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Research Article

Nonlinear Stochastic SIS Epidemic Model Incorporating Lévy Process

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In this work, we study a stochastic SIS epidemic model with Lévy jumps and nonlinear incidence rates. Firstly, we present our proposed model and its parameters. We establish sufficient conditions for the extinction and persistence of the disease in the population using some stochastic analysis background. We illustrate our theoretical results by numerical simulations. We conclude that the white noise and Lévy jump influence the transmission of the epidemic.

1. Introduction and Preliminary

For a long time, infectious diseases have been the cause of disappointment of many people in the world, and only very few of these diseases have disappeared, despite the development of medicine and the change in the lifestyle of human beings. Therefore, several scientists have concentrated their research on the study of the transmission mechanisms of these diseases and have proposed relevant solutions in order to reduce the contamination by these infectious diseases. Also, several mathematical epidemic models are proposed to describe the dynamics of infectious diseases in human populations and to study the complex behavior of these diseases. Among the models proposed, the classic SIR epidemic model of Kermack and McKendrick is widely used [1] which divides the population into three classes, namely, susceptible (S), infected (I), and recovered (R). As a result, other works have generalized the Kermack–McKendrick (see, for example, [2–8]) model. On the other hand, for some diseases such as bacterial diseases and some sexually transmitted diseases, the SIR model is not suitable because the individuals infected with these diseases start to be susceptible, at a certain stage get the disease, and after a short infectious period become susceptible again [9, 10]. Therefore, the SIS epidemic model [11–13] is often used to model the dynamics of these specific diseases. Then, the SIS epidemic model is represented by the following ordinary differential equations:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = A - \rho S - \beta SI + \delta I, \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta SI - (\rho + \theta + \delta)I, \end{cases} \tag{1}$$

where $S(t)$ and $I(t)$ represent the number of susceptible and infected individuals, respectively. A represents the recruitment rate of susceptible, β denotes the transmission coefficient of diseases, ρ represents the natural death rate for susceptible and infected classes, θ is the disease-related death rate, and δ denotes the recovery rate.

The quantity βSI is the disease incidence rate, which represents the number of new cases per unit of time. Many authors have used the bilinear incidence to model disease transmission. But, in many cases, the bilinear incidence is not preferable (for example, when the population is saturated [14]). So, the nonlinear incidence can better model the nonlinear transmission of epidemics. Swati in [15] proposed a fractional-order epidemic model and modeled the transmission of disease by the Beddington–DeAngelis incidence rate. In [16], Lu et al. introduced a nonmonotone incidence

rate into an epidemic model composed of three classes of individuals (susceptible, infectious, and recovered). Rajasekar and Zhu [17] examined the impact of media coverage on a SIRS epidemic model with relapse. Therefore, several nonlinear incidences have been proposed (see Table 1). In the present paper, we model the disease transmission by a nonlinear incidence $\beta\phi(S,I)$, where ϕ satisfies the following conditions.

$(C)\phi(S,I)$ is two-order continuously differentiable for any $S(t)$, $I(t)\geq 0$. For each fixed $I\geq 0$, $\phi(S,I)$ is increasing for $S>0$ and for each fixed $S\geq 0$, $\phi(S,I)/I$ is decreasing for $I>0$. $\phi(S,0)=\phi(0,I)=0$ for any $S,I>0$, and $\partial\phi(S_0,0)/\partial I>0$, with $S_0=A/\rho$.

In mathematical modeling, the stochastic systems show more precisely the reality by including the environmental effects, which are an essential aspect in biological environments. So, epidemic models are often subject to random noises (see [4]). For this reason, many works have studied the effect of white noise on deterministic systems. Tornatore et al. in [22] studied the effect of white noise on the SIR epidemic model, and they presented the model by a stochastic differential system. In [23], the author has examined the effect of environmental fluctuations on an epidemic model by affecting some parameters in the model by the white noise. Hussain et al. [24] investigated a stochastic epidemic model with white noise for the transmission of coronavirus. They showed sufficient conditions for the extinction and existence of stationary distribution by employing some stochastic calculus background. To reasonably measure the influence of environmental noise on disease transmission, we assume that parameter β is perturbed by the white noise as follows:

$$\beta\longrightarrow\beta+\sigma\dot{M}_B(t), \tag{2}$$

where $M_B(t)$ is a standard Brownian motion and σ represent the intensities of white noise. Then, we represent the stochastic model corresponding to deterministic model (1) by the following stochastic differential equation system:

$$\begin{cases} \mathrm{d}S(t)=(A-\rho S-\beta\phi(S,I)+\delta I)\mathrm{d}t-\sigma\phi(S,I)\mathrm{d}M_B(t),\\ \mathrm{d}I(t)=(\beta\phi(S,I)-(\rho+\theta+\delta)I)\mathrm{d}t+\sigma\phi(S,I)\mathrm{d}M_B(t). \end{cases} \tag{3}$$

Stochastic differential equations with white noise represent many advantages in modeling infectious diseases. But, in reality, the biological systems are frequently attacked by abrupt and massive disturbances such as natural disasters: volcanoes, tsunamis, earthquakes, and pandemics (SARS, COVID-19, Ebola, and so on). These events may break the continuity of the solution [4, 25, 26]. Then, to describe these events, it is necessary to integrate a jump process [27] in the stochastic system (3).

Thus, to properly describe the reality, we use the Lévy jump process which can well model the sudden and massive fluctuations; also, we perturb the parameter β by two environmental noises (white noise and Lévy noise) as follows:

$$\beta\longrightarrow\beta+\sigma\dot{M}_B(t)+\dot{\mathscr{Y}}(t), \tag{4}$$

where $M_B(t)$ is an independent standard Brownian motion, σ is the intensity of $M_B(t)$, and $\dot{\mathscr{Y}}(t)=\int_0^t\int_{\mathbb{E}}\eta(t)\tilde{N}(\mathrm{d}t,\mathrm{d}l)$. Then, we present the stochastic version corresponding to model (3) by the following stochastic differential equation system driven with Lévy jumps:

$$\begin{cases} \mathrm{d}S(t)=(A-\rho S-\beta\phi(S,I)+\delta I)\mathrm{d}t-\sigma\phi(S,I)\mathrm{d}M_B(t)-\displaystyle\int_{\mathbb{E}}\eta(l)\phi(S((t-),I(t-)\tilde{N}(\mathrm{d}t,\mathrm{d}l)),\\ \mathrm{d}I(t)=(\beta\phi(S,I)-(\rho+\theta+\delta)I)\mathrm{d}t+\sigma\phi(S,I)\mathrm{d}M_B(t)+\displaystyle\int_{\mathbb{E}}\eta(l)\phi(S(t-),I((t-))\tilde{N}(\mathrm{d}t,\mathrm{d}l), \end{cases} \tag{5}$$

where $S(t-)$ and $I(t-)$ are the left limits of $S(t)$ and $I(t)$, respectively. $\tilde{N}(\mathrm{d}t,\mathrm{d}l)=N(\mathrm{d}t,\mathrm{d}l)-\nu(l)\mathrm{d}t$, N is a Poisson counting measure with characteristic measure ν on measurable subset $\mathbb{E}$ of $[0,\infty)$, with $\nu(\mathbb{E})<\infty$, and $\eta\colon\mathbb{E}\times\Omega\longrightarrow R$ represents the effect of random jumps; it is bounded and continuous with respect to ν and $\mathfrak{B}(\mathbb{E})\times\mathscr{F}_t$-measurable.

Throughout this paper, let $(\Omega,\mathscr{F},\{\mathscr{F}\}_{t\geq 0},\mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets), and we suppose that the Brownian motion $M_B(t)$ is defined on the complete probability space $(\Omega,\mathscr{F},\{\mathscr{F}\}_{t\geq 0},\mathbb{P})$.

For equation (5) to admit a unique global solution, it must satisfy the linear growth condition and the local Lipschitz condition [28]. In effect, equation (5) satisfies the local Lipschitz condition and not the linear growth condition. Therefore, the solution of system (3) will explode in finite time. So, to ensure the global existence and uniqueness of the solution, we propose as in [4] the following assumptions:

$(\mathbf{C}_1)$ For each $N>0$, there exists $L_N>0$ such that

$$\int_{\mathbb{E}}|K(x,\alpha)-K(y,\alpha)|^2\nu(dl)\leq L_N|x-y|^2, \tag{6}$$

and $|x|\vee|y|\leq N$, with $K(x',l)=\eta(l)x'$ for $x'=S(t-)I(t-)$.

$(\mathbf{C}_2)$ $0\leq A/\rho\eta(l)<1$, for $l\in\mathbb{E}$.

The following region:

$$\mathscr{I}=\left\{(S,I)\in\mathbb{R}_+^2\colon S+I\leq\frac{A}{\rho}\right\}, \tag{7}$$

is almost surely positively invariant by stochastic system (3), namely, if $(S(0),I(0))\in\mathscr{I}$, then $(S(t),I(t))\in\mathscr{I}\ \forall t\geq 0\ a.s.$

Table 1: Some nonlinear incidence rates.

Incidence name	Expression	Reference
Standard incidence rate	$\beta SI/N$	[18]
Saturated incidence rate	$\beta SI/1+kI$	[14]
Beddington–DeAngelis functional response	$\beta SI/1+k_1S+k_2I$	[19]
Crowley–Martin functional response	$\beta SI/1+k_1S+k_2I+k_1k_2SI$	[20]
Incidence with media coverage effect	$\beta_1-\beta_2 I/I+m$	[21]

Theorem 1. *For any initial condition $(S(0),I(0))\in\mathcal{I}$, there exists a unique positive solution $(S(t),I(t))\in\mathcal{I}\forall t\geq 0$ a.s.*

Let $V_m=\inf_{(S,I)\in\mathcal{T}}\phi(S,I)/I$.

Definition 1. System (3) is said to be persistent in the mean, if

$$\liminf_{t\longrightarrow\infty}\frac{1}{t}\int_0^t I(r)\mathrm{d}r>0\ a.s. \tag{8}$$

Lemma 1. *Let $f\in\mathcal{C}([0,\infty)\times t\Omega n;q(0,\infty))$. If there exist positive constants m_1, m_2, and T, such that*

$$\ln f(t)\geq m_1t-m_2\int_0^t f(x)\mathrm{d}x+F(t)\ a.s.\quad for all\ t\geq T, \tag{9}$$

where $F\in([0,\infty)\times\Omega;\mathbb{R})$ and $\lim_{t\longrightarrow\infty}F(t)/t=0$ a.s., then $\liminf_{t\longrightarrow\infty}\langle f(t)\rangle\geq m_1/m_2$ a.s.

Lemma 2 (see [29]). *Suppose that (**C**) hold. For all $s>z>0$, define*

$$\mathbb{H}=\left\{(S,I)\in\mathbb{R}_+^2|z\leq S+I\leq s\right\}. \tag{10}$$

Then,

$$\begin{aligned}&\max_{(S,I)\in\mathbb{H}}\left\{\frac{\phi((S,I))}{S},\frac{\phi((S,I))}{I}\right\}<\infty,\\&\max_{(S,I)\in\mathbb{H}}\left\{\left|\frac{1}{I}\frac{\partial\phi(S,I)}{\partial I}-\frac{\phi(S,I)}{I^2}\right|,\left|\frac{1}{I}\frac{\partial\phi(S,I)}{\partial S}\right|\right\}<\infty.\end{aligned} \tag{11}$$

The differential operator $\mathscr{L}$ (see [30]) associated with the following stochastic differential equation with Lévy process:

$$\mathrm{d}x(t)=f(x(t),t)\mathrm{d}t+g(x(t),t)\mathrm{d}M_B(t)+\int_{\mathbb{E}}h(x(t-),l)\widetilde{N}(\mathrm{d}t,\mathrm{d}l), \tag{12}$$

is defined by

$$\begin{aligned}\mathscr{L}x(t-)=&\frac{\partial x(t-)}{\partial t}+\sum_{i=1}^n\frac{\partial x(t-)}{\partial x_i}f_i(x,t)+\frac{1}{2}\sum_{i,j=1}^n\frac{\partial^2x(t-)}{\partial x_i\partial x_j}\left[g^T(x,t)g(x,t)\right]_{ij}\\&+\int_{\mathbb{E}}\left[(x(t-)+h(x(t-),l))-x(t-)-\frac{\partial x(t-)}{\partial x_i}h(x(t-),l)\right]\nu(\mathrm{d}l).\end{aligned} \tag{13}$$

If $\mathscr{L}$ acts on a function $F\in\mathcal{C}^{1,2}(\mathbb{R}^n\times\mathbb{R}_+;\mathbb{R}_+)$, then

$$\begin{aligned}\mathscr{L}F(x(t))=&F_t(x(t-))+F_x(x(t-))f(x(t-),t)\\&+\frac{1}{2}\mathrm{trace}\left[g^T(x(t-),t)F_{xx}(x(t-))g(x(t-),t)\right]_{ij}\\&+\int_{\mathbb{E}}\left[F(x(t-)+h(x(t-),l)-F(x(t-))-F_x(x(t-))h(x(t-),l)\right]\nu(\mathrm{d}l),\end{aligned} \tag{14}$$

where $F_t=\partial F/\partial t$, $F_x=(\partial F/\partial x_1,\ldots,\partial F/\partial x_n)$, $F_{xx}=(\partial^2F/\partial x_i\partial x_j)_{nn}$.

Then, generalized Itô's formula (for more details, see [31]) is presented by

$$dH(x(t)) = \mathscr{L}F(x(t-))dt + F_x(x(t-)g(x(t),t)dM_B(t) + \int_Z [F(x(t-) + h(x(t-),l)) - F(x(t-))]\tilde{N}(dt,dl). \tag{15}$$

The goal of this work is the proposition of conditions for the extinction and persistence of diseases. For this, we define a threshold number that coincides with the basic reproduction number of the deterministic model when the stochastic terms are absent and determine the extinction or persistence of disease. Moreover, it is important to note that our system (3) generalizes many models existing in the literature (for example, see [32–34]). In addition, our model (3) represents the impact of massive events on the transmission of disease and gives an additional degree of realism compared with the deterministic model and stochastic model with white noise. The organization of this paper is as follows. In Section 2, we give sufficient conditions for the extinction of the disease. Persistence in mean results is explored in Section 3. In Section 4, the analytical results are illustrated with the support of numerical examples. Finally, we close the article with a conclusion.

2. Extinction

In this section, we show sufficient conditions for the extinction of the disease of system (3) with the Lévy process.

We know that for deterministic systems, we should determine the extinction or persistence of disease according to the value of $\mathscr{R}_0$ (basic reproduction number). That is, if $\mathscr{R}_0$ is less than one, the disease dies out. In contrast, if $\mathscr{R}_0$ is greater than one, the disease persists. Likewise, we express the following threshold of our stochastic SIS epidemic model (3) with Lévy jumps as follows:

$$\mathscr{R}_{lj} = \beta\frac{\partial\phi(S_0,0)}{\partial I}\frac{1}{(\rho+\theta+\delta)} - \overline{\eta}\left(\frac{\partial\phi(S_0,0)}{\partial I}\right)^2\frac{1}{(\rho+\theta+\delta)} = \mathscr{R}_0 - \overline{\eta}\left(\frac{\partial\phi(S_0,0)}{\partial I}\right)^2\frac{1}{(\rho+\theta+\delta)}, \tag{16}$$

where $\overline{\eta} = (\sigma^2/2 + \int_{\mathbb{E}}\eta^2(l)/2(1+\eta(l)\partial\phi(S_0,0)/\partial I)^2\nu(dl))$.

Remark 1. The threshold $\mathscr{R}_{lj}$ coincides with the basic reproduction number $\mathscr{R}_0$ of the corresponding deterministic system in the absence of the noise coefficient.

Theorem 2. *Under the assumptions* (**C**). *Let* $(S(t),I(t))$ *be the solution of model (3) with any initial value* $(S(0),I(0)) \in \mathscr{I}$:

(i) If $\mathscr{R}_{lj} < 1$ *and* $\partial\phi(S_0,0)/\partial I \le \beta/\overline{\eta}$, *then*

$$\limsup_{t\longrightarrow\infty}\frac{\log I(t)}{t} \le (\rho+\theta+\delta)\left[\mathscr{R}_{lj}-1\right] < 0 \ a.s. \tag{17}$$

(ii) If $\sigma^2/2 + \int_{\mathbb{E}}\eta^2(l)/2(1+\eta(l)\partial\phi(S_0,0)/\partial I)^2\nu(dl) > \beta^2/2(\rho+\theta+\delta)$, *then*

$$\limsup_{t\longrightarrow\infty}\frac{\log I(t)}{t} \le \frac{\beta^2}{2}\left(\frac{\sigma^2}{2} + \int_{\mathbb{E}}\frac{\eta^2(l)}{2(1+\eta(l)\partial\phi(S_0,0)/\partial I)^2}\nu(dl)\right)^{-1} - (\rho+\theta+\delta) < 0 \ a.s. \tag{18}$$

In others word, $I(t)$ will go to zero almost surely. That is, the disease will be extinct almost surely.

Proof

(i) Using generalized Itô's formula, one can see that

$$d\log I = \left[\beta\frac{\phi(S,I)}{I} - (\rho+\theta+\delta) - \frac{\sigma^2}{2}\left(\frac{\phi(S,I)}{I}\right)^2\right]dt + \int_{\mathbb{E}}\left[\log\left(1+\eta(l)\frac{\phi(S,I)}{I}\right) - \eta(l)\frac{\phi(S,I)}{I}\right]\nu(dl) + \sigma\frac{\phi(S,I)}{I}dM_B(t) + \int_{\mathbb{E}}\log\left(1+\eta(l)\frac{\phi(S,I)}{I}\right)\tilde{N}(dt,dl). \tag{19}$$

Integrating both *sides* from 0 to t and dividing by t, we get

$$\frac{\log I(t)}{t} = \frac{\log I(0)}{t} + \frac{1}{t}\int_0^t\left[\beta\frac{\phi(S(r),I(r))}{I(r)} - (\rho+\theta+\delta) - \frac{\sigma^2}{2}\left(\frac{\phi(S(r),I(r))}{I(r)}\right)^2\right]dr + \frac{1}{t}\int_0^t\int_{\mathbb{E}}\left[log\left(1+\eta(l)\frac{\phi(S(r-),I(r-))}{I(r-)}\right) - \eta(l)\frac{\phi(S(r-),I(r-))}{I(r-)}\right]\nu(dl)dr + \frac{1}{t}\int_0^t\int_{\mathbb{E}}\log\left(1+\eta(l)\frac{\phi(S(r-),I(r-))}{I(r-)}\right)\tilde{N}(dr,dl) + \frac{1}{t}\int_0^t\sigma\frac{\phi(S(r),I(r))}{I(r)}dM_B(r). \tag{20}$$

Using the Taylor–Lagrange formula, one can see that

$$\begin{aligned}\log\left(1+\eta(l)\frac{\phi(S,I)}{I}\right)-\eta(l)\frac{\phi(S,I)}{I}&=-\frac{\eta^2(l)(\phi(S,I)/I)^2}{2(1+\eta(l)\phi(S,I)/I)^2}\\&\le-\frac{\eta^2(l)}{2\left(1+\eta(l)\partial\phi(S_0,0)/\partial I\right)^2}\left(\frac{\phi(S,I)}{I}\right)^2.\end{aligned}\tag{21}$$

Therefore,

$$\begin{aligned}\frac{\log I(t)}{t}\le\frac{\log I(0)}{t}+\frac{1}{t}\int_0^t\Bigg[\beta\frac{\phi(S(r),I(r))}{I(r)}-(\rho+\theta+\delta)\\-\left(\frac{\sigma^2}{2}+\int_{\mathbb{E}}\frac{\eta^2(l)}{2\left(1+\eta(0)\partial\phi(S_0,0)/\partial I\right)^2}\nu(\mathrm{d}l)\right)\left(\frac{\phi(S(r),I(r))}{I(r)}\right)^2\Bigg]\mathrm{d}r\\+\frac{1}{t}\int_0^t\int_{\mathbb{E}}\log\left(1+\eta(l)\frac{\phi(S(r-),I(r-))}{I(r-)}\right)\tilde{N}(\mathrm{d}r,\mathrm{d}l)\\+\frac{1}{t}\int_0^t\sigma\frac{\phi(S(r),I(r))}{I(r)}\mathrm{d}M_B(r).\end{aligned}\tag{22}$$

Since the function

$$h(z)=-(\rho+\theta+\delta)+\beta z-\left(\frac{\sigma^2}{2}+\int_{\mathbb{E}}\frac{\eta^2(l)}{2\left(1+\eta(l)\partial\phi(S_0,0)/\partial I\right)^2}\nu(\mathrm{d}l)\right)z^2,\tag{23}$$

is monotone *increasing* for all $z\in[0,\beta/(\sigma^2/2+\int_{\mathbb{E}}\eta^2(l)/2(1+\eta(l)\partial\phi(S_0,0)/\partial I)^2\nu(dl))]$, employing condition (i) and the inequality $\phi(S,I)/I\le\partial\phi(S_0,0)/\partial I$, we obtain

$$\begin{aligned}\frac{\log I(t)}{t}\le\frac{\log I(0)}{t}+\frac{1}{t}\int_0^t\Bigg[\beta\frac{\partial\phi(S_0,0)}{\partial I}-(\rho+\theta+\delta)\\-\left(\frac{\sigma^2}{2}+\int_{\mathbb{E}}\frac{\eta^2(l)}{2\left(1+\eta(l)\partial\phi(S_0,0)/\partial I\right)^2}\nu(\mathrm{d}l)\right)\left(\frac{\partial\phi(S_0,0)}{\partial I}\right)^2\Bigg]\mathrm{d}r\\+\frac{\mathscr{M}(t)}{t}+\frac{\mathscr{G}(t)}{t},\end{aligned}\tag{24}$$

where

$$\begin{aligned}\mathscr{M}(t)&=\int_0^t\int_{\mathbb{E}}\log\left(1+\eta((l))\frac{\phi(S(r-),I(r-))}{I(r-)}\right)\tilde{N}(\mathrm{d}r,\mathrm{d}l),\\\mathscr{G}(t)&=\int_0^t\sigma\frac{\phi(S(r-),I(r-))}{I(r-)}\mathrm{d}M_B(r).\end{aligned}\tag{25}$$

Then,

$$\begin{aligned}\langle\mathscr{M}(t),\mathscr{M}(t)\rangle&=\int_0^t\int_{\mathbb{E}}\left[\log\left(1+\eta(l)\frac{\phi(S(r-),I(r-))}{I(r-)}\right)\right]^2\nu(\mathrm{d}l)\mathrm{d}r\\&\le t\left[\log\left(1+\eta(l)\frac{\partial\phi(S_0,0)}{\partial I}\right)\right]^2\nu(\mathbb{E})<\infty,\end{aligned}\tag{26}$$

and

$$\langle \mathcal{G}(t), \mathcal{G}(t) \rangle = \int_0^t \left[\sigma \frac{\phi(S(r), I(r))}{I(r)} \right]^2 \mathrm{d}r \leq t \left[\sigma \frac{\partial \phi(S_0, 0)}{\partial I} \right]^2 < \infty. \tag{27}$$

According to the strong law of large numbers for martingales [28], we have

$$\begin{aligned} \lim_{t \longrightarrow \infty} \frac{\mathcal{M}(t)}{t} &= 0, \\ \lim_{t \longrightarrow \infty} \frac{\mathcal{G}(t)}{t} &= 0 \, a.s. \end{aligned} \tag{28}$$

Taking the limit superior on the both sides of (24) and combining with (28), we get

$$\begin{aligned} &\operatorname*{limsup}_{\mathrm{t} \longrightarrow \infty} \frac{\log \mathrm{I}(\mathrm{t})}{\mathrm{t}} \\ &\leq \beta \frac{\partial \phi(\mathrm{S}_0, 0)}{\partial \mathrm{I}} - (\rho + \theta + \delta) - \left(\frac{\sigma^2}{2} + \int_{\mathbb{E}} \frac{\eta^2(\mathrm{l})}{2(1 + \eta(\mathrm{l}) \partial \phi(\mathrm{S}_0, 0)/\partial \mathrm{I})^2} \nu(\mathrm{dl}) \right) \left(\frac{\partial \phi((\mathrm{S}_0, 0))}{\partial \mathrm{I}} \right)^2 \\ &\triangleq (\rho + \theta + \delta) \left[\mathscr{R}_{\mathrm{lj}} - 1 \right] < 0 \, a.s, \end{aligned} \tag{29}$$

which implies that

$$\lim_{t \longrightarrow \infty} I(t) = 0 \, a.s. \tag{30}$$

(ii) Using (4), we have

$$\begin{aligned} \frac{\log I(t)}{t} \leq & \frac{\log I(0)}{t} + \frac{1}{t} \int_0^t \left[\beta \frac{\phi(S(r), I(r))}{I(r)} - (\rho + \theta + \delta) \right. \\ & \left. - \left(\frac{\sigma^2}{2} + \int_{\mathbb{E}} \frac{\eta^2(l)}{2(1 + \eta(l) \partial \phi(S_0, 0)/\partial I)^2} \nu(dl) \right) \left(\frac{\phi(S(r), I(r))}{I(r)} \right)^2 \right] \mathrm{d}r \\ & + \frac{1}{t} \int_0^t \int_{\mathbb{E}} \log \left(1 + \eta(l) \frac{\phi(S(r-), I(r-))}{I(r-)} \right) \tilde{N}(\mathrm{d}r, \mathrm{d}l) \\ & + \frac{1}{t} \int_0^t \sigma \frac{\phi(S(r), I(r))}{I(r)} \mathrm{d}M_B(r) \\ = & \frac{\log I(0)}{t} + \frac{1}{t} \int_0^t \left[-\frac{1}{2} \xi \left(Z - \frac{\beta}{\xi} \right)^2 + \frac{\beta^2}{2\xi} - (\rho + \theta + \delta) \right] \mathrm{d}r, \end{aligned} \tag{31}$$

where $\xi = (\sigma^2/2 + \int_{\mathbb{E}} \eta^2(l)/2(1 + \eta(l) \partial \phi(S_0, 0)/\partial I)^2 \nu(\mathrm{d}l))$ and $Z = \phi(S, I)/I$. Then,

$$\frac{\log I(t)}{t} \leq \frac{\log I(0)}{t} + \frac{\beta^2}{2\xi} - (\rho + \theta + \delta) + \frac{1}{t}\int_0^t \int_{\mathbb{E}} \log\left(1 + \eta(l)\frac{\phi((S(r-), I(r-))}{I(r-)}\right)\widetilde{N}(\mathrm{d}r, \mathrm{d}l) + \frac{1}{t}\int_0^t \sigma \frac{\phi(S(r), I(r))}{I(r)}\mathrm{d}M_B(r). \tag{32}$$

By taking the superior on both sides of (32), we obtain

$$\operatorname{limsup}\frac{\text{limits}}{t} \longrightarrow \infty \frac{\log I(t)}{\mathrm{t}} \leq \frac{\beta^2}{2}\left(\frac{\sigma^2}{2} + \int_{\mathbb{E}} \frac{\eta^2(\mathrm{l})}{2\left(1 + \eta(\mathrm{l})\partial\phi\left(\mathrm{S}_0, 0\right)/\partial \mathrm{I}\right)^2}\nu(\mathrm{d}l)\right)^{-1} - (\rho + \theta + \delta)\ a.s. \tag{33}$$

This completes the proof of the theorem. □

3. Persistence

In this section, we present sufficient conditions for the persistence in mean of disease in model (3). So, we have the following result.

Theorem 3. *Assume that* (**C**) *hold. If* $\mathscr{R}_{I'j} > 1$, *then for any given initial value* $(S(0), I(0)) \in \mathscr{F}$, *the solution of (3) satisfies*

$$\liminf_{\mathrm{t}\longrightarrow\infty} \frac{1}{\mathrm{t}}\int_0^t \mathrm{I}(\mathrm{r})\mathrm{d}r \geq \mathrm{K}_3^{-1}(\rho + \theta + \delta)\left[\mathscr{R}_{\mathrm{I'j}} - 1\right] > 0\ a.s., \tag{34}$$

where K_3 *is a positive constant.*

Proof. From system (3), we have

$$\frac{A}{\rho} - \frac{1}{t}\int_0^t S(r)\mathrm{d}r = \frac{(\rho + \theta)}{\rho}\frac{1}{t}\int_0^t I(r)\mathrm{d}r + \frac{\varpi((t))}{\rho}, \tag{35}$$

where $\varpi(t) = S(t) - S(0)/t + I(t) - I(0)/t$. Using Itô's formula and the fact that $\phi(S, I)/I \leq \partial\phi(S_0, 0)/\partial I$, we get

$$\begin{aligned} d\log\ I &\geq \left[\beta\frac{\phi(S, I)}{I} - (\rho + \theta + \delta) - \left(\frac{\sigma^2}{2} + \int_{\mathbb{E}} \frac{\eta^2(l)}{2\left(1 + \eta(l)V_m\right)^2}\nu(\mathrm{d}l)\right)\left(\frac{\partial\phi(S_0, 0)}{\partial I}\right)^2\right]\mathrm{d}t \\ &\quad + \sigma\frac{\phi(S, I)}{I}\mathrm{d}M_B(t) + \int_{\mathbb{E}} \log\left(1 + \eta(l)\frac{\phi(S, I)}{I}\right)\widetilde{N}(\mathrm{d}t, \mathrm{d}l) \\ &= \left[\beta\frac{\partial\phi(S_0, 0)}{\partial I} - (\rho + \theta + \delta) - \left(\frac{\sigma^2}{2} + \int_{\mathbb{E}} \frac{\eta^2(l)}{2\left(1 + \eta(l)V_m\right)^2}\nu(dl)\right)\left(\frac{\partial\phi(S_0, 0)}{\partial I}\right)^2\right]\mathrm{d}t \\ &\quad + \left[\beta\frac{\phi(S, I)}{I} - \beta\frac{\partial\phi(S_0, 0)}{\partial I}\right]\mathrm{d}t + \sigma\frac{\phi(S, I)}{I}\mathrm{d}M_B(t) \\ &\quad + \int_{\mathbb{E}} \log\left(1 + \eta(l)\frac{\phi(S, I)}{I}\right)\widetilde{N}(\mathrm{d}t, \mathrm{d}l). \end{aligned} \tag{36}$$

Using Lagrange's mean value *theorem*, we obtain

$$\frac{\phi(S(t),I(t))}{I(t)}-\frac{\partial\phi(S_0,0)}{\partial I}=\left[\frac{1}{\Lambda_2(t)}\frac{\partial\phi(\Lambda_1(t),\Lambda_2(t))}{\partial I}-\frac{\phi(\Lambda_1(t),\Lambda_2(t))}{\Lambda_2^2(t)}\right]I(t)+\frac{1}{\Lambda_2(t)}\frac{\partial\phi(\Lambda_1(t),\Lambda_2(t))}{\partial S}(S(t)-S_0), \tag{37}$$

with $\Lambda_1(t)\in(S(t),S_0)$ and $\Lambda_2(t)\in(0,I(t))$. *Consequently*, from (37), one can derive that

$$\begin{aligned}\mathrm{dlog}\ I\geq\Bigg[&\beta\frac{\partial\phi(S_0,0)}{\partial I}-(\rho+\theta+\delta)-\left(\frac{\sigma^2}{2}+\int_{\mathbb{E}}\frac{\eta^2(l)}{2(1+\eta(l)V_m)^2}\nu(\mathrm{dl})\right)\left(\frac{\partial\phi(S_0,0)}{\partial I}\right)^2\\&+\beta\left(\frac{1}{\Lambda_2(t)}\frac{\partial\phi(\Lambda_1(t),\Lambda_2(t))}{\partial I}-\frac{\phi(\Lambda_1(t),\Lambda_2(t))}{\Lambda_2^2(t)}\right)I(t)\\&+\beta\frac{1}{\Lambda_2(t)}\frac{\partial\phi(\Lambda_1(t),\Lambda_2(t))}{\partial S}(S(t)-S_0)\Bigg]\mathrm{d}t+\sigma\frac{\phi(S,I)}{I}\mathrm{d}M_B(t)\\&+\int_{\mathbb{E}}\log\left(1+\eta(l)\frac{\phi(S,I)}{I}\right)\widetilde{N}(\mathrm{d}t,\mathrm{d}l).\end{aligned} \tag{38}$$

According to Lemma 2 and *since* $(\Lambda_1(t),\Lambda_2(t))\in\mathcal{F}$ *a.s.*, then

$$\begin{aligned}\frac{1}{\Lambda_2(t)}\frac{\partial\phi(\Lambda_1(t),\Lambda_2(t))}{\partial I}-\frac{\phi(\Lambda_1(t),\Lambda_2(t))}{\Lambda_2^2(t)}&\geq-K_1,\\\frac{1}{\Lambda_2(t)}\frac{\partial\phi(\Lambda_1(t),\Lambda_2(t))}{\partial S}(S(t)-S_0)&\leq K_2,\end{aligned} \tag{39}$$

with

$$\max_{(S,I)\in\mathbb{L}}\left\{\left|\frac{1}{I}\frac{\partial\phi(S,I)}{\partial I}-\frac{\phi(S,I)}{I}\right|\right\}=K_1, \tag{40}$$

and

$$\max_{(S,I)\in\mathbb{L}}\left\{\left|\frac{1}{I}\frac{\partial\phi(S,I)}{\partial S}\right|\right\}=K_2. \tag{41}$$

Injecting (11) in (10), we get

$$\begin{aligned}\mathrm{dlog}\ I\geq\Bigg\{&\beta\frac{\partial\phi(S_0,0)}{\partial I}-(\rho+\theta+\delta)-\left(\frac{\sigma^2}{2}+\int_{\mathbb{E}}\frac{\eta^2(l)}{2(1+\eta(l)V_m)^2}\nu(\mathrm{dl})\right)\left(\frac{\partial\phi(S_0,0)}{\partial I}\right)^2\\&-\beta[K_1I(t)+K_2(S_0-S(t))]\Bigg\}\mathrm{d}t+\sigma\frac{\phi(S,I)}{I}\mathrm{d}M_B(t)\\&+\int_{\mathbb{E}}\log\left(1+\eta(l)\frac{\phi(S,I)}{I}\right)\widetilde{N}(\mathrm{d}t,\mathrm{d}l).\end{aligned} \tag{42}$$

Integrating both sides of the *above* inequality from 0 to t and dividing by t, we have

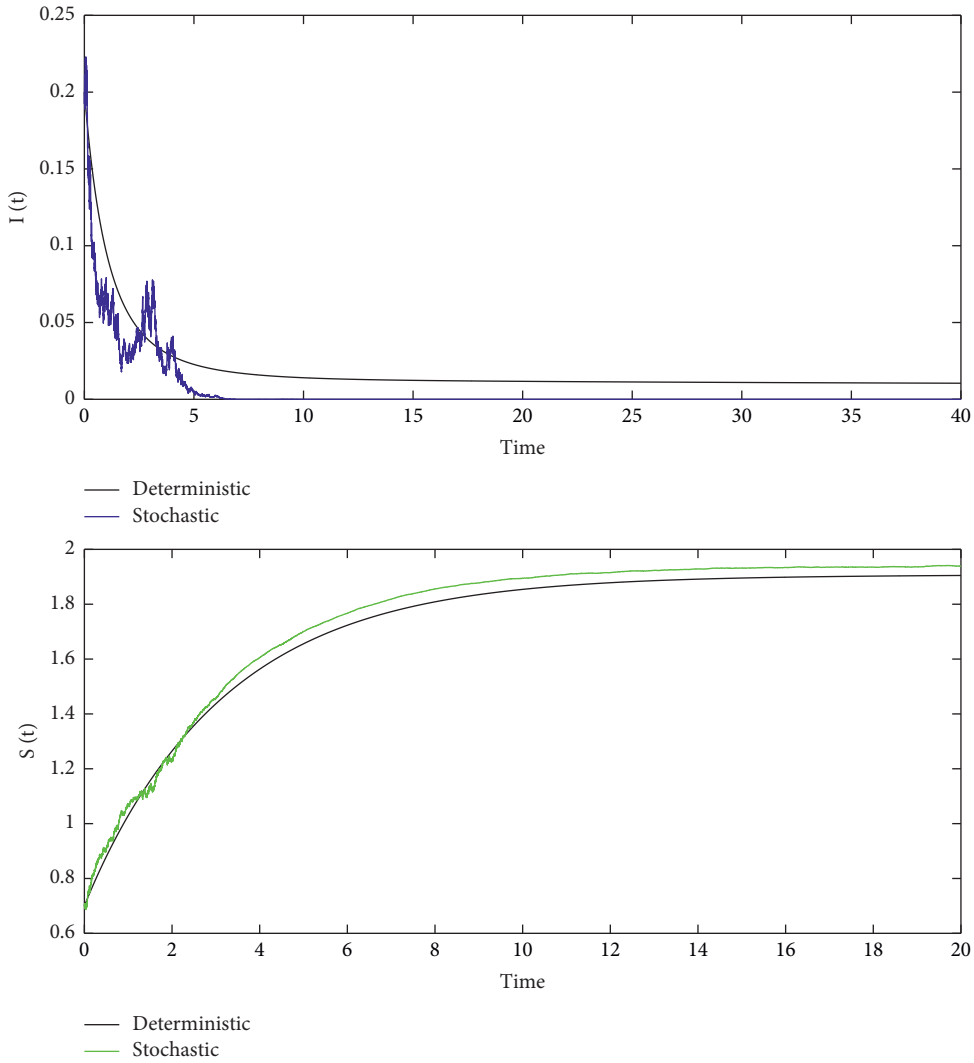

FIGURE 1: Comparison of the trajectory in stochastic system (3) ($I(t)$: blue graph and $S(t)$: green graph) and deterministic system (1) ($I(t)$: black graph and $S(t)$: black graph) for the extinction case.

$$\begin{aligned}\frac{\log I(t)}{t}-\frac{\log I(0)}{t}&\geq\beta\frac{\partial\phi(S_0,0)}{\partial I}-(\rho+\theta+\delta)\\&-\left(\frac{\sigma^2}{2}+\int_{\mathbb{E}}\frac{\eta^2(l)}{2(1+\eta(l)V_m)^2}\nu(\mathrm{d}l)\right)\left(\frac{\partial\phi(S_0,0)}{\partial I}\right)^2\\&-\beta\left[K_1\frac{1}{t}\int_0^t I(r)\mathrm{d}r+K_2\left(S_0-\frac{1}{t}\int_0^t S(r)dr\right)\right]\\&+\frac{1}{t}\int_0^t\sigma\frac{\phi(S,I)}{I}\mathrm{d}M_B(r)+\frac{1}{t}\int_0^t\\&\int_{\mathbb{E}}\log\left(1+\eta(l)\frac{\phi(S,I)}{I}\right)\widetilde{N}(\mathrm{d}r,\mathrm{d}l).\end{aligned}\tag{43}$$

In view of (35), we obtain

$$\begin{aligned}\frac{\log I(t)}{t}&\geq\beta\frac{\partial\phi(S_0,0)}{\partial I}-(\rho+\theta+\delta)\\&-\left(\frac{\sigma^2}{2}+\int_{\mathbb{E}}\frac{\eta^2(l)}{2(1+\eta(l)V_m)^2}\nu(\mathrm{d}l)\right)\left(\frac{\partial\phi(S_0,0)}{\partial I}\right)^2\\&-\beta\left[K_1+K_2\frac{\rho+\theta}{\rho}\right]\frac{1}{t}\int_0^t I(r)\mathrm{d}r+\pi(t),\end{aligned}\tag{44}$$

where

$$\begin{aligned}\pi(t)=&-\frac{\beta K_2}{\rho}\varpi(t)+\frac{\log I(0)}{t}+\frac{1}{t}\int_0^t\sigma\frac{\phi(S,I)}{I}\mathrm{d}M_B(r)\\&+\frac{1}{t}\int_0^t\int_{\mathbb{E}}\log\left(1+\eta(l)\frac{\phi(S,I)}{I}\right)\widetilde{N}(\mathrm{d}r,\mathrm{d}l).\end{aligned}\tag{45}$$

According to the large number theorem for local martingales [28] and the fact that $S, I\in\mathcal{I}$, we have

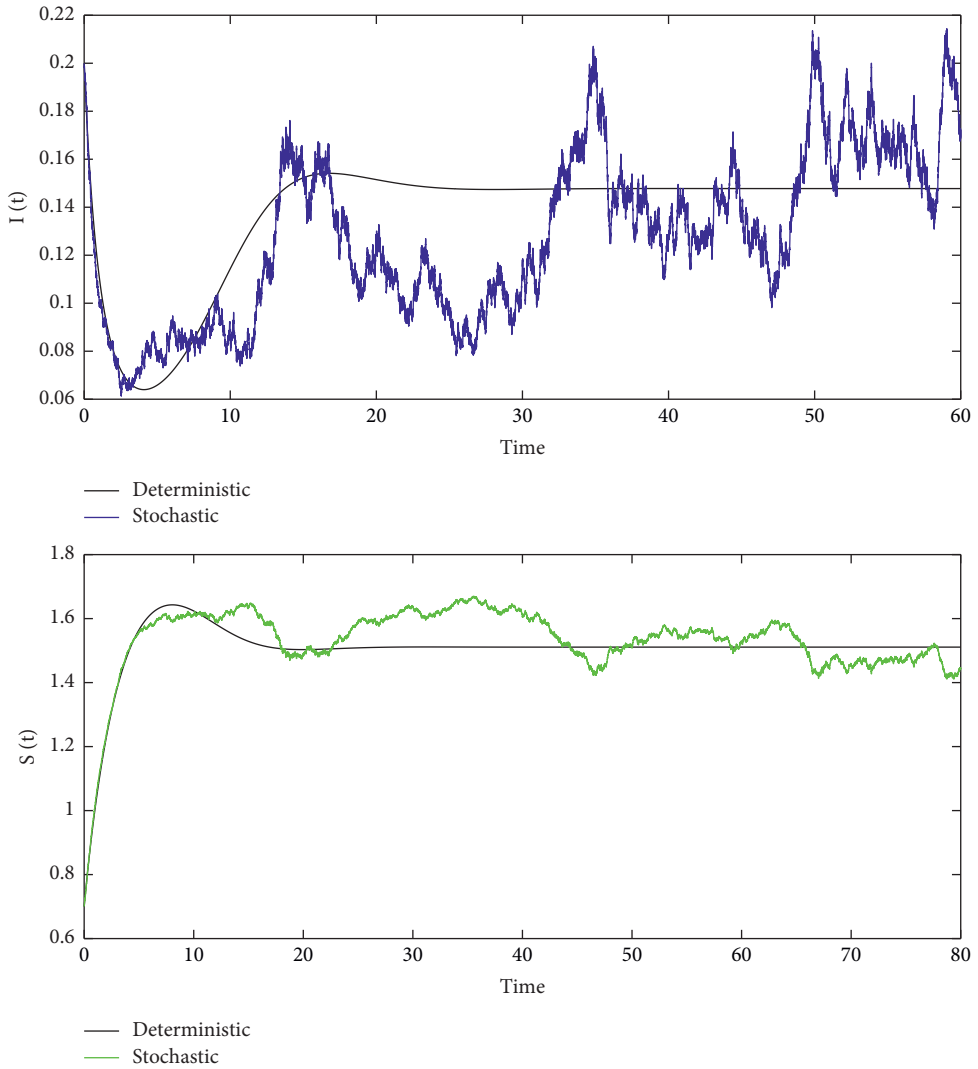

Figure 2: Comparison of the trajectory in stochastic system (3) ($I(t)$: blue graph and $S(t)$: green graph) and deterministic system (1) ($I(t)$: black graph and $S(t)$: black graph) for the persistence case.

$$\lim_{t\longrightarrow\infty} \pi(t) = 0 \; a.s. \tag{46}$$

According to Lemma 1, we obtain the following inequality:

$$\begin{aligned}\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t I(r)dr \geq K_3^{-1}\left[\beta\frac{\partial\phi(S_0,0)}{\partial I} - (\rho+\theta+\delta)\right.\\ \left.-\left(\frac{\sigma^2}{2} + \int_{\mathbb{E}} \frac{\eta^2(l)}{2(1+\eta(l)V_m)^2}\nu(dl)\right)\left(\frac{\partial\phi(S_0,0)}{\partial I}\right)^2\right],\end{aligned} \tag{47}$$

where $K_3 = \beta[K_1 + K_2\rho + \theta/\rho]$. Hence,

$$\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t I(r)dr \geq K_3^{-1}(\rho+\theta+\delta)\left[\mathscr{R}_{1'j} - 1\right] > 0 \; a.s. \tag{48}$$

□

Remark 2. The condition $\mathscr{R}_{1'j} > 1$ implies that the reproduction number is also greater than one, and this means that when the disease in stochastic system (3) persists, it can also persist in deterministic system (1).

4. Numerical Application

In this section, we give some simulations to support the theoretical results presented in this paper. For this, we use the Euler scheme described in [35]. In the figures, the black lines represent solutions of a deterministic system (1), the blue lines are the paths of $S(t)$ for stochastic system (3) with Lévy jumps, and the green lines are the paths of $I(t)$ for stochastic system (3) with Lévy jumps. In model (3), we take $\phi(S, I) = \beta SI/1 + kI$, which is the saturated incidence rate introduced by Capasso and Serio [14]. We can easily show that ϕ satisfies the assumptions (**C**). Then, we have

$$\mathscr{R}_{lj} = \frac{\beta S_0}{(\rho+\theta+\delta)} - \overline{\eta} S_0^2 \frac{1}{(\rho+\theta+\delta)} = \mathscr{R}_0 - \overline{\eta} S_0^2 \frac{1}{(\rho+\theta+\delta)}. \tag{49}$$

Hence, we have the following corollary of Theorem 3.

Corollary 1. *Under the assumptions* (**C**)*, let* $(S(t), I(t))$ *be the solution of model (3) with any initial value* $(S(0), I(0)) \in$*:*

(i) If $\mathscr{R}_{lj} < 1$ *and* $\partial\phi(S_0, 0)/\partial I \le \beta/\overline{\eta}$*, then*

$$\limsup_{t\longrightarrow\infty} \frac{\log I(t)}{t} \le (\rho+\theta+\delta)\left[\mathscr{R}_{lj} - 1\right] < 0 \; a.s. \tag{50}$$

(ii) If $\sigma^2/2 + \int_{\mathbb{E}} \eta^2(l)/2(1+\eta(l)\partial\phi(S_0,0)/\partial I)^2 \nu(dl) > \beta^2/2(\rho+\theta+\delta)$*, then*

$$\limsup_{t\longrightarrow\infty} \frac{\log I(t)}{t} \le \frac{\beta^2}{2}\left(\frac{\sigma^2}{2} + \int_{\mathbb{E}} \frac{\eta^2(l)}{2(1+\eta(l)\partial\phi(S_0,0)/\partial I)^2}\nu(dl)\right)^{-1} - (\rho+\theta+\delta) < 0 \; a.s. \tag{51}$$

In others word, $I(t)$ will go to zero almost surely. That is, the disease will be extinct almost surely.

4.1. Extinction Case. Take the parameters in stochastic system (3) as follows: $A = 0.66$, $\rho = 0.34$, $\beta = 0.7$, $k = 0.1$, $\theta = 0.65$, $\delta = 0.35$, $\sigma = 0.7$, and $\eta(l) = 0.05$. By simple computation, we obtain $\mathscr{R}_{lj} = 0.8295 < 1$ and $\mathscr{R}_0 = 1.0140$. Then, the condition of Theorem 2 holds. Hence, one can observe that disease is extinct. Figure 1 demonstrates this result. From a comparative point of view, we remark that in Figure 1, epidemic I tends to zero for the stochastic system (blue graph) and not for the deterministic system (black graph). Thus, the epidemic does not disappear from the population if there is no Lévy process effect. Deduce that Lévy jumps can significantly influence the properties of the system and can drive the disease to disappear (see Figure 1).

4.2. Persistence Case. In this case, we save the same parameter values employed in the extinction case. Also, we choose the noise values as follows: $\sigma = 0.1$ and $\eta(l) = 0.02$. By calculation, we get $\mathscr{R}_{l'j} = 1.2998 > 1$. Therefore, it follows from Theorem 3 that disease $I(t)$ persists in the mean with probability one. Figure 2 shows this result. So, the disease disappears when the values of the noise terms are not interesting.

Finally, the numerical simulation in Figures 1 and 2 clarifies the dynamics of the diseases as a function of time for two different values of the noise parameters. Then, you can see that the large value of noises parameters can remove the disease from the population.

5. Conclusion

This paper studies a stochastic SIS epidemic model with nonlinear incidence rate and Lévy jumps. Under assumption (**C**), we prove the following results:

(1) If $\mathscr{R}_{lj} < 1$ and $\partial\phi(S_0, 0)/\partial I \le \beta/\overline{\eta}$, then

$$\limsup_{t\longrightarrow\infty} \frac{\log I(t)}{t} \le (\rho+\theta+\delta)\left[\mathscr{R}_{lj} - 1\right] < 0 \; a.s. \tag{52}$$

Thus, the disease$I$ dies out with probability one.

(2) If $\sigma^2/2 + \int_{\mathbb{E}} \eta^2(l)/2(1+\eta(l)\partial\phi(S_0,0)/\partial I)^2 \nu(dl) > \beta^2/2(\rho+\theta+\delta)$, then

$$\limsup_{t\longrightarrow\infty} \frac{\log I(t)}{t} \le \frac{\beta^2}{2}\left(\frac{\sigma^2}{2} + \int_{\mathbb{E}} \frac{\eta^2(l)}{2(1+\eta(l)\partial\phi(S_0,0)/\partial I)^2}\nu(dl)\right)^{-1} - (\rho+\theta+\delta) < 0 \; a.s. \tag{53}$$

Thus, the disease I dies out with probability one.

(3) If $\mathscr{R}_{l'j} > 1$, then the disease persists in mean.

For our epidemic model (3), we have established the generalized basic reproduction number noted $\mathscr{R}_{lj}$ and concluded that the noise coefficient can eliminate the disease, that is, if the white noise value is large and $\eta(l) > 0$, the disease goes extinct. On the other hand, if the value of the noise parameters is very low, the disease persists in the population. So, white noise and Lévy noise can control the spread of disease in the population.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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