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Research Article

Wiener Index of Intuitionistic Fuzzy Graphs with an Application to Transport Network Flow

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The Wiener index ($\mathscr{W}\mathscr{I}$) is one of the connectivity parameters used to know the biochemical and physicochemical properties of compounds depending upon their molecular structures. Intuitionistic fuzzy graphs (IFG*s*) are a convenient tool to represent the objects and relations between them with two types of information using truth membership degree and falsity membership degree. This research work presents the concept of $\mathscr{W}\mathscr{I}$ under the structure IFG*s*, *IF* trees, and *IF* cycles. Some bounds on $\mathscr{W}\mathscr{I}$ are investigated. The relationship between $\mathscr{W}\mathscr{I}$ and connectivity index ($\mathscr{C}\mathscr{I}$) is also studied. In the end of this study, an application of the $\mathscr{W}\mathscr{I}$ in transport network flow is proposed.

1. Introduction

Zadeh [1] was the first person who presented his first publication on fuzzy sets (FSs) in 1965. He generalized crisp set by giving membership grades to every object of the set in the interval $[0, 1]$. Various concepts of ordinary sets were established for FSs. Further explorations in the work of Zadeh were made by Goguen [2] in 1967. Zadeh [3] discussed models including constraints and goals for the betterment and development of different sections of a country's society in 1970. Interval-valued FSs were presented bySambuc [4] in 1975 under the name of H-fuzzy set, which is proposed by Zadeh [5] taken as a special case of type 2 FSs. In 1999, the authors in [6] studied a method for the calculation of correlation coefficient under fuzzy data. The value obtained by this formula provides the strength of correlation between the sets and positivity or negativity of this correlation. Ashraf et al. introduced spherical FSs and proposed their applications [7]. Mahmood et al. [8] introduced spherical and T-spherical FSs along with some operations on them. They discussed some problems using these sets. There have been a lot of publications on FSs and more to come. There are various areas in which FSs are applied, for example, pattern recognition [9], psychology [10], and decision-making [11].

The graph is a convenient source by which objects and their relationships are presented by vertices and edges. When these objects and relationships contain vague information, it is necessary to present them by a FG. The first step in the field of FGs was taken by Rosenfeld [12] in 1975. In 1994, J.N. Mordeson [13] defined some operations on FGs. Bhattacharya [14] showed an association between FG and fuzzy group in 1987. The researchers completed a lot of work on FGs in the duration of 1981 to 1990. After that, the concept of domination in the environment of FGs was introduced by Somasundaram [15] in 1998. Akram [16] developed the concept of bipolar FGs in 2011, discussed different ways of their construction, and investigated some properties on them. By the same author [17], irregular bipolar FGs of different types were introduced in 2013. They also developed the concept of regular bipolar FGs [18]. FGs have applications in various fields such as controversial problems, decision-making, and social networks [19–21].

FSs contain only membership information. Atanassov [22] was the first one who made the generalization of FSs by considering nonmembership information and invented a new set called intuitionistic fuzzy set (IFS). After this consideration, researchers started to generalize various concepts of FSs into the frame of IFSs. In 1991, Gerstenkorn and Manko [23] defined a function to measure the correlation of IFSs and introduced a coefficient of this correlation to analyze some of its properties. Angelov [24] extended the concept of fuzzy optimization and introduced IF optimization in 1997. The authors in [25] proposed the concept of distance between IFSs in 2000. Atanassov [26] introduced interval-valued IFs and discussed their properties. Li [27] investigated multiattribute decision-making under IFSs. From application point of view, IFSs have application in medicine [28], pattern classification [29], decision-making [30], etc.

IF relations and IFG*s* are the foundations of Atanassov [31]. Akram et al. defined IFG*s* in a precise way and presented various concepts such as strong IFG*s*, balanced IFG*s*, IF hypergraphs, IF trees, and IF cycles [32–35]. Karunambigai et al. [36] introduced arcs in IFG*s* and classified them. The idea of nth type of IFG*s* was proposed by the authors in [37]. The concept of domination was developed by Parvathi [38]. Akram [39] introduced double domination for IFG*s*. Sun et al. [40] proposed the framework to increase the degree of consensus in the group of decision-making. The authors also made an analysis, and a numerical example is given to illustrate this work. Xing et al. [41] studied the concept of a new Choquet integral for the purpose to obtain the matrix of consensus of all individual experts. The authors also presented an illustration to show the effectiveness of the proposed work. IFG*s* have a variety of applications, for instance, decision support system, communication networks, and water supplier system [42–44].

Mathematical chemistry deals with molecular structure in terms of mathematical techniques. In mathematical chemistry, molecular descriptors play a vital role. Chemical graph theory as a subject develops a connection among chemistry, graph theory, and mathematics. A graph representing atoms and bonds of a compound by vertices and edges is known as a molecular graph. Topological indices (TIs) are scalars connected with a molecular graph used to know the correlation of a chemical structure with many physical properties, biological activity, and chemical reactivity. $\mathscr{W}\mathscr{I}$ [45] is the first TI proposed by Wiener in 1947. This TI is distance-based. There are many degree-based TIs discovered to know various properties of drugs and compounds, and some of them are Randic index, harmonic index, Zagreb indices, atomic bond connectivity index, geometric arithmetic index, and augmented Zagreb index. Binu et al. introduced $\mathscr{C}\mathscr{I}s$ and $\mathscr{W}\mathscr{I}$ for FGs with applications [46, 47]. The TIs based on vertex degrees such as Zagreb indices, Randic index, and harmonic index are defined in [48] for FGs with the discussion on two applications. Naeem et al. [49] proposed the concept of $\mathscr{C}\mathscr{I}s$ under IFG*s* environment with applications in two types of networks. TIs are widely used in chemistry, mathematics, and pharmacy engineering [50–52]. Besides, TIs also have applications in human trafficking, Internet routing, and transport network flow [46–49].

The setting of our research article is given as follows. Preliminary requirements are arranged in Section 2. Throughout Section 3, the concept of $\mathscr{W}\mathscr{I}$ with illustrations and some bounds for it is proved. The relationship between the $\mathscr{W}\mathscr{I}$ and $\mathscr{C}\mathscr{I}$ including supporting examples is presented in Section 4. An application of $\mathscr{W}\mathscr{I}$ for IFG*s* is proposed in Section 5. In the last two sections, the conclusion and advantages of our proposed work are given.

2. Preliminaries

To understand this research paper in a better way, we give some definitions along with some results throughout the preliminary section.

Definition 1. (see [32]). A pair $\chi = (N, M)$ under V and E known as the sets of vertices and edges is an IFG satisfying the following:

(1) $T_N, F_N\colon V \longrightarrow [0,1]$ are functions that stand for the truth membership degree, and $F_N(w_i)$ indicates the falsity membership degree of every $w_i \in V$ under the condition that $T_N(w_i) + F_N(w_i) \in [0,1]$.

(2) $T_M, F_M\colon E \longrightarrow [0,1]$ are functions, $T_M(w_i, w_j)$ represents the truth membership degree, and $F_M(w_i, w_j)$ denotes falsity membership degree of $(w_i, w_j) \in E$ with the following:

$$\begin{aligned} T_M\left(w_i, w_j\right) &\leq \min\left\{T_N(w_i), T_N\left(w_j\right)\right\},\\ F_M\left(w_i, w_j\right) &\geq \max\left\{F_N(w_i), F_N\left(w_j\right)\right\}, \end{aligned} \tag{1}$$

such that $T_M(w_i, w_j) + F_M(w_i, w_j) \in [0,1]$.

Definition 2. (see [53]). A partial subgraph of an IFG is also an IFG $H = (N\prime, M\prime)$ such that

(1) $T_{N'}(w_i) \leq T_N(w_i)$ and $F_{N'}(w_i) \geq F_N(w_i)$

(2) $T_{M'}(w_i, w_j) \leq T_M(w_i, w_j)$ and $F_{M'}(w_i, w_j) \geq F_M(w_i, w_j)$

Definition 3. (see [36]). An IFG is known as complete if $T_M(w_i, w_j) = \wedge\left\{T_N(w_i), T_N(w_j)\right\}$ and $F_M(w_i, w_j) = \vee\left\{F_N(w_i), F_N(w_j)\right\}$.

Definition 4. (see [36]). A vertex sequence $v_1, v_2, v_3, \ldots, v_n$ with $v_i \neq v_j$ for all $i \neq j$ is known as an IF path if one of the following holds for at least one i and j.

(1) $T_M(w_i, w_j) > 0$ and $F_M(w_i, w_j) = 0$

(2) $T_M(w_i, w_j) = 0$ and $F_M(w_i, w_j) > 0$

(3) $T_M(w_i, w_j) > 0$ and $F_M(w_i, w_j) > 0$

Definition 5. (see [36]). The T strength and F strength of a path $P = v_1 v_2 v_3 \ldots v_n$ are defined by $S_T = \min\left\{T_M(v_i, v_j)\right\}$ and $S_F = \max\left\{F_M(v_i, v_j)\right\}$ for every i and j. The pair

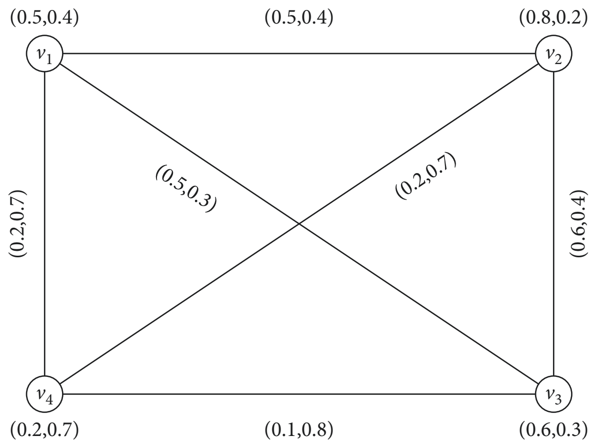

Figure 1: An intuitionistic graph χ.

containing both the strengths is known as the strength of P provided both of them lie on the common arc.

Definition 6. (see [36]). The T strength and F strength of connectedness between two vertices v_i and v_j are denoted and defined by $\mathrm{CONN}_{T(\chi)}(v_i, v_j) = \max\{S_T\}$ and $\mathrm{CONN}_{F(\chi)}(v_i, v_j) = \min\{S_F\}$ for all paths P: $v_i - v_j$.

Definition 7. (see [36]). An arc (w_i, w_j) is known as a bridge in χ if it has one of the following:

(1) $\mathrm{CONN}_{T(\chi)-(w_i,w_j)}(w_i, w_j) < \mathrm{CONN}_{T(\chi)}(w_i, w_j)$ and $\mathrm{CONN}_{F(\chi)-(w_i,w_j)}(w_i, w_j) \geq \mathrm{CONN}_{F(\chi)}(w_i, w_j)$

(2) $\mathrm{CONN}_{T(\chi)-(w_i,w_j)}(w_i, w_j) \leq \mathrm{CONN}_{T(\chi)}(w_i, w_j)$ and $\mathrm{CONN}_{F(\chi)-(w_i,w_j)}(w_i, w_j) > \mathrm{CONN}_{F(\chi)}(w_i, w_j)$

Definition 8. (see [36]). An edge (w_i, w_j) is strong if $T_M(w_i, w_j) \geq \mathrm{CONN}_{T(\chi)}(w_i, w_j)$ and $F_M(w_i, w_j) \leq \mathrm{CONN}_{F(\chi)}(w_i, w_j)$, and (w_i, w_j) is weakest if $T_M(w_i, w_j) < \mathrm{CONN}_{T(\chi)}(w_i, w_j)$ and $F_M(w_i, w_j) > \mathrm{CONN}_{F(\chi)}(w_i, w_j)$.

Definition 9. (see [36]). If P is a path between any two nodes v_i and v_j such that $S_T(P) = \mathrm{CONN}_{T(\chi)}(v_i, v_j)$ and $S_F(P) = \mathrm{CONN}_{F(\chi)}(v_i, v_j)$, then it is called strongest path between v_i and v_j.

Definition 10. (see [36]). A path P from vertex v_i to vertex v_j in an IFG is known as strong provided it has just strong arcs.

Example 1. In Figure 1, the arcs (v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), and (v_2, v_4) are all strong arcs, while (v_3, v_4) is the only weak arc. The path P: $v_3v_2v_1v_4$ is a strong path, as well as strongest path between the vertices v_3 and v_4.

Definition 11. (see [36]). An arc (v_i, v_j) in an IFG is such that

(1) $T_M(v_i, v_j) > \mathrm{CONN}_{T(\chi)-(v_i,v_j)}(v_i, v_j)$ and $F_M(v_i, v_j) < \mathrm{CONN}_{F(\chi)-(v_i,v_j)}(v_i, v_j)$ are known as α-strong

(2) $T_M(v_i, v_j) = \mathrm{CONN}_{T(\chi)-(v_i,v_j)}(v_i, v_j)$ and $F_M(v_i, v_j) = \mathrm{CONN}_{F(\chi)-(v_i,v_j)}(v_i, v_j)$ are known as β-strong

(3) $T_M(v_i, v_j) < \mathrm{CONN}_{T(\chi)-(v_i,v_j)}(v_i, v_j)$ and $F_M(v_i, v_j) > \mathrm{CONN}_{F(\chi)-(v_i,v_j)}(v_i, v_j)$ are known as δ-weak

Definition 12. (see [35]). A connected IFG $\chi = (N, M)$ is an IF tree if χ has an IF spanning subgraph $H = (N, M')$, which is a tree, where for all arcs (v_i, v_j) not in H, $T_M(v_i, v_j) < \mathrm{CONN}_{T(H)}(v_i, v_j)$, $F_M(v_i, v_j) > \mathrm{CONN}_{F(H)}(v_i, v_j)$.

Example 2. Here, we have taken $(T_N(u), F_N(u)) = (1, 0)$ for all $u \in N$. As $\mathrm{CONN}_{T(H)}(v_2, v_3) = 0.7$ and $\mathrm{CONN}_{F(H)}(v_2, v_3) = 0.3$, we see that $T_M(v_2, v_3) = 0.6 < \mathrm{CONN}_{T(H)}(v_2, v_3)$ and $F_M(v_2, v_3) = 0.4 > \mathrm{CONN}_{F(H)}(v_2, v_3)$. Thus, χ is an IF tree, and H is spanning tree of χ. The graph is shown in Figure 2.

Definition 13. (see [49]). The $\mathscr{C}\mathscr{I}$ of an IFG, $\chi = (N, M)$, is defined by $\mathscr{C}\mathscr{I}(\chi) = \mathscr{T}\mathscr{C}\mathscr{I}(\chi) + \mathscr{F}\mathscr{C}\mathscr{I}(\chi)$, where $\mathscr{T}\mathscr{C}\mathscr{I}(\chi)$ and $\mathscr{F}\mathscr{C}\mathscr{I}(\chi)$ are T-connectivity and F-connectivity indices of χ, respectively, given as follows:

$$\begin{aligned}\mathscr{T}\mathscr{C}\mathscr{I}(\chi)(\chi) &= \sum_{v,v\prime \in V(\chi)} T_N(v)T_N(v\prime)\mathrm{CONN}_{T(\chi)}(v, v),\\ \mathscr{F}\mathscr{C}\mathscr{I}(\chi) &= \sum_{v,v\prime \in V(\chi)} F_N(v)F_N(v\prime)\mathrm{CONN}_{F(\chi)}(v, v\prime).\end{aligned} \tag{2}$$

Theorem 1. *(see [49]). Let $\chi = (N, M)$ be a complete IFG, and $\chi = (N, M)$ with $N^* = \{w_1, w_2, w_3, \ldots, w_n\}$ such that $r_1 \leq r_2 \leq \ldots \leq r_n$ and $s_1 \geq s_2 \geq s_3 \geq \ldots \geq s_n$, where $r_i = T_N(w_i)$ and $s_i = F_N(w_i)$. Then,*

$$\mathscr{C}\mathscr{I}(\chi) = \sum_{i=1}^{n-1} r_i^2 \sum_{j=i+1}^{n} r_j + \sum_{i=1}^{n-1} s_i^2 \sum_{j=i+1}^{n} s_j. \tag{3}$$

FGs contain less information than IFGs. There are certain situations where we need information from both sides. IFGs are more useful in such type of situations. So, to fulfill these requirements, we aim to establish the concept of $\mathscr{W}\mathscr{I}$ for IFGs settings. We have made the extension in the results for $\mathscr{W}\mathscr{I}$ of FGs. One thing is interesting to see that we have two $\mathscr{W}\mathscr{I}$s, one for truth membership values and other for falsity membership values, which produce a comparison between them. To get single value, we take the sum of both the values of these indices.

3. Wiener Index for Intuitionistic Fuzzy Graphs

The concepts of geodesic path and $\mathscr{W}\mathscr{I}$ for IFGs are presented in this section. Some relevant illustrations are given for these concepts, and some bounds of $\mathscr{W}\mathscr{I}$ are investigated.

Definition 14. A path P between two vertices is known as geodesic if:

(1) P is strong

(2) There does not exist any strong path Q shorter than P

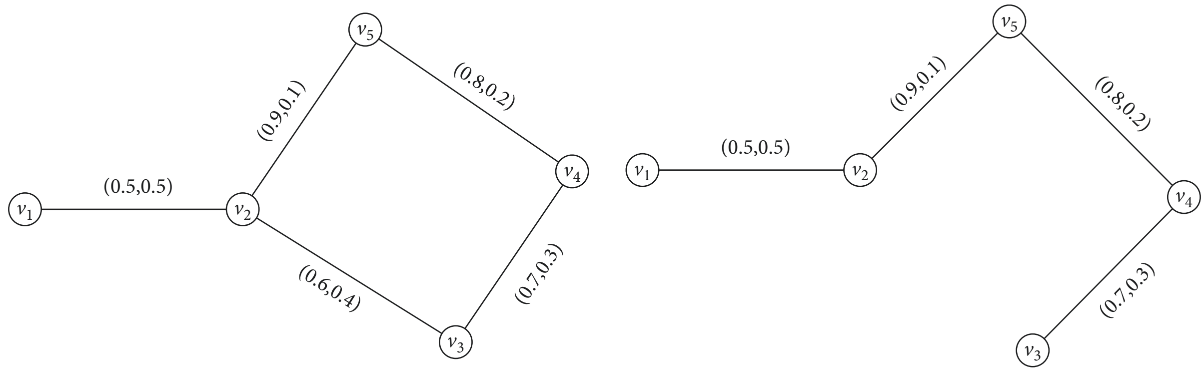

Figure 2: An IF tree χ and its spanning tree H.

The T weight and F weight are the sum of truth and falsity membership values assigned to all the arcs in the geodesic.

Example 3. In Figure 1, all arcs except (v_3, v_4) are strong. The strongest path between v_3 and v_4 is P: $v_3v_2v_1v_4$, which is mentioned already in Example 1, and therefore, $\mathrm{CONN}_{T(\chi)}(V_3, V_4) = 0.2$ and $\mathrm{CONN}_{F(\chi)}(V_3, V_4) = 0.7$. Moreover, the strong paths between v_3 and v_4 are $v_3v_2v_1v_4, v_3v_1v_4$ and $v_3v_2v_4$. The latter two of them are shortest, and hence, both are geodesic.

Definition 15. The $\mathscr{W}\mathscr{I}$ of an IFG, $\chi = (N, M)$ is defined as follows:

$$\begin{aligned}
\mathscr{W}\mathscr{I}(\chi) &= \sum_{\zeta,\zeta' \in N^*} (T_N(\zeta), F_N(\zeta))\Big(T_N(\zeta'), F_N(\zeta')\Big) d_s(\zeta, \zeta') \\
&= \sum_{\zeta,\zeta' \in N^*} (T_N(\zeta), F_N(\zeta))\Big(T_N(\zeta'), F_N(w)\Big)\Big(Td_s(\zeta, \zeta'), Fd_s(\zeta, \zeta')\Big) \\
&= \sum_{\zeta,\zeta' \in N^*} T_N(\zeta)T_N(\zeta')Td_s(\zeta, \zeta') + \sum_{\zeta,\zeta' \in N^*} F_N(\zeta)F_N(\zeta')Fd_s(\zeta, \zeta') \\
&= \mathscr{F}\mathscr{W}\mathscr{I}(\chi) + \mathscr{F}\mathscr{W}\mathscr{I}(\chi),
\end{aligned} \tag{4}$$

where $Td_s(\zeta, \zeta')$ is the minimum T weight sum and $Fd_s(\zeta, \zeta')$ is the maximum F weight sum of the geodesics from ζ to w and both lie on the same geodesic.

Remark 1. In 3.3, if both $Td_s(v, v')$ and $Fd_s(v, v')$ do not lie on the same geodesic, we left it for the user to choose the geodesic according to his will. In that case, one of the two geodesics must be compromised.

Example 4. From Figure 1, we have the following:

$$\begin{aligned}
&\mathscr{T}\mathscr{W}\mathscr{I}(\chi) = (0.5)(0.8)(0.5) + (0.5)(0.6)0.5) + (0.5)(0.2)(0.2) + (0.8)(0.6)(0.6) + (0.8)(0.2)(0.2) + (0.6)(0.2)(0.7) = 0.774, \\
&\mathscr{F}\mathscr{W}\mathscr{I}(\chi) = (0.4)(0.2)(0.4) + (0.4)(0.3)(0.3) + (0.4)(0.7)(0.7) + (0.2)(0.3)(0.4) + (0.2)(0.7)(0.7) + (0.3)(0.7)(1) = 0.596, \\
&\text{Thus,} \\
&\mathscr{W}\mathscr{I}(\chi) = \mathscr{T}\mathscr{W}\mathscr{I}(\chi) + \mathscr{F}\mathscr{W}\mathscr{I}(\chi) = 0.774 + 0.596 = 1.37.
\end{aligned} \tag{5}$$

Note 1. It may be noted that $\mathscr{F}\mathscr{W}\mathscr{I}(\chi) < \mathscr{T}\mathscr{W}\mathscr{I}(\chi)$, which shows that this problem has more truth level.

Theorem 2. *Let $\chi = (N, M)$ be an IFG with $|N^*| = n$ and $(T_N(w), F_N(w)) = (\lambda, \mu)$ for all $w \in V(\chi)$ with $0 \le \lambda + \mu \le 1$. Let $p = \wedge\{T_M(w, w') | (w, w') \in M^*$ and (w, w') is not a $\delta -$ arc$\}$ and $q = \wedge\{F_M(w, w') | (w, w') \in M*$ and (w, w') is not a $\delta -$ arc$\}$. Then,*

$$\mathscr{W}\mathscr{I}(\chi) \ge \left(p\lambda^2 + q\mu^2\right)\left[n(n-1) - \left|M^*\right|\right]. \tag{6}$$

Proof. Let $\chi = (N, M)$ be an IFG. Suppose that $p = \wedge\{T_M(w, w\prime) \mid (w, w\prime) \in M^*$ and $(w, w\prime)$ is not a δ − arc$\}$. Let $w, w\prime \in N^*$. If $(w, w\prime) \in M^*$, then $Td_s(w, w\prime) \geq p$. The number of unordered pairs (v, w) satisfying $Td_s(w, w\prime) \geq p$ is $|M^*|$, and the number of unordered vertex pairs $(v, v\prime)$ satisfying $Td_s(v, v\prime) \geq 2p$ is $\binom{n}{2} - |M^*|$. Then,

$$\begin{aligned}\mathcal{TWI}(\chi) &= \sum_{c,d\in N^*} T_N(c)T_N(d)Td_s(c,d)\\ &= \lambda^2 \sum_{c,d\in N^*} Td_s(c,d)\\ &= \lambda^2\left[\sum_{w,w'\in N^*} Td_s(w,w\prime) + \sum_{v,v'\in N^*} Td_s(v,v\prime)\right]\\ &\geq \lambda^2\left[\left|M^*\right|p + \left\{\binom{n}{2} - \left|M^*\right|\right\}2p\right]\\ &= p\lambda^2\left[\left|M^*\right| + 2\frac{n(n-1)}{2} - 2\left|M^*\right|\right]\\ &= p\lambda^2\left[n(n-1) - \left|M^*\right|\right].\end{aligned} \tag{7}$$

Now, let $q = \wedge\left\{F_M(w, w\prime) \mid (w, w\prime) \in M^*\right.$ and $(w, w\prime)$ is not a δ − arc$\}$. Then, as argued before, we have the following:

$$\mathcal{FWI}(\chi) \geq q\mu^2\left[n(n-1) - \left|M^*\right|\right]. \tag{8}$$

Thus, we have the following:

$$\begin{aligned}\mathcal{WI}(\chi) &= \mathcal{TWI}(\chi) + \mathcal{FWI}(\chi)\\ &\geq \left(p\lambda^2 + q\mu^2\right)\left[n(n-1) - \left|M^*\right|\right].\end{aligned} \tag{9}$$

□

Theorem 3. *Let $\chi_1 = (N_1, M_1)$ and $\chi_2 = (N_2, M_2)$ be two IFGs such that χ_1 is isomorphic to χ_2. Then, $\mathcal{WI}(\chi_1) = \mathcal{WI}(\chi_2)$.*

Proof. Suppose that $\chi = (N_1, M_1)$ and $\chi_2 = (N_2, M_2)$ are two isomorphic IFGs. Then, there exists a bijective mapping $f: N_1^* \longrightarrow N_2^*$ such that $T_{N_1}(w) = T_{N_2}(f(w))$ and $F_{N_1}(w) = F_{N_2}(f(w))$ for each $w \in N_1^*$ and $T_{M_1}(w, w\prime) = T_{M_2}(f(w), f(w\prime)), F_{M_1}(w, w\prime) = F_{M_2}(f(w), tfn(w\prime))$ for every edge $(w, w\prime) \in M_1^*$. Suppose $P_{w,w_\prime}$ be the path for each $w, w\prime \in N_1^*$ serving as $Td_s(w, w\prime)$ and $Fd_s(w, w\prime)$. Then, there will be the corresponding path $P'_{(f(w),f(w'))}$ in χ_2 serving as $Td_s(f(w), tfn(w\prime))$ and $Fd_s(f(w), tfn(w\prime))$ such that $T_{M_1}(v, v\prime) = T_{M_2}(f(v), f(v\prime)), F_{M_1}(v, v\prime) = F_{M_2}(f(v), tfn(v\prime))$ for each edge $(v, v\prime) \in P_{w,w_\prime}$. Since all properties are preserved between both the graphs, $Td_s(w, w\prime) = Td_s(f(w), tfn(w\prime))$ and $Fd_s(w, w\prime) = Fd_s(f(w), tfn(w\prime))$. Then,

$$\begin{aligned}\mathcal{TWI}(\chi_1) &= \sum_{w,w_\prime\in N_1^*} T_{N_1}(w)T_{N_1}(w')Td_s(w,w\prime)\\ &= \sum_{f(w),f(w_\prime)\in N_2^*} T_{N_2}(f(w))T_{N_2}(f(w'))\\ &\quad\cdot Td_s(f(w),f(w\prime))\\ &= \mathcal{TWI}(\chi_2),\\ \mathcal{FWI}(\chi_1) &= \sum_{w,w_\prime\in N_1^*} F_{N_1}(w)F_{N_1}(w\prime)Fd_s(w,w\prime)\\ &= \sum_{f(w),f(w_\prime)\in N_2^*} F_{N_2}(f(w))F_{N_2}(f(w\prime))\\ &\quad\cdot Fd_s(f(w),f(w\prime))\\ &= \mathcal{FWI}(\chi_2).\end{aligned} \tag{10}$$

Hence, $\mathcal{TWI}(\chi_1) + \mathcal{FWI}(\chi_1) = \mathcal{TWI}(\chi_2) + \mathcal{FWI}(\chi_2)$, which implies that $\mathcal{WI}(\chi_1) = \mathcal{WI}(\chi_2)$. □

4. Relationship between the Connectivity Index and Weiner Index

In the forthcoming example, it can be noted that $\mathcal{CI}(\chi) = \mathcal{WI}(\chi)$, but this is not always the case. Examples 5 and 6 are given to show this matter.

Example 5. Since every arc in Figure 3 is strong, therefore $\mathrm{CONN}_{T(\chi)}(v, w) = Td_s(v, w)$ and $\mathrm{CONN}_{F(\chi)}(v, w) = Fd_s(v, w)$. We see that for all $v, w \in V(\chi)$, $\mathrm{CONN}_{T(\chi)}(v, w) = T_M(v, w)$ and $\mathrm{CONN}_{F(\chi)}(v, w) = F_M(v, w)$. Thus, $Td_s(v, w) = T_M(v, w)$ and $Fd_s(v, w) = F_M(v, w)$. Then,

$$\begin{aligned}\mathcal{TWI}(\chi) &= (0.5)(0.8)(0.5) + (0.5)(0.6)(0.5)\\ &\quad+(0.5)(0.2)(0.2) + (0.8)(0.6)(0.6)\\ &\quad+(0.8)(0.2)(0.2) + (0.6)(0.2)(0.2)\\ &= 0.714,\\ \mathcal{FWI}(\chi) &= (0.4)(0.2)(0.5) + (0.4)(0.3)(0.5)\\ &\quad+(0.4)(0.7)(0.7) + (0.2)(0.3)(0.4)\\ &\quad+(0.2)(0.7)(0.7) + (0.3)(0.7)(0.7)\\ &= 0.565\end{aligned} \tag{11}$$

Thus, $\mathcal{WI}(\chi) = \mathcal{TWI}(\chi) + \mathcal{FWI}(\chi) = 0.714 + 0.565 = 1.279$. Obviously, $\mathcal{CI}(\chi) = 1.279$, and thus, $\mathcal{WI}(\chi) = \mathcal{CI}(\chi)$.

Note 2. In last example, we can observe that

(1) Every arc is strong

(2) Every pair of vertices is connected by an edge

(3) There is no δ-arc

Example 6. Consider χ is a graph depicted in Figure 4. After calculations, we get $\mathcal{TCI}(\chi) = 0.714$, $\mathcal{FCI}(\chi) = 0.565$,

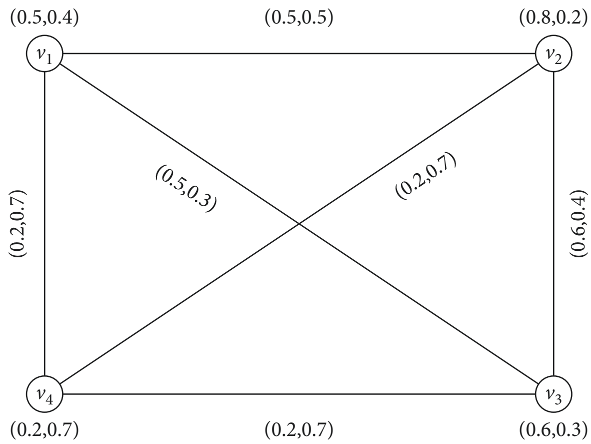

Figure 3: IFG with $\mathrm{WI}(\chi) = 1.279$.

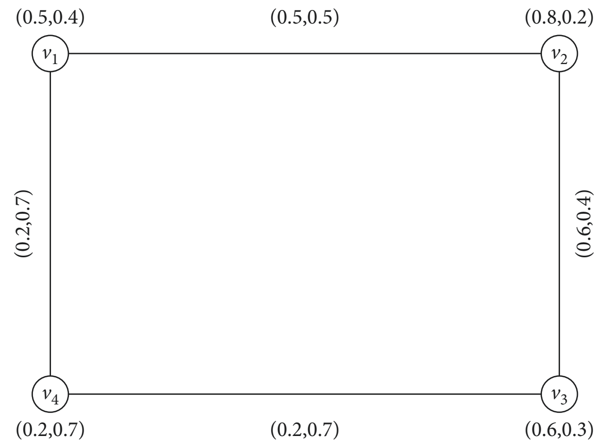

Figure 4: IFG with $\mathscr{C}\mathscr{I}(\chi) = 1.279$ and $\mathscr{W}\mathscr{I}(\chi) = 1.507$.

and $\mathscr{C}\mathscr{I}(\chi) = 0.714 + 0.565 = 1.279$. Since each arc is strong,

$$\begin{aligned}\mathrm{CONN}_\chi(v_1, v_2) &= d_s(v_1, v_2)\\ &= (0.5, 0.5),\\ \mathrm{CONN}_\chi(v_1, v_4) &= d_s(v_1, v_4)\\ &= (0.2, 0.7),\\ \mathrm{CONN}_\chi(v_2, v_3) &= d_s(v_2, v_3)\\ &= (0.6, 0.4),\\ \mathrm{CONN}_\chi(v_3, v_4) &= d_s(v_3, v_4)\\ &= (0.2, 0.7).\end{aligned} \tag{12}$$

The paths $v_1v_2v_3$ and $v_1v_4v_3$ are strong between v_1 and v_3 and of equal length. The sum of membership values of $v_1v_2v_3$ and $v_1v_4v_3$ is $(1.1, 0.9)$ and $(0.4, 1.4)$. Thus $d_s(v_1, v_3) = (0.4, 1.4)$. This implies that $Td_s(v_1, v_3) = 0.4$ and $Fd_s(v_1, v_3) = 1.4$. Similarly, we have $Td_s(v_2, v_4) = 0.7$ and $Fd_s(v_2, v_4) = 1.2$.

$$\begin{aligned}\mathscr{T}\mathscr{W}\mathscr{I}(\chi) &= (0.5)(0.8)(0.5) + (0.5)(0.6)(0.4) + (0.5)(0.2)(0.2)\\ &\quad + (0.8)(0.6)(0.6) + (0.8)(0.2)(0.7) + (0.6)(0.2)(0.2)\\ &= 0.764,\\ \mathscr{F}\mathscr{W}\mathscr{I}(\chi) &= (0.4)(0.2)(0.5) + (0.4)(0.3)(1.4) + (0.4)(0.7)(0.7)\\ &\quad + (0.2)(0.3)(0.4) + (0.2)(0.7)(1.2) + (0.3)(0.7)(0.7)\\ &= 0.743.\end{aligned} \tag{13}$$

Hence, $\mathscr{W}\mathscr{I}(\chi) = 0.764 + 0.743 = 1.507$.

Note 3. Note that in Example 6, $\mathscr{C}\mathscr{I}(\chi) < \mathscr{W}\mathscr{I}(\chi)$ although every arc is strong. This inequality does not happen always.

Theorem 4. *Let $\chi = (N, M)$ be an IFG with the following two conditions.*

(1) χ has no δ-arcs

(2) For every pair ζ, $w \in N^*$, $(\zeta, w) \in M^*$

Then,

$$\mathscr{C}\mathscr{I}(\chi) = \mathscr{W}\mathscr{I}(\chi). \tag{14}$$

Proof. Consider $\chi = (N, M)$ is an IFG satisfying (i) and (ii). Since χ has no δ-arcs, all arcs will be either α- or β-strong. In each of the case, we have for all $\zeta, \zeta\prime \in N^*$:

$$\begin{aligned}T_M(\zeta, w) &= \mathrm{CONN}_{T(\chi)}(\zeta, \zeta\prime)\\ &= Td_s(\zeta, \zeta'),\\ F_M(\zeta, \zeta\prime) &= \mathrm{CONN}_{F(\chi)}(\zeta, \zeta\prime)\\ &= Fd_s(\zeta, \zeta\prime).\end{aligned} \tag{15}$$

So, we have the following:

$$\begin{aligned}\mathscr{T}\mathscr{C}\mathscr{I}(\chi) &= \sum_{\zeta,\zeta\prime \in N^*} T_N(\zeta)T_N(\zeta\prime)\mathrm{CONN}_{T(\chi)}(\zeta, \zeta\prime)\\ &= \sum_{\zeta,\zeta' \in N^*} T_N(\zeta)T_N(\zeta\prime)T_M(\zeta, \zeta\prime)\\ &= \sum_{\zeta,\zeta' \in N^*} T_N(\zeta)T_N(\zeta\prime)Td_s(\zeta, \zeta\prime)\\ &= \mathscr{T}\mathscr{W}\mathscr{I}(\chi),\\ \mathscr{F}\mathscr{C}\mathscr{I}(\chi) &= \sum_{\zeta,\zeta' \in N^*} F_N(\zeta)F_N(\zeta\prime)\mathrm{CONN}_{F(\chi)}(\zeta, \zeta\prime)\\ &= \sum_{\zeta,\zeta' \in N^*} F_N(\zeta)F_N(\zeta\prime)F_M(\zeta, \zeta\prime)\\ &= \sum_{\zeta,\zeta' \in N^*} F_N(\zeta)F_N(\zeta\prime)Fd_s(\zeta, \zeta\prime)\\ &= \mathscr{F}\mathscr{W}\mathscr{I}(\chi).\end{aligned} \tag{16}$$

Hence,

$$\begin{aligned}\mathscr{C}\mathscr{I}(\chi) &= \mathscr{T}\mathscr{C}\mathscr{I}(\chi) + \mathscr{F}\mathscr{C}\mathscr{I}(\chi)\\ &= \mathscr{T}\mathscr{W}\mathscr{I}(\chi) + \mathscr{F}\mathscr{W}\mathscr{I}(\chi)\\ &= \mathscr{W}\mathscr{I}(\chi).\end{aligned}$$

□

Theorem 5. *Let $\chi = (N, M)$ be a complete IFG with $|N^*| = n$. Also, $r_1 \le r_2 \le r_3 \le \cdots \le r_n$ such that $T_N(w_i) = t_i$ for $i = 1, 2, 3, \ldots, n$ and $s_1 \ge s_2 \ge s_3 \ge \cdots \ge s_n$ be such that $F_N(w_i) = s_i$ for $i = 1, 2, 3, \ldots, n$. Then,*

$$\mathcal{WI}(\chi) = \sum_{i=1}^{n-1} r_i^2 \sum_{j=i+1}^{n} r_j + \sum_{i=1}^{n-1} s_i^2 \sum_{j=i+1}^{n} s_j. \tag{17}$$

Proof. Suppose $\chi = (N, M)$ is a complete IFG. Then, χ does not have any δ-arc. By the previous theorem, $\mathcal{CI}(\chi) = \mathcal{WI}(\chi)$. By Theorem 1, we obtain the required result. □

Corollary 1. *Let $\chi = (N, M)$ be a complete IFG. Then, $\mathcal{WI}(\chi) = \mathcal{CI}(\chi)$.*

Theorem 6. *If $\chi = (N, M)$ is an IF tree and $|N^*| \ge 3$, then $\mathcal{WI}(\chi) = \mathcal{CI}(\chi)$.*

Proof. Assume that $\chi = (N, M)$ is an IF tree with $|N^*| \ge 3$. Then, a unique strong path exits between every two vertices. Indeed, it is also unique and strongest. For all $v, w \in N^*$, $Td_S(v, w) = \sum_{(c,d)\in P} T_M(c, d)$ and $Fd_S(v, w) = \sum_{(c,d)\in P} F_M(c, d)$, while $\mathrm{CONN}_{T(\chi)}(v, w) = \min_{(c,d)\in P} T_M(c, d)$ and $\mathrm{CONN}_{F(\chi)}(v, w) = \max_{(c,d)\in P} F_M(c, d)$. So, we have the following:

$$\mathrm{CONN}_{T(\chi)}(v, w) \le Td_S(v, w) \text{ and } \mathrm{CONN}_{F(\chi)}(v, w) \le Fd_S(v, w). \tag{18}$$

If (v, w) is a strong arc, then $\mathrm{CONN}_{T(\chi)}(v, w) = Td_s(v, w)$ and $\mathrm{CONN}_{F(\chi)}(v, w) = Fd_S(v, w)$, but if $T_M(v, w) = 0$ or $(v, w) \in M^*$ is not strong, then $\mathrm{CONN}_{T(\chi)}(v, w) < Td_S(v, w)$ and $\mathrm{CONN}_{F(\chi)}(v, w) < Fd_S(v, w)$. Thus, $\sum \mathrm{CONN}_{T(\chi)}(v, w) < \sum Td_s(v, w)$, which implies that $\mathcal{TCI}(\chi) < \mathcal{TWI}(\chi)$. Similarly, we have $\sum \mathrm{CONN}_{F(\chi)}(v, w) < \sum Fd_s(v, w)$, which gives $\mathcal{FCI}(\chi) < \mathcal{FWI}(\chi)$. Hence:

$$\mathcal{TCI}(\chi) + \mathcal{FCI}(\chi) < \mathcal{TWI}(\chi) + \mathcal{FWI}(\chi)\mathcal{CI}(\chi) < \mathcal{WI}(\chi). \tag{19}$$

□

Corollary 2. *Let $\chi = (N, M)$ be an IF tree. Then, $\mathcal{CI}(\chi) \le \mathcal{WI}(\chi)$.*

Proof. Suppose $\chi = (N, M)$ is an IF tree and $|N^*| = 2$. Then, χ has only one arc joining its two vertices. Obviously, this arc is strong, and therefore, $\mathcal{CI}(\chi) < \mathcal{WI}(\chi)$. By Theorem 6 with $|N^*| \ge 3$, we have $\mathcal{CI}(\chi) < \mathcal{WI}(\chi)$. By combining both $\mathcal{CI}(\chi) < \mathcal{WI}(\chi)$ and $\mathcal{CI}(\chi) < \mathcal{WI}(\chi)$, we get $\mathcal{CI}(\chi) < \mathcal{WI}(\chi)$ as desired. □

Example 7. From Figure 2, we calculate connectivity index as follows:

$$\begin{aligned}
\mathcal{TCI}(\chi) &= \sum_{u,v\in V(\chi)} T_N(u) T_N(v) \mathrm{CONN}_{T(\chi)}(u, v) \\
&= (0.9)^2 [0.5 + 0.5 + 0.5 + 0.5 + 0.7 + 0.8 + 0.9 + 0.7 + 0.7 + 0.8] \\
&= 5.346, \\
\mathcal{FCI}(\chi) &= \sum_{u,v\in V(\chi)} F_N(u) F_N(v) \mathrm{CONN}_{F(\chi)}(u, v) \\
&= (0.1)^2 [0.5 + 0.5 + 0.5 + 0.5 + 0.3 + 0.2 + 0.1 + 0.3 + 0.3 + 0.2] \\
&= 0.034.
\end{aligned} \tag{20}$$

Thus, $\mathcal{CI}(\chi) = \mathcal{TCI}(\chi) + \mathcal{FCI}(\chi) = 5.346 + 0.034 = 5.38$. All the arcs except (v_2, v_3) are strong. Now, we calculate $\mathcal{WI}$ as follows:

$$\begin{aligned}
\mathcal{TW}(\chi) &= \sum_{u,v\in V(\chi)} T_N(u) T_N(v) Td_s(u, v) \\
&= (0.9)^2 [0.5 + 2.9 + 2.2 + 1.4 + 2.4 + 1.7 + 0.9 + 0.7 + 1.5 + 0.8] \\
&= 12.15, \\
\mathcal{FW}(\chi) &= \sum_{u,v\in V(\chi)} F_N(u) F_N(v) Fd_s(u, v) \\
&= (0.1)^2 [0.5 + 1.1 + 0.8 + 0.6 + 0.6 + 0.3 + 0.1 + 0.3 + 0.5 + 0.2] \\
&= 0.05.
\end{aligned} \tag{21}$$

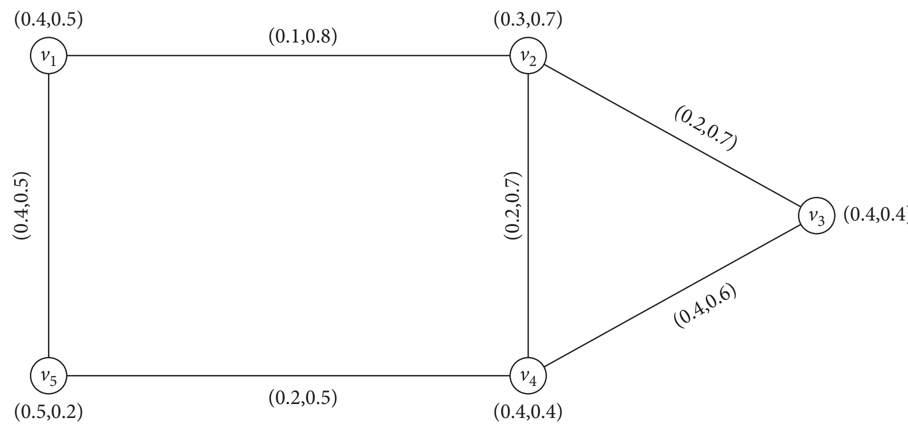

Figure 5: An IFG with WI(χ) = 2.798.

Table 1: Values of wiener index for χ and $\chi - (v_i, v_j)$.

Arcs	Types of arcs	Graphs	$\mathcal{TWI}$	$\mathcal{FWI}$	$\mathcal{WI}$
—	—	χ	0.764	2.034	2.798
(v_1, v_2)	δ-weak	$\chi - (v_1, v_2)$	0.764	2.034	2.798
(v_2, v_3)	β-strong	$\chi - (v_2, v_3)$	0.812	2.202	3.014
(v_2, v_4)	β-strong	$\chi - (v_2, v_4)$	0.921	2.496	3.416
(v_3, v_4)	α-strong	$\chi - (v_3, v_4)$	0.764	2.386	3.15
(v_4, v_5)	α-strong	$\chi - (v_4, v_5)$	0.192	0.548	0.74
(v_1, v_5)	α-strong	$\chi - (v_1, v_5)$	0.332	0.784	1.116

So, $\mathcal{WI}(\chi) = \mathcal{TWI}(\chi) + \mathcal{FWI}(\chi) = 12.15 + 0.05 = 12.2$. Clearly, we can observe that $\mathcal{WI}(\chi) > \mathcal{CI}(\chi)$.

Theorem 7. *If $H = (N\prime, M\prime)$ is an partial IF subgraph of an IFG $\chi = (N, M)$, then $CONN_{T(H)}(v, w) \leq CONN_{T(\chi)}(v, w)$ and $CONN_{F(H)}(v, w) \geq CONN_{F(\chi)}(v, w)$.*

Proof. (i) By the definition of partial IF subgraph, we have $T_{M'}(v_i, v_j) \leq T_M(v_i, v_j)$ for each i, j. If $S_{T'}$ denotes the T strength of a path P' in H and S_T denotes the T strength of a path P in χ, then $\min\left\{T_{M\prime}(v_i, v_j)\right\} \leq \min\left\{T_M(v_i, v_j)\right\}$ implies $S_{T'} \leq S_T$. So, we have $\max\left\{S_{T'}\right\} \leq \max\left\{S_T\right\}$ for all possible paths P' in H and P in χ between V and W. Thus, we obtain $\mathrm{CONN}_{T(H)}(v, w) \leq \mathrm{CONN}_{T(\chi)}(v, w)$.

(ii) Again by the definition of partial IF subgraph, we have $F_{M'}(v_i, v_j) \geq F_M(v_i, v_j)$, which implies $\mathrm{CONN}_{F(H)}(v, w) \geq \mathrm{CONN}_{F(\chi)}(v, w)$. ☐

Example 8. Using Figure 5, we have prepared the following table after routine computations. From Table 1, we see that $\mathcal{WI}$ of χ has been increased and decreased by deleting α-strong edges. We also have made the observation that $\mathcal{WI}$ of χ is increased by removing β-strong edges, but by deleting the δ-arc (v_1, v_2), the strength of connectedness between every pair of vertices will remain the same. So, that is the reason we have $\mathcal{WI}(\chi) = \mathcal{WI}(\chi - (v_1, v_2))$.

Theorem 8. *Let $\chi = (N, M)$ be an IFG. For $c, d \in N^*$, let $P_{c,d}$ be the path with minimum sum of truth values and maximum total of falsity values between the entire shortest strong paths among c & d. Let $(v, w) \in M^*$.*

(1) If (v, w) is an α-strong or β-strong arc and (v, w) is not a part of any $P_{c,d}$ for $c, d \in N^*$ with $\{c, d\} \neq \{v, w\}$, then $\mathcal{WI}(\chi - (v, w)) \neq \mathcal{WI}(\chi)$

(2) If (v, w) is a δ-edge, then $\mathcal{WI}(\chi - (u, v)) = \mathcal{WI}(\chi)$

Proof. Suppose $\chi = (N, M)$ is an IFG and $(v, w) \in M^*$. (i) Let (v, w) be an α-strong or β-strong arc. Then, (v, w) arc itself is the shortest strong path between v and w. Therefore, $Td_s(v, w) = T_M(v, w)$ and $Fd_s(v, w) = F_M(v, w)$. ☐

Case 1. If (v, w) is a β-strong arc, then one or more strong paths connecting v and w having length of 2 or more will exist. Suppose that (v, w) is not a part of any $P_{c,d}$ for $c, d \in N^*$ with $\{c, d\} \neq \{v, w\}$. Then, one of the following will hold according to the path $P_{c,d}$ in $\chi - (v, w)$:

(1) $Td_s(v, w)|_{\chi-(v,w)} > Td_s(v, w)|_\chi = T_M(v, w)$ and $Fd_s(v, w)|_{\chi-(v,w)} > Fd_s(v, w)|_\chi = F_M(v, w)$

(2) $Td_s(v, w)|_{\chi-(v,w)} < Td_s(v, w)|_\chi = T_M(v, w)$ and $Fd_s(v, w)|_{\chi-(v,w)} < Fd_s(v, w)|_\chi = F_M(v, w)$

(3) $Td_s(v, w)|_{\chi-(v,w)} > Td_s(v, w)|_\chi = T_M(v, w)$ and $Fd_s(v, w)|_{\chi-(v,w)} < Fd_s(v, w)|_\chi = F_M(v, w)$

(4) $Td_s(v, w)|_{\chi-(v,w)} < Td_s(v, w)|_\chi = T_M(v, w)$ and $Fd_s(v, w)|_{\chi-(v,w)} > Fd_s(v, w)|_\chi = F_M(v, w)$

Let $\{s, t\} \neq \{v, w\}$. By supposition, (v, w) is not a part of any $P_{s,t}$. So, we have $Td_s(v, w)|_{\chi-(v,w)} = Td_s(v, w)|_\chi$ and

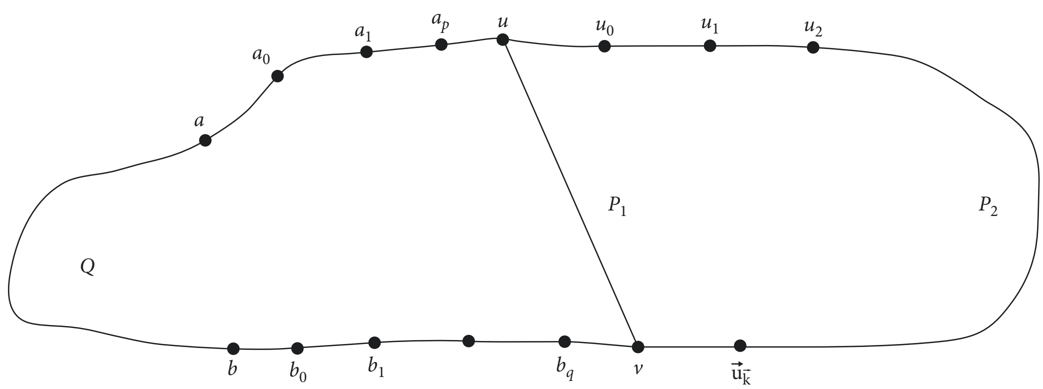

Figure 6: Sketch of Case I.

$Fd_s(s,t)|_{\chi-(v,w)} = Fd_s(s,t)|_{\chi}$. So, either $\mathscr{WI}(\chi-(v,w)) < \mathscr{WI}(\chi)$ or $\mathscr{WI}(\chi-(v,w)) > \mathscr{WI}(\chi)$ implies that $\mathscr{WI}(\chi-(v,w)) \neq \mathscr{WI}(\chi)$.

Case 2. Let (v,w) be an α-strong arc. If v and w are not connected by any path on $\chi-(v,w)$, then $Td_s(v,w)|_{\chi-(v,w)} = 0 = Fd_s(v,w)|_{\chi-(v,w)}$. Therefore, $Td_s(v,w)|_{\chi-(v,w)} < Td_s(v,w)|_{\chi}$ and $Fd_s(v,w)|_{\chi-(v,w)} < Fd_s(v,w)|_{\chi}$. If v and w are connected in $\chi-(v,w)$, then there exits one or more strong paths connecting v and w of length of 2 or more. So, using the case (1), we get $\mathrm{WI}(\chi-(v,w)) \neq \mathrm{WI}(\chi)$.

(ii) If $(v,w) \in M^*$ is a δ-arc, then no geodesic from c to d for any $c,d \in N^*$ will contain the arc (v,w). Therefore, the deletion of (v,w) from χ has no effect on $Td_s(c,d)$ and $Fd_s(c,d)$ for any $c,d \in N^*$. So, we must have $\mathscr{WI}(\chi-(v,w)) = \mathscr{WI}(\chi)$.

Theorem 9. *Let $\chi=(N,M)$ be an IFG with the given condition. If $(u,v) \in M^*$ is β-strong, then a unique strong cycle of strength $(T_M(u,v), F_M(u,v))$ exits through (u,v). Then, $\mathscr{WI}(\chi-(u,v)) > \mathscr{WI}(\chi)$.*

Proof. Suppose $\chi=(N,M)$ is an IFG with the given condition. Further, suppose $(u,v) \in M^*$ is β-strong arc. Then, by the given condition, there exits only one strong cycle C through (u,v) having strength $(T_M(u,v), F_M(u,v))$, which implies that $T_M(c,d) \geq T_M(u,v)$ and $F_M(c,d) \geq F_M(u,v)$ for all arcs (c,d) in C.

Any strong path from u to v in $\chi-(u,v)$ will have two or more arcs with $\mathrm{CONN}_{T(\chi)}(c,d) \geq T_M(u,v)$ and $\mathrm{CONN}_{F(\chi)}(c,d) \geq F_M(u,v)$ for each arc (c,d) in the path. Thus,

$$Td_s(u,v)|_{\chi-uv} > Td_s(u,v)|_{\chi} \& Fd_s(u,v)|_{\chi-uv} > Fd_s(u,v)|_{\chi}. \tag{22}$$

Suppose $u, u_0, u_1, \ldots, u_k, v$ is the strong path from u to v having strength $(T_M(u,v), F_M(u,v))$. Let $a,b \in N^*$ such that (u,v) is an arc of the shortest strong path P_1: $a, a_0, a_1, \ldots, a_p, u, v, b_q, \ldots, b_0, b$ from a to b with the following:

$$\left(\sum_{(x,y)\in P_1} T_M(x,y), \sum_{(x,y)\in P_1} F_M(x,y)\right) = (Td_s(a,b), Fd_s(a,b)). \tag{23}$$

Also, P_1 and P_2: $a, a_0, a_1, \ldots, a_p, u, u_0, \ldots, u_k, v, b$ are only two strong paths exist from $a \longrightarrow b$. Now, we prove the assertion. If possible, there will be a strong path Q between $a \longrightarrow b$ other than P_1 and P_2. Let the strong cycle be C: $u, u_0, u_1, \ldots, u_k, v, u$. Now, consider three cases. □

Case 3. Figure 6 shows that P_1 and P_2 are internally disjoint with Q. Then, Q together with P_1 makes a strong cycle through (u,v). This contradicts the assumption that C: $u, u_0, u_1, \ldots, u_k, v, u$ is the only strong cycle through (u,v).

Case 4. Figure 7 shows that Q is not internally disjoint with either P_1 or P_2. Suppose Q and P_1 are not internally disjoint. Let Q: $a, a_0, \ldots, a_1, c_0, c_1, \ldots, c_m, b_d, \ldots, b_0, b$. Then, $a_1, \ldots, a_p, u, v, b_q, b_{q-1}, \ldots, b_d, c_m, \ldots, c_1, c_0, a_1$ is a strong cycle through (u,v) different from C, which is again contradiction to the assumption that C is the only strong cycle through (u,v). The case when Q have common vertices with P_2 also provides a contradiction.

Case 5. P_1 and P_2 are not internally disjoint with Q (see Figure 8).

Suppose that both P_1 and P_2 are not internally disjoint with Q. Let Q: $a, a_0, \ldots, a_x, d_0, d_1, \ldots, d_r, u_y, u_{y-1}, \ldots, u_k, v$. Then, the path $u, a_p, a_{p-1}, \ldots, a_x, d_r, \ldots, d_1, d_0, u_y, \ldots, u_k, v, u$ acts as a strong cycle through (u,v) other than C, which is impossible. Thus, there are only two strong paths P_1 and P_2 between a and b. Thus,

$$Td_s(a,b)|_{\chi-uv} > Td_s(a,b)|_{\chi} \& Fd_s(a,b)|_{\chi-uv} > Fd_s(a,b)|_{\chi}. \tag{24}$$

If $p,q \in N^*$ such that (u,v) is not an arc of any strong path from p to q, then $Td_s(p,q)|_{\chi-uv} = Td_s(p,q)|_{\chi}$ and $Fd_s(p,q)|_{\chi-uv} = Fd_s(p,q)|_{\chi}$. Hence, $\mathscr{WI}(\chi-uv) > \mathscr{WI}(\chi)$.

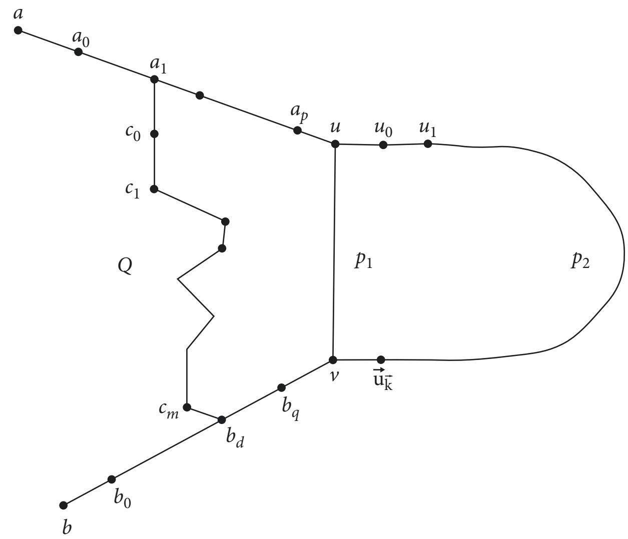

Figure 7: Sketch of Case II.

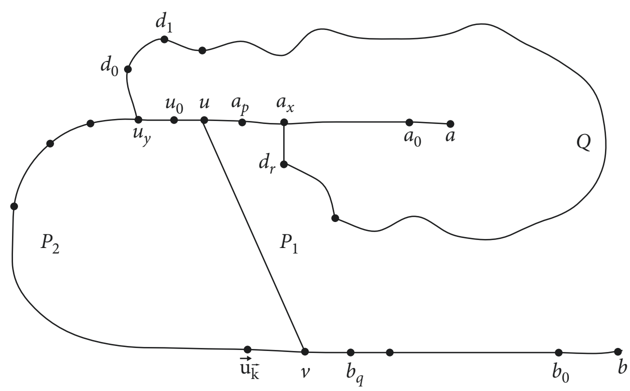

Figure 8: Sketch of Case III.

5. Application: Transport Network Flow

An application of Wiener index is proposed in this section. The Wiener index is applied to a transport network flow to know different situations created by the removal or closing of certain roads.

Consider a transport network flow as shown in the Figure 9. In it, vertices and edges represent cities and roads. The values given to the vertices and edges are incoming and outgoing flows of traffic. The maximum values for incoming and outgoing flows are 25 and 20, and vertices are fuzzified by dividing their values by 45 (sum of 25 and 20). The edges are also fuzzified by the same technique. The fuzzified network is shown in Figure 10.

In this network, each arc is strong. By simple calculations, we have prepared the following table.

From Table 2, we see that cities v_1, v_4, v_5, v_8 become isolated after the removal of arcs or roads (v_1, v_2), (v_3, v_4), (v_5, v_6), and (v_7, v_8). The people of these cities will be disconnected from the cities v_2, v_3, v_6 and v_7. The deletion of these edges reduces the overall Wiener index of the network, but still path number for incoming flow is greater than the path number for outgoing flow in other places. From all of these observations, we have arrived at the result that a given network has more incoming flow.

Table 2 shows that $\mathscr{W}\mathscr{I}(\chi - (v_2, v_3)) > \mathscr{W}\mathscr{I}(\chi)$. This is why that elimination of (v_2, v_3) makes some routes longer than before. For example, the geodesic route from v_1 to v_3 was $v_1 v_2 v_3$, which is of length 2, but now it has become $v_1 v_2 v_7 v_6 v_3$ of 4 in length. Besides its length, its sum of weights has increased also. People who used to travel from city v_1 to v_3 will now have to spend more money and time. Similarly, the routes from v_1 to v_4 and from v_2 to v_3 are also affected by this elimination. All other routes are not affected by this removal, for example, the route from v_2 to v_5. There are two geodesics between v_2 and v_5 of the same length and the same sum of incoming and outgoing flows. So, alternative route with the same geodetic distances is available after the removal of (v_2, v_3) and the people who used to travel between these cities will not be affected much. The removal of (v_2, v_3) is shown in the Figure 11. Similar information can be obtained for other three removals (v_3, v_6), (v_6, v_7), and (v_2, v_7). The purpose of this application is to analyze the results produced by the

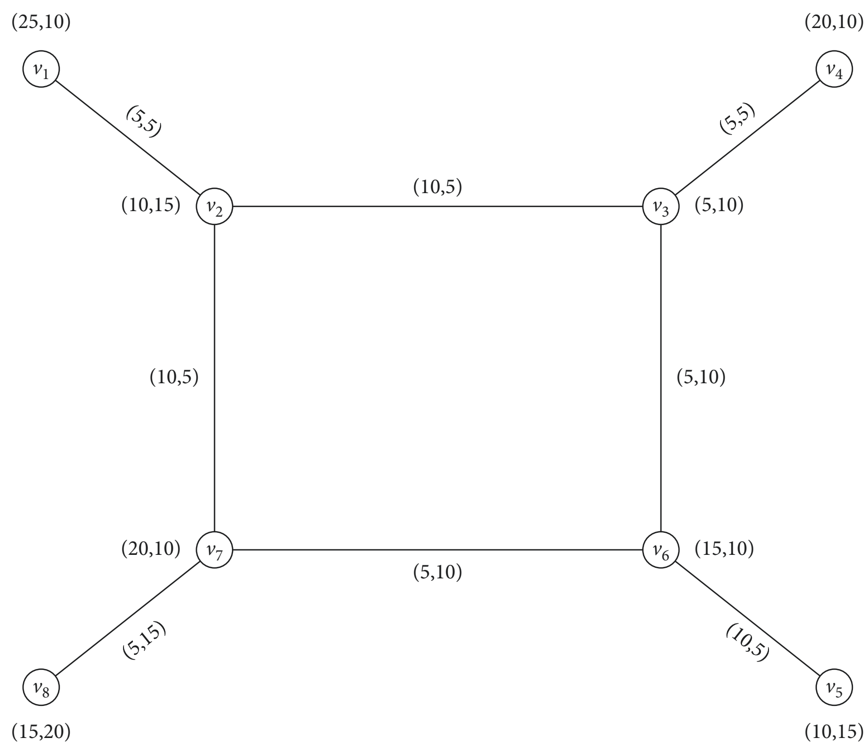

Figure 9: A transport network.

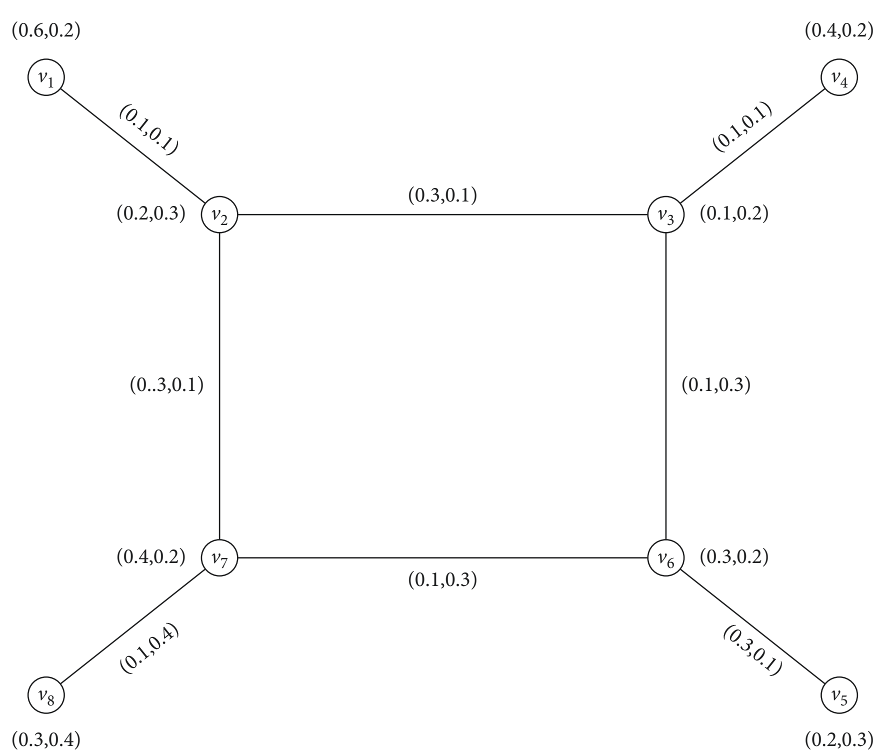

Figure 10: Intuitionistic fuzzified transport network flow.

closing or removal of certain routes connecting different cities.

By comparison between the Wiener index and the connectivity index, we can observe from Table 2 that the values of both the indices do not agree with each other for χ and $\chi - (v_i, v_j)$, although each road is strong. The reason is that there is no road between every two cities; for example, the cities v_2 and v_6 are not connected directly by a road.

TABLE 2: $\mathscr{WI}(\chi)$, $\mathscr{WI}(\chi-(v_i,v_j))$, and $\mathscr{CI}$.

Arcs	Types of arcs	Graphs	$\mathscr{TWI}$	$\mathscr{FWI}$	$\mathscr{WI}$	$\mathscr{CI}$
—	—	χ	0.98	0.811	1.791	0.78
(v_1,v_2)	Strong arc	$\chi-(v_1,v_2)$	0.452	0.673	1.125	0.586
(v_2,v_3)	α-strong	$\chi-(v_2,v_3)$	1.048	0.943	1.991	0.806
(v_3,v_4)	Strong arc	$\chi-(v_3,v_4)$	0.664	0.621	1.285	0.584
(v_3,v_6)	β-strong	$\chi-(v_3,v_6)$	1.27	0.767	2.037	0.78
(v_5,v_6)	Strong arc	$\chi-(v_5,v_6)$	0.728	0.562	1.29	0.569
(v_6,v_7)	β-strong	$\chi-(v_6,v_7)$	1.33	0.787	2.117	0.78
(v_7,v_8)	Strong arc	$\chi-(v_7,v_8)$	0.749	0.363	1.112	0.458
(v_2,v_7)	α-strong	$\chi-(v_2,v_7)$	1.308	0.991	2.299	0.792

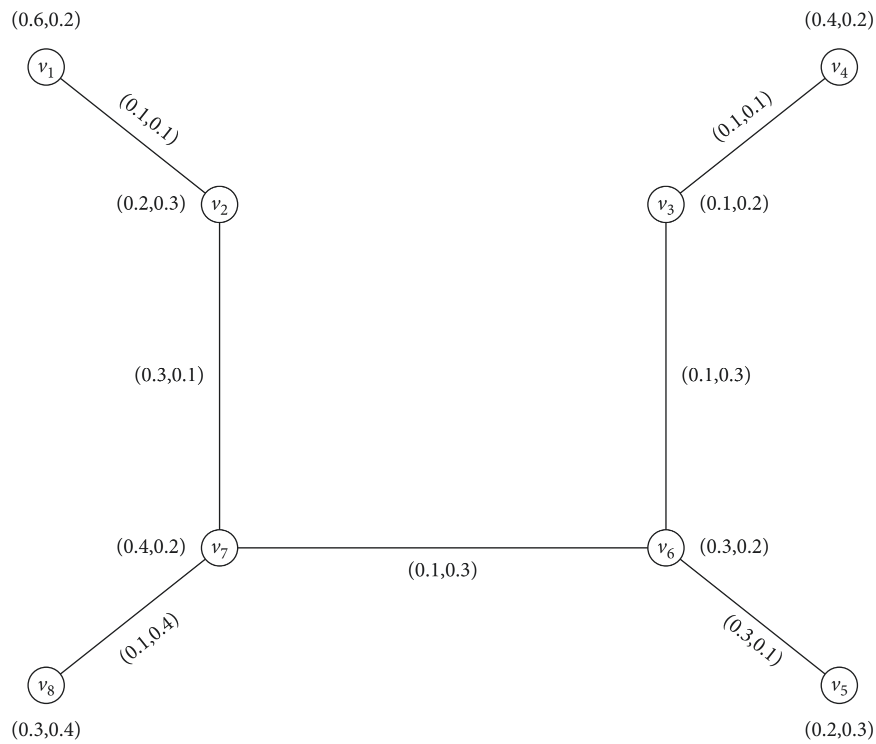

FIGURE 11: $\chi-(v_2,v_3)$.

Another thing to note is that the values for both the indices are lower than before by deleting strong roads; that is, the behavior of both the indices is same, and the values become higher than before in case of deleting α-strong roads, but the values of the Wiener index rise than before while the values of the connectivity index remain the same by deleting β-strong roads.

6. Conclusion

Mainly, our goal for the setting of this study was to introduce the concept of $\mathscr{WI}$ for IFG*s*, which deals with two types of degrees named as truth membership degree and falsity membership degree.

(i) We have purposed the definition of $\mathscr{WI}$ for IFG*s* structure with numeral examples.

(ii) Some bounds of $\mathscr{WI}$ are investigated for IFG*s* settings.

(iii) We have discussed the relationship between $\mathscr{CI}$ and $\mathscr{WI}$ with examples. Also, some related results are provided.

(iv) An application in transport network flow is given.

In the future, we will do this work for T-spherical fuzzy graphs and picture fuzzy graphs. We will also extend this study for interval-valued IFG*s* and explore its applications.

6.1. Advantages. Our study has the following advantages:

(i) The main advantage of our work lies in the fact that we have developed $\mathscr{WI}$ for IFG*s* settings, which are defined by two membership degrees.

(ii) FG*s* are defined by only one degree that is membership degree and therefore failed to give more information in some situations. Instead of FG*s*, IFG*s* give two types of information that is membership and nonmembership and hence are more suitable in various situations.

(iii) Our study and results are in fact of the generalization of the corresponding results of FG*s*. Our results will be shifted into the corresponding results of FGS by ignoring the second membership grades. In this way, the corresponding results of FG*s* will be taken as a special case of the results for IFG*s* developed by us.

(iv) Our work shows that IFG*s* have more information than FG*s*.

Data Availability

All data and materials used in this research are included in the study, and there are no other supplementary data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] J. A. Goguen, "L-fuzzy sets," *Journal of Mathematical Analysis and Applications*, vol. 18, no. 1, pp. 145–174, 1967.

[3] R. E. Bellman and L. A. Zadeh, "Decision-making in a fuzzy environment," *Management Science*, vol. 17, no. 4, pp. B–141, 1970.

[4] R. Sambuc, *Fonctions ϕ-floues. Application l'aide au diagnostic en pathologie thyroidienne*, Ph. D. Thesis Univ, Marseille, France, 1975.

[5] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-I," *Information Sciences*, vol. 8, no. 3, pp. 199–249, 1975.

[6] D.-A. Chiang and N. P. Lin, "Correlation of fuzzy sets," *Fuzzy Sets and Systems*, vol. 102, no. 2, pp. 221–226, 1999.

[7] S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani, and T. Mahmood, "Spherical fuzzy sets and their applications in multi-attribute decision making problems," *Journal of Intelligent and Fuzzy Systems*, vol. 36, no. 3, pp. 2829–2844, 2019.

[8] T. Mahmood, K. Ullah, Q. Khan, and N. Jan, "An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets," *Neural Computing & Applications*, vol. 31, no. 11, pp. 7041–7053, 2019.

[9] K. Ullah, T. Mahmood, Z. Ali, and N. Jan, "On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition," *Complex & Intelligent Systems*, vol. 6, no. 1, pp. 15–27, 2020.

[10] M. Kochen, "Applications of Fuzzy Sets in Psychology," in *Proceedings of the fuzzy Sets and Their Applications to Cognitive and Decision Processes*, July 1975.

[11] B. Farhadinia, S. Effati, and F. Chiclana, "A family of similarity measures for q-rung orthopair fuzzy sets and their applications to multiple criteria decision making," *International Journal of Intelligent Systems*, vol. 36, no. 4, pp. 1535–1559, 2021.

[12] A. Rosenfeld, "Fuzzy Graphs," *Fuzzy Graphs and Fuzzy Hypergraphs. Studies in Fuzziness and Soft Computing*, Physica, Heidelberg, Baden-Württemberg, Germany, 1975.

[13] J. N. Mordeson and P. Chang-Shyh, "Operations on fuzzy graphs," *Information sciences*, vol. 79, no. 3-4, pp. 159–170, 1994.

[14] P. Bhattacharya, "Some remarks on fuzzy graphs," *Pattern Recognition Letters*, vol. 6, no. 5, pp. 297–302, 1987.

[15] A. Somasundaram and S. Somasundaram, "Domination in fuzzy graphs - I," *Pattern Recognition Letters*, vol. 19, no. 9, pp. 787–791, 1998.

[16] M. Akram, "Bipolar fuzzy graphs," *Information Sciences*, vol. 181, no. 24, pp. 5548–5564, 2011.

[17] M. Akram, "Bipolar fuzzy graphs with applications," *Knowledge-Based Systems*, vol. 39, pp. 1–8, 2013.

[18] M. Akram and W. A. Dudek, "Regular bipolar fuzzy graphs," *Neural Computing & Applications*, vol. 21, no. 1, pp. 197–205, 2012.

[19] M. Sitara, M. Akram, and M. Yousaf Bhatti, "Fuzzy graph structures with application," *Mathematics*, vol. 7, no. 1, p. 63, 2019.

[20] A. N. A. Koam, M. Akram, and P. Liu, "Decision-making analysis based on fuzzy graph structures," *Mathematical Problems in Engineering*, vol. 2020, Article ID 6846257, 30 pages, 2020.

[21] M. Pal, S. Samanta, and G. Ghorai, "Few Applications of Fuzzy Graphs," *Modern Trends in Fuzzy Graph Theory*, Springer, Singapore, pp. 275–305, 2020.

[22] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[23] T. Gerstenkorn and J. Mańko, "Correlation of intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 44, no. 1, pp. 39–43, 1991.

[24] P. P. Angelov, "Optimization in an intuitionistic fuzzy environment," *Fuzzy Sets and Systems*, vol. 86, no. 3, pp. 299–306, 1997.

[25] E. Szmidt and J. Kacprzyk, "Distances between intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 114, no. 3, pp. 505–518, 2000.

[26] K. T. Atanassov, "Interval Valued Intuitionistic Fuzzy Sets," *Intuitionistic Fuzzy Sets*, Physica, Heidelberg, Baden-Württemberg, Germany, pp. 139–177, 1999.

[27] D.-F. Li, "Multiattribute decision making models and methods using intuitionistic fuzzy sets," *Journal of Computer and System Sciences*, vol. 70, no. 1, pp. 73–85, 2005.

[28] B. Davvaz and E. Hassani Sadrabadi, "An application of intuitionistic fuzzy sets in medicine," *International Journal of Biomathematics*, vol. 09, no. 03, Article ID 1650037, 2016.

[29] F. Xiao, "A distance measure for intuitionistic fuzzy sets and its application to pattern classification problems," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 6, pp. 3980–3992, 2019.

[30] D. Rani and H. Garg, "Distance measures between the complex intuitionistic fuzzy sets and their applications to the decision-making process," *International Journal for Uncertainty Quantification*, vol. 7, no. 5, 2017.

[31] K. T. Atanassov, *On intuitionistic fuzzy sets theory*, Springer, Heidelberg, Germany, 2012.

[32] M. Akram and B. Davvaz, "Strong intuitionistic fuzzy graphs," *Filomat*, vol. 26, no. 1, pp. 177–196, 2012.

[33] M. G. Karunambigai, M. Akram, S. Sivasankar, and K. Palanivel, "Balanced intuitionistic fuzzy graphs," *Applied Mathematical Sciences*, vol. 7, no. 51, pp. 2501–2514, 2013.

[34] M. Akram and W. A. Dudek, "Intuitionistic fuzzy hypergraphs with applications," *Information Sciences*, vol. 218, pp. 182–193, 2013.

[35] M. Akram and N. O. Alshehri, "Intuitionistic fuzzy cycles and intuitionistic fuzzy trees," *The Scientific World Journal*, vol. 2014, Article ID 305836, 11 pages, 2014.

[36] M. G. Karunambigai, R. Parvathi, and R. Buvaneswari, "Arcs in intuitionistic fuzzy graphs," *Notes on Intuitionistic fuzzy sets*, vol. 18, no. 4, pp. 48–58, 2012.

[37] B. Davvaz, N. Jan, T. Mahmood, and K. Ullah, "Intuitionistic fuzzy graphs of nth type with applications," *Journal of Intelligent and Fuzzy Systems*, vol. 36, no. 4, pp. 3923–3932, 2019.

[38] R. Parvathi and G. Thamizhendhi, "Domination in intuitionistic fuzzy graphs," *Notes on Intuitionistic Fuzzy Sets*, vol. 16, no. 2, pp. 39–49, 2010.

[39] A. Nagoorgani, M. Akram, and S. Anupriya, "Double domination on intuitionistic fuzzy graphs," *Journal of Applied Mathematics and Computing*, vol. 52, no. 1, pp. 515–528, 2016.

[40] Q. Sun, J. Wu, F. Chiclana, H. Fujita, and E. Herrera-Viedma, "A dynamic feedback mechanism with attitudinal consensus threshold for minimum adjustment cost in group decision making," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 5, p. 1, 2021.

[41] Y. Xing, M. Cao, Y. Liu, M. Zhou, and J. Wu, "A Choquet integral based interval type-2 trapezoidal fuzzy multiple attribute group decision making for sustainable supplier selection," *Computers & Industrial Engineering*, vol. 165, Article ID 107935, 2022.

[42] M. Akram, A. Ashraf, and M. Sarwar, "Novel applications of intuitionistic fuzzy digraphs in decision support systems," *The Scientific World Journal*, vol. 2014, Article ID 904606, 11 pages, 2014.

[43] S. Shahzadi and M. Akram, "Intuitionistic fuzzy soft graphs with applications," *Journal of Applied Mathematics and Computing*, vol. 55, no. 1, pp. 369–392, 2017.

[44] Z. Shao, S. Kosari, H. Rashmanlou, and M. Shoaib, "New concepts in intuitionistic fuzzy graph with application in water supplier systems," *Mathematics*, vol. 8, no. 8, p. 1241, 2020.

[45] H. Wiener, "Structural determination of paraffin boiling points," *Journal of the American Chemical Society*, vol. 69, no. 1, pp. 17–20, 1947.

[46] M. Binu, S. Mathew, and J. N. Mordeson, "Connectivity index of a fuzzy graph and its application to human trafficking," *Fuzzy Sets and Systems*, vol. 360, pp. 117–136, 2019.

[47] M. Binu, S. Mathew, and J. N. Mordeson, "Wiener index of a fuzzy graph and application to illegal immigration networks," *Fuzzy Sets and Systems*, vol. 384, pp. 132–147, 2020.

[48] S. Kalathian, S. Ramalingam, S. Raman, and N. Srinivasan, "Some topological indices in fuzzy graphs," *Journal of Intelligent and Fuzzy Systems*, vol. 39, no. 5, pp. 6033–6046, 2020.

[49] T. Naeem, A. Gumaei, M. Kamran Jamil, A. Alsanad, and K. Ullah, "Connectivity indices of intuitionistic fuzzy graphs and their applications in internet routing and transport network flow," *Mathematical Problems in Engineering*, vol. 2021, Article ID 4156879, 16 pages, 2021.

[50] A. R. Ashrafi, H. Shabania, and M. V. Diudeab, "Balaban index of dendrimers," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 69, pp. 151–158, 2013.

[51] W. Gao, M. K. Siddiqui, M. Imran, M. K. Jamil, and M. R. Farahani, "Forgotten topological index of chemical structure in drugs," *Saudi Pharmaceutical Journal*, vol. 24, no. 3, pp. 258–264, 2016.

[52] W. Gao, W. F. Wang, and M. R. Farahani, "Topological indices study of molecular structure in anticancer drugs," *Journal of Chemistry*, vol. 2016, Article ID 3216327, 8 pages, 2016.

[53] W. A. Alzoubi, "AS'AD mahmoud, "A study on connectivity concepts in intuitionistic fuzzy graphs"," *WSEAS Transactions on Systems and Control*, vol. 16, 2021.