \chi_0) - z_2].
\end{align*}
\tag{4}
\]

When a current passes through the induction coil, a time-varying magnetic field will be generated in the induction coil, and the magnetic field strength \(B\) can be expressed as:

\[
B = \mu n I_L, \tag{5}
\]

where \(n\) is the number of turns per unit length of the coil and \(\mu\) denotes the magnetic dielectric coefficient of the induction coil. The terminal voltages of the two neuronal circuits coupled by the induction coil are different if the two neurons are in an asynchronous state. Then the current passing through the induction coil will produce a time-varying magnetic field, and the induction coil will store the magnetic field energy.

And there must be a lag in the transmission of information in the coupling process, so the time delay is added to the coupling term. Then the dynamic equation of two HR neurons can be expressed as:

\[
\begin{align*}
\dot{x}_1 &= y_1 - ax_1^3 + bx_2^2 - z_1 + I_1 + D \int (x_2(t - \tau_1) - x_1) \, dt, \\
\dot{y}_1 &= c - dx_1^2 - y_1, \\
\dot{z}_1 &= r[s(x_1 - \chi_0) - z_1], \\
\dot{x}_2 &= y_2 - ax_2^3 + bx_1^2 - z_2 + I_2 + D \int (x_1(t - \tau_2) - x_2) \, dt, \\
\dot{y}_2 &= c - dx_2^2 - y_2, \\
\dot{z}_2 &= r[s(x_2 - \chi_0) - z_2],
\end{align*}
\tag{6}
\]

where \(\tau_1\) and \(\tau_2\) indicate time delay, which is given \(\tau_1 = \tau_2 = \tau\) for simplicity.

3. Bifurcation Analysis

It is difficult to keep the system parameters fixed when the system changes under external influence. Usually, two or more parameters in neurons change in a specific range at the same time. Therefore, it will be more practical to study the firing activity of membrane voltage on the two-parameter plane. In this section, the fourth-order Runge–Kutta algorithm is used for numerical calculation with a constant time step size of 0.01. Assuming \(I_1 = I_2\), the values of other parameters are the same as above. The membrane voltage bifurcation behavior of the first neuron is analyzed in the two-parameter plane under different time delays and coupling strengths. Taking two groups of different parameter combinations, the bifurcation diagrams of neurons in the two-parameter space are shown in Figures 4 and 5. Different firing states of the membrane voltage are drawn in different colors, and the color column on the right side of the figure indicates the corresponding period number (for example, 0, 1, and 2 represent the quiescent state, period-1 firing, and period-2 firing, respectively, and the white area means the periods state equal to or greater than 20 or chaos).

When \(I\) and \(d\) are taken as variable parameters, the corresponding two-parameter bifurcation diagrams are shown in Figures 4(a)–4(f) on the parameter plane of \(I \in [3, 5]\) and \(d \in [4, 6]\). System (4) reveals rich and complex firing characteristics and looks horizontally at Figure 4. Two-parameter bifurcation diagrams with coupling strengths of 3 and 5 when \(\tau = 0.01\) are displayed in Figures 4(a) and 4(b), respectively. The membrane voltage is in the firing state of period-1 in the largest color area (light blue area), and there is a little yellow region in the lower left corner, which is a quiescent state. The upper left region is connected to the chaotic window by a period-doubling bifurcation structure and the corresponding chaotic window becomes smaller or even disappears with the increase of the number of periods. Compared with Figure 4(a), the chaotic region at the lower
right corner of Figure 4(b) moves up and becomes larger, which is caused by increasing coupling strength. Figures 4(c) and 4(d) are two-parameter bifurcation diagrams at $\tau = 0.05$. In Figure 4(c), the period-doubling bifurcation with chaos is exhibited on the left, while the period-adding bifurcation without chaos on the right is extended from the lower left. Increasing the coupling strength to 5, there is only period-adding bifurcation without chaos in Figure 4(d), and the region in the quiescent state increases. When $\tau = 0.1$, Figures 4(e) and 4(f) indicate the corresponding bifurcation diagrams. Figure 4(e) is similar to Figure 4(d), revealing a regular period-adding bifurcation structure. However, the chaos in Figure 4(e) completely disappears, presenting a large-scale quiescent state and period-1 firing, mixed with a little high-period firing. Then, look longitudinally at Figure 4. Two-parameter bifurcation diagrams with different time delays are shown in Figures 4(a), 4(c), and 4(e) when $D = 3$. Compared with Figures 4(a) and 4(c) which have an extra region of period-adding bifurcation without chaos, the chaotic window in Figure 4(e) is greatly reduced, and there is only a period-adding bifurcation structure. Figures 4(b), 4(d), and 4(f) are bifurcation diagrams at $D = 5$. Figure 4(d) has less chaotic windows than Figures 4(b) and 4(f) which have less chaotic and high period windows, but the quiescent window gradually increases. Evidently, these results are caused by coupling strength and time delay.

Taking $I$ as the variable and $d = 1.444444I + 0.366667$, $D = 3$, and $\tau = 0.01$, the bifurcation diagram of membrane
voltage along the direction of the black line in Figure 4(a) is shown in Figure 6(a). Obviously, it is in good agreement with the two-parameter bifurcation diagram. With the increase of $I$, the bifurcation mode of membrane voltage is: period-2 firing $\rightarrow$ chaotic state through period-doubling bifurcation $\rightarrow$ period-3 firing $\rightarrow$ chaotic state through period-doubling bifurcation $\rightarrow$ period-4 firing $\rightarrow$ chaotic state through period-doubling bifurcation $\rightarrow$ ... $\rightarrow$ period-adding firing. The firing state is 1 period larger than before after each chaos, and the corresponding chaotic window gradually narrows with the increase of the period. The chaotic window almost disappears when the firing period is greater than 8, and the membrane voltage enters the period-adding bifurcation mode. When the parameters are $d = 2I - 2$, $D = 3$, and $\tau = 0.1$, following the direction of the black line in Figure 4(e), the bifurcation diagram is exhibited in Figure 6(b). The bifurcation mode of $I$ is: quiescent state $\rightarrow$ period-1 firing $\rightarrow$ period-2 firing $\rightarrow$ period-3 firing $\rightarrow$ ... $\rightarrow$ period-adding firing. It can be observed that neurons directly enter the adjacent periodic state from a certain periodic state, and the corresponding periodic window decreases with the addition of the number of periods.

Then Figure 5 displays the two-parameter bifurcation diagrams of $I \in [2, 4.2]$ and $b \in [2.5, 3.25]$. The period-doubling bifurcation with chaos is exhibited in Figures 5(a)~5(c). It should be noted in particular that the yellow and blue areas in Figure 5(a) represent the quiescent state and period-
1 firing, respectively. However, there is no quiescent state in Figures 5(b) and 5(c), the yellow area stands for period-1 firing, and so on. With the increase of coupling strength and time delay, the chaos window gradually decreases to disappear, and the high-period window in the lower right corner also gradually decreases, as shown in Figures 5(d)∼5(f). This result is similar to Figure 4.

It can be seen from the above discussion that the coupled neuron model (6) has different bifurcation modes under different delays and coupling strengths. In fact, the field energy between two neurons spreads and swaps along the coupling channel through the connection of the induction coil, and part of the energy was stored and saved. The energy transfer between neurons can be carried out with effect by changing the propagation delay and coupling gain in the coupling channel.

4. Synchronization Pattern

In this section, the transient period for calculation is about 1000 time units, and the parameter value is the same as above. The coupling between two neurons with different external stimulation currents is considered when the initial value is the same as shown in Table 1. And to evaluate synchronization, the average synchronization error \( E \) is calculated by the following formula:

\[
E = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.
\]
neurons implement synchronization when synchronization mode is a quiescent state as long as the two increasing coupling strength. Figure 10(c) displays that the synchronization is realized. But the quiescent synchronization is reached with the coupling strength is small, spiking firings synchronization lead to different synchronization patterns when \( \tau = 0.5 \) and \( \tau = 3 \), as shown in Figure 10(d).

The external stimulation is further increased to study the coupling of a quiescent neuron with a periodical firing neuron. The synchronization error diagrams are exhibited in Figure 11. Next, the time series and the error diagrams are plotted in Figure 12. Once the coupling strength is greater than the threshold, neuron 1 and neuron 2 reach bursting firing together when \( \tau = 0.01 \), as shown in Figure 12(a). Different coupling strengths can lead to different synchronization patterns when \( \tau = 0.1 \). If the coupling strength is small, neuron 1 and neuron 2 realize bursting firing together. When the coupling strength is increased, the two neurons achieve irregular spiking firing synchronization, and then further increase the coupling strength to implement quiescent synchronization. Figures 12(b)–12(d) demonstrate the above conclusions fully. Similarly, the smaller coupling strength leads to spiking firing synchronization when \( \tau = 0.5 \), and the larger coupling strength realizes quiescent synchronization, as shown in Figures 12(e) and 12(f). The two neurons also reach synchronization when \( \tau = 3 \), and the synchronization pattern changes to periodical firing, as illustrated in Figure 12(g).

Next, case 4 is explored, where one bursting neuron is coupled with another spiking neuron, and synchronization errors are plotted in Figure 13. Compared with Figures 11 and 13, it is found that the coupling strength of case 3 and case 4 to achieve synchronization under the same time delay is similar. The time series of the coupled model and the variation of errors are calculated, and the results are

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**Table 1:** Two HR neurons with different firing patterns are coupled under six kinds of cases.

<table>
<thead>
<tr>
<th>Neuron 1</th>
<th>Neuron 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quiescent state ((I_1 = 0))</td>
<td>Spiking firing ((I_2 = 2))</td>
</tr>
<tr>
<td>Spiking firing ((I_1 = 2))</td>
<td>Case 1</td>
</tr>
<tr>
<td>Bursting firing ((I_1 = 3))</td>
<td>Case 2</td>
</tr>
<tr>
<td>Periodical firing ((I_1 = 6))</td>
<td>Case 3</td>
</tr>
</tbody>
</table>

For case 1, a quiescent neuron is coupled to a spiking neuron. Firstly, Figure 7 shows the synchronization error diagrams of the coupled model varying with the coupling strengths under different time delays. As long as the coupling strength is greater than a threshold, synchronization can be achieved when \( \tau = 0.5 \), but only reach the synchronization if the coupling strength is within a certain interval, when \( \tau = 3 \). Figure 8 takes the coupling strength to realize synchronization under different time delays, and displays the time series of two neurons and the change of error functions. The synchronization mode only has a quiescent state when \( \tau = 0.5 \), as shown in Figure 8(a). When the time delay is increased to 3, the final synchronization mode is periodical firing, and the result is presented in Figure 8(b).

Now, the external stimulation is augmented to discuss the coupling of a bursting neuron with a quiescent neuron. The synchronization error diagrams varying with the coupling strength is shown in Figure 9. It can be observed that synchronization can be achieved at a threshold of coupling strength when \( \tau = 0.01 \) and 0.5. And the greater \( \tau \) is, the greater the threshold of coupling strength is. The time series of the coupled model and the errors are calculated, and the results are displayed in Figure 10. By observing Figures 10(a) and 10(b), it is found that different coupling strengths can lead to different synchronization patterns when \( \tau = 0.01 \). If the coupling strength is small, spiking firing synchronization is realized. But the quiescent synchronization is reached with increasing coupling strength. Figure 10(c) displays that the synchronization mode is a quiescent state as long as the two neurons implement synchronization when \( \tau = 0.5 \). The synchronization pattern changes to periodical firing when \( \tau = 3 \), as shown in Figure 10(d).
presented in Figure 14. It can be noted that the pattern of reaching synchronization in case 4 is also consistent with case 3.

Then, we study the coupling between one spiking neuron and another periodical neuron. Figure 15 exhibits the synchronization error diagrams varying with coupling strength. The time series of the coupled model and the variation of errors are plotted in Figure 16. By observing Figures 16(a) and 16(b), it is found that when $\tau = 0.01$ and the coupling strength is small, the interaction between neuron 1 and neuron 2 is synchronized, and the synchronized pattern is periodical firing. But two neurons can reach...
Figure 10: The time series of membrane potentials $x_1$ (red line) and $x_2$ (blue line), and error functions of the coupled model at case 2 (bursting and quiescent neurons). For (a) spiking firing synchronization with $\tau = 0.01$ and $D = 1$; (b) quiescent state synchronization with $\tau = 0.01$ and $D = 15$; (c) quiescent state synchronization with $\tau = 0.5$ and $D = 5$; and (d) periodical firing synchronization with $\tau = 3$ and $D = 5$. The initial values are selected with (0.1, 0.01, 1.05, 0.1, 0.01, 1.05).

Figure 11: The synchronization errors of the coupled model with respect to coupling strength at case 3 (periodical and quiescent neurons). For (a) $\tau = 0.01$; (b) $\tau = 0.1$; (c) $\tau = 0.5$; and (d) $\tau = 3$. Partially enlarged interval for partial synchronization, and the initial values are selected with (0.1, 0.01, 1.05, 0.1, 0.01, 1.05).
Figure 12: The time series of membrane potentials $x_1$ (red line) and $x_2$ (blue line), and error functions of the coupled model at case 3 (periodical and quiescent neurons). For (a) bursting firing synchronization with $\tau = 0.01$ and $D = 5$; (b) bursting firing synchronization with $\tau = 0.1$ and $D = 1.5$; (c) spiking firing synchronization with $\tau = 0.1$ and $D = 11$; (d) quiescent state synchronization with $\tau = 0.1$ and $D = 13$; (e) spiking firing synchronization with $\tau = 0.5$ and $D = 2$; (f) quiescent state synchronization with $\tau = 0.5$ and $D = 5$; and (g) periodical firing synchronization with $\tau = 3$ and $D = 4$. The initial values are selected with $(0.1, 0.01, 1.05, 0.1, 0.01, 1.05)$.
Figure 13: The synchronization errors of the coupled model with respect to coupling strength at case 4 (bursting and spiking neurons). For (a) $\tau = 0.01$; (b) $\tau = 0.1$; (c) $\tau = 0.5$; and (d) $\tau = 3$. Partially enlarged interval for partial synchronization, and the initial values are selected with (0.1, 0.01, 1.05, 0.1, 0.01, 1.05).

Figure 14: Continued.
Figure 14: The time series of membrane potentials $x_1$ (red line) and $x_2$ (blue line), and error functions of the coupled model at case 4 (bursting and spiking neurons). For (a) bursting firing synchronization with $\tau = 0.01$ and $D = 5$; (b) bursting firing synchronization with $\tau = 0.1$ and $D = 1$; (c) spiking firing synchronization with $\tau = 0.1$ and $D = 11$; (d) quiescent state synchronization with $\tau = 0.1$ and $D = 13$; (e) spiking firing synchronization with $\tau = 0.5$ and $D = 2$; (f) quiescent state synchronization with $\tau = 0.5$ and $D = 5$; and (g) periodic firing synchronization with $\tau = 3$ and $D = 4$. The initial values are selected with $(0.1, 0.01, 1.05, 0.1, 0.01, 1.05)$.

Figure 15: The synchronization errors of the coupled model with respect to coupling strength at case 5 (periodical and spiking neurons). For (a) $\tau = 0.01$; (b) $\tau = 0.1$; (c) $\tau = 0.3$; (d) $\tau = 0.5$; and (e) $\tau = 3$. Partially enlarged interval for partial synchronization, and the initial values are selected with $(0.1, 0.01, 1.05, 0.1, 0.01, 1.05)$. 

Complexity
bursting firing synchronization with increasing the coupling strength. It can be seen from Figures 16(c)~16(e) that if the coupling strength is small, bursting firing synchronization is realized when $\tau = 0.1$. If the coupling strength is increased, the spiking firing synchronization is implemented, and then the coupling strength is further increased to achieve quiescent synchronization. In addition, the spiking firing synchronization is reached if the coupling strength is small, and quiescent synchronization is achieved if the coupling strength is large when $\tau = 0.3$, as shown in Figures 16(f) and 16(g). When the time delay is increased to 0.5 and 3, the quiescent state and periodical firing of the coupled model can be observed in Figures 16(h) and 16(i), respectively.

Finally, the synchronization patterns of coupling between a periodical neuron and a bursting neuron in case 6 are discussed, and the synchronization errors are presented in Figure 17. The time series of the two neurons and the variation of errors are plotted in Figure 18. Notice that the pattern of reaching synchronization in case 6 is consistent with case 5. Separately, because Figure 18(f) takes a long time to achieve synchronization, the calculated transition period is set as 1500 time units for a better display effect.

The corresponding induced magnetic field will be generated in the coupling channel when the coupling current passes through the induction coil, and the induction coil stores part of the energy of the system in the form of a magnetic field. The current of the coupling channel decays to zero after the two neurons achieve complete synchronization, and the energy of the two neurons reaches balance. According to cases 1 to 6, the neurons will achieve synchronization when two neurons in different firing patterns are coupled by a magnetic field. There are rich firing sequences with the change of time delay and coupling strength, as shown in Table 2. Firstly, the results with small-time delays ($\tau \leq 0.5$) are discussed. Neuron 1 in the excited state ($I = 2, 3, 6$) and neuron 2 in the quiescent state ($I = 0$) achieve synchronization under different time delays. And with the increase of coupling strength, the synchronization patterns shift to those states where the stimulation current is smaller than that of excitatory neurons. However, there is a state with a large stimulation current of an excitatory neuron when two neurons in the excited state ($I = 2, 3, 6$) are synchronized. Secondly, it can be found that only periodic synchronization can be realized when the time delay is large ($\tau = 3$) no matter which coupling case. In particular, the synchronization state of periodical-quiescent coupling and bursting-spiking coupling are consistent, and periodical-spiking coupling and periodical-bursting coupling are also identical.

5. Open Problems

At present, there are many improved functional HR neuron models. For instance, the four-dimensional HR model considers the magnetic flux and electromagnetic effect passing through the cell membrane [19, 21], and the five-dimensional HR model introduces the electric field variable and magnetic flux variable to describe the induced electric field and induced magnetic field, respectively [20, 22]. Indeed, functional areas of the brain contain a large number of neurons, making it important to further investigate the collective behavior of electrical activity on neuronal networks. Therefore, these functional neurons can be extended to networks to study the dynamic behavior of various networks. A variety of networks with different topological connection types are discussed, such as chain networks [51, 52], ring networks [53], star networks [54], small-world networks [37, 55], and scale-free networks [37]. And, researchers have also designed multilayer networks, such as two-layer networks [56, 57] and three-layer networks [49].

Here a chain network is built with induction coils coupling four-dimensional HR neurons [19] and taking into account the field interactions between neurons. Thus, the mathematical equations of the neuronal network under field coupling are shown as follows:

$$
\begin{align*}
\dot{x}_i &= y_i - ax_i^3 + bx_i^2 - z_i + I_i - k_{0\rho}(\phi_i)x_i + D \int (x_{i+1} + x_{i-1} - 2x_i)dt, \\
\dot{y}_i &= c - dx_i^2 - y_i, \\
\dot{z}_i &= r|\varepsilon(x_i - \chi_i) - z_i|, \\
\dot{\phi}_i &= k_1 x_i - k_2 \phi_i + D_0 \left( \phi_i - \sum_{j \neq i} W_{ij} \frac{\phi_j}{|1 - j\phi_j|^2} \right) \rho \phi_i = \alpha + \beta \phi^2.
\end{align*}
$$

(8)

where the variable $\phi$ denotes the magnetic flux through the cell membrane. $k_1 x$ and $k_2 \phi$ are the change rate of the membrane voltage with respect to the external magnetic flux electromagnetic induction and the leakage flux, respectively, and $k_1$ and $k_2$ are the coefficients of variation. $k_{0\rho}(\phi)x$ represents the feedback current of the magnetic flux to the membrane voltage and $k_0$ is the modulation intensity. $\rho(\phi) = \alpha + 3\beta\phi^2$ is the memory conduction of magnetron memristor, where $\alpha$ and $\beta$ are the determined constants. The subscript $i$ indicates the position of the neuron in the network, $D$ is the coupling coefficient between adjacent neurons via electric synapse coupling, $D_0$ describes the field interaction between neurons. $W$ represents the intensity of the field-effect associated with the distance between neurons. Interestingly, the coexistence of synchronization and unsynchronization can form chimeric states [53] in neuronal networks, which coincides with the cooperation and competition between different modes in neuronal activity. For a variety of different neuronal networks, the pattern formation and selection phase synchronization can be investigated, furthermore, the coupling time delay, the external noise, and electromagnetic radiation can be considered. Moreover, each neuron can be involved in field coupling even in the long-range distribution of the network, so that each neuron can be modulated by isolated fields triggered by other neurons in other layers. In the same layer, the weight of the field coupling from different neurons should also be considered, and the open problems are worthy of further investigation.
Figure 16: Continued.
Figure 16: The time series of membrane potentials $x_1$ (red line) and $x_2$ (blue line), and error functions of the coupled model at case 5 (periodical and spiking neurons). For (a) periodical firing synchronization with $\tau = 0.01$ and $D = 2$; (b) bursting firing synchronization with $\tau = 0.01$ and $D = 14$; (c) bursting firing synchronization with $\tau = 0.1$ and $D = 2$; (d) spiking firing synchronization with $\tau = 0.1$ and $D = 9$; (e) quiescent state synchronization with $\tau = 0.1$ and $D = 14$; (f) spiking firing synchronization with $\tau = 0.3$ and $D = 3$; (g) quiescent state synchronization with $\tau = 0.3$ and $D = 4$; (h) quiescent state synchronization with $\tau = 0.5$ and $D = 3$; and (i) periodical firing synchronization with $\tau = 3$ and $D = 4$. The initial values are selected with $(0.1, 0.01, 1.05, 0.1, 0.01, 1.05)$.

Figure 17: The synchronization errors of the coupled model with respect to coupling strength at case 6 (periodical and bursting neurons). For (a) $\tau = 0.01$; (b) $\tau = 0.1$; (c) $\tau = 0.3$; (d) $\tau = 0.5$; and (e) $\tau = 3$. Partially enlarged interval for partial synchronization, and the initial values are selected with $(0.1, 0.01, 1.05, 0.1, 0.01, 1.05)$. 
Figure 18: Continued.
6. Conclusions

The coupling mode based on induction coil connection is a magnetic field coupling essentially because a time-varying magnetic field will be generated in the coupling channel. And part of the energy in the neuron circuit will be retained as magnetic field energy in the coupling coil. This article studies the bifurcation modes and synchronization states of two HR neurons coupled with an induction coil under the influence of propagation delay and coupling gain. The transitions rules of firing rhythm affected by time delay and coupling strength are analyzed when two identical HR neurons are coupled. The structures of period-doubling bifurcation with chaos or without chaos and period-adding bifurcation without chaos are revealed. And the chaotic window disappears and the high-period window decreases are found due to large delays and strong coupling. In addition, it is proposed that field coupling can be used to synchronize neurons with different firing patterns. It is found that the synchronized patterns are related not only to the initial firing state of the two neurons but also to the time delay and coupling strength. The coupling strength and time delay act on the synchronous pattern together when the time delay is small. However, the coupling strength has no effect if the time delay is large, and the two neurons can only realize periodic firing synchronization.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.
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References


