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Research Article

Analytical Investigation of Some Dynamical Systems by ZZ Transform with Mittag–Leffler Kernel

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In this work, ZZ transformation is combined with the Adomian decomposition method to solve the dynamical system of fractional order. The derivative of fractional order is represented in the Atangana–Baleanu derivative. The numerical examples are combined for their approximate-analytical solution. It is explored using graphs that indicate that the actual and approximation results are close to each other, demonstrating the method's usefulness. Fractional-order solutions are the most in line with the dynamics of the targeted problems, and they provide an endless number of options for an optimal mathematical model solution for a particular physical phenomenon. This analytical approach produces a series form solution that is quickly convergent to exact solutions. The acquired results suggest that the novel analytical solution technique is simple to use and very successful at assessing complicated problems that arise in related fields of research and technology.

1. Introduction

Because of their extensive applications in many science and engineering disciplines, fractional differential equations have sparked much attention in recent years. Critical phenomena well characterize differential equations of fractional order in electromagnetics, finance, viscoelasticity, acoustics, material science, and electrochemistry. Barkai et al. [1], Mainardi [2], Tadjeran and Meerschaert [3], Meerschaert et al. [4], and Magin et al. [5] just released a review article on fractional signals and systems, including control theory applications. The edited volume of Machado contains several applications of fractional calculus, such as image processing [6]. The importance and necessity of fractional calculus can be seen in several applications in transdisciplinary disciplines. Miller and Ross [7], Oldham and Spanier [8], Podlubny [9], Kilbas et al. [10], Samko et al. [11], Caponetto [12], and Diethelm [13] have all authored essential studies on the fractional derivative and fractional differential equations. A review study on the recent history of fractional calculus was written by Machado et al. [14]. An article on recent developments in the theory of abstract differential equations with fractional derivatives was published by Hernandez et al. [15]. These publications provide a systematic explanation of fractional calculus, including the existence and uniqueness of solutions and various analytical methods for solving fractional differential equations, such as Green's function method, power series approach, Mellin transform method, and others. No method in the literature produces a precise solution for nonlinear fractional differential equations (16) and (17). Using linearization or perturbation approaches, only approximate answers can be obtained. All of these push us to develop a numerical approach for fractional differential equations that is both efficient and accurate [18–21]. Chaos theory, heat transfer, variational issues, and other fields have used the Atangana–Baleanu fractional differential extensively. Recently, a fractional-differential mask based on a fractional Gaussian kernel with Atangana–Baleanu fractional differential has been published in the literature for the detection of blood vessels in retinal pictures, with the suggested method's efficacy compared to other well-known approaches. Furthermore, it discusses the underlying differences between power-law, exponential-law, and Mittag–Leffler kernels, as well as their potential applications in diverse domains.

This paper establishes a connection between the Aboodh transformation (AT), ZZ transformation (ZZT), and Laplace transformation (LT), with several applications mentioned in [22–24]. The ZZT was then employed to define fractional Atangana–Baleanu Caputo operators and characterize Riemann–Liouville senses using theorems. Later, we solved several test problems stated in the Atangana–Baleanu sense using this ZZ transform. The current author's contributions to this study are (i) applying the ZZ transform to solve fractional differential equations expressed in the Atangana–Baleanu derivative and (ii) establishing the connection between the Laplace, Aboodh, and ZZ transformations. A few well-known transforms that the ZZ transform generalizes can be related to other well-known transforms. Divide the ZZ transform by the adjusted variable to get the natural transform. Relationships with other integral transformations are also included in this work in terms of theorems. This transformation has the advantage of converging to the Sumudu transformation, which is advantageous when solving fractional differential equations with variable coefficients, such as [25–27].

Adomian (1980) established the Adomian decomposition technique (ADM), an efficient method for finding explicit and numerical solutions to a larger and more general class of differential systems representing real-world issues [28–30]. This strategy effectively addresses initial and boundary value problems, linear and nonlinear, ordinary and partial differential equations, and stochastic systems. Furthermore, this approach does not require any linearization or perturbation. ADM has been used extensively in the last two decades since it yields approximate analytical solutions for nonlinear problems, and there has been much interest in utilizing it to solve fractional differential equations (31)–(33).

The ZZ decomposition method has the following advantages with respect to Adomian decomposition method:

(i) ZZ decomposition method required small calculations as compared to the Adomian decomposition method

(ii) The fractional derivative is simplified by using the ZZ transformation first and then applying decomposition method while it is not the case if we use the Adomian decomposition method directly

(iii) The initial conditions/boundary conditions are used directly in the ZZ decomposition method, and it mostly avoids the extra calculations of Adomian polynomials

The novelty of the present work is to deal with the analytical solutions of important fractional-order some dynamical systems. The fractional-order of some dynamical systems have many applications in physical sciences, and therefore, different graphs are presented to show various dynamics of fractional parabolic equations. The solutions are obtained in rather simpler way as compared to other techniques. The current study has been structured as follows: in Section 2, some basic notions of basic definitions of ZZ transformation are described. In Section 3, we give an analysis of the suggested technique. In Section 4, we provide current solutions which suggested equations explaining how to implemented the suggested technique. Finally, the conclusion is provided.

2. Preliminaries

Definition 1. The function set of the Aboodh transform (AT) is defined as

$$B = \{h(\rho)\colon \exists M, n_1, n_2 > 0, |h(\rho)| < Me^{-s\rho}\}, \tag{1}$$

and is given as [22, 23]

$$A\{h(\rho)\} = \frac{1}{\varsigma}\int_0^\infty h(\rho)e^{-\varsigma\rho}d\rho \ , \rho > 0 \text{ and } n_1 \le \varsigma \le n. \tag{2}$$

Theorem 1. *Now, we consider G and F as the Aboodh and Laplace transforms of $h(\rho) \in B$; then [24, 25],*

$$G(\varsigma) = \frac{F(\varsigma)}{\varsigma}. \tag{3}$$

Zafar [26] was the first to develop the ZZ transform. It is mixture of the Laplace and Aboodh integral transforms. The ZZ transform is expressed in the following.

Definition 2 (ZZ transform). Suppose that $h(\rho)\forall\rho \ge 0$ is a function, then the ZZ transformation $Z(\varrho, \varsigma)$ of $h(\rho)$ is defined as [26]

$$ZZ(h(\rho)) = Z(\varrho, \varsigma) = \varsigma\int_0^\infty h(\varrho\rho)e^{-\varsigma\rho}d\rho. \tag{4}$$

The ZZ transformation is linear, just as the Aboodh and Laplace transforms. The MLF is a function that is given as an extension of the exponential term.

$$E_\delta(z) = \sum_{m=0}^\infty \frac{z^m}{\Gamma(1 + m\delta)}, \operatorname{Re}(\delta) > 0. \tag{5}$$

Definition 3. The Atangana–Baleanu Caputo derivative of a function $\nu(\varphi, \rho) \in H^1(a, b)$; then, for $\delta \in (0, 1)$, it is defined as [27]

$$ABC_a D_\rho^\delta \nu(\varphi, \rho) = \frac{\psi(\delta)}{1 - \delta}\int_a^\rho \nu'(\varphi, \rho)E_\delta\left(\frac{-\delta(\rho - \eta)^\delta}{1 - \delta}\right)d\eta. \tag{6}$$

Definition 4. Let the Riemann–Liouville Atangana–Baleanu derivative $\nu(\varphi, \rho) \in H^1(a, b)$; then, for $\delta \in (0, 1)$, it is given as [27]

$$ {}_{a}^{ABR}D_{\rho}^{\delta}\nu(\varphi,\eta) = \frac{\psi(\delta)}{1-\delta}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_{a}^{\rho}\nu(\varphi,\eta)E_{\delta}\left(\frac{-\delta(\rho-\eta)^{\delta}}{1-\delta}\right)\mathrm{d}\eta, \tag{7} $$

with the condition $\psi(0)=\psi(1)=1$, $\psi(\delta)$ is a function, and $b>a$.

Theorem 2. *The Laplace transform of Riemann–Liouville Atangana–Baleanu derivative and Atangana–Baleanu Caputo are, respectively, defined as [27]*

$$ \mathrm{L}\left\{{}_{a}^{ABC}D_{\rho}^{\delta}\nu(\varphi,\rho)\right\}(\varsigma) = \frac{\psi(\delta)}{1-\delta}\frac{\varsigma^{\delta}L\{\nu(\varphi,\rho)\}-\varsigma^{\delta-1}\nu(\varphi,0)}{\varsigma^{\delta}+\delta/1-\delta} \tag{8} $$

and

$$ L\left\{{}_{a}^{ABR}D_{\rho}^{\delta}\nu(\varphi,\rho)\right\}(\varsigma) = \frac{\psi(\delta)}{1-\delta}\frac{\varsigma^{\delta}L\{\nu(\varphi,\rho)\}}{\varsigma^{\delta}+\delta/1-\delta}. \tag{9} $$

The theorems that follow are based on the idea that $h(\rho)\in H^{1}(a,b), b>a$, and $\delta\in(0,1)$.

Theorem 3. *The Aboodh transformation of Atangana–Baleanu Riemann–Liouville derivative is defined as [25]*

$$ G(\varsigma) = A\left\{{}_{a}^{ABR}D_{\rho}^{\delta}\nu(\varphi,\rho)\right\}(\varsigma) = \frac{1}{\varsigma}\left[\frac{\psi(\delta)}{1-\delta}\frac{\varsigma^{\delta}L\{\nu(\varphi,\rho)\}}{\varsigma^{\delta}+\delta/1-\delta}\right]. \tag{10} $$

Proof 1. Using Theorem 1 and equation (3), we arrive to the required solution. The relationship among the transforms of ZZ and Aboodh is given in the following theorem. □

Theorem 4. *The Aboodh transform of Atangana–Baleanu Caputo derivative is defined as [25]*

$$ \begin{aligned} G(\varsigma) &= A\left\{{}_{a}^{ABC}D_{\rho}^{\delta}\nu(\varphi,\rho)\right\}(\varsigma) \\ &= \frac{1}{\varsigma}\left[\frac{\psi(\delta)}{1-\delta}\frac{\varsigma^{\delta}L\{\nu(\varphi,\rho)\}-\varsigma^{\delta-1}\nu(\varphi,0)}{\varsigma^{\delta}+\delta/1-\delta}\right]. \end{aligned} \tag{11} $$

Proof 2. Using Theorem 1 and equation (2), we can discover the desired solution. □

Theorem 5. *If $Z(\varrho,\varsigma)$ and $G(\varsigma)$ are the Aboodh and ZZ transforms of $h(\rho)\in B$, then we obtain the following [25]:*

$$ Z(\varrho,\varsigma) = \frac{\varsigma^{2}}{\varrho^{2}}G\left(\frac{\varsigma}{\varrho}\right). \tag{12} $$

Proof 3. (The *ZZ* transform definitions). We get

$$ Z(\varrho,\varsigma) = \varsigma\int_{0}^{\infty}h(\varrho\rho)e^{-\varsigma\rho}d\rho. \tag{13} $$

Put $\varrho\rho=\rho$ in (13); we get

$$ Z(\varrho,\varsigma) = \frac{\varsigma}{\varrho}\int_{0}^{\infty}h(\rho)e^{-\frac{\pi z}{\varrho}}d\rho\ . \tag{14} $$

The right-hand side of (14) may be expressed as

$$ Z(\varrho,\varsigma) = \frac{\varsigma}{\varrho}F\left(\frac{\varsigma}{\varrho}\right), \tag{15} $$

where $F(.)$ expresses the Laplace transformation of $h(\rho)$. Using Theorem 1, (15) can be defined as

$$ Z(\varrho,\varsigma) = \frac{\varsigma}{\varrho}F\left(\frac{\varsigma}{\varrho}\right)\left(\frac{\varsigma}{\varrho}\right)\times\left(\frac{\varsigma}{\varrho}\right) = \left(\frac{\varsigma}{\varrho}\right)^{2}G\left(\frac{\varsigma}{\varrho}\right), \tag{16} $$

where $G(.)$ defines the Aboodh transform of $h(\rho)$. □

Theorem 6. *ZZ transformation of $h(\rho)=\rho^{\delta-1}$ is defined as*

$$ Z(\varrho,\varsigma) = \Gamma(\delta)\left(\frac{\varrho}{\varsigma}\right)^{\delta-1} \tag{17} $$

Proof 4. The Aboodh transformation of $h(\rho)=\rho^{\delta}, \delta\geq 0$, is

$$ \begin{aligned} G(s) &= \frac{\Gamma(\delta)}{\varsigma^{\delta+1}}, \\ \text{now}, G\left(\frac{\varsigma}{\varrho}\right) &= \frac{\Gamma(\delta)\varrho^{\delta+1}}{\varsigma^{\delta+1}}. \end{aligned} \tag{18} $$

Applying (17), we achieve

$$ Z(\varrho,\varsigma) = \frac{\varsigma^{2}}{\varrho^{2}}G\left(\frac{\varsigma}{\varrho}\right) = \frac{\varsigma^{2}}{\varrho^{2}}\frac{\Gamma(\delta)\varrho^{\delta+1}}{\varsigma^{\delta+1}} = \Gamma(\delta)\left(\frac{\varrho}{\varsigma}\right)^{\delta-1}. \tag{19} $$

□

Theorem 7. *Let $\delta,\omega\in C$ and $Re(\delta)>0$; then, the ZZ transformation of $E_{\delta}(\omega\rho^{\delta})$ is defined as [25]*

$$ ZZ\left\{\left(E_{\delta}\left(\omega\rho^{\delta}\right)\right)\right\} = Z(\varrho,\varsigma) = \left(1-\omega\left(\frac{\varrho}{\varsigma}\right)^{\delta}\right)^{-1}. \tag{20} $$

Proof 5. We know that Aboodh transform of $E_{\delta}(\omega\rho^{\delta})$ is defined as

$$ G(\varsigma) = \frac{F(\varsigma)}{\varsigma} = \frac{\varsigma^{\delta-1}}{\varsigma\left(\varsigma^{\delta}-\omega\right)}. \tag{21} $$

So,

$$ G\left(\frac{\varsigma}{\varrho}\right) = \frac{(\varsigma/\varrho)^{\delta-1}}{(\varsigma/\varrho)\left((\varsigma/\varrho)^{\delta}-\omega\right)}. \tag{22} $$

Applying Theorem 9, we achieve

$$Z(\varrho,\varsigma)=\left(\frac{\varsigma}{\varrho}\right)^{2}G\left(\frac{\varsigma}{\varrho}\right)=\left(\frac{\varsigma}{\varrho}\right)^{2}\frac{(\varsigma/\varrho)^{\delta-1}}{(\varsigma/\varrho)\left((\varsigma/\varrho)^{\delta}-\omega\right)}$$
$$=\frac{(\varsigma/\varrho)^{\delta}}{(\varsigma/\varrho)^{\delta}-\omega}=\left(1-\omega(\varrho/\varsigma)^{\delta}\right)^{-1}. \quad (23)$$

□

Theorem 8. *If $Z(\varrho,\varsigma)$ and $G(s)$ are the Aboodh and ZZ transforms of $h(\rho)$, then the Atangana–Baleanu Caputo ZZ transformation derivative is defined as [25]*

$$ZZ\left\{{}_{0}^{ABC}D_{\rho}^{\delta}h(\rho)\right\}=\left[\frac{\psi(\delta)}{1-\delta}\frac{\left(\left(\varsigma^{\delta+2}/\varrho^{\delta+2}\right)\right)G(\varsigma/\varrho)-\left(\varsigma^{\delta}/\varrho^{\delta}\right)f(0)}{\left(\varsigma^{\delta}/\varrho^{\delta}\right)+(\delta/1-\delta)}\right]. \quad (24)$$

Proof 6. Applying equations (1) and (5), we get

$$G\left(\frac{\varsigma}{\varrho}\right)==\frac{\varrho}{\varsigma}\left[\frac{\psi(\delta)}{1-\delta}\frac{(\varsigma/\varrho)^{\delta+1}G(\varsigma/\varrho)-(\varsigma/\varrho)^{\delta-1}f(0)}{(\varsigma/\varrho)^{\delta}+\delta/1-\delta}\right]. \quad (25)$$

So, the Atangana–Baleanu Caputo of ZZ transformation is defined as

$$Z(\varrho,\varsigma)=\left(\frac{\varsigma}{\varrho}\right)^{2}G\left(\frac{\varsigma}{\varrho}\right)$$
$$=\left(\frac{\varsigma}{\varrho}\right)^{2}\frac{\varrho}{\varsigma}\left[\frac{\psi(\delta)}{1-\delta}\frac{(\varsigma/\varrho)^{\delta+1}G(\varsigma/\varrho)-(\varsigma/\varrho)^{\delta-1}f(0)}{(\varsigma/\varrho)^{\delta}+(\delta/1-\delta)}\right]$$
$$=\left[\frac{\psi(\delta)}{1-\delta}\frac{(\varsigma/\varrho)^{\delta+2}G(\varsigma/\varrho)-(\varsigma/\varrho)^{\delta}f(0)}{(\varsigma/\varrho)^{\delta}+(\delta/1-\delta)}\right]. \quad (26)$$

□

Theorem 9. *Let us suppose that $Z(\varrho,\varsigma)$ and $G(s)$ are the Aboodh and ZZ transforms of $h(\rho)$. Then, the Atangana–Baleanu Riemann–Liouville ZZ transform derivative is defined as [25]*

$$ZZ\left\{{}_{0}^{ABR}D_{\rho}^{\delta}f(\rho)\right\}=\left[\frac{\psi(\delta)}{1-\delta}\frac{5^{\delta+2}/\varrho^{\delta+2}G(\varsigma/\varrho)}{\varsigma^{\mu}/\varrho^{\mu}+(\delta/1-\delta)}\right]. \quad (27)$$

Proof 7. Applying equations (1) and (4), we get

$$G\left(\frac{\varsigma}{\varrho}\right)=\frac{\varrho}{\varsigma}\left[\frac{\psi(\delta)}{1-\delta}\frac{(\varsigma/\varrho)^{\delta+1}G(\varsigma/\varrho)}{(\varsigma/\varrho)^{\delta}+\delta/1-\delta}\right]. \quad (28)$$

From (16), the ZZ transform of Riemann–Liouville Atangana–Baleanu is defined as

$$Z(\varrho,\varsigma)=\left(\frac{\varsigma}{\varrho}\right)^{2}G\left(\frac{\varsigma}{\varrho}\right)$$
$$=\left(\frac{\varsigma}{\varrho}\right)^{2}\left(\frac{\varrho}{\varsigma}\right)\left[\frac{\psi(\delta)}{1-\delta}\frac{(\varsigma/\varrho)^{\delta+1}G(\varsigma/\varrho)}{(\varsigma/\varrho)^{\delta}+(\delta/1-\delta)}\right] \quad (29)$$
$$=\left[\frac{\psi(\delta)}{1-\delta}\frac{(\varsigma/\varrho)^{\delta+2}G(\varsigma/\varrho)}{(\varsigma/\varrho)^{\delta}+(\delta/1-\delta)}\right].$$

□

3. Idea of MDM

Consider the fractional order partial differential equation by MDM:

$$D_{\rho}^{\delta}\nu(\varphi,\rho)=\overline{\mathscr{H}}_{1}(\nu,\varphi)+\mathscr{N}_{1}(\nu,\varphi),0<\delta\leq 1, \quad (30)$$

with the initial condition

$$\nu(\varphi,0)=\xi(\varphi), \quad (31)$$

where $D_{\rho}^{\delta}=\partial^{\delta}/\partial\rho^{\delta}$ is the Atangana–Baleanu fractional derivative of order δ; $\overline{\mathscr{H}}_{1}$ is linear and $\mathscr{N}_{1}$ nonlinear terms, respectively. On both sides, we use ZZ transformation of (30), to achieve

$$\mathscr{Z}\left[D_{\rho}^{\delta}\nu(\varphi,\rho)\right]=\mathscr{Z}\left[\overline{\mathscr{H}}_{1}(\nu,\varphi)+\mathscr{N}_{1}(\nu,\varphi)\right]. \quad (32)$$

By the differentiation property of ZZ transformation, we get

$$\psi(\delta)\left(1-\delta+\delta\left(\frac{\varrho}{\varsigma}\right)^{\delta}\right)\{\nu(\varphi,\rho)\}-\frac{\varrho}{\varsigma}\nu(\varphi,0)$$
$$=\mathscr{Z}\left[\overline{\mathscr{H}}_{1}(\nu,\varphi)+\mathscr{N}_{1}(\nu,\varphi)\right]. \quad (33)$$

(33) implies that

$$\nu(\varphi,\rho)=\frac{\varrho}{\varsigma}\nu(\varphi,0)$$
$$+\left(1-\delta+\delta\left(\frac{\varrho}{\varsigma}\right)^{\delta}\right)\psi(\delta)\mathscr{Z}\left[\overline{\mathscr{H}}_{1}(\nu,\varphi)+\mathscr{N}_{1}(\nu,\varphi)\right]. \quad (34)$$

Applying the ZZ inverse transformation of (34), we get

$$\nu(\varphi,\rho)=\nu(\varphi,0)$$
$$+\mathscr{Z}^{-1}\left[\left(1-\delta+\delta\left(\frac{\varrho}{\varsigma}\right)^{\delta}\right)\psi(\delta)\mathscr{Z}\left\{\overline{\mathscr{H}}_{1}(\nu,\varphi)+\mathscr{N}_{1}(\nu,\varphi)\right\}\right]. \quad (35)$$

MDM determines the infinite sequence's result of $\nu(\varphi,\rho)$:

$$\nu(\varphi,\rho)=\sum_{m=0}^{\infty}\nu_{m}(\varphi,\rho). \quad (36)$$

The nonlinear functions can be found with the help of Adomian polynomials $\mathscr{N}_{1}$ which is expressed as

$$\mathscr{N}_{1}(\nu,\varphi)=\sum_{m=0}^{\infty}\mathscr{A}_{m}. \quad (37)$$

The Adomian polynomials can show all types of nonlinearity as

$$\mathscr{A}_{m}=\frac{1}{m!}\left[\frac{\partial^{m}}{\partial\ell^{m}}\left\{\mathscr{N}_{1}\left(\sum_{k=0}^{\infty}\ell^{k}\nu_{k},\sum_{k=0}^{\infty}\ell^{k}\varphi_{k}\right)\right\}\right]_{\ell=0}. \quad (38)$$

Putting (36) and (38) into (35), it gives

$$\sum_{m=0}^{\infty} \nu_m(\varphi,\rho) = \nu(\varphi,0) + \mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varsigma/\varrho)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left\{\overline{\mathscr{H}}_1\left(\sum_{m=0}^{\infty}\nu_m, \sum_{m=0}^{\infty}\varphi_m\right) + \sum_{m=0}^{\infty}\mathscr{A}_m\right\}\right]. \tag{39}$$

The following terms are described:

$$\nu_0(\varphi,\rho) = \nu(\varphi,0),$$
$$\nu_1(\varphi,\rho) = \mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left\{\overline{\mathscr{H}}_1(\nu_0,\varphi_0) + \mathscr{A}_0\right\}\right]. \tag{40}$$

The general form for $m \geq 1$ is determined as

$$\nu_{m+1}(\varphi,\rho) = \mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left\{\overline{\mathscr{H}}_1(\nu_m,\varphi_m) + \mathscr{A}_m\right\}\right]. \tag{41}$$

4. Numerical Examples

Example 1. Here, we take the following FPDE:

$$\begin{cases} D_{\rho}^{\delta}(\mu) - \dfrac{\partial \nu}{\partial \zeta} + \nu + \mu = 0, \\ \\ D_{\rho}^{\delta}(\nu) - \dfrac{\partial \mu}{\partial \zeta} + \nu + \mu = 0, \delta, \in (0,1], \end{cases}. \tag{42}$$

with initial source

$$\begin{cases} \mu(\zeta,0) = \sinh(\zeta) \\ \nu(\zeta,0) = \cosh(\zeta) \end{cases}. \tag{43}$$

The exact result at $\delta = 1$ is (1) $\mu(\varphi,\rho) = \sinh(\varphi-\rho)$ and (2) $\nu(\varphi,\rho) = \cosh(\varphi+\rho)$.

Applying ZZT (42), we get

$$\mathscr{Z}\left\{\frac{\partial^{\delta}\mu}{\partial\rho^{\delta}}\right\} = \mathscr{Z}\left\{\frac{\partial\nu}{\partial\varphi} - \nu - \mu\right\}, \mathscr{Z}\left\{\frac{\partial^{\delta}\nu}{\partial\rho^{\delta}}\right\} = \mathscr{Z}\left\{\frac{\partial\mu}{\partial\varphi} - \nu - \mu\right\},$$

$$\frac{\psi(\delta)}{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}\mathscr{Z}\{\mu(\varphi,\rho)\} - \frac{\varrho}{\varsigma}\mu(\varphi,0) = \mathscr{Z}\left\{\frac{\partial\nu}{\partial\varphi} - \nu - \mu\right\}, \tag{44}$$

$$\frac{\psi(\delta)}{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}\mathscr{Z}\{\nu(\varphi,\rho)\} - \frac{\varrho}{\varsigma}\nu(\varphi,0) = \mathscr{Z}\left\{\frac{\partial\mu}{\partial\varphi} - \nu - \mu\right\}.$$

We get

$$\mathscr{Z}\{\mu(\varphi,\rho)\} = \frac{\varrho}{\varsigma}\{\mu(\varphi,0)\} + \frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left\{\frac{\partial\nu}{\partial\varphi} - \nu - \mu\right\},$$

$$\mathscr{Z}\{\nu(\varphi,\rho)\} = \frac{\varrho}{\varsigma}\{\nu(\varphi,0)\} + \frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left\{\frac{\partial\mu}{\partial\varphi} - \nu - \mu\right\}. \tag{45}$$

Applying the inverse ZZT to (45), we get

$$\mu(\varphi,\rho) = \mu(\varphi,0) + \mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left[\frac{\partial\nu}{\partial\varphi} - \nu - \mu\right]\right],$$

$$\nu(\varphi,\rho) = \nu(\varphi,0) + \mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left[\frac{\partial\mu}{\partial\varphi} - \nu - \mu\right]\right]. \tag{46}$$

Decomposition results for $\mu(\varphi,\rho)$ and $\nu(\varphi,\rho)$ can be expressed as

$$\mu(\varphi,\rho) = \sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\rho), \text{and} \nu(\varphi,\rho) = \sum_{\aleph=0}^{\infty}\nu_{\aleph}(\varphi,\rho),$$

$$\sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\rho) = \mu(\varphi,0) + \mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left[\frac{\partial\sum_{\aleph=0}^{\infty}\nu_{\aleph}(\varphi,\rho)}{\partial\varphi} - \sum_{\aleph=0}^{\infty}\nu_{\aleph}(\varphi,\rho) - \sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\rho)\right]\right], \tag{47}$$

$$\sum_{\aleph=0}^{\infty}\nu_{\aleph}(\varphi,\varphi,\rho) = \nu(\varphi,0) + \mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left[\frac{\partial\sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\rho)}{\partial\varphi} - \sum_{\aleph=0}^{\infty}\nu_{\aleph}(\varphi,\rho) - \sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\rho)\right]\right].$$

Furthermore,

$$\sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\rho)=\sinh(\varphi)+\mathscr{L}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{L}\left[\frac{\partial\sum_{\aleph=0}^{\infty}\nu_{\aleph}(\varphi,\rho)}{\partial\varphi}-\sum_{\aleph=0}^{\infty}\nu_{\aleph}(\varphi,\rho)-\sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\rho)\right]\right],$$
$$\sum_{\aleph=0}^{\infty}\nu_{\aleph}(\varphi,\rho)=\cosh(\varphi)+\mathscr{L}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{L}\left[\frac{\partial\sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\rho))}{\partial\varphi}-\sum_{\aleph=0}^{\infty}\nu_{\aleph}(\varphi,\rho)-\sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\rho)\right]\right]. \tag{48}$$

The component comparison in (48) provides the following recursive MDM algorithm:

$$\mu_0(\varphi,\rho)=\sinh(\varphi),\ \nu_0(\varphi,\rho)=\cosh(\varphi). \tag{49}$$

For $\aleph=0$,

$$\mu_1(\varphi,\rho)=-\cosh(\varphi)\frac{1}{\psi(\delta)}\left[1-\delta+\frac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right],\ \nu_1(\varphi,\rho)=-\sinh(\varphi)\frac{1}{\psi(\delta)}\left[1-\delta+\frac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right]. \tag{50}$$

For $\aleph=1$,

$$\begin{aligned}
\mu_2(\varphi,\rho)&=-\cosh(\varphi)\frac{(1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]\\
&+\sinh(\varphi)\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]\\
&+\cosh(\varphi)\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right],\\
\nu_2(\varphi,\rho)&=-\sinh(\varphi)\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]\\
&+\cosh(\varphi)\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]\\
&+\sinh(\varphi)\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right].
\end{aligned} \tag{51}$$

For $\aleph=2$,

$$\begin{aligned}
&\mu_3(\varphi,\rho)=-\cosh(\varphi)\frac{\rho^{3\delta}}{\Gamma(3\delta+1)},\ \nu_3(\varphi,\rho)=\sinh(\varphi)\frac{\rho^{3\delta}}{\Gamma(3\delta+1)}.\\
&\vdots
\end{aligned} \tag{52}$$

Similar to $\aleph>2$, MDM can be used to determine the remaining terms of μ_m and ν_m. In general, MDM's solution is as follows:

$$
\begin{aligned}
\mu(\varphi,\rho) &= \sum_{\aleph=0}^{\infty} \mu_{\aleph}(\varphi,\rho) = \mu_0(\varphi) + \mu_1(\varphi) + \mu_2(\varphi) + \mu_3(\varphi) + \cdots, \\
\nu(\varphi,\rho) &= \sum_{\aleph=0}^{\infty} \nu_{\aleph}(\varphi,\rho) = \nu_0(\varphi) + \nu_1(\varphi) + \nu_2(\varphi) + \nu_3(\varphi) + \cdots, \\
\mu(\varphi,\rho) &= \sum_{\aleph=0}^{\infty} \mu_{\aleph}(\varphi) \\
&\quad = \sinh(\varphi) - \cosh(\varphi)\frac{1}{\psi(\delta)}\left[1-\delta+\frac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right] - \cosh(\varphi)\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right] \\
&\quad\quad + \sinh(\varphi)\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right] + \cosh(\varphi)\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2\right. \\
&\quad\quad \left.+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]\cdots, \\
\nu(\varphi,\rho) &= \sum_{\aleph=0}^{\infty} \nu_{\aleph}(\varphi) \\
&\quad = \cosh(\varphi) - \sin(\varphi)\frac{1}{\psi(\delta)}\left[1-\delta+\frac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right] - \sinh(\varphi)\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right] \\
&\quad\quad + \cosh(\varphi)\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right] + \sinh(\varphi)\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2\right. \\
&\quad\quad \left.+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]\cdots, \\
\mu(\varphi,\rho) &= \sinh(\varphi)\left[1+\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]+\cdots\right] \\
&\quad\quad - \cosh(\varphi)\left[\frac{1}{\psi(\delta)}\left[1-\delta+\frac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right]+\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]\right. \\
&\quad\quad \left.-\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]+\cdots\right], \\
\nu(\varphi,\rho) &= \cosh(\varphi)\left[1+\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]+\cdots\right] \\
&\quad\quad - \sinh(\varphi)\left[\frac{1}{\psi(\delta)}\left[1-\delta+\frac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right]+\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]\right. \\
&\quad\quad \left.-\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]+\cdots\right].
\end{aligned}
\tag{53}
$$

Set $\delta = 1$ in (42); we get

$$\begin{aligned}\mu(\varphi,\rho) &= \sinh(\varphi)\left[1+\frac{\rho^2}{(2)!}+\frac{\rho^4}{(4!)}+\cdots\right] - cosh(\varphi)\left[\frac{\rho}{(1!)}+\frac{\rho^3}{(3!)}+\frac{\rho^5}{(5!)}+\cdots\right] = \sinh(\varphi-\rho),\\ \nu(\varphi,\rho) &= \cosh(\varphi)\left[1+\frac{\rho^2}{(2)!}+\frac{\rho^4}{(4!)}+\cdots\right] - sinh(\varphi)\left[\frac{\rho}{(1!)}+\frac{\rho^3}{(3!)}+\frac{\rho^5}{(5!)}+\cdots\right] = \cosh(\varphi+\rho).\end{aligned} \tag{54}$$

The exact results are at $\delta = 1$.

$$\begin{aligned}\mu(\varphi,\rho) &= sinh(\varphi-\rho),\\ \nu(\varphi,\rho) &= cosh(\varphi+\rho).\end{aligned} \tag{55}$$

We analyze the solution figures of the problem, which have been investigated by applying the ZZ decomposition method in the sense of the Atangana–Baleanu operator. Figure 1 represents the three-dimensional solution-figures for variables μ of example 1 at fractional order $\delta = 1$ and 0.8, respectively; Figure 2 represents different fractional order of $\delta = 0.6$ and 0.4; and Figure 3 represents that at δ. In Figure 4, different fractional order with respect to φ and ρ. It is observed that the ZZ decomposition method solution-figures are identical and in close contact with each other. In the same way, Figures 5–8 show different fractional order graphs of δ at ν of Example 1.

Example 2. Here, we take the following FPDE:

$$\begin{cases} D_\rho^\delta(\mu) + \nu_\varphi\omega_\chi - \nu_\chi\omega_\varphi = -\mu,\\ D_\rho^\delta(\nu) + \mu_\chi\omega_\varphi + \mu_\chi\omega_\varphi = \nu,\\ D_\rho^\delta(\omega) + \mu_\varphi\nu_\chi + \mu_\chi\nu_\varphi = \omega, \delta \in (0,1],\end{cases} \tag{56}$$

with initial sources

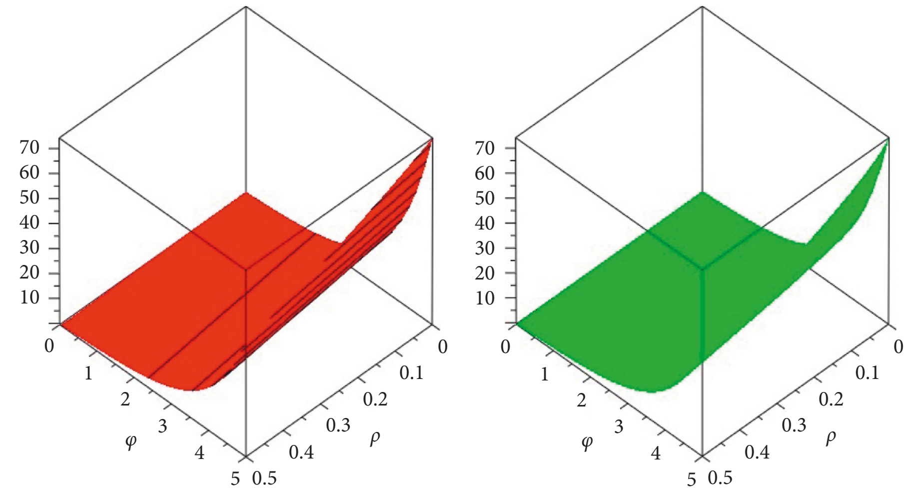
$$\begin{cases} \mu(\varphi,\chi,0) = \exp^{\varphi+\chi},\\ \nu(\varphi,\chi,0) = \exp^{\varphi-\chi},\\ \omega(\varphi,\chi,0) = \exp^{-\varphi+\chi}.\end{cases} \tag{57}$$

The exact solution at $\delta = 1$ is

$$\begin{cases} \mu(\varphi,\chi,\rho) = \exp^{\varphi+\chi-\rho},\\ \nu(\varphi,\chi,\rho) = \exp^{\varphi-\chi+\rho},\\ \omega(\varphi,\chi,\rho) = \exp^{-\varphi+\chi+\rho}.\end{cases} \tag{58}$$

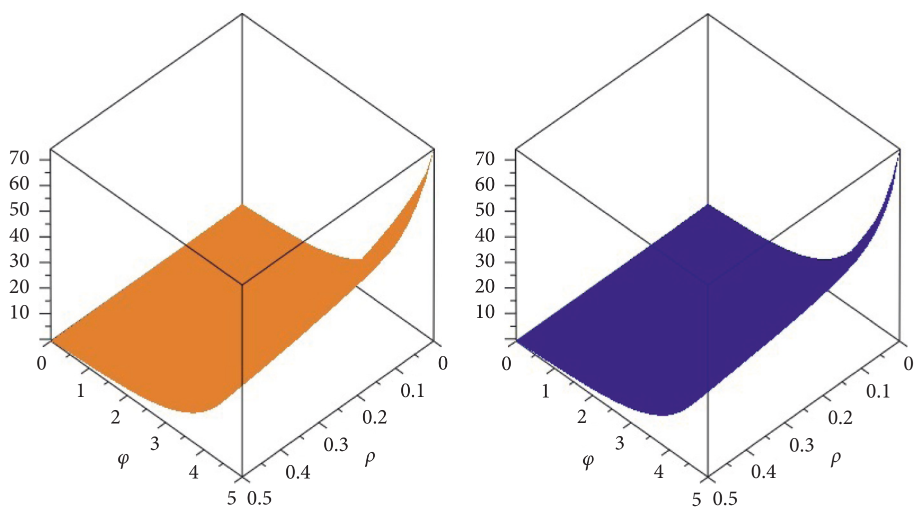
Using ZZT equation (33), it can be written as

$$\begin{aligned}
&\mathcal{Z}\left\{\frac{\partial^\delta\mu}{\partial\rho^\delta}\right\} = \mathcal{Z}\left\{-\mu+\nu_\varphi\omega_\chi-\nu_\chi\omega_\varphi\right\}, \mathcal{Z}\left\{\frac{\partial^\delta\nu}{\partial\rho^\delta}\right\} = \mathcal{Z}\left\{\nu-\mu_\chi\omega_\varphi-\mu_\chi\omega_\varphi\right\},\\
&\mathcal{Z}\left\{\frac{\partial^\delta\omega}{\partial\rho^\delta}\right\} = \mathcal{Z}\left\{\omega-\mu_\varphi\nu_\chi-\mu_\chi\nu_\varphi\right\},\\
&\frac{\psi(\delta)}{\left(1-\delta+\delta(\varrho/\varsigma)^\delta\right)}\mathcal{Z}\{\mu(\varphi,\chi,\rho)\} - \frac{\varrho}{\varsigma}\mu(\varphi,\chi,0) = \mathcal{Z}\left\{-\mu+\nu_\varphi\omega_\chi-\nu_\chi\omega_\varphi\right\},\\
&\frac{\psi(\delta)}{\left(1-\delta+\delta(\varrho/\varsigma)^\delta\right)}\mathcal{Z}\{\nu(\varphi,\chi,\rho)\} - \frac{\varrho}{\varsigma}\nu(\varphi,\chi,0) = \mathcal{Z}\left\{\nu-\mu_\chi\omega_\varphi-\mu_\chi\omega_\varphi\right\},\\
&\frac{\psi(\delta)}{\left(1-\delta+\delta(\varrho/\varsigma)^\delta\right)}\mathcal{Z}\{\omega(\varphi,\varphi,\rho)\} - \frac{\varrho}{\varsigma}\omega(\varphi,\chi,0) = \mathcal{Z}\left\{\omega-\mu_\varphi\nu_\chi-\mu_\chi\nu_\varphi\right\}.
\end{aligned} \tag{59}$$



(a) (b)

Figure 1: (a) The exact and approximate solution at $\delta = 1$ and (b) second fractional order at $\delta = 0.8$.



(a) (b)

Figure 2: The graph shows the fractional order at $\delta = 0.6$ and 0.4.

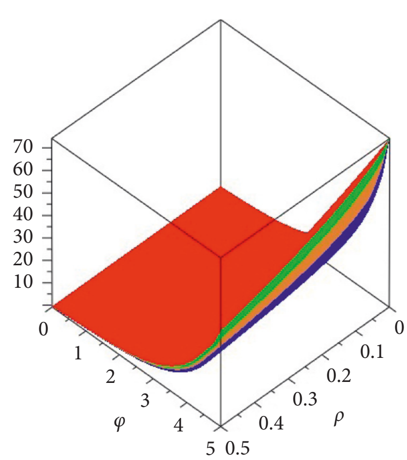


Figure 3: The graph shows different fractional orders of δ.

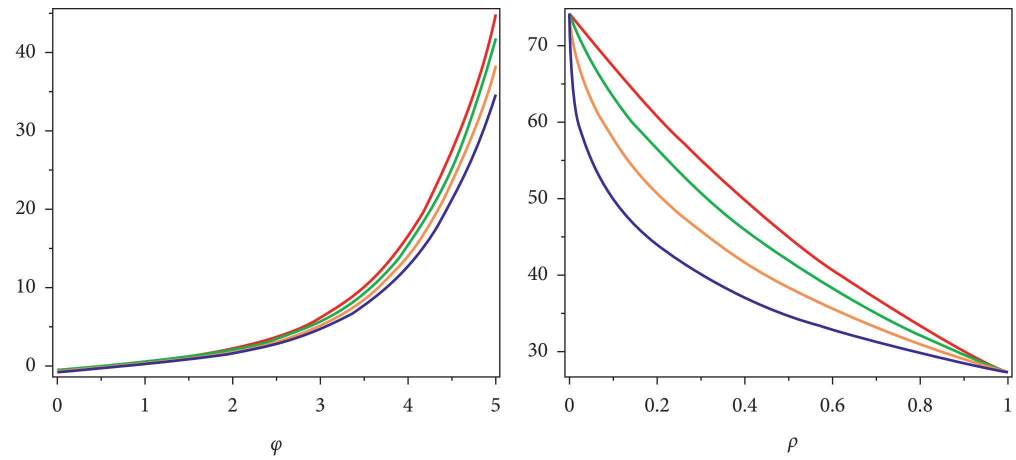
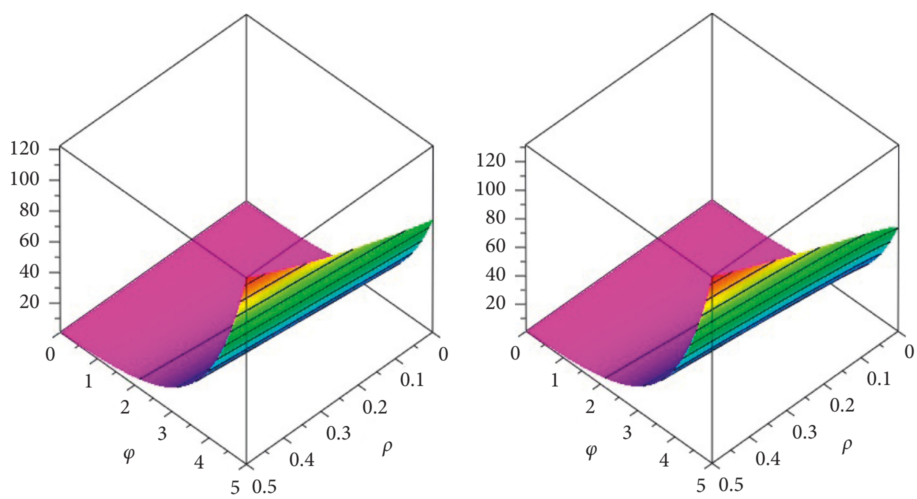
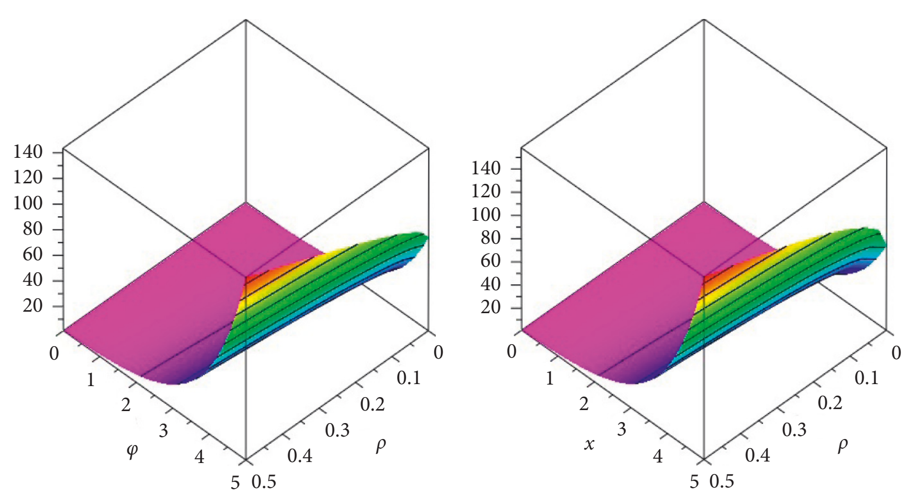


FIGURE 4: The graph of the two dimensions of different fractional orders at δ with respect to φ and ρ.



(a) (b)

FIGURE 5: (a) The exact and approximate solution at $\delta = 1$ and (b) second fractional order at $\delta = 0.8$.



(a) (b)

FIGURE 6: The graph shows the fractional order at $\delta = 0.6$ and 0.4.

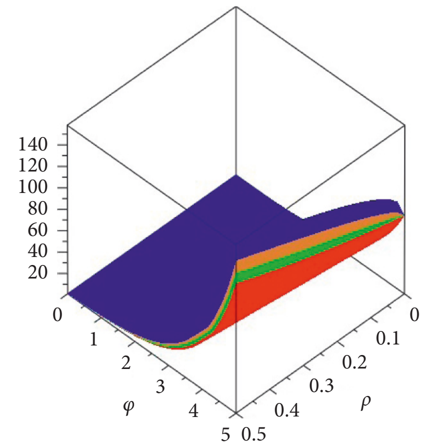


Figure 7: The graph shows different fractional orders of δ.

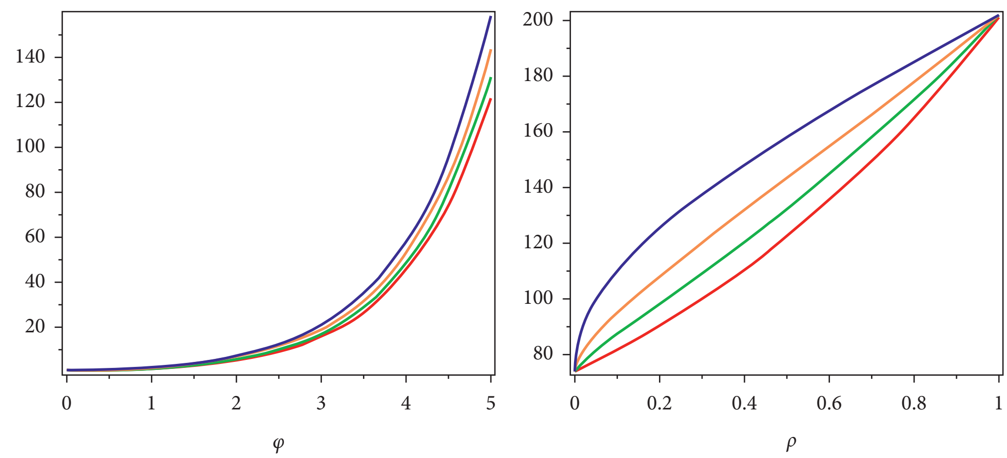


Figure 8: The graph of the two dimensions of different fractional orders at δ with respect to φ and ρ.

After simplification, we obtain

$$
\begin{aligned}
&\frac{\psi(\delta)}{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}\mathscr{Z}\{\mu(\varphi,\chi,\rho)\} = \frac{\varrho}{\varsigma}\mu(\varphi,\chi,0) + \mathscr{Z}\left\{-\mu + \nu_{\varphi}\omega_{\chi} - \nu_{\chi}\omega_{\varphi}\right\},\\
&\frac{\psi(\delta)}{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}\mathscr{Z}\{\nu(\varphi,\chi,\rho)\} = \frac{\varrho}{\varsigma}\nu(\varphi,\chi,0) + \mathscr{Z}\left\{\nu - \mu_{\chi}\omega_{\varphi} - \mu_{\chi}\omega_{\varphi}\right\},\\
&\frac{\psi(\delta)}{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}\mathscr{Z}\{\omega(\varphi,\varphi,\rho)\} = \frac{\varrho}{\varsigma}\omega(\varphi,\chi,0) + \mathscr{Z}\left\{\omega - \mu_{\varphi}\nu_{\chi} - \mu_{\chi}\nu_{\varphi}\right\},\\
&\mathscr{Z}\{\mu(\varphi,\chi,\rho)\} = \frac{\varrho}{\varsigma}\mu(\varphi,\chi,0) + \frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left\{-\mu + \nu_{\varphi}\omega_{\chi} - \nu_{\chi}\omega_{\varphi}\right\},\\
&\mathscr{Z}\{\nu(\varphi,\chi,\rho)\} = \frac{\varrho}{\varsigma}\nu(\varphi,\chi,0) + \frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left\{\nu - \mu_{\chi}\omega_{\varphi} - \mu_{\chi}\omega_{\varphi}\right\},\\
&\mathscr{Z}\{\omega(\varphi,\varphi,\rho)\} = \frac{\varrho}{\varsigma}\omega(\varphi,\chi,0) + \frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left\{\omega - \mu_{\varphi}\nu_{\chi} - \mu_{\chi}\nu_{\varphi}\right\}.
\end{aligned}
\tag{60}
$$

Taking inverse ZZT of (60), we obtain

$$\begin{aligned}\mu(\varphi,\chi,\rho) &= \mu(\varphi,\chi,0) + \mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left\{-\mu+\nu_{\varphi}\omega_{\chi}-\nu_{\chi}\omega_{\varphi}\right\}\right],\\ \nu(\varphi,\chi,\rho) &= \nu(\varphi,\chi,0) + \mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left\{\nu-\mu_{\chi}\omega_{\varphi}-\mu_{\chi}\omega_{\varphi}\right\}\right],\\ \omega(\varphi,\chi,\rho) &= \omega(\varphi,\chi,0) + \mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left\{\omega-\mu_{\varphi}\nu_{\chi}-\mu_{\chi}\nu_{\varphi}\right\}\right].\end{aligned} \tag{61}$$

Assume decomposition solutions for variables $\mu(\varphi,\chi,\rho)$, $\nu(\varphi,\chi,\rho)$, and $\omega(\varphi,\chi,\rho)$, it can be written as

$$\mu(\varphi,\chi,\rho) = \sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\chi,\rho), \nu(\varphi,\chi,\rho) = \sum_{\aleph=0}^{\infty}\nu_{\aleph}(\varphi,\chi,\rho), \text{and}\omega(\varphi,\chi,\rho) = \sum_{\aleph=0}^{\infty}\omega_{\aleph}(\varphi,\chi,\rho). \tag{62}$$

Remember that $\nu_{\varphi}\omega_{\chi} = \sum_{\aleph=0}^{\infty}\mathscr{A}_m$, $\nu_{\chi}\omega_{\varphi} = \sum_{\aleph=0}^{\infty}\mathscr{B}_{\aleph}$, $\mu_{\varphi}\omega_{\chi} = \sum_{\aleph=0}^{\infty}\mathscr{C}_{\aleph}$, $\mu_{\chi}\omega_{\varphi} = \sum_{\aleph=0}^{\infty}\mathscr{D}_{\aleph}, \mu_{\varphi}\nu_{\chi} = \sum_{\aleph=0}^{\infty}\mathscr{E}_{\aleph}$, and $\mu_{\chi}\nu_{\varphi} = \sum_{\aleph=0}^{\infty}\mathscr{F}_{\aleph}$ are the Adomian polynomials and the nonlinear terms were characterized, which can be further simplified as

$$\begin{aligned}\sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\chi,\rho) &= \mu(\varphi,\chi,0) + \mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left[-\sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\chi,\rho)+\left(\sum_{\aleph=0}^{\infty}\mathscr{A}_{\aleph}-\sum_{\aleph=0}^{\infty}\mathscr{B}_{\aleph}\right)\right]\right],\\ \sum_{\aleph=0}^{\infty}\nu_{\aleph}(\varphi,\chi,\rho) &= \nu(\varphi,\chi,0) + \mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left[\sum_{\aleph=0}^{\infty}\nu_{\aleph}(\varphi,\chi,\rho)-\left(\sum_{\aleph=0}^{\infty}\mathscr{C}_{\aleph}+\sum_{\aleph=0}^{\infty}\mathscr{D}_{\aleph}\right)\right]\right],\\ \sum_{\aleph=0}^{\infty}\omega_{\aleph}(\varphi,\chi,\rho) &= \omega(\varphi,\varphi,\delta,0) + \mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left[\sum_{\aleph=0}^{\infty}\omega_{\aleph}(\varphi,\chi,\rho)-\left(\sum_{\aleph=0}^{\infty}\mathscr{E}_{\aleph}+\sum_{\aleph=0}^{\infty}\mathscr{F}_{\aleph}\right)\right]\right].\end{aligned} \tag{63}$$

Using (38), the nonlinearity in the given problem can be expressed as

$$\begin{aligned}&\mathscr{A}_0 = \frac{\partial\nu_0}{\partial\varphi}\frac{\partial\omega_0}{\partial\chi}, \mathscr{A}_1 = \frac{\partial\nu_0}{\partial\varphi}\frac{\partial\omega_1}{\partial\chi}+\frac{\partial\nu_1}{\partial\varphi}\frac{\partial\omega_0}{\partial\chi}, \mathscr{B}_0 = \frac{\partial\nu_0}{\partial\chi}\frac{\partial\omega_0}{\partial\varphi}, \mathscr{B}_1 = \frac{\partial\nu_0}{\partial\chi}\frac{\partial\omega_1}{\partial\varphi}+\frac{\partial\nu_0}{\partial\varphi}\frac{\partial\omega_1}{\partial\chi},\\ &\mathscr{C}_0 = \frac{\partial\mu_0}{\partial\varphi}\frac{\partial\omega_0}{\partial\chi}, \mathscr{C}_1 == \frac{\partial\mu_1}{\partial\varphi}\frac{\partial\omega_0}{\partial\chi}+\frac{\partial\mu_0}{\partial\varphi}\frac{\partial\omega_1}{\partial\chi}, \mathscr{D}_0 = \frac{\partial\mu_0}{\partial\chi}\frac{\partial\omega_0}{\partial\varphi}, \mathscr{D}_1 = \frac{\partial\mu_0}{\partial\chi}\frac{\partial\omega_1}{\partial\varphi}+\frac{\partial\mu_1}{\partial\chi}\frac{\partial\omega_0}{\partial\varphi},\\ &\mathscr{E}_0 = \frac{\partial\mu_0}{\partial\varphi}\frac{\partial\nu_0}{\partial\varphi}, \mathscr{E}_1 = \frac{\partial\mu_1}{\partial\varphi}\frac{\partial\nu_0}{\partial\chi}+\frac{\partial\mu_0}{\partial\varphi}\frac{\partial\nu_1}{\partial\chi}, \mathscr{F}_0 = \frac{\partial\mu_0}{\partial\chi}\frac{\partial\nu_0}{\partial\varphi}, \mathscr{F}_1 = \frac{\partial\mu_1}{\partial\chi}\frac{\partial\nu_0}{\partial\varphi}+\frac{\partial\mu_0}{\partial\varphi}\frac{\partial\nu_1}{\partial\chi}.\end{aligned} \tag{64}$$

The component comparison provides the following recursive MDM algorithm:

$$\begin{aligned}
&\mu_0(\varphi,\chi,\rho)=\mu(\varphi,\chi,0),\nu_0(\varphi,\chi,\rho)=\nu(\varphi,\chi,0),\omega_0(\varphi,\chi,\rho)=\omega(\varphi,\chi,0),\\
&\mu_1(\varphi,\chi,\rho)=\mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left[-\mu_0(\varphi,\chi,\rho)+\left[\mathscr{A}_0-\mathscr{B}_0\right]\right]\right],\\
&\nu_1(\varphi,\chi,\rho)=\mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left[\nu_0(\varphi,\chi,\rho)-\left[\mathscr{C}_0+\mathscr{D}_0\right]\right]\right],\\
&\omega_1(\varphi,\chi,\rho)=\mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left[\omega_0(\varphi,\chi,\rho)-\left[\mathscr{E}_0+\mathscr{F}_0\right]\right]\right],\\
&\mu_{\aleph+1}(\varphi,\chi,\rho)=\mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left[-\mu_{\aleph}(\varphi,\chi,\rho)+\left[\mathscr{A}_{\aleph}-\mathscr{B}_{\aleph}\right]\right]\right],\\
&\nu_{\aleph+1}(\varphi,\chi,\rho)=\mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left[\nu_{\aleph}(\varphi,\chi,\rho)-\left[\mathscr{C}_{\aleph}+\mathscr{D}_{\aleph}\right]\right]\right],\\
&\omega_{\aleph+1}(\varphi,\chi,\rho)=\mathscr{Z}^{-1}\left[\frac{\left(1-\delta+\delta(\varrho/\varsigma)^{\delta}\right)}{\psi(\delta)}\mathscr{Z}\left[\omega_{\aleph}(\varphi,\chi,\rho)-\left[\mathscr{E}_{\aleph}+\mathscr{F}_{\aleph}\right]\right]\right],\\
&\mu_0(\varphi,\chi,\rho)=\exp^{\varphi+\chi},\nu_0(\varphi,\chi,\rho)=\exp^{\varphi-\chi},\omega_0(\varphi,\chi,\rho)=\exp^{-\varphi+\chi}.
\end{aligned}\tag{65}$$

For $\aleph=0$,

$$\begin{aligned}
\mu_1(\varphi,\chi,\rho)&=-\exp^{\varphi+\chi}\frac{1}{\psi(\delta)}\left[1-\delta+\frac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right],\\
\nu_1(\varphi,\chi,\rho)&=\exp^{\varphi-\chi}\frac{1}{\psi(\delta)}\left[1-\delta+\frac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right],\\
\omega_1(\varphi,\chi,\rho)&=\exp^{-\varphi+\chi}\frac{1}{\psi(\delta)}\left[1-\delta+\frac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right].
\end{aligned}\tag{66}$$

For $\aleph=1$,

$$\begin{aligned}
\mu_2(\varphi,\chi,\rho)&=\exp^{\varphi+\chi}\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right],\\
\nu_2(\varphi,\chi,\rho)&=\exp^{\varphi-\chi}\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right],\\
\omega_2(\varphi,\chi,\rho)&=\exp^{-\varphi+\chi}\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right].
\end{aligned}\tag{67}$$

For $\aleph=2$,

$$\begin{aligned}
\mu_3(\varphi,\chi,\rho)&=-\exp^{\varphi+\chi}\frac{1}{(\mathrm{B}(\delta))^3}\left[(1-\delta)^3+\frac{3\delta(1-\delta)^2\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2(1-\delta)\rho^{2\delta+1}}{\Gamma(2\delta+2)}+\frac{2\delta^2(1-\delta)\rho^{2\delta}}{\Gamma(2\delta+1)}+\frac{\delta^3\rho^{2\delta+1}}{\Gamma(2\delta+2)}\right],\\
\nu_3(\varphi,\chi,\rho)&=\exp^{\varphi-\chi}\frac{1}{(\mathrm{B}(\delta))^3}\left[(1-\delta)^3+\frac{3\delta(1-\delta)^2\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2(1-\delta)\rho^{2\delta+1}}{\Gamma(2\delta+2)}+\frac{2\delta^2(1-\delta)\rho^{2\delta}}{\Gamma(2\delta+1)}+\frac{\delta^3\rho^{2\delta+1}}{\Gamma(2\delta+2)}\right],\\
\omega_3(\varphi,\chi,\rho)&=\exp^{-\varphi+\chi}\frac{1}{(\mathrm{B}(\delta))^3}\left[(1-\delta)^3+\frac{3\delta(1-\delta)^2\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2(1-\delta)\rho^{2\delta+1}}{\Gamma(2\delta+2)}+\frac{2\delta^2(1-\delta)\rho^{2\delta}}{\Gamma(2\delta+1)}+\frac{\delta^3\rho^{2\delta+1}}{\Gamma(2\delta+2)}\right].\\
&\vdots
\end{aligned}\tag{68}$$

In same manner, the remaining terms of $\mu_{\aleph}$, $\nu_{\aleph}$, and $\omega_{\aleph}$ for $(\aleph>3)$ can be calculated easily by using MDM. The general solution of MDM is given by

$$
\begin{aligned}
\mu(\varphi,\chi,\rho) &= \sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\chi,\rho) = \mu_0(\varphi,\chi,\rho)+\mu_1(\varphi,\chi,\rho)+\mu_2(\varphi,\chi,\rho)+\mu_3(\varphi,\chi,\rho)+\cdots,\\
\nu(\varphi,\chi,\rho) &= \sum_{\aleph=0}^{\infty}\nu_{\aleph}(\varphi,\chi,\rho) = \nu_0(\varphi,\chi,\rho)+\nu_1(\varphi,\chi,\rho)+\nu_2(\varphi,\chi,\rho)+\nu_3(\varphi,\chi,\rho)+\cdots,\\
\omega(\varphi,\chi,\rho) &= \sum_{\aleph=0}^{\infty}\omega_{\aleph}(\varphi,\chi,\rho) = \omega_0(\varphi,\chi,\rho)+\omega_1(\varphi,\chi,\rho)+\omega_2(\varphi,\chi,\rho)+\omega_3(\varphi,\chi,\rho)+\cdots,\\
\mu(\varphi,\chi,\rho) &= \sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\chi,\rho) = \exp^{\varphi+\chi}-\exp^{\varphi+\chi}\frac{1}{\psi(\delta)}\left[1-\delta+\frac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right]\\
&\quad+\exp^{\varphi+\chi}\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]-\exp^{\varphi+\chi}\frac{1}{(\mathrm{B}(\delta))^3}\\
&\quad\cdot\left[(1-\delta)^3+\frac{3\delta(1-\delta)^2\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2(1-\delta)\rho^{2\delta+1}}{\Gamma(2\delta+2)}+\frac{2\delta^2(1-\delta)\rho^{2\delta}}{\Gamma(2\delta+1)}+\frac{\delta^3\rho^{2\delta+1}}{\Gamma(2\delta+2)}\right]\\
\cdots,\nu(\varphi,\chi,\rho) &= \sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\chi,\rho) = \exp^{\varphi-\chi}+\exp^{\varphi-\chi}\frac{1}{\psi(\delta)}\left[1-\delta+\frac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right]+\exp^{\varphi-\chi}\frac{1}{(\mathrm{B}(\delta))^2}\\
&\quad\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]\\
&\quad+\exp^{\varphi-\chi}\frac{1}{(\mathrm{B}(\delta))^3}\left[\begin{array}{c}(1-\delta)^3\\ +\dfrac{3\delta(1-\delta)^2\rho^{\delta}}{\Gamma(\delta+1)}+\dfrac{\delta^2(1-\delta)\rho^{2\delta+1}}{\Gamma(2\delta+2)}+\dfrac{2\delta^2(1-\delta)\rho^{2\delta}}{\Gamma(2\delta+1)}+\dfrac{\delta^3\rho^{2\delta+1}}{\Gamma(2\delta+2)}\end{array}\right]\\
+\cdots,\omega(\varphi,\chi,\rho) &= \sum_{\aleph=0}^{\infty}\mu_{\aleph}(\varphi,\chi,\rho) = \exp^{-\varphi+\chi}+\exp^{-\varphi+\chi}\frac{1}{\psi(\delta)}\left[1-\delta+\frac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right]\\
&\quad+\exp^{-\varphi+\chi}\frac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]\\
&\quad+\exp^{-\varphi+\chi}\frac{1}{(\mathrm{B}(\delta))^3}\left[(1-\delta)^3+\frac{3\delta(1-\delta)^2\rho^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2(1-\delta)\rho^{2\delta+1}}{\Gamma(2\delta+2)}+\frac{2\delta^2(1-\delta)\rho^{2\delta}}{\Gamma(2\delta+1)}+\frac{\delta^3\rho^{2\delta+1}}{\Gamma(2\delta+2)}\right]+\cdots,\\
\mu(\varphi,\chi,\rho) &= \exp^{\varphi+\chi}\left[\begin{array}{c}1-\dfrac{1}{\psi(\delta)}\left[1-\delta+\dfrac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right]+\dfrac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\dfrac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\dfrac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]-\\ \dfrac{1}{(\mathrm{B}(\delta))^3}\left[(1-\delta)^3+\dfrac{3\delta(1-\delta)^2\rho^{\delta}}{\Gamma(\delta+1)}+\dfrac{\delta^2(1-\delta)\rho^{2\delta+1}}{\Gamma(2\delta+2)}+\dfrac{2\delta^2(1-\delta)\rho^{2\delta}}{\Gamma(2\delta+1)}+\dfrac{\delta^3\rho^{2\delta+1}}{\Gamma(2\delta+2)}\right]+\cdots\end{array}\right],\\
\nu(\varphi,\chi,\rho) &= \exp^{\varphi-\chi}\left[\begin{array}{c}1+\dfrac{1}{\psi(\delta)}\left[1-\delta+\dfrac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right]+\dfrac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\dfrac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\dfrac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]+\\ \dfrac{1}{(\mathrm{B}(\delta))^3}\left[(1-\delta)^3+\dfrac{3\delta(1-\delta)^2\rho^{\delta}}{\Gamma(\delta+1)}+\dfrac{\delta^2(1-\delta)\rho^{2\delta+1}}{\Gamma(2\delta+2)}+\dfrac{2\delta^2(1-\delta)\rho^{2\delta}}{\Gamma(2\delta+1)}+\dfrac{\delta^3\rho^{2\delta+1}}{\Gamma(2\delta+2)}\right]+\cdots\end{array}\right],\\
\omega(\varphi,\chi,\rho) &= \exp^{-\varphi+\chi}\left[\begin{array}{c}1+\dfrac{1}{\psi(\delta)}\left[1-\delta+\dfrac{\delta\rho^{\delta}}{\Gamma(\delta+1)}\right]+\dfrac{1}{(\mathrm{B}(\delta))^2}\left[(1-\delta)^2+\dfrac{2\delta(1-\delta)\rho^{\delta}}{\Gamma(\delta+1)}+\dfrac{\delta^2\rho^{2\delta}}{\Gamma(2\delta+1)}\right]+\\ \dfrac{1}{(\mathrm{B}(\delta))^3}\left[(1-\delta)^3+\dfrac{3\delta(1-\delta)^2\rho^{\delta}}{\Gamma(\delta+1)}+\dfrac{\delta^2(1-\delta)\rho^{2\delta+1}}{\Gamma(2\delta+2)}+\dfrac{2\delta^2(1-\delta)\rho^{2\delta}}{\Gamma(2\delta+1)}+\dfrac{\delta^3\rho^{2\delta+1}}{\Gamma(2\delta+2)}\right]\cdots\end{array}\right].
\end{aligned}
\tag{69}
$$

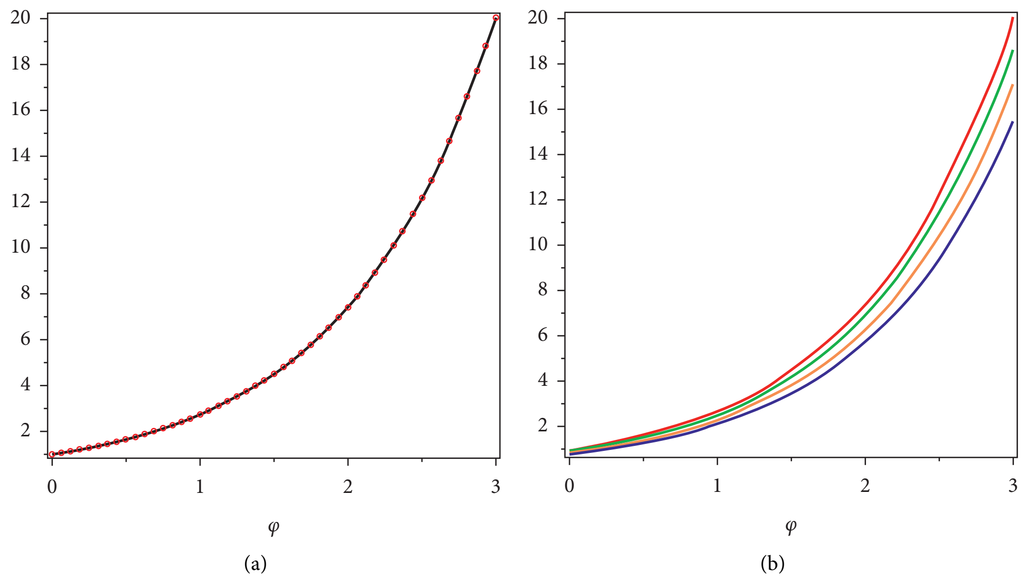


Figure 9: (a) The exact and approximate solution at $\delta = 1$ and (b) the different fractional order of δ.

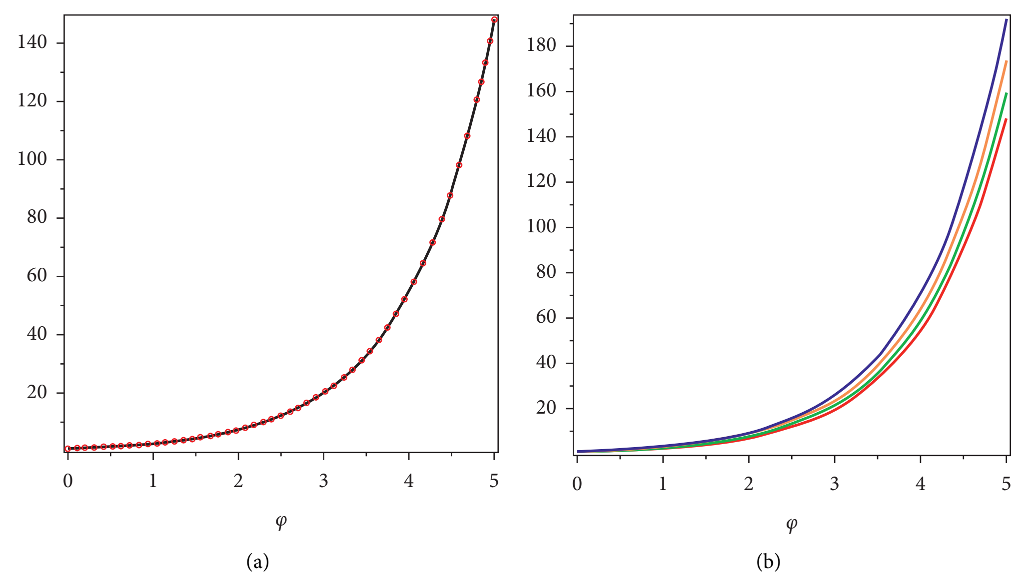


Figure 10: (a) The exact and approximate solution at $\delta = 1$ and (b) the different fractional order of δ.

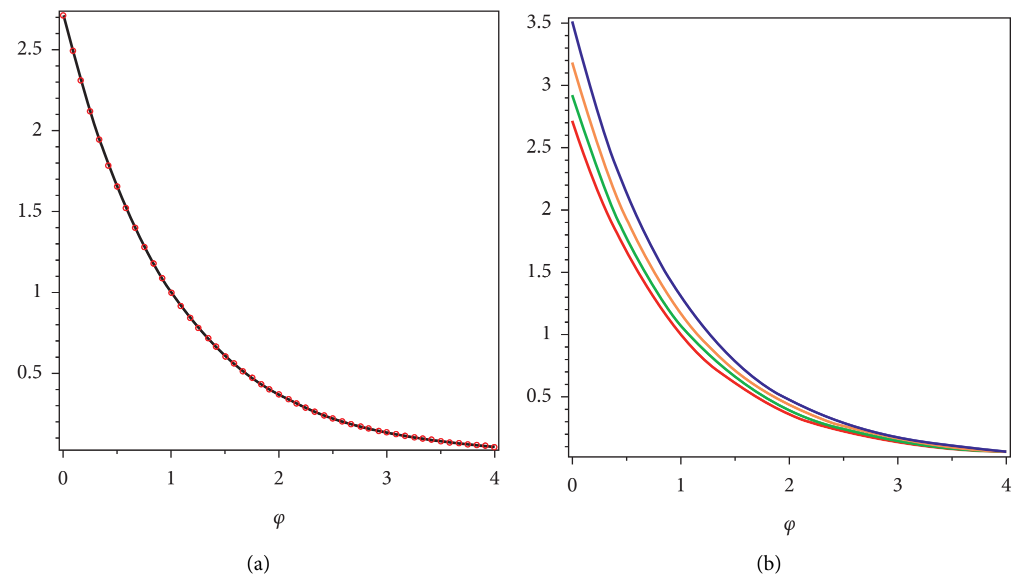


Figure 11: (a) The exact and approximate solution at $\delta = 1$ and (b) the different fractional order of δ.

Setting $\delta = 1$ in (69), we get

$$\begin{aligned}
\mu(\varphi,\chi,\rho) &= \exp^{\varphi+\chi}\left[1 - \frac{\rho}{\Gamma(2)} + \frac{\rho^2}{\Gamma(3)} - \frac{\rho^3}{\Gamma(4)}\cdots\right],\\
\nu(\varphi,\chi,\rho) &= \exp^{\varphi-\chi}\left[1 + \frac{\rho}{\Gamma(2)} + \frac{\rho^2}{\Gamma(3)} + \frac{\rho^3}{\Gamma(4)}\cdots\right],\\
\omega(\varphi,\chi,\rho) &= \exp^{-\varphi+\chi}\left[1 + \frac{\rho}{\Gamma(2)} + \frac{\rho^2}{\Gamma(3)} + \frac{\rho^3}{\Gamma(4)}\cdots\right].\\
\mu(\varphi,\chi,\rho) &= \exp^{\varphi+\chi}\left[1 - \frac{\rho}{1!} + \frac{\rho^2}{2!} - \frac{\rho^3}{3!}\cdots\right],\\
\nu(\varphi,\chi,\rho) &= \exp^{\varphi-\chi}\left[1 + \frac{\rho}{1!} + \frac{\rho^2}{2!} + \frac{\rho^3}{3!}\cdots\right],\\
\omega(\varphi,\chi,\rho) &= \exp^{-\varphi+\chi}\left[1 + \frac{\rho}{1!} + \frac{\rho^2}{2!} + \frac{\rho^3}{3!}\cdots\right],
\end{aligned} \tag{70}$$

which is the MDM solution in closed form of equation (34). When $\delta = 1$,

$$\begin{aligned}
\mu(\varphi,\chi,\rho) &= \exp^{\varphi+\chi-\rho},\\
\nu(\varphi,\chi,\rho) &= \exp^{\varphi-\chi+\rho},\\
\omega(\varphi,\chi,\rho) &= \exp^{-\varphi+\chi+\rho}.
\end{aligned} \tag{71}$$

We analyze the solution-figures of the problem, which have been investigated by applying the ZZ decomposition method in the sense of the Atangana–Baleanu operator. Figure 9 represents the two-dimensional solution-figures for variables μ of example 2 and second graph of different fractional order δ. Figure 10 represents the two-dimensional solution-figures for variables ν of example 2 and second graph of different fractional-order δ. Figure 11 represents the two-dimensional solution-figures for variables ω of example 2 and second graph of different fractional-order δ. It is observed that the ZZ decomposition method solution-figures are identical and in close contact with each other.

5. Conclusion

In this paper, some important system of fractional partial differential equations is considered for its analytical solution using the ZZ decomposition method. It has been demonstrated from the figures that the present techniques have the greater tendency to analyze the results of the given models. The problems results at different time fractional are investigated which cover the various aspects of the proposed models and proposed method. The results at different fractional orders are suggested and shown a very closed convergence phenomena of the fractional results towards integer order solutions. The graph has shown a very consistent relation between the integer and fractional orders results. It is noted that the effective and straight-forward solution of the ZZ decomposition method implies its applicability to solve other fractional partial differential equations.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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References

[1] E. Barkai, R. Metzler, and J. Klafter, "From continuous time random walks to the fractional Fokker-Planck equation," *Physical Review A*, vol. 61, no. 1, pp. 132–138, 2000.

[2] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, World Scientific Imperial College Press, London, 2010.

[3] C. Tadjeran and M. M. Meerschaert, "A second-order accurate numerical method for the two-dimensional fractional diffusion equation," *Journal of Computational Physics*, vol. 220, no. 2, pp. 813–823, 2007.

[4] M. M. Meerschaert, D. A. Benson, H. P. Scheffler, and P. Becker-Kern, "Governing equations and solutions of anomalous random walk limits," *Physical Review A*, vol. 66, no. 6, 2002.

[5] R. Magin, M. D. Ortigueira, I. Podlubny, and J. Trujillo, "On the fractional signals and systems," *Signal Processing*, vol. 91, no. 3, pp. 350–371, 2011.

[6] J. T. Machado, D. Baleanu, and A. C. Luo, *Discontinuity and Complexity in Nonlinear Physical Systems*, Springer International Publishing, New York City, 2014.

[7] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, NY, USA, 1993.

[8] K. Oldham and J. Spanier, *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*, Elsevier, California, 1974.

[9] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations to Methods of Their Solution and Some of Their Applications*, Elsevier, Amsterdam, Netherlands, 1998.

[10] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, vol. 204, 2006.

[11] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives*Vol. 1, Gordon and breach science publishers, Yverdon; Gordan and Breach, Switzerland, 1993.

[12] R. Caponetto, *Fractional order systems: modeling and control applications*, Vol. 72, World Scientific Publishing Company, Singapore, 2010.

[13] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer Science & Business Media, Berlin, Germany, 2010.

[14] J. T. Machado, V. Kiryakova, and F. Mainardi, "Recent history of fractional calculus," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1140–1153, 2011.

[15] E. Hernandez, D. ORegan, and K. Balachandran, "On recent developments in the theory of abstract differential equations with fractional derivatives," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 10, pp. 3462–3471, 2010.

[16] N. Iqbal, A. Akgul, R. Shah, A. Bariq, M. Mossa Al-Sawalha, and A. Ali, "On solutions of fractional-order gas dynamics equation by effective techniques," *Journal of Function Spaces*, pp. 1–14, 2022.

[17] M. K. Alaoui, R. Fayyaz, A. Khan, and M. S. Abdo, "Analytical Investigation of Noyes-Field Model for Time-Fractional Belousov-Zhabotinsky Reaction," *Complexity*, 2021.

[18] M. Meddahi, H. Jafari, and M. N. Ncube, "New general integral transform via Atangana-Baleanu derivatives," *Advances in Difference Equations*, vol. 2021, no. 1, pp. 385–414, 2021.

[19] H. Jafari, "A new general integral transform for solving integral equations," *Journal of Advanced Research*, vol. 32, pp. 133–138, 2021.

[20] H. Jafari, H. K. Jassim, D. Baleanu, and Y. M. Chu, "On the approximate solutions for a system of coupled Korteweg-de Vries equations with local fractional derivative," *Fractals*, vol. 29, no. 05, Article ID 2140012, 2021.

[21] H. Jafari, J. G. Prasad, P. Goswami, and R. S. Dubey, "Solution of the local fractional generalized KDV equation using homotopy analysis method," *Fractals*, vol. 29, no. 05, Article ID 2140014, 2021.

[22] K. S. Aboodh, "Application of new transform "Aboodh Transform" to partial differential equations," *Global Journal of Pure and Applied Mathematics*, vol. 10, no. 2, pp. 249–254, 2014.

[23] K. Suliman Aboodh, "Solving fourth order parabolic PDE with variable coefficients using Aboodh transform homotopy perturbation method," *Pure and Applied Mathematics Journal*, vol. 4, no. 5, pp. 219–224, 2015.

[24] R. M. Jena, S. Chakraverty, D. Baleanu, and M. M. Alqurashi, "New aspects of ZZ transform to fractional operators with Mittag-Leffler kernel," *Frontiers in Physics*, vol. 8, p. 352, 2020.

[25] L. Riabi, K. Belghaba, M. H. Cherif, and D. Ziane, "Homotopy perturbation method combined with ZZ transform to solve some nonlinear fractional differential equations," *International Journal of Analysis and Applications*, vol. 17, no. 3, pp. 406–419, 2019.

[26] Z. U. A. Zafar, "Application of ZZ transform method on some fractional differential equations," *International Journals of Advanced Engineering & Global Technology*, vol. 4, pp. 1355–1363, 2016.

[27] A. Atangana and D. Baleanu, "New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model," *Thermal Science*, vol. 20, no. 2, pp. 763–769, 2016.

[28] K. Nonlaopon, M. Naeem, A. M. Zidan, A. Alsanad, and A. Gumaei, "Numerical Investigation of the Time-Fractional Whitham-Broer-Kaup Equation Involving without Singular Kernel Operators," *Complexity*, 2021.

[29] N. H. Aljahdaly, A. Akgul, R. Shah, I. Mahariq, and J. Kafle, "A comparative analysis of the fractional-order coupled Korteweg-De Vries equations with the Mittag-Leffler law," *Journal of Mathematics*, pp. 1–30, 2022.

[30] N. A. Shah, H. A. Alyousef, S. A. El-Tantawy, R. Shah, and J. D. Chung, "Analytical investigation of fractional-order korteweg-de-vries-type equations under atangana-baleanu-caputo operator: modeling nonlinear waves in a plasma and fluid," *Symmetry*, vol. 14, no. 4, p. 739, 2022.

[31] A. M. Wazwaz, "A reliable modification of Adomian decomposition method," *Applied Mathematics and Computation*, vol. 102, no. 1, pp. 77–86, 1999.

[32] R. Shah, H. Khan, M. Arif, and P. Kumam, "Application of Laplace Adomian decomposition method for the analytical solution of third-order dispersive fractional partial differential equations," *Entropy*, vol. 21, no. 4, p. 335, 2019.

[33] H. Thabet and S. Kendre, "New modification of Adomian decomposition method for solving a system of nonlinear fractional partial differential equations," *International Journal of Advances in Applied Mathematics and Mechanics*, vol. 6, pp. 1–13, 2019.