WILEY | Hindawi

Research Article

Some Novel Solutions to a Quadratically Damped Pendulum Oscillator: Analytical and Numerical Approximations

Alvaro H. Salas [ID],[1] **Wedad Albalawi,**[2] **M. R. Alharthi,**[3] and **S. A. El-Tantawy** [ID][4,5]

[1]*Department of Mathematics and Statistics, Universidad Nacional de Colombia, FIZMAKO Research Group, Bogota, Colombia*
[2]*Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia*
[3]*Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia*
[4]*Department of Physics, Faculty of Science, Port Said University, Port Said 42521, Egypt*
[5]*Research Center for Physics (RCP), Department of Physics, Faculty of Science and Arts, Al-Baha University, Al-Mikhwah, Saudi Arabia*

Correspondence should be addressed to Alvaro H. Salas; ahsalass@unal.edu.co

In this paper, some novel analytical and numerical techniques are introduced for solving and analyzing nonlinear second-order ordinary differential equations (ODEs) that are associated to some strongly nonlinear oscillators such as a quadratically damped pendulum equation. Two different analytical approximations are obtained: for the first approximation, the ansatz method with the help of Chebyshev approximate polynomial is employed to derive an approximation in the form of trigonometric functions. For the second analytical approximation, a novel hybrid homotopy with Krylov–Bogoliubov–Mitropolsky method (HKBMM) is introduced for the first time for analyzing the evolution equation. For the numerical approximation, both the finite difference method (FDM) and Galerkin method (GM) are presented for analyzing the strong nonlinear quadratically damped pendulum equation that arises in real life, such as nonlinear phenomena in plasma physics, engineering, and so on. Several examples are discussed and compared to the Runge–Kutta (RK) numerical approximation to investigate and examine the accuracy of the obtained approximations. Moreover, the accuracy of all obtained approximations is checked by estimating the maximum residual and distance errors.

1. Introduction

Duffing-type equation is one of the most important second-degree differential equations that is used to describe many different phenomena [1–6]. The Duffing equation can be used for describing a nonlinear oscillator with a cubic nonlinearity, and the standard form of this equation reads as $\ddot{x}(t) + f(x) = 0$, with $f(x) = \sum_{i=1}^{\infty} \alpha_i x^i$ being the only odd polynomial where $i = 1, 3, 5, \ldots$. George Duffing, a German engineer, is the first person who did arrive at this equation and used it in the study of many different oscillators [3]. He also prepared a book in this regard and explained in it many applications that use this equation in the interpretation of many natural phenomena. Since then, there has been a tremendous amount of research works done about this equation of motion and some related equations, including (un)damped Duffing oscillator $\ddot{x}(t) + \beta\dot{x}(t) + f(x) = 0$, forced Duffing oscillator $\ddot{x}(t) + f(x) = F(t)$, forced damped Duffing oscillator $\ddot{x}(t) + \beta\dot{x}(t) + f(x) = F(t)$, and many other oscillators with odd polynomials and complicated damping term [7–10]. Moreover, there is another type of oscillator that combines both odd and even polynomials, which is called the Helmholtz–Duffing (HD) oscillator $\ddot{x}(t) + f(x) + g(x) = 0$ (here, $f(x) = \sum_{i=1}^{\infty} \alpha_i x^i$ is only odd polynomial where $i = 1, 3, 5, \ldots$ and $g(x) = \sum_{i=2}^{\infty} \gamma_i x^i$ is only even polynomial where $i = 2, 4, 6, \ldots$) and some related oscillators such as (un)damped HD oscillator $\ddot{x}(t) + \beta\dot{x}(t) + f(x) + g(x) = 0$, forced HD oscillator

$\ddot{x}(t) + f(x) + g(x) = F(t)$, forced damped HD oscillator $\ddot{x}(t) + \beta\dot{x}(t) + f(x) + g(x) = F(t)$ and many other HD oscillators with complicated damping term and complicated polynomials [11–15]. All these oscillations have several applications in various fields of science, e.g., oscillations in electronic circuits, oscillations in different plasma models, pendulum oscillator, etc. Due to the importance of these equations, many studies have been conducted to find some analytical and numerical solutions to accurately describe the engineering and physical systems associated with these oscillations [17–19].

As a contribution to the literature, in this article, we present some novel analytical and numerical solutions to the complicated damped HD-type oscillator for a given arbitrary initial conditions by means of both elliptic (exact solution) and trigonometric functions (approximate solution). First, we follow the work of Sugie [20] where the author obtained the equation of motion of underwater pendulum and studied the stability of this oscillator. This equation is called the quadratically damped pendulum equation [20]:

$$\ddot{\theta} + 2\varepsilon\,\dot{\theta}|\dot{\theta}| + \omega_0^2 \sin\,\theta = 0, \tag{1}$$

where ε represents the coefficient of the damping term and ω_0 indicates restoring coefficient per unit of the moment of inertia. For small θ, equation (1) can be approximated as follows:

$$\ddot{\theta} + 2\varepsilon\dot{\theta}|\dot{\theta}| + \omega_0^2\theta = 0. \tag{2}$$

Numerous oscillators with quadrature damping have been investigated over a wide range of different fields [21–26]. There are many methods for solving nonlinear differential equations. There are many analytical and numerical methods that dealt with solving different differential equations, and some of these methods can be found in Refs. [28–34]. In this paper, we will consider four different methods for solving and analyzing equation (1). First, we will solve this equation using the effective ansatz method in order to find some analytical approximations. In the second method, the hybrid homotopy Krylov–Bogoliubov–Mitropolsky method (HKBMM) will be employed to find an approximate solution with high accuracy. On the other hand, two highly accurate numerical schemes which are called the finite difference method (FDM) and Galerkin Hats method (GHM) will be introduced for analyzing evolution equation (1).

2. Analytical Approximations

In this section, two different approximations will be obtained. For the first approximation, the ansatz method with the help of Chebyshev approximate polynomial is employed to obtain an approximation in trigonometric form. For the second approximation, the new HKBMM is introduced.

2.1. First Approach: Ansatz Method and Trigonometric Solution. Let us rewrite evolution equation (1) in the form of the initial value problem (i.v.p.):

$$\begin{cases} \ddot{\theta} + 2\varepsilon\,\dot{\theta}|\dot{\theta}| + \omega_0^2 \sin\,\theta = 0, \\ \theta(0) = \theta_0 \text{ and } \theta'(0) = \dot{\theta}_0. \end{cases} \tag{3}$$

Based on Chebyshev polynomial approximation, the value of $\sin\,\theta$ can be expanded as

$$\begin{aligned} \sin\,\theta &\approx \theta - \lambda\theta^3, \\ \dot{\theta}|\dot{\theta}| &\approx r_0\dot{\theta} + r_1\dot{\theta}^3, \text{for}|\dot{\theta}| \le M, \end{aligned} \tag{4}$$

where

$$\begin{aligned} r_0 &= \frac{M}{2}\sqrt{1 - \frac{1}{\sqrt{2}}}, \\ r_1 &= \frac{1}{M}\sqrt{2 - \sqrt{2}}, \\ \lambda &= \frac{2}{13}. \end{aligned} \tag{5}$$

Other possible choices for (r_0, r_1, λ) can be considered as

$$\begin{aligned} r_0 &= \frac{5M}{16}, \\ r_1 &= \frac{3}{5}48M, \\ \lambda &= \frac{1}{6}. \end{aligned} \tag{6}$$

For $(\omega_0, \dot{\theta}_0) = (1, 0)$, the following approximation is obtained:

$$M = -0.900775\varepsilon\theta_0 + 0.240105\varepsilon + 0.905583\theta_0. \tag{7}$$

Next, we replace the original i.v.p. (3) by the following approximate i.v.p.

$$\begin{cases} \mathbb{R} \equiv \ddot{\theta} + 2\varepsilon\left(r_0\dot{\theta} + r_1\dot{\theta}^3\right) + \omega_0^2\left(\theta - \lambda\theta^3\right) = 0, \\ \theta(0) = \theta_0\theta'(0) = \dot{\theta}_0. \end{cases} \tag{8}$$

Assume that the solutions to the i.v.p. (8) have the following formulas:

$$\begin{aligned} \theta_{\text{Trigon}} &= c_0\exp\left(-\rho t\right)\cos\left(f(t) + \cos^{-1}\left(\frac{\theta_0}{c_0}\right)\right), \\ \theta_{\text{Trigon}} &= c_0\exp\left(-\rho t\right)\sin\left(f(t) + \sin^{-1}\left(\frac{\theta_0}{c_0}\right)\right), \end{aligned} \tag{9}$$

with the initial conditions (ICs)

$$\begin{aligned} f(0) &= 0, \\ \theta(0) &= \theta_0, \\ \theta'(0) &= \dot{\theta}_0, \end{aligned} \tag{10}$$

where the number ρ and the function $f \equiv f(t)$ are chosen in order to get the smallest possible residual error $R(t)$:

$$R(t) = \theta'' + 2\varepsilon\,\theta'\left|\theta'\right| + \omega_0^2\left(\theta - \lambda\theta^3\right). \tag{11}$$

Now by substituting ansatz (9) into the i.v.p. (8), we get

$$\mathbb{R} = \mathsf{F}_1\ \sin(\phi) + \mathsf{F}_2\cos(\phi) + \mathsf{F}_3\sin(3\phi) \\ + \mathsf{F}_4\cos(3\phi),\ \sqrt{b^2 - 4ac}, \tag{12}$$

with

$$\mathsf{F}_1 = \frac{1}{2}c_0 e^{-3\varepsilon\rho t}\left[-3c_0^2\varepsilon\left(f'\right)^3 r_1 - \left(3c_0^2\varepsilon^3\rho^2 r_1 + 4\varepsilon r_0 e^{2\varepsilon\rho t}\right.\right.$$
$$\left.\left. - 4\varepsilon\rho e^{2\varepsilon\rho t}\right)f' - 2f''e^{2\varepsilon\rho t}\right],$$

$$\mathsf{F}_2 = \frac{1}{4}c_0 e^{-3\varepsilon\rho t}\left[\begin{array}{c}-\left(6c_0^2\varepsilon^2\rho r_1 + 4e^{2\varepsilon\rho t}\right)\left(f'\right)^2 - 3c_0^2\lambda\omega_0^2 - 6c_0^2\varepsilon^4\rho^3 r_1 \\ -8\varepsilon^2\rho r_0 e^{2\varepsilon\rho t} + 4\varepsilon^2\rho^2 e^{2\varepsilon\rho t} + 4\omega_0^2 e^{2\varepsilon\rho t}\end{array}\right],$$

$$\mathsf{F}_3 = -\frac{1}{2}c_0^3\varepsilon r_1 e^{-3\varepsilon\rho t}\left(3\varepsilon^2\rho^2 - \left(f'\right)^2\right)f',$$

$$\mathsf{F}_4 = -\frac{1}{4}c_0^3 e^{-3\varepsilon\rho t}\left(-6\varepsilon^2\rho r_1\left(f'\right)^2 + \lambda\omega_0^2 + 2\varepsilon^4\rho^3 r_1\right), \tag{13}$$

where $\phi = f(t) + \cos^{-1}(\theta_0/c_0)$.

For $\mathsf{F}_2 = 0$, we get

$$\left(f'\right)^2 = \frac{4e^{2\varepsilon\rho t}\left(\varepsilon^2\ \rho\left(\rho - 2r_0\right) + \omega_0^2\right) - 3c_0^2\left(\lambda\omega_0^2 + 2\varepsilon^4\rho^3 r_1\right)}{6c_0^2\varepsilon^2\rho r_1 + 4e^{2\varepsilon\rho t}}$$

$$= A + \frac{B}{1 + C\ \exp\left(2\rho\varepsilon t\right)}, \tag{14}$$

with

$$A = \varepsilon^2\rho\left(\rho - 2r_0\right) + \omega_0^2,$$

$$B = -2\varepsilon^2\rho^2 - \omega_0^2\left(\frac{\lambda}{2\varepsilon^2\rho r_1} + 1\right) + 2\varepsilon^2\rho r_0, \tag{15}$$

$$C = \frac{2}{3c_0^2\varepsilon^2\rho r_1}.$$

Integrating equation (14) leads to

$$f = F(t) - F(0), \tag{16}$$

with

$$F(t) = \frac{1}{\varepsilon\rho}\left[\sqrt{A}\tanh^{-1}\left(\frac{\sqrt{A + B/Ce^{2\varepsilon\rho t} + 1}}{\sqrt{A}}\right)\right.$$
$$\left. - \sqrt{A + B}\tanh^{-1}\left(\frac{\sqrt{A + B/Ce^{2\varepsilon\rho t} + 1}}{\sqrt{A + B}}\right)\right]. \tag{17}$$

Inserting the value of f given in equation (16) into ansatz (9) and applying the ICs $\theta'(0) = \theta_0$, the value of c_0 can be determined from the following quartic equation:

$$4\left(2\varepsilon^2\rho^2\theta_0^2 - 2\varepsilon^2\rho r_0\theta_0^2 + \theta_0^2\omega_0^2 + 2\varepsilon\rho\theta_0\dot{\theta}_0 + \dot{\theta}_0^2\right)$$
$$+ \left(-4\varepsilon^2\rho^2 + 8\varepsilon^2\rho r_0 - 4\omega_0^2 - 3\lambda\theta_0^2\omega_0^2\right.$$
$$\left. + 12\varepsilon^3\rho^2 r_1\theta_0\dot{\theta}_0 + 6\varepsilon^2\rho r_1\dot{\theta}_0^2\right)c_0^2$$
$$+ 3\left(2\varepsilon^4\rho^3 r_1 + \lambda\omega_0^2\right)c_0^4 = 0. \tag{18}$$

Solution (9) θ_{Trigon} is presented in Figures 1(a) and 1(b) for $\theta_0 = 0$ and $\theta_0 = \pi/6$ at $(\varepsilon, \omega_0, \theta_0) = (0.2, 1, 0.1)$. Moreover, in the same figure, solution (9) is compared to the RK numerical approximation and the maximum distance error according to the following relation is calculated:

$$\text{Error} \equiv L_\infty = \max_{0\leq t\leq T}\left|\text{RK} - \theta_{\text{Trigon}}\right|. \tag{19}$$

The maximum distance error according to relation (19) for $\theta_0 = 0$ and $\theta_0 = \pi/6$ at $(\varepsilon, \omega_0, \theta_0) = (0.2, 1, 0.1)$ is, respectively, estimated as

$$L_\infty|_{\theta_0=0} = 0.00515945,$$

$$L_\infty|_{\theta_0 = \frac{\pi}{6}} = 0.0511852. \tag{20}$$

It is noted that the accuracy of solution (9) becomes good and acceptable for small θ_0, but for large value of θ_0, the accuracy of solution (9) reduces as shown in Figure 1.

Also, the maximum residual error is defined as

$$E_T(\theta) = \max_{0\leq t\leq T}\left|\theta''(t) + 2\varepsilon\,\theta'(t)\left|\theta'(t)\right| + \omega_0^2\sin(\theta(t))\right|. \tag{21}$$

This is another form for the error to check the accuracy of the obtained approximations.

2.2. Second Approach: HKBMM. Let us consider the i.v.p.

$$\left\{\ddot{x} + \omega_0^2 x = F(t, x, \dot{x}), x(0) = x_0 \text{ and } x'(0) = \dot{x}_0 \text{ for } 0\leq t\leq T.\right. \tag{22}$$

Suppose that the physical problem described by (22) involves some small parameters $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$. Let $x \equiv x(t)$ be the solution to the i.v.p. (22) and assume that

$$F(t, x, \dot{x}) \equiv 0 \text{ when } \varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_r = 0. \tag{23}$$

The solution $x \equiv x(t)$ depends not only on t but also on the parameters $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$, so that we can rewrite $x \equiv x(t)$ as

$$x = x(t; \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r). \tag{24}$$

Let us multiply each parameter by some other parameter p and consider the following p–parametric solution:

$$x_p = x_p(t; p\varepsilon_1, p\varepsilon_2, \ldots, p\varepsilon_r). \tag{25}$$

Accordingly, the function x_p may be written in a power series as follows:
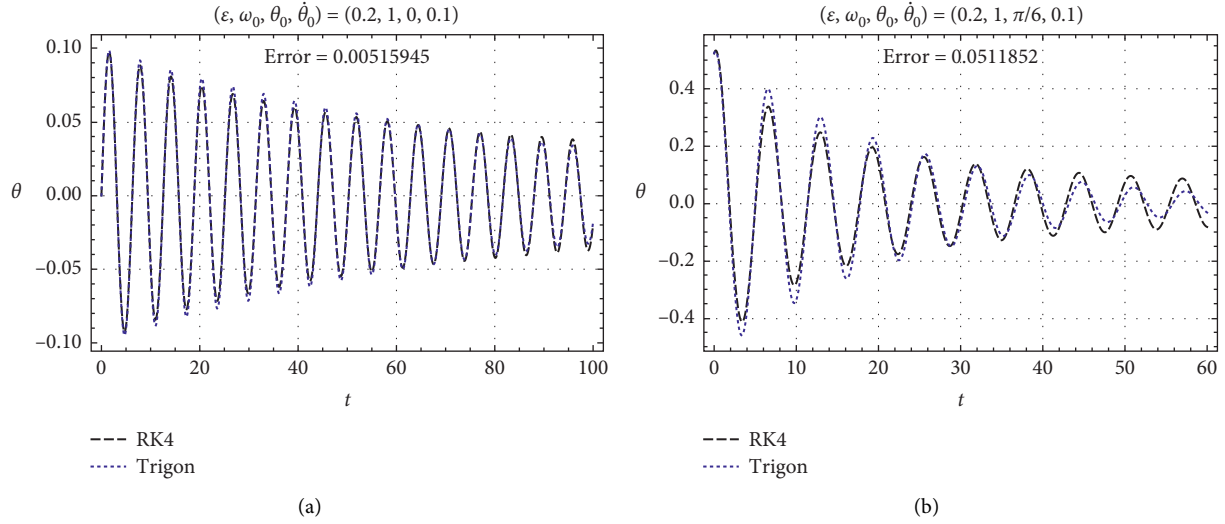
$$x_p = u_0 + pu_1 + p^2 u_2 + \cdots, \tag{26}$$

Figure 1: Both trigonometric solution (9) and RK numerical approximation are plotted in (θ, t)–plane.

where u_k depends on t only, say $u_k = u_k(t)$.

Based on Krylov–Bogoliubov–Mitropolsky method (KBMM), the solution of equation (22) is assumed to be

$$x_p = a \cos(\psi) + \sum_{n=1}^{N} p^n u_n(a, \psi) + O(p^{N+1}), \qquad (27)$$

where each u_n is a periodic function of ψ and a and ψ are assumed to vary with time according to

$$\frac{\mathrm{d}a}{\mathrm{d}t} \equiv \dot{a} = \sum_{n=1}^{N} p^n A_n(a) + O(p^{N+1}), \qquad (28)$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} \equiv \dot{\psi} = \omega_0 + \sum_{n=1}^{N} p^n \psi_n(a) + O(p^{N+1}), \qquad (29)$$

where $a \equiv a(t)$ and $\psi \equiv \psi(t)$.

Moreover, the hybrid homotopy KBMM (HKBMM) is suggested to be

$$H(x_p, t) = \ddot{x}_p + \omega_0^2 x_p - pF(t, x_p, \dot{x}_p). \qquad (30)$$

The next step is to write the residual $H_p(x, t)$ as a power series in p:

$$H(x_p, t) = \ddot{x}_p + \omega_0^2 x_p + p\Upsilon_1 + p^2\Upsilon_2 + p^3\Upsilon_3 + \cdots. \qquad (31)$$

For the determination of the unknown functions u_n, ψ_n, A_n, and a, we equate to zero the coefficients Υ_n in equation (22) and then we can get a system of ODEs. To avoid the so-called secularity, we choose only the solutions that do not

contain $\cos \psi$ nor $\sin \psi$. For $N = 2$ (the first approximation), we may use the following formulas (we neglected all terms containing p^j for $j \geq 2$):

$$\dot{x} = p\left(\omega_0 u_{1,\psi} - a\psi_1 \sin(\psi) + A_1 \cos(\psi)\right) - a\omega_0 \sin(\psi),$$

$$x = p\left(\omega_0^2 u_{1,\psi,\psi} - 2a\psi_1\omega_0 \cos(\psi)\right.$$
$$\left. -2A_1\omega_0 \sin(\psi)\right) - a\omega_0^2 \cos(\psi),$$

$$x^2 = \frac{1}{2}a^2 \cos(2\psi) + \frac{a^2}{2} + 2apu_1 \cos(\psi),$$

$$x^3 = \frac{3}{4}a^3 \cos(\psi) + \frac{1}{4}a^3 \cos(3\psi) + p\left(\frac{3}{2}a^2 u_1 \cos(2\psi) + \frac{3a^2 u_1}{2}\right),$$

$$x^4 = \frac{1}{2}a^4 \cos(2\psi) + \frac{1}{8}a^4 \cos(4\psi) + \frac{3a^4}{8}$$
$$+ p\left(3a^3 u_1 \cos(\psi) + a^3 u_1 \cos(3\psi)\right),$$

$$x^5 = \frac{5}{8}a^5 \cos(\psi) + \frac{5}{16}a^5 \cos(3\psi) + \frac{1}{16}a^5 \cos(5\psi)$$
$$+ p\left(\frac{5}{2}a^4 u_1 \cos(2\psi) + \frac{5}{8}a^4 u_1 \cos(4\psi) + \frac{15a^4 u_1}{8}\right),$$

$$(32)$$

and

$$x\dot{x} = p\left(\begin{array}{c} a\omega_0 \cos(\psi)u_{1,\psi} + a^2\psi_1 \sin(\psi)(-\cos(\psi)) \\ \\ +\dfrac{1}{2}aA_1 \cos(2\psi) + \dfrac{aA_1}{2} - au_1\omega_0 \sin(\psi) \end{array} \right), -a^2\omega_0 \sin(\psi)\cos(\psi),$$

$$\dot{x}^2 = p \left(\begin{array}{c} -2a\omega_0^2 \sin(\psi)u_{1,\psi} + a^2\psi_1\omega_0 \\ +a^2\psi_1\omega_0(-\cos(2\psi)) - aA_1\omega_0\sin(2\psi) \end{array} \right) - \frac{1}{2}a^2\omega_0^2\cos(2\psi) + \frac{1}{2}a^2\omega_0^2,$$

$$\dot{x}^3 = p \left(\begin{array}{c} \frac{3}{2}a^2\omega_0^3 u_{1,\psi} - \frac{3}{2}a^2\omega_0^3\cos(2\psi)u_{1,\psi} \\ -\frac{9}{4}a^3\psi_1\omega_0^2\sin(\psi) + \frac{3}{4}a^3\psi_1\omega_0^2\sin(3\psi) \\ +\frac{3}{4}a^2 A_1\omega_0^2\cos(\psi) - \frac{3}{4}a^2 A_1\omega_0^2\cos(3\psi) \end{array} \right) - \frac{3}{4}a^3\omega_0^3\sin(\psi) + \frac{1}{4}a^3\omega_0^3\sin(3\psi),$$

$$x^2\dot{x} = p \left(\begin{array}{c} \frac{1}{2}a^2\omega_0 u_{1,\psi} + \frac{1}{2}a^2\omega_0\cos(2\psi)u_{1,\psi} - \frac{1}{4}a^3\psi_1\sin(\psi) \\ -\frac{1}{4}a^3\psi_1\sin(3\psi) + \frac{3}{4}a^2 A_1\cos(\psi) \\ +\frac{1}{4}a^2 A_1\cos(3\psi) - a^2 u_1\omega_0\sin(2\psi) \end{array} \right) - \frac{1}{4}a^3\omega_0\sin(\psi) - \frac{1}{4}a^3\omega_0\sin(3\psi), \tag{33}$$

with

$$\begin{aligned} u_{1,\psi,\psi} &= \partial_\psi^2 u_1, \\ u_{1,\psi} &= \partial_\psi u_1. \end{aligned} \tag{34}$$

The approximate analytical solution is obtained by putting $p = 1$. However, we may keep the parameter p and then we may use it as a residual minimization parameter. The optimal value to p will be near $p = 1$.

Now, the proposed method can be applied for investigating the i.v.p. (8):

$$\begin{cases} \ddot{\theta} + \omega_0^2\theta = F(t, \theta, \dot{\theta}), \\ \theta(0) = \theta_0 \text{ and } \theta'(0) = \dot{\theta}_0, \end{cases} \tag{35}$$

where in our case, $x = \theta$ and

$$F(t, \theta, \dot{\theta}) = -2\varepsilon\left(r_0\dot{\theta} + r_1\dot{\theta}^3 \right) + \lambda\omega_0^2\theta^3. \tag{36}$$

Observe that when $\varepsilon_1 = \varepsilon \longrightarrow 0$ and $\varepsilon_2 = \lambda \longrightarrow 0$, we get $F = 0$.

In equations (27)–(29), for $N = 1$ and $\lambda = 2/13$, we have

$$\begin{aligned} x_p &= a\cos(\psi) + pu_1(a, \psi), \\ \dot{a} &= A_1(a), \\ \dot{\psi} &= \omega_0 + \psi_1(a). \end{aligned} \tag{37}$$

The homotopy to equation (35) is written as

$$H_p(\theta, t) = \ddot{\theta}_p + \omega_0^2\theta_p - p\left[-2\varepsilon\left(r_0\dot{\theta} + r_1\dot{\theta}^3 \right) + \frac{2}{13}\omega_0^2\theta^3\right]. \tag{38}$$

The substitution of equation (37) into equation (38) leads to

$$H_p(\theta, t) = \left[\frac{1}{4}\left(4\omega_0^2 u_{1,\psi,\psi} + 4u_1\omega_0^2\right)\right.$$

$$+ \frac{1}{4}\left(-3a^3\lambda\omega_0^2 - 8a\psi_1\omega_0\right)\cos(\psi)$$

$$+ \frac{1}{4}\left(-6a^3\varepsilon r_1\omega_0^3 - 8a\varepsilon r_0\omega_0 - 8A_1\omega_0\right)\sin(\psi)$$

$$\left. -\frac{1}{4}a^3\lambda\omega_0^2\cos(3\psi) + \frac{1}{2}a^3\varepsilon r_1\omega_0^3\sin(3\psi)\right]p + \cdots. \tag{39}$$

We must have

$$\frac{1}{4}\left(4\omega_0^2 u_{1,\psi,\psi} + 4u_1\omega_0^2\right)$$

$$+ \frac{1}{4}\left(-3a^3\lambda\omega_0^2 - 8a\psi_1\omega_0\right)\cos(\psi)$$

$$+ \frac{1}{4}\left(-6a^3\varepsilon r_1\omega_0^3 - 8a\varepsilon r_0\omega_0 - 8A_1\omega_0\right)\sin(\psi)$$

$$- \frac{1}{4}a^3\lambda\omega_0^2\cos(3\psi) + \frac{1}{2}a^3\varepsilon r_1\omega_0^3\sin(3\psi) = 0. \tag{40}$$

The coefficients of $\cos(\psi)$ and $\sin(\psi)$ must be vanished to eliminate the secularity. Accordingly, we have

$$\begin{aligned} \psi_1(a) &= -\frac{3}{52}a^2\omega_0, \\ A_1(a) &= -\frac{3}{4}a^3\varepsilon r_1\omega_0^2 - a\varepsilon r_0. \end{aligned} \tag{41}$$

Thus, equation (40) reduces to

$$\frac{1}{4}\left(4\omega_0^2 u_{1,\psi,\psi} + 4u_1\omega_0^2\right) + \frac{1}{4}a^3\lambda\omega_0^2\cos(3\psi)$$

$$+ \frac{1}{2}a^3\varepsilon r_1\omega_0^3\sin(3\psi) = 0. \tag{42}$$

Solving equation (42), the following particular solution without any secularity terms is obtained:

$$u_1(a, \psi) = -\frac{1}{208}a^3\left(\cos(3\psi) - 13\varepsilon r_1\omega_0\sin(3\psi)\right). \tag{43}$$

From equations (37) and (41), the functions $a \equiv a(t)$ and $\psi \equiv \psi(t)$ can be determined:

$$\begin{cases} \dot{a} = -\dfrac{3}{4}\varepsilon r_1 \omega_0^2 a^3 - \varepsilon r_0 a, \\[4mm] \dot{\psi} = \omega_0 - \dfrac{3}{52}\omega_0 a^2. \end{cases} \tag{44}$$

By solving system (44), we get

$$a = \frac{2\sqrt{r_0/3r_1}}{\sqrt{\left(1 + 4r_0/3A^2 r_1 \omega_0^2\right)e^{2t\varepsilon r_0} - 1} \cdot \omega_0}, \tag{45}$$

and

$$\psi = \left(\omega_0 + \frac{r_0}{13\omega_0 r_1}\right)t + B - \frac{1}{26\varepsilon r_1 \omega_0}\log \left(e^{2\varepsilon \rho r_0 t} + \frac{3A^2 r_1 \omega_0^2}{4r_0}\left(e^{2t\varepsilon r_0 t} - 1\right)\right). \tag{46}$$

We finally get the analytical approximation in its first approximation ($p = 1$).

$$\theta = a\,\cos(\psi) - \frac{1}{208}a^3\left(\cos(3\psi) - 13\varepsilon r_1 \omega_0 \sin(3\psi)\right). \tag{47}$$

We now introduce three optimization parameters by replacing $(\varepsilon, \psi) = (\rho\varepsilon, \kappa\psi)$ and then the approximation (47) can be modified to be

$$\theta(p, \rho, \kappa) = a_\rho \cos(\psi_{\rho,\kappa}) - \frac{p}{208}a_\rho^3 \left(\cos(3\psi_{\rho,\kappa}) - 13\varepsilon r_1 \omega_0 \sin(3\psi_{\rho,\kappa})\right), \tag{48}$$

with

$$a_\rho = \frac{2\sqrt{r_0/3r_1}}{\sqrt{\left(1 + 4r_0/3A^2 r_1 \omega_0^2\right)e^{2t\varepsilon \rho r_0} - 1}\,\omega_0}, \tag{49}$$

and

$$\psi_{\rho,\kappa} = \kappa\left(\omega_0 t + B - \frac{\log\left(e^{2\varepsilon \rho r_0 t} + 3A^2 r_1 \omega_0^2/4r_0\left(e^{2t\varepsilon \rho r_0 t} - 1\right)\right) - 2\varepsilon \rho r_0 t}{26\varepsilon \rho r_1 \omega_0}\right). \tag{50}$$

The numbers ρ, κ, and p are free parameters that we choose in order to minimize the residual error

$$R(t) = \theta''(t) + 2\varepsilon \theta'(t)\left|\theta'(t)\right| + \omega_0^2 \sin \theta(t). \tag{51}$$

The default parameter values are $\rho = \kappa = p = 1$. The constants A and B are determined from the initial conditions (ICs) $\theta(0) = \theta_0$ and $\theta'(0) = \theta_0$.

Following the same procedure above, we can get some higher-order approximations. For example, for $N = 3$, the following solution is obtained:

$$\theta = a\,\cos(\psi) + S_1 a^3 + S_2 a^5 + S_3 a^7, \tag{52}$$

where the coefficients $S_{1,2,3}$ are defined in Appendix. The values of (a, ψ) associated to this solution can be determined from the following equations:

$$\dot{a} = -\varepsilon r_0 a - \frac{3}{52}\varepsilon\left(r_0 + 13r_1 \omega_0^2\right)a^3 - \frac{3\varepsilon\left(5r_0 - 16r_1 \omega_0^2 + 429\varepsilon^2 r_0 r_1^2 \omega_0^2\right)}{1664}a^5 - \frac{27\varepsilon r_1 \omega_0^2\left(21 + 2197\varepsilon^2 r_1^2 \omega_0^2\right)}{173056}a^7, \tag{53}$$

and

$$\dot{\psi} = \omega_0 - \frac{\varepsilon^2 r_0^2}{2\omega_0} + \frac{3\left(\varepsilon^2 r_0^2 - 2\omega_0^2\right)}{104\omega_0}a^2 + \frac{3\omega_0\left(494\varepsilon^2 r_0 r_1 + 507\varepsilon^2 r_1^2 \omega_0^2 - 5\right)}{10816}a^4 + \frac{3\omega_0\left(37687\varepsilon^2 r_1^2 \omega_0^2 - 41\right)}{2249728}a^6. \tag{54}$$

The approximate solution of the i.v.p. (35) using the HKBMM is introduced in Figures 2(a) and 2(b) for $\theta_0 = 0$ and $\theta_0 = \pi/6$. Also, the maximum distance error L_∞ is estimated for the two cases as follows:

$$L_\infty|_{\theta_0 = 0} = 0.00145602,$$
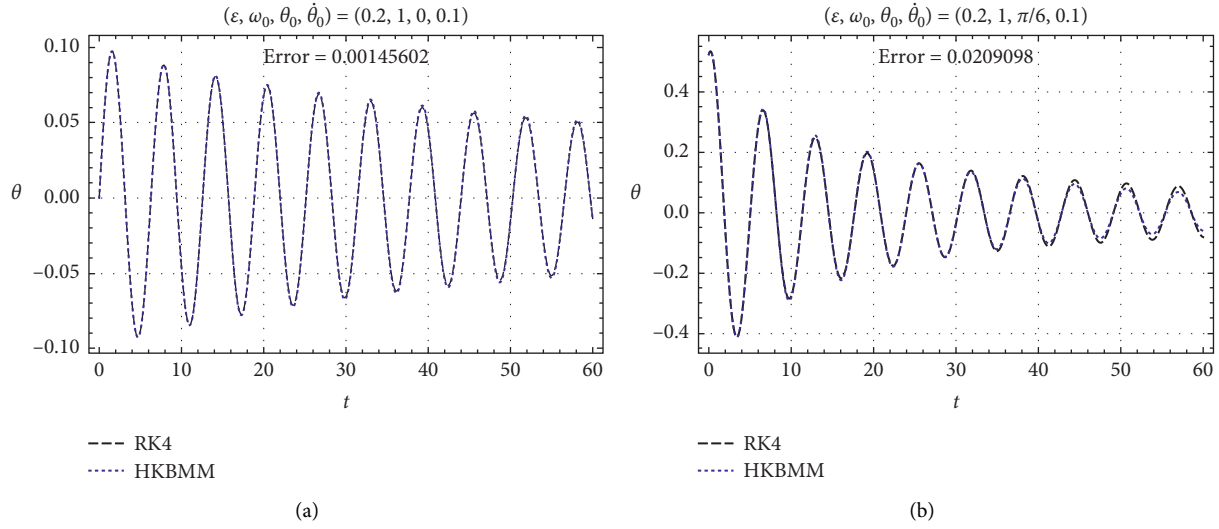$$L_\infty|_{\theta_0 = \frac{\pi}{6}} = 0.0209098. \tag{55}$$

FIGURE 2: The approximate solutions using both HKBMM and RK numerical method are plotted in $(\theta, t)-$ plane.

It is observed that approximate solution of the i.v.p. (35) using the HKBMM is characterized by high accuracy and more stability for long time and for arbitrary values of θ_0. Also, this approximation is better than the trigonometric solution (9) θ_{Trigon} as shown from Figures 1 and 2 as well as from the values of the errors.

3. Numerical Solution

In this section, some effective and highly accurate numerical schemes will be introduced for analyzing evolution equation (3). Both the finite difference method (FDM) and Galerkin Hats method (GHM) are presented below.

3.1. Numerical Approximation via FDM. First, let us discuss and apply the FDM on the general second-order ODE. Thus, the following general second-order is introduced:

$$\{ \ddot{x} = F(t, x, \dot{x}), x(0) = x_0, \ x'(0) = \dot{x}_0 \text{ and } 0 \le t \le T. \quad (56)$$

Choose some positive integer $n \ge 6$ and divide the interval $[0, T]$ into n-subintervals by means of the knots $t_i = ih$, where $h = T/n/ (i = 0, 1, 2, \dots, n)$. Then, the first and second-order derivatives can be approximated as follows:

$$x'(t_i) \approx \frac{-12x_{i-5} + 75x_{i-4} - 200x_{i-3} + 300x_{i-2} - 300x_{i-1} + 137x_i}{60h},$$

$$x''(t_i) \approx \frac{-10x_{i-5} + 61x_{i-4} - 156x_{i-3} + 214x_{i-2} - 154x_{i-1} + 45x_i}{12h^2}. \quad (57)$$

Consequently, the following discrete version to ODE (56) for $i = 5, 6, \dots$ is obtained:

$$\frac{-10x_{i-5} + 61x_{i-4} - 156x_{i-3} + 214x_{i-2} - 154x_{i-1} + 45x_i}{12h^2}$$

$$= F\left(t_i, x_i, \frac{-12x_{i-5} + 75x_{i-4} - 200x_{i-3} + 300x_{i-2} - 300x_{i-1} + 137x_i}{60h}\right). \quad (58)$$

The values of x_1, x_2, x_3, and x_4 are obtained from some numerical or approximate analytical solution to the i.v.p. (56). System (58) may be solved recursively.

The above algorithm can be applied for analyzing the i.v.p. (note here $\theta(t) \equiv x(t)$ without loss of generality):

$$\begin{cases} \ddot{\theta} = F(t, \theta, \dot{\theta}), \\ \theta(0) = \theta_0 \& \theta'(0) = \dot{\theta}_0, \end{cases} \quad (59)$$

with

$$F(t, \theta, \dot{\theta}) = -2\varepsilon \dot{\theta}|\dot{\theta}| - \omega_0^2 \sin \theta. \quad (60)$$

The numerical approximation using FDM is plotted in Figures 3 and 4 for different values of (n, θ_0). In Figures 3(a) and 3(b), the FDM numerical approximation is plotted against $n = 150$ and $n = 300$, respectively. Moreover, the effect of θ_0 on the numerical approximation is illustrated in Figures 4(a) and 4(b) for $\theta_0 = 0$ and $\theta_0 = \pi/6$, respectively. Furthermore, the maximum distance error is calculated for all mentioned cases as follows:
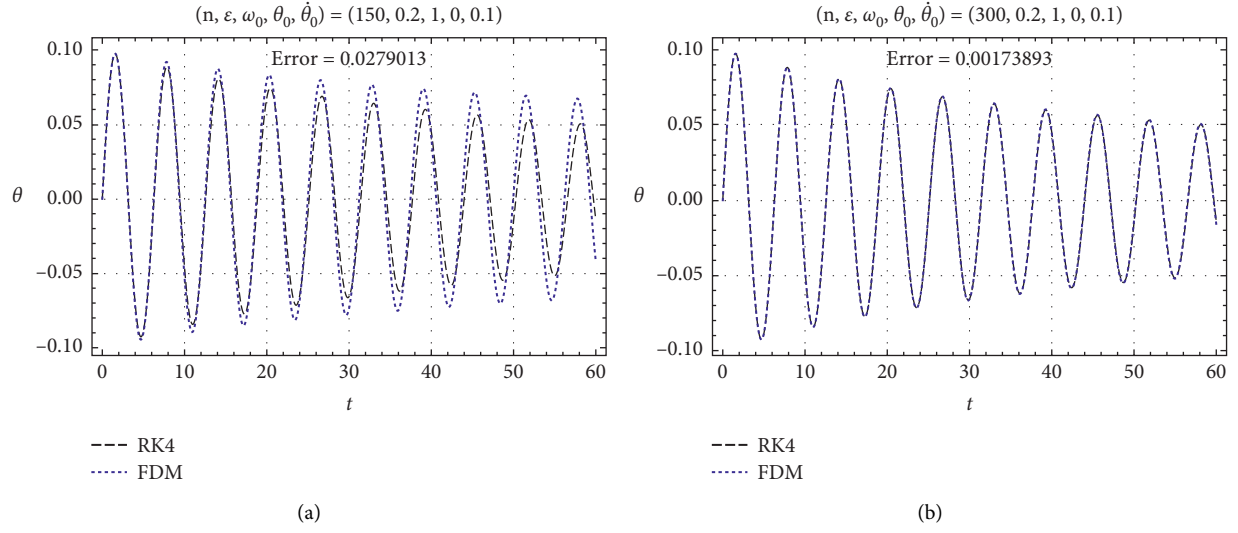
FIGURE 3: The approximate solutions using both FDM and RK numerical method are plotted in (θ, t)–plane for (a) $(n, \theta_0) = (150, 0)$ and (b) $(n, \theta_0) = (300, 0)$.
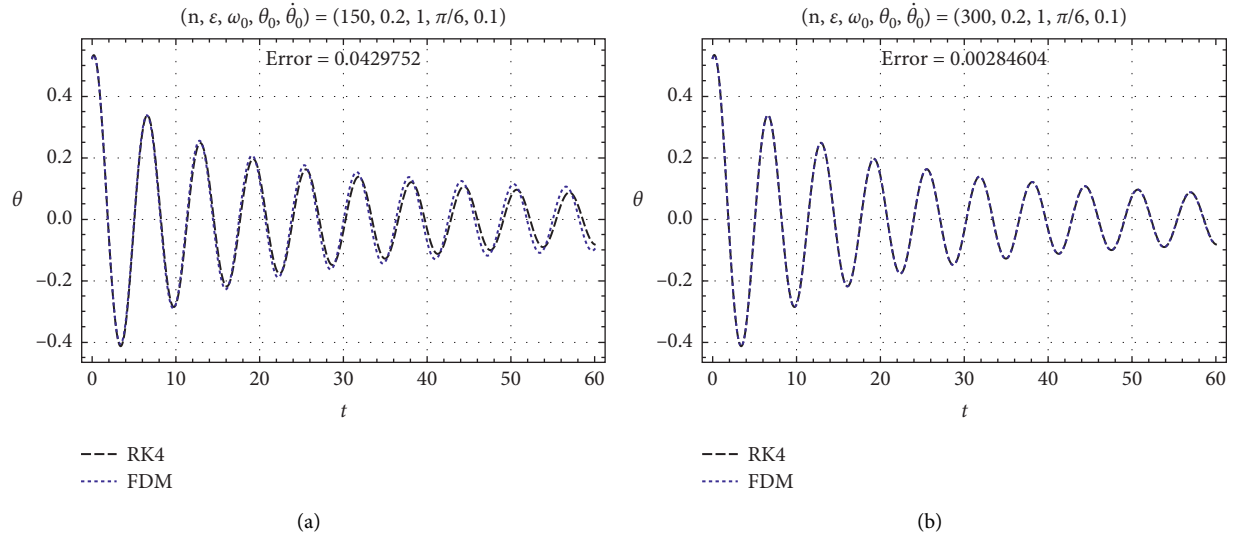

FIGURE 4: The approximate solutions using both FDM and RK numerical method are plotted in (θ, t)–plane for (a) $(n, \theta_0) = (150, \pi/6)$ and (b) $(n, \theta_0) = (300, \pi/6)$.

$$L_\infty = 0.0279013, \text{for} (n, \theta_0) = (150, 0),$$

$$L_\infty = 0.00173893, \text{for} (n, \theta_0) = (300, 0),$$

$$L_\infty = 0.0429752, \text{for} (n, \theta_0) = \left(150, \frac{\pi}{6}\right), \qquad (61)$$

$$L_\infty = 0.00284604, \text{for} (n, \theta_0) = \left(300, \frac{\pi}{6}\right).$$

It is clear that the accuracy of the FDM numerical approximation increases with increasing n. Also, this approximation is stable against the long time intervals and arbitrary angle θ_0. Moreover, this approximation is better than the trigonometric solution (9) θ_{Trigon}.

3.2. Numerical Approximation via Galerkin Hats Method. First, let us consider a polynomial second-order forced damped i.v.p.

$$\{ \ddot{x} + 2\varepsilon \dot{x} + P(x) = 0, x(0) = 0 \text{and} x'(0) = \dot{x}_0, \qquad (62)$$

where $x \equiv x(t)$.

Let us consider the i.v.p.

$$\{\ddot{y} + 2\varepsilon\dot{y} + P(y) = 0, y(0) = y_0 \text{ and } y'(0) = \dot{y}_0. \qquad (63)$$

Suppose that $x = x(t; c_1)$ is the solution to the i.v.p.

$$\{\ddot{x} + 2\varepsilon\dot{x} + P(x) = 0, x(0) = 0 \text{ and } x'(0) = c_1. \qquad (64)$$

Define

$$y(t) = x(t + c_0; c_1). \qquad (65)$$

The constants c_0 and c_1 are determined from the system

$$x(0 + c_0; c_1) = y_0 \text{ and } x'(0 + c_0; c_1) = \dot{y}_0. \qquad (66)$$

Thus, problemproblem (61) reduces to (60) by problem (63) reduces to (62).

Some particular cases to the i.v.p. (62) are defined as

$$\{\ddot{x} + 2\varepsilon\dot{x} + n(t) + p(t)x = f(t), \ddot{x} + 2\varepsilon\dot{x} + n(t) + p(t)x + q(t)x^2$$
$$= f(t), \ddot{x} + 2\varepsilon\dot{x} + n(t) + p(t)x + q(t)x^2 + r(t)x^3 = f(t).$$
$$(67)$$

We will use the same idea for the linear case.

Here, we start to discuss the linear case in system (67):

$$\{\ddot{x} + 2\varepsilon\dot{x} + p(t)x = 0, x(0) = 0 \text{ and } x'(0) = \dot{x}_0. \qquad (68)$$

An approximate solution to the i.v.p. (68) is assumed to be in the following ansatz form:

$$x = \sum_{k=1}^{n} c_k \varphi_k(t), \qquad (69)$$

where the functions $\varphi_k(t)$ are the so-called linear Galerkin hats.

Let us investigate the present problem in the interval $0 \le t \le T$ and by choosing some positive integer $n \ge 2$ and define the step $h = T/n$ and let $\xi_j = jh = jT/n$ for $j = 0, 1, 2, \cdots$. The functions $\varphi_k(t)$ for $k = 1, 2, \ldots, n$ are defined as

$$\varphi_k(t) = \begin{cases} c\dfrac{t - \xi_{k-1}}{h}, & \text{if } \xi_{k-1} \le t \le \xi_k, \\ \\ -\dfrac{t - \xi_{k+1}}{h}, & \text{if } \xi_k \le t \le \xi_{k+1}. \end{cases} \qquad (70)$$

For an illustration, see Figure 1.

Figure 1 Galerkin hats for $n = 7$ and $T = 5$.

Some properties of these functions can be illustrated as follows:

$$\varphi_j(t)\varphi_k(t)dt = 0, \qquad (71)$$

for $|j - k| \ge 2$ and $t \in [0, T]$, and

$$\int_0^T \varphi_j^p(t) = \frac{2T}{(p + 1)n}, \qquad (72)$$

for $j \ge 1$ and $p = 1, 2, 3, \ldots$.

In general, for $|j - k| = 1$ and $r, s = 0, 1, 2, 3, \cdots$, we have

$$\int_0^T \varphi_j^r(t)\varphi_k^s(t) = \frac{Tr!s!}{n(r + s + 1)!}. \qquad (73)$$

Using the formula

$$\int_0^T \varphi_j(t)x(t)^N dt = \frac{h}{(N + 1)(N + 2)}$$
$$\left(\sum_{k=0}^{N-1} (k + 1)\left(c_{j-1}^{N-k} + c_{j+1}^{N-k}\right)c_j^k + 2(N + 1)c_j^N \right),$$
$$(74)$$

for any $N \ge 0$ and $c_0 = c_{n+1} = 0$, and assuming that $a_j(t) \equiv a_j = \text{const}$, we may evaluate easily the following integration:

$$\int_0^T P(x)\varphi_j(t)dt = \sum_{N=0}^m a_N \int_0^T \varphi_j(t)x(t)^N dt \text{ for any } j,$$

$$\int_0^T x''(t)\varphi_j(t)dt = -\int_0^T x'(t)\varphi_j'(t)dt$$
$$= \frac{c_{j-1} - 2c_j + c_{j+1}}{h} \text{ for any } j,$$

$$\int_0^T x'(t)|x'(t)|\varphi_j(t)dt = \frac{n}{2T}\left[\sqrt{(c_{j-1} - c_{j-2})^2} + \sqrt{(c_j - c_{j-1})^2} \right].$$
$$(75)$$

Now, let us return to the original i.v.p. (3):

$$\begin{cases} \ddot{\theta} + 2\varepsilon\dot{\theta}|\dot{\theta}| + \omega_0^2 \sin \theta = 0, \\ \theta(0) = 0 \text{ and } \theta'(0) = \dot{\theta}_0. \end{cases} \qquad (76)$$

Using the Chebyshev approximation,

$$\sin \theta \approx \theta - \frac{4}{0}241\theta^3 + \frac{\theta^5}{131} \text{ for } |\theta| \le \frac{\pi}{2}. \qquad (77)$$

Another approximation may be obtained by minimizing the integral

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(a\theta + b\theta^3 + c\theta^5 - \sin \theta\right)^2 d\theta \longrightarrow \min. \qquad (78)$$

The minimization procedure yields the values

$$\min = \frac{\pi}{2} - \frac{3360\left(11975040 - 2661120\pi^2 + 171720\pi^4 - 2664\pi^6 + 13\pi^8\right)}{\pi^{11}} \approx 5.5 \times 10^{-9}, \qquad (79)$$

with

$$a = \frac{105\left(23760 - 2592\pi^2 + 19\pi^4\right)}{\pi^7} = 0.99977,$$

$$b = -\frac{2520\left(18480 + 17\pi^2\left(\pi^2 - 120\right)\right)}{\pi^9} = -0.16583,$$

$$c = \frac{166320\left(1008 - 112\pi^2 + \pi^4\right)}{\pi^{11}} = 7.5742 \times 10^{-3},$$

$$\sin \theta \approx \theta - \frac{1}{6}\theta^3 + \frac{\theta^5}{132}.$$

Then, i.v.p. (76) can be reduced to the following approximate i.v.p.

$$\begin{cases} \ddot{\theta} + 2\varepsilon \dot{\theta}|\dot{\theta}| + \omega_0^2\left(\theta - \frac{4}{0}241\theta^3 + \frac{\theta^5}{131}\right) = 0, \\ \\ \theta(0) = 0 \text{ and } \theta'(0) = \dot{\theta}_0. \end{cases} \tag{81}$$

Assume that the solution to i.v.p. (81) is given by

$$\theta = \sum_{k=1}^{n} c_k \varphi_k(t). \tag{82}$$

The Galerkin method solves the following algebraic system of nonlinear equations:

$$\frac{n}{T}\left(c_{j-2} - 2c_{j-1} + c_j\right) + \frac{n}{T}$$
$$\left(\sqrt{\left(c_{j-2} - c_{j-1}\right)^2} + \sqrt{\left(c_j - c_{j-1}\right)^2}\right)\varepsilon + \omega_0^2 S_j = 0, \tag{83}$$

with

$$S_j = \frac{T\omega_0^2}{5544n}c_j^5 + \frac{T\omega_0^2 c_{j-1}}{2772n}c_j^4 + \frac{T\omega_0^2\left(5c_{j-1}^2 - 77\right)}{9240n}c_j^3$$

$$+ \frac{T\omega_0^2 c_{j-1}\left(10c_{j-1}^2 - 231\right)}{13860n}c_j^2 + \frac{T\omega_0^2\left(25c_{j-1}^4 - 693c_{j-1}^2 + 4620\right)}{27720n}c_j$$

$$+ \frac{T\omega_0^2}{27720n}\left(\begin{array}{c} 5c_{j-2}^5 + 10c_{j-1}c_{j-2}^4 + 3\left(5c_{j-1}^2 - 77\right)c_{j-2}^3 \\ +\left(20c_{j-1}^3 - 462c_{j-1}\right)c_{j-2}^2 + \left(25c_{j-1}^4 - 693c_{j-1}^2 + 4620\right)c_{j-2} \\ +12c_{j-1}\left(5c_{j-1}^4 - 154c_{j-1}^2 + 1540\right) \end{array} \right), \tag{84}$$

for $j = 2, 3, \ldots$, where $c_0 = 0$ and $c_1 = T/n\dot{\theta}_0$. System (83) can be solved recursively.

For $j = 2$, the value of c_2 can be determined from the following quintic equation:

$$\frac{2310n^6\varepsilon\left|\dot{\theta}_0\right| - 4620n^6\dot{\theta}_0 \mp 2310n^6\varepsilon\dot{\theta}_0 + 1540n^4T^2\omega_0^2\dot{\theta}_0 - 154n^2T^4\omega_0^2\dot{\theta}_0^3 + 5T^6\omega_0^2\dot{\theta}_0^5}{2310n^6}$$

$$+ \frac{27720n^6 \pm 27720n^6\varepsilon + 4620n^4T^2\omega_0^2 - 693n^2T^4\omega_0^2\dot{\theta}_0^2 + 25T^6\omega_0^2\dot{\theta}_0^4}{27720n^5T}c_2 - \frac{T^2\omega_0^2\dot{\theta}_0\left(231n^2 - 10T^2\dot{\theta}_0^2\right)}{13860n^4}c_2^2 \tag{85}$$

$$- \frac{T\omega_0^2\left(77n^2 - 5T^2\dot{\theta}_0^2\right)}{9240n^3}c_2^3 + \frac{T^2\omega_0^2\dot{\theta}_0}{2772n^2}c_2^4 + \frac{T\omega_0^2}{5544n}c_2^5 = 0.$$

We choose the real root to equation (86) that is closest to c_1. Suppose we already found the values $c_2, c_3, \ldots, c_{k-1}$ for some $k \geq 2$. Then, the value of c_k is found by solving the quintic in equation (82) by letting $j = k$. However, since $\sqrt{(c_k - c_{k-1})^2} = \pm(c_k - c_{k-1})$, we must solve two quintics in c_k. We choose the real root to the two quintics that is closest to c_{k-1}. Let us introduce the quintic

$$\lambda_5 c_k^5 + \lambda_4 c_k^4 + \lambda_3 c_k^3 + \lambda_2 c_k^2 + \lambda_1 c_k + \lambda_0 = 0, \qquad (86)$$

Any of the two quintics (82) for $j = k$. Then, for sufficiently large n, the value of c_k may be estimated by means of the formula

$$
\begin{aligned}
&\left(\lambda_1 + 5\lambda_5 z_0^4 + 4\lambda_4 z_0^3 + 3\lambda_3 z_0^2 + 2\lambda_2 z_0\right) \\[4pt]
z \approx z_0 - &\frac{\left(\lambda_0 + \lambda_5 z_0^5 + \lambda_4 z_0^4 + \lambda_3 z_0^3 + \lambda_2 z_0^2 + \lambda_1 z_0\right)}{\lambda_1^2 - \lambda_0\lambda_2 + 15\lambda_5^2 z_0^8 + 24\lambda_4\lambda_5 z_0^7 + 10\lambda_4^2 z_0^6} \\[4pt]
&+ 17\lambda_3\lambda_5 z_0^6 + 15\lambda_3\lambda_4 z_0^5 + 9\lambda_2\lambda_5 z_0^5 + 6\lambda_3^2 z_0^4 \qquad (87) \\[4pt]
&+ 9\lambda_2\lambda_4 z_0^4 + 8\lambda_2\lambda_3 z_0^3 + 2\lambda_1\lambda_4 z_0^3 \\[4pt]
&- 10\lambda_0\lambda_5 z_0^3 + 3\lambda_2^2 z_0^2 + 3\lambda_1\lambda_3 z_0^2 \\[4pt]
&- 6\lambda_0\lambda_4 z_0^2 + 3\lambda_1\lambda_2 z_0 - 3\lambda_0\lambda_3 z_0
\end{aligned}
$$

where $z_0 = c_{k-1}$.

For arbitrary ICs, i.e., for any values to $(\theta_0, \dot{\theta}_0)$, the following ansatz is assumed:

$$\theta(t) = \theta_0 \varphi_0(t) + \frac{T}{n}\dot{\theta}_0 \varphi_2(t) + \sum_{k=2}^{n} c_k \varphi_k(t). \qquad (88)$$

Then, we get

$$\int_0^T \varphi_j(t)\theta''(t)\,dt = \frac{n\left(c_{j-2} - 2c_{j-1} + c_j\right)}{T}, \qquad (89)$$

and

$$
\begin{aligned}
2\varepsilon \int_0^T \varphi_j(t)\theta'(t)|\theta'(t)|\,dt = \frac{\varepsilon n}{T}\Big(&\left(c_j - c_{j-1}\right)^2 \mathrm{sgn}\left(c_j - c_{j-1}\right) \\
&- \left(c_{j-2} - c_{j-1}\right)^2 \mathrm{sgn}\left(c_{j-2} - c_{j-1}\right)\Big).
\end{aligned}
$$
$$(90)$$

Using the approximation

$$\sin\theta \approx \theta - \frac{1}{6}\theta^3 + \frac{1}{132}\theta^5, \qquad (91)$$

gives

$$\int_0^T \varphi_j(t)\sin(\theta(t))\,dt \approx \frac{T\omega_0^2}{27720n}\left(\begin{array}{l} 5c_{j-2}^5 - 231c_{j-2}^3 + 4620c_{j-2} + 60c_{j-1}^5 + 5c_j^5 - 231c_j^3 + 4620c_j + \\[4pt] 25c_{j-1}^4\left(c_{j-2} + c_j\right) + 4c_{j-1}^3\left(5c_{j-2}^2 + 5c_j^2 - 462\right) + \\[4pt] 3c_{j-1}^2\left(5c_{j-2}^3 - 231c_{j-2} + 5c_j^3 - 231c_j\right) + \\[4pt] 2c_{j-1}\left(5c_{j-2}^4 - 231c_{j-2}^2 + 5c_j^4 - 231c_j^2 + 9240\right) \end{array}\right). \qquad (92)$$

The following definitions are used in our analysis:

$$c_0 = \theta_0,$$

$$c_1 = \theta_0 + \frac{T}{n}\dot{\theta}_0, \qquad (93)$$

$$c_j = 0 \text{ for } j < 0 \text{ or } j > n.$$

The algebraic system to be solved for $j = 2, 3, \ldots, n$ reads as

$$
\frac{n\left(c_{j-2} - 2c_{j-1} + c_j\right)}{T} + \frac{\varepsilon n}{T}\left(\left(c_j - c_{j-1}\right)^2 \mathrm{sgn}\left(c_j - c_{j-1}\right) - \left(c_{j-2} - c_{j-1}\right)^2 \mathrm{sgn}\left(c_{j-2} - c_{j-1}\right)\right)
$$
$$
+ \frac{T\omega_0^2}{27720n}\left(\begin{array}{l} 5c_{j-2}^5 - 231c_{j-2}^3 + 4620c_{j-2} + 60c_{j-1}^5 + 5c_j^5 - 231c_j^3 + 4620c_j + \\[4pt] 25c_{j-1}^4\left(c_{j-2} + c_j\right) + 4c_{j-1}^3\left(5c_{j-2}^2 + 5c_j^2 - 462\right) + \\[4pt] 3c_{j-1}^2\left(5c_{j-2}^3 - 231c_{j-2} + 5c_j^3 - 231c_j\right) + \\[4pt] 2c_{j-1}\left(5c_{j-2}^4 - 231c_{j-2}^2 + 5c_j^4 - 231c_j^2 + 9240\right) \end{array}\right) = 0. \qquad (94)
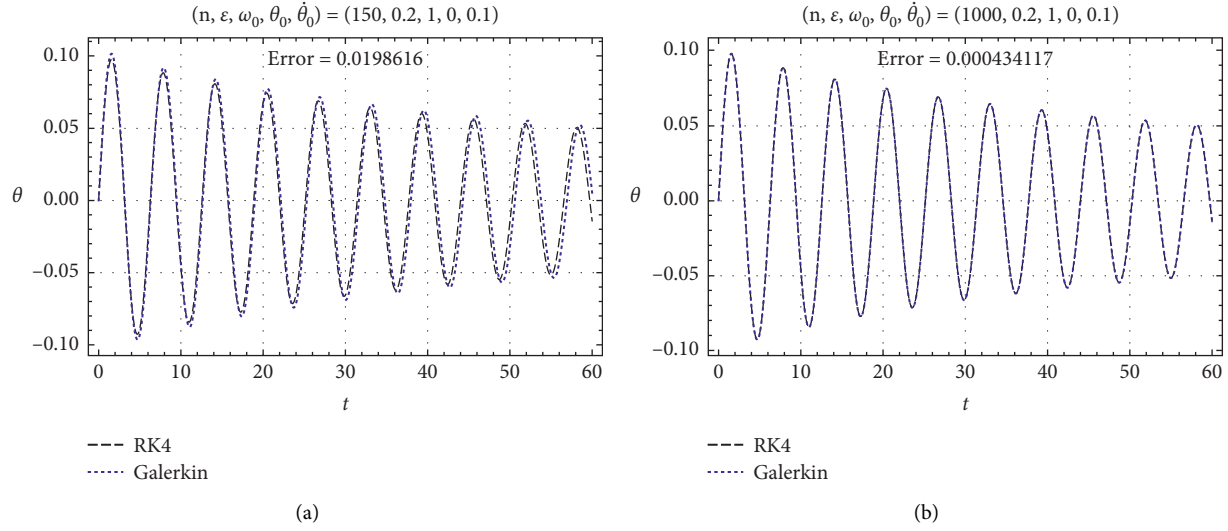$$

FIGURE 5: The approximate solutions using both GM and RK numerical method are plotted in (θ, t)−plane for (a) $(n, \theta_0) = (150, 0)$ and (b) $(n, \theta_0) = (1000, 0)$.
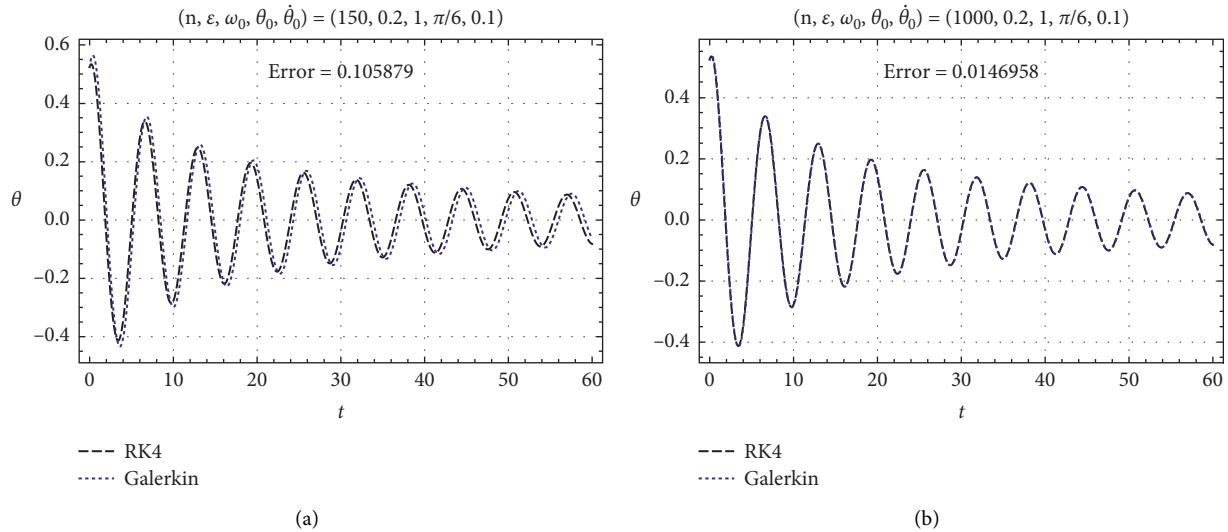

FIGURE 6: The approximate solutions using both GM and RK numerical method are plotted in (θ, t)−plane for (a) $(n, \theta_0) = (150, \pi/6)$ and (b) $(n, \theta_0) = (1000, \pi/6)$.

The Galerkin numerical approximation to i.v.p. (3) is introduced in Figures 5 and 6 for different values of $(n, \theta_0, \dot{\theta}_0)$. Also, the maximum distance error is estimated for different values of $(n, \theta_0, \dot{\theta}_0)$ as follows:

$$L_\infty = 0.0198616, \quad \text{for } (n, \theta_0) = (150, 0),$$

$$L_\infty = 0.000434117, \quad \text{for } (n, \theta_0) = (1000, 0),$$

$$L_\infty = 0.105879, \quad \text{for } (n, \theta_0) = \left(150, \frac{\pi}{6}\right), \tag{95}$$

$$L_\infty = 0.0146958, \quad \text{for } (n, \theta_0) = \left(1000, \frac{\pi}{6}\right).$$

It is clear that increasing the number of hats n leads to the increase of approximation accuracy, i.e., the error shrinks with the enhancement of the number of hats n. Moreover, it is observed that the Galerkin numerical approximation is characterized by high accuracy compared to RK numerical approximation.

4. Conclusion

In this paper, the quadratically damped pendulum equation with strong nonlinearity has been solved and analyzed using some novel and effective analytical and numerical techniques. In the beginning, the ansatz method was devoted to

find an analytical approximation to the mentioned equation in the form of trigonometric functions. Also, in this study, for the first time, a new hybrid homotopy with Krylov–Bogoliubov–Mitropolsky method (HKBMM) was applied for analyzing the evolution equation and deriving an analytical approximation with high accuracy. Moreover, both the finite difference method (FDM) and Galerkin method (GM) were employed for analyzing the present evolution equation as well as some related oscillators. The obtained approximations were graphically compared with each other. Furthermore, the maximum residual error for each approximation was estimated. In the GM, we derived the iterative schemes for finding the coefficients that appear in the linear Galerkin hat combination in the ansatz form solution for the evolution equation. These coefficients may be found iteratively. It was found that the numerical approximations are more accurate than analytical ones, but both give good accuracy. Also, it was observed that the obtained results become reasonably good for small initial

speed. One of the most important features of Galerkin method is that it gives more stable approximations for any values to the physical parameters and for long time. Thus, this method can be devoted for studying and investigating different pendulum oscillators for any nonlinearity [16, 27].

4.1. Future Work.
There are many oscillators that may be solved using the proposed method. Examples include

$$
\begin{aligned}
&\ddot{x} + 2\varepsilon\dot{x} + \alpha x + \beta x^2 + \gamma x^3 = F\cos(\Omega t),\\
&x + 2\varepsilon\dot{x} + \alpha x + \beta x + \gamma x^5 = F\cos(\Omega t),\\
&x - \varepsilon\left(1 - x^2\right)\dot{x} + \alpha x + \beta x^2 + \gamma x^3 = F\cos(\Omega t),\\
&x + 2\varepsilon\dot{x} + \left(\alpha - Q_0\cos(\gamma t)\right)\sin(x) = F\cos(\Omega t).
\end{aligned}
\tag{96}
$$

Appendix

The coefficients $S_{1,2,3}$ of solution (46):

$$
S_1 = -\frac{1}{208}\left(\cos(3\psi) - 13\varepsilon\,\sin(3\psi)r_1\omega_0\right) + \frac{312\varepsilon r_0}{43264\omega_0}\left(-\sin(3\psi) + 13\varepsilon\,\cos(3\psi)r_1\omega_0\right)
$$

$$
- \frac{97344\varepsilon^2 r_0^2}{26996736\omega_0^2}\left(-\cos(3\psi) + 13\varepsilon\,\sin(3\psi)r_1\omega_0\right),
$$

$$
S_2 = \frac{1}{43264}\begin{pmatrix} -52\varepsilon(6\,\sin(3\psi) + \sin(5\psi))r_1\omega_0 \\ +\cos(5\psi)\left(1 - 507\varepsilon^2 r_1^2\omega_0^2\right) \\ +3\,\cos(3\psi)\left(-7 + 1521\varepsilon^2 r_1^2\omega_0^2\right) \end{pmatrix} + \frac{104\varepsilon r_0}{26996736\omega_0}\begin{pmatrix} -567\,\sin(3\psi) + 19\,\sin(5\psi) \\ +5616\varepsilon\,\cos(3\psi)r_1\omega_0 + 52\varepsilon\,\cos(5\psi)r_1\omega_0 \\ +31941\varepsilon^2\,\sin(3\psi)r_1^2\omega_0^2 + 8619\varepsilon^2\,\sin(5\psi)r_1^2\omega_0^2 \end{pmatrix},
$$

$$
S_3 = \frac{3}{26996736}\begin{pmatrix} -\cos(7\psi) - 10647\varepsilon\,\sin(3\psi)r_1\omega_0 - 195\varepsilon\,\sin(5\psi)r_1\omega_0 \\ +130\varepsilon\,\sin(7\psi)r_1\omega_0 + 3549\varepsilon^2\,\cos(7\psi)r_1^2\omega_0^2 \\ +731601\varepsilon^3\,\sin(3\psi)r_1^3\omega_0^3 + 336141\varepsilon^3\,\sin(5\psi)r_1^3\omega_0^3 \\ -26364\varepsilon^3\,\sin(7\psi)r_1^3\omega_0^3 + \cos(5\psi)\left(43 + 3211\varepsilon^2 r_1^2\omega_0^2\right) \\ +3\,\cos(3\psi)\left(-139 + 8957\varepsilon^2 r_1^2\omega_0^2\right) \end{pmatrix}.
\tag{A.1}
$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and approved the final version of the manuscript.

Acknowledgments

References

[1] N. Nayfeth and D. T. Mook, *Non-linear Oscillations*, John Wiley, New York, NY, USA, 1973.

[2] I. Kovacic and M. J. Brennan, *The Duffing Equation: Nonlinear Oscillators and Their Behaviour*, John Wiley & Sons, New Jersey, NJ, USA, 2011.

[3] G. Duffing, *Erzwungene Schwingungen bei veranderlicher Eigenfrequenz und ihre technische Bedeutung*, Vieweg & Sohn, Braunschweig, Germany, 1918.

[4] A. H. Salas, S. A. El-Tantawy, and J. H Castillo, "Analytical solution to the damped cubic-quintic duffing equation,"

International Journal of Mathematics and Computer Science, vol. 17, no. 1, pp. 425–430, 2022.

[5] A. H. Salas, S. A. El-Tantawy, and E. C. H. Jairo, "On the approximate and analytical solutions to the fifth-order Duffing oscillator and its physical applications," *Waves in Random and Complex Media,* pp. 1–21, 2021.

[6] A. H. Salas and A. E. El-Tantawy, "Analytical solutions of some strong nonlinear oscillators," *Optimization Problems in Engineering [Working Title],* Springer, Berlin, Germnay, 2021.

[7] S. A. E. Tantawy, A. H. Salas, and M. R. Alharthi, "On the analytical solutions of the forced damping duffing equation in the form of weierstrass elliptic function and its applications," *Mathematical Problems in Engineering,* vol. 2021, Article ID 6678102, 9 pages, 2021.

[8] N. H. Aljahdaly and S. A. El-Tantawy, "On the multistage differential transformation method for analyzing damping Duffing oscillator and its applications to plasma physics," *Mathematics,* vol. 9, no. 4, p. 432, 2021.

[9] S. Nourazar and A. Mirzabeigy, "Approximate solution for nonlinear Duffing oscillator with damping effect using the modified differential transform method," *Scientia Iranica B,* vol. 20, pp. 364–368, 2013.

[10] R. E. Mickens, "Mathematical and numerical study of the Duffing-harmonic oscillator," *Journal of Sound and Vibration,* vol. 244, no. 3, pp. 563–567, 2001.

[11] H. S. Alvaro, S. A. El-Tantawy, and M. R. Alharthi, "Novel solutions to the (un) damped Helmholtz-Duffing oscillator and its application to plasma physics: moving boundary method," *Physica Scripta,* vol. 96, Article ID 104003, 2021.

[12] H. S. Alvaro, S. A. El-Tantawy, and M. R. Alharthi, "A new approach for modelling the damped Helmholtz oscillator: applications to plasma physics and electronic circuits," *Communications in Theoretical Physics,* vol. 73, Article ID 035501, 2021.

[13] M. A. Hosen and M. S. H. Chowdhury, "Analytical approximate solutions for the helmholtz-duffing oscillator," *ARPN J. Eng. App. Sci.* vol. 10, Article ID 17363, 2015.

[14] Y. Geng, "Exact solutions for the quadratic mixed-parity Helmholtz-Duffing oscillator by bifurcation theory of dynamical systems," *Chaos, Solitons & Fractals,* vol. 81, pp. 68–77, 2015.

[15] D.-Q. Wei, X.-S. Luo, and S.-Y. Zeng, "Noise-triggered escapes in Helmholtz oscillator," *Modern Physics Letters B,* vol. 28, Article ID 1450047, 2014.

[16] J. A. Almendral, "Integrability and symmetries for the Helmholtz oscillator with friction," *Journal of physics a general physics,* vol. 36, 2033.

[17] W. Albalawi, A. H. Salas, S. A. El-Tantawy, and A. A. A.-R. Youssef, "Approximate analytical and numerical solutions to the damped pendulum oscillator: Newton–Raphson and moving boundary methods," *Journal of Taibah University for Science,* vol. 15, pp. 479–485, 2021.

[18] E. Gluskin, "A nonlinear resistor and nonlinear inductor using a nonlinear capacitor," *Journal of the Franklin Institute,* vol. 336, pp. 1035–1047, 1999.

[19] A. H. Salas and S. A. El-Tantawy, "On the approximate solutions to a damped harmonic oscillator with higher-order nonlinearities and its application to plasma physics: semi-analytical solution and moving boundary method," *Eur. Phys. J. Plus,* vol. 135, 2020.

[20] J. Sugie, "Asymptotic stability of a pendulum with quadratic damping," *Zeitschrift für Angewandte Mathematik und Physik,* vol. 65, pp. 865–884, 2014.

[21] D. W. Bass and M. R. Haddara, "Nonlinear models of ship roll damping," *International Shipbuilding Progress,* vol. 38, pp. 51–71, 1991.

[22] A. Cardo, A. Francescutto, and R. Nabergoj, "On damping models in free and forced rolling motion," *Ocean Engineering,* vol. 9, pp. 171–179, 1982.

[23] L. Cvetićanin, "Oscillator with strong quadratic damping force," *Publications de l'Institut Mathématique,* vol. 85, no. 99, pp. 119–130, 2009.

[24] I. Kovacic and Z. Rakaric, "Study of oscillators with a non-negative real-power restoring force and quadratic damping," *Nonlinear Dynamics,* vol. 64, pp. 293–304, 2011.

[25] M. A. S. Neves, N. A. Pérez, and L. Valerio, "Stability of small fishing vessels in longitudinal waves," *Ocean Engineering,* vol. 26, pp. 1389–1419, 1999.

[26] M. Taylan, "The effect of nonlinear damping and restoring in ship rolling," *Ocean Engineering,* vol. 27, pp. 921–932, 2000.

[27] A. H. Salas, J. E. Castillo, and J. G. E. Lugo, "Exact solutions for a nonlinear model," *Applied Mathematics and Computation,* vol. 217, pp. 1646–1651, 2010.

[28] P. J. Jolmes, "A nonlinear oscillator with a strange attractor," *Proc. R. Sot.* vol. 292, pp. 419–448, 1979.

[29] S. A. Khuri and S. S. Xie, "On the numerical verification of the asymptotic expansion of Duffing 's equation," *International Journal of Computer,* vol. 72, pp. 325–330, 1999.

[30] F. Mirzaee and N. Samadyar, "Combination of nite difference method and meshless method based on radial basis functions to solve fractional stochastic advection diffusion equations," *Engineering with Computers,* vol. 36, 2019.

[31] N. Samadyar and F. Mirzaee, "Numerical solution of two-dimensional weakly singular stochastic integral equations on non-rectangular domains via radial basis functions," *Engineering Analysis with Boundary Elements,* vol. 101, pp. 27–36, 2019.

[32] F. Mirzaee and N. Samadyar, "Numerical solution based on two-dimensional orthonormal Bernstein polynomials for solving some classes of two-dimensional nonlinear integral equations of fractional order," *Applied Mathematics and Computation,* vol. 344, pp. 191–203, 2019.

[33] Y. F. Liu and R. L. Guo, "Matter wave soliton solution of the cubic-quintic nonlinear Schrodinger equation with an anharmonic potential," *Applied Mathematics and Computation,* vol. 219, pp. 4847–4852, 2013.

[34] Y. Geng, J. Li, and L. Zhang, "Exact explicit traveling wave solutions for two nonlinear Schrodinger type equatios," *Applied Mathematics and Computation,* vol. 217, pp. 1509–1521, 2010.