Hindawi
Complexity
Volume 2022, Article ID 8153249, 12 pages
https://doi.org/10.1155/2022/8153249

*Research Article*

# A Brief Survey of the Graph Wavelet Frame

**Jie Zhou [1] and Zeze Zhang [2]**

[1] *School of Science, Xi'an Polytechnic University, Xi'an 710048, Shaanxi, China*
[2] *Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710072, Shaanxi, China*

Correspondence should be addressed to Jie Zhou; zhoujie0506@126.com

Received 30 May 2022; Accepted 23 August 2022; Published 3 October 2022

Academic Editor: Sigurdur F. Hafstein



In recent years, the research of wavelet frames on the graph has become a hot topic in harmonic analysis. In this paper, we mainly introduce the relevant knowledge of the wavelet frames on the graph, including relevant concepts, construction methods, and related theory. Meanwhile, because the construction of graph tight framelets is closely related to the classical wavelet framelets on $\mathbb{R}$, we give a new construction of tight frames on $\mathbb{R}$. Based on the pseudosplines of type II, we derive an MRA tight wavelet frame with three generators $\psi_1, \psi_2$, and $\psi_3$ using the oblique extension principle (OEP), which generate a tight wavelet frame in $L_2(\mathbb{R})$. We analyze that three wavelet functions have the highest possible order of vanishing moments, which matches the order of the approximation order of the framelet system provided by the refinable function. Moreover, we introduce the construction of the Haar basis for a chain and analyze the global orthogonal bases on a graph $\mathscr{G}$. Based on the sequence of framelet generators in $L_2(\mathbb{R})$ and the Haar basis for a coarse-grained chain, the decimated tight framelets on graphs can be constructed. Finally, we analyze the detailed construction process of the wavelet frame on a graph.

## 1. Introduction

In the past two centuries, harmonic analysis, including Fourier analysis and wavelet analysis, has been intensively researched and widely applied in areas, such as signal processing, representation theory, and number theory. Since Daubechies constructed the compactly supported orthonormal wavelet bases in 1988 [1], wavelet analysis has been extensively studied and successfully applied to more fields. For an orthonormal wavelet basis, the corresponding refinable function $\phi$ and its mask $a$ must satisfy the following conditions:

$$\langle \phi(\cdot - k), \phi \rangle = \int_{\mathbb{R}} \phi(x-k)\overline{\phi(x)}\mathrm{d}x = \delta_k, \quad k \in \mathbb{Z}, \tag{1}$$

$$|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 = 1, \quad \forall \xi \in \mathbb{R}. \tag{2}$$

These two conditions enforce very restrictive constraints for the refinable function, and many refinable functions do not satisfy the conditions. By introducing redundancy into a wavelet system, a tight wavelet frame is considered as a generalization of an orthonormal wavelet basis [2–4]. It is much easier and more flexible to construct tight wavelet frames than orthonormal wavelet bases. This theory of the construction of tight wavelet frames on regular Euclidean domains is relatively mature [5–11]. In recent years, driven by the rapid progress of deep learning and their successful applications in an interdisciplinary area, data in deep learning are typically from social networks, biology, physics, finance, etc., and can be naturally organized as graphs [12–22]. There has been a great interest in developing harmonic analysis for data defined on non-Euclidean domains such as manifold data or graph data [23–26]. Such data are usually regarded as random samples from a smooth manifold. The matrix is used to organize it as an undirected graph, where the graph Laplacian approximates the manifold Laplacian. The underlying manifold encodes the geometric information of the data, which has been widely used in various machine learning and statistical models [27–32].

Different from that on Euclidean domains, the construction systems and the corresponding fast algorithm for tight framelets on graphs are less studied. The main reason is that we do not know how to define the operators of translation and dilation on the manifolds or graphs similar to the classical wavelet framelet systems. In order to further study the construction of tight framelets on non-Euclidean domains, some alternative approaches are proposed. By introducing diffusion operators, Coifman and Maggioni in [33] constructed orthogonal diffusion wavelets on a smooth manifold. Maggioni and Mhaskar in [34] further extended the construction from diffusion wavelets to diffusion polynomial frames on manifolds. In recent years, with the development of computer science and mathematics, the classical spectral theory has been introduced into the construction of wavelet framework on the graph. The eigenvalues and eigenvectors of the graph Laplacian matrix can be effectively used to define translation and dilation transformation of graph functions, and the construction of tight framelets on graphs can be obtained [23, 24, 35–37]. Based on spectral theory and graph Laplacian [38], some work about the construction of tight framelets on graphs has been derived [23–25, 35–37, 39–44]. In this paper, we focus on the review of a number of representative construction methods of tight framelets on graphs and provide a specific introduction for wavelet frame on graphs.

We organize the rest of the paper as follows: in Section 2, we introduce some basic theoretical knowledge, including the classical wavelet frame theory, graph, chain, and tight framelets on the graph, and review some construction methods of tight framelets on graphs; in Section 3, we construct three wavelet functions $\psi_1, \psi_2$, and $\psi_3$, which generate a tight wavelet frame in $L_2(\mathbb{R})$, and give a specific example of the construction process of wavelet frame on the graph; and in Section 4, we conclude this paper and discuss future challenges of the graph wavelet frame.

## 2. Wavelet Frame on the Graph

The construction method of wavelet frames on the graph is to design the wavelet function on the graph and its corresponding translation and dilation transformation of graph functions. With the spectral graph theory, by considering the spectral decomposition of the graph Laplacian, Hammond in [35] defined translation and dilation transformation on a graph and constructed the spectral graph wavelets and spectral graph scaling functions based on an indicator function. However, the spectral scaling functions in this way did not generate the graph wavelets, which is analogous to the construction of traditional wavelets through the two-scale relation. Hence, the associated construction of wavelet frames on the graph did not have the filter bank. Dong in [36] defined the quasi-affine systems on a given manifold by considering generalized translation and dilation transformation of wavelet functions in $L_2(\mathbb{R})$, and constructed wavelet frames on both compact manifold and discrete graphs. These wavelet functions are defined from the scaling functions, which implies that the associated construction of wavelet frames on the graph has the filter bank.

However, the graph wavelet frames constructed by the above two methods are the undecimated wavelet frames. When the level number of framelet transforms is big, the undecimated wavelet frames may result in a high redundancy rate. In contrast, using clustering techniques, the decimated framelet system can be constructed by embedding edge relation and clustering feature in a chain-based orthonormal system and would achieve a low redundancy rate. Chui [23] first defined an orthogonal system based on a local filtration and constructed the different orthogonal system at the vertices at each level. Later, Chui [24] further extended the work to directed graphs and established an orthogonal system with localization properties on the tree. Based on graph clustering algorithms [45–50], a suitable orthonormal eigenpair can be constructed for a coarse-grained chain on the graph, and decimated tight framelets can be constructed based on the orthonormal eigenpair in [37]. By introducing a Haar basis for a coarse-grained chain on a graph, the graph neural networks and graph pooling under the Haar basis can be defined in [25, 39], and achieved low computational cost when the graph size is large.

The construction of decimated tight framelets on the graph first utilizes graph clustering to obtain the coarse-grained chain. Once a chain of nested graphs is obtained, we can build the multiscale structure on a graph by framelet filter banks, which bridges with the classical wavelet framelets systems on $\mathbb{R}$. In this paper, we focus on the construction of decimated tight framelets on the graph. In the following, we will review relevant knowledge for the construction methods of decimated tight framelets on the graph.

*2.1. Tight Wavelet Frames on* $\mathbb{R}$. In this subsection, we first recall some basic theories of the tight wavelet frame on the $L_2(\mathbb{R})$, which are the basis for the construction of decimated tight framelets [5, 8, 9]. A function $\phi$ is a refinable function if it satisfies the following refinement equation:

$$\phi = 2\sum_{k\in\mathbb{Z}} a_k \phi(2\cdot - k), \tag{3}$$

where $a_k$ is called the mask for the refinable function $\phi$, which is a finitely supported sequence on $\mathbb{Z}$. The Fourier series of a sequence $a_k$ is defined to be

$$\widehat{a}(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{-i\xi k}, \quad \xi \in \mathbb{R}. \tag{4}$$

The Fourier transform of a function $f \in L_1(\mathbb{R})$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} \mathrm{d}x, \quad \xi \in \mathbb{R}, \tag{5}$$

and can be naturally extended to $L_2(\mathbb{R})$ functions. In terms of the Fourier transform, the refinement equation in (3) can be rewritten as follows:

$$\widehat{\phi}(\xi) = \widehat{a}\left(\frac{\xi}{2}\right)\widehat{\phi}\left(\frac{\xi}{2}\right), \quad \xi \in \mathbb{R}. \tag{6}$$

Throughout this paper, we assume $\widehat{a}(0) = \widehat{\phi}(0) = 1$. According to (6), we have $\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$.

We say that a set $\{\psi^1, \ldots, \psi^r\}$ of functions in $L_2(\mathbb{R})$ generates a tight wavelet frame in $L_2(\mathbb{R})$ if

$$\|f\|^2 = \sum_{l=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left|\langle f, \psi^l_{j,k}\rangle\right|^2, \quad \forall f \in L_2(\mathbb{R}), \tag{7}$$

where $\psi^l_{j,k} = 2^{(j/2)}\psi^l(2^j \cdot -k)$, $\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\mathrm{d}x$ and $\|f\|^2 = \langle f, f\rangle$. The set $\{\psi^1, \ldots, \psi^r\}$ is called a set of generators for the corresponding tight wavelet frame. For any function $f \in L_2(\mathbb{R})$, we have the wavelet expansion

$$f = \sum_{l=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi^l_{j,k}\rangle \psi^l_{j,k}. \tag{8}$$

By introducing redundancy into a wavelet system, it has a lot of freedom in the construction of tight wavelet frames derived from refinable function. Petukhov [7] showed that if the mask $a$ satisfies $|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 \le 1$ for all $\xi \in \mathbb{R}$, then a symmetric tight wavelet frame with three generators can be obtained. Jiang [8] systematically analyzed the construction of hexagonal tight wavelet frame filter banks with three high-pass filters. For the construction of the wavelet frame, the oblique extension principle is considered as a popular approach, which was first proposed in the literature [5], and the specific contents are as follows.

**Theorem 1.** *Let $\phi$ be a compactly supported refinable function in $L_2(\mathbb{R})$ with a finitely supported mask a on $\mathbb{Z}$ such that $\widehat{\phi}(0) = \widehat{a}(0) = 1$ and $\widehat{a}(\pi) = 0$ . Suppose that there exist finitely supported sequences $b_1, \ldots, b_r$ and a $2\pi$ -periodic trigonometric polynomial $\Theta$ such that $\Theta(0) = 1$ and*

$$\begin{bmatrix} \widehat{b}_1(\xi) & \cdots & \widehat{b}_r(\xi) \\ \widehat{b}_1(\xi+\pi) & \cdots & \widehat{b}_r(\xi+\pi) \end{bmatrix} \begin{bmatrix} \overline{\widehat{b}_1(\xi)} & \overline{\widehat{b}_1(\xi+\pi)} \\ \vdots & \vdots \\ \overline{\widehat{b}_r(\xi)} & \overline{\widehat{b}_r(\xi+\pi)} \end{bmatrix} = M_\Theta(\xi), \tag{9}$$

*where*

$$M_\Theta(\xi) = \begin{bmatrix} \Theta(\xi) - \Theta(2\xi)|\widehat{a}(\xi)|^2 & -\Theta(2\xi)\widehat{a}(\xi)\overline{\widehat{a}(\xi+\pi)} \\ -\Theta(2\xi)\widehat{a}(\xi+\pi)\overline{\widehat{a}(\xi)} & \Theta(\xi+\pi) - \Theta(2\xi)|\widehat{a}(\xi+\pi)|^2 \end{bmatrix}. \tag{10}$$

The *wavelet functions* $\psi^1, \ldots, \psi^r$ is defined as follows:

$$\widehat{\psi^l}(\xi) = \widehat{b}_l\left(\frac{\xi}{2}\right)\widehat{\phi}\left(\frac{\xi}{2}\right), \quad l = 1, \ldots, r. \tag{11}$$

Then, $\{\psi^1, \ldots, \psi^r\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$. Moreover, $\{\psi^1, \ldots, \psi^r\}$ has $m$ vanishing moments if and only if

$$\Theta(\xi) - \Theta(2\xi)|\widehat{a}(\xi)|^2 = O\left(|\xi|^{2m}\right), \quad \xi \longrightarrow 0. \tag{12}$$

The construction in Theorem 1 is called the oblique extension principle. Daubechies et al. in [5] discussed the importance of the nonconstant $\Theta$ in this principle, which provides a tight wavelet frame with good vanishing moments. The order of vanishing moments is an important property of a tight wavelet frame. For a set $\{\psi^1, \ldots, \psi^r\}$ of compactly supported functions in $L_2(\mathbb{R})$, it has vanishing moments of order $m$ if

$$\int_{\mathbb{R}} t^j \psi^l(t)\mathrm{d}t = 0 \quad \forall l = 1, \ldots r;\ j = 0, \ldots, m-1. \tag{13}$$

The related theory of compactly supported functions with good vanishing moments can be found in references [9, 51]. Moreover, some scholars have also studied the construction of multiwavelet and multiwavelet frames [52–56]. Based on two-direction refinable functions, Li and Yang [52] studied the construction of dual multiwavelet frames with symmetry and discussed the vanishing moment of the constructed multiwavelet frames. Atreas et al. [53] discussed homogeneous dual multiwavelet frames from a pair of refinable function vectors and derived that the mixed oblique extension principle can be described by dual multiwavelet frames. For OEP-based tight framelets, Han [54] also extended to high dimensional case and constructed compactly supported tight $M$-wavelet frames from compactly supported $M$-refinable functions for any $d \times d$ dilation matrix $M$. Based on the approach of polyphase matrix extension of multiscaling vectors, Cen et al. [55] presented a novel approach for the construction of symmetric compactly supported bi-orthogonal multiwavelets with multiplicity of 2. Li and Peng [56] analyzed the sampling property of bi-orthogonal multiwavelets and discussed the application of bi-orthogonal multiwavelets in image compression filter bank. In addition, some researchers investigated the construction and application of multivariate wavelet frames. The interested readers are referred to the literature [57–65].

*2.2. Graph and Chain.* In this subsection, we introduce some basic conceptions about a graph and chain [6, 37].

A graph $\mathcal{G} = (V, E, \omega)$ is made up of a set of vertices $V = \{v_1, \ldots, v_n\}$ and a set of edges $E \subseteq V \times V$ between vertices. The non-negative function $\omega$: $E \longrightarrow \mathbb{R}$ indicates weight of edges between vertices. $\omega(v_i, v_j) \ne 0$ if there is an edge from the vertex $v_i$ to vertex $v_j$; otherwise, 0. If we ignore the directionality of the edges, the graph is called an undirected graph. In this case, weight $\omega$ is symmetric, that is, $\omega(v_i, v_j) = \omega(v_j, v_i)$ for all $v_i, v_j \in V$. Otherwise, the graph $\mathcal{G}$ is said to be a directed graph. The degree of a vertex $v \in V$ is denoted as $d(v) = \sum_{p\in V} \omega(v, p)$. The sum of degrees of all vertices of $\mathcal{G}$ is denoted as vol $(\mathcal{G}) :=$ vol $(V) = \sum_{v\in V} d(v)$, which is the volume of the graph [6, 37]. For a subset $V_0$ of $V$, the volume $V_0$ is the sum of degrees of all nodes in $V_0$.

Let $(e_1, e_2, \ldots, e_n)$ be a sequence of edges in $\mathcal{G}$. If there exist distinct vertices $v_0, \ldots v_n$ in $V$ such that any pair of consecutive nodes is connected by the edges of $\mathcal{G}$, that is, $e_i = (v_{i-1}, v_i)$ for $i = 1, 2, \ldots, n$, then the sequence $(e_1, e_2, \ldots, e_n)$ is called a path of the graph $\mathcal{G}$ between $v_0$ and $v_n$. The length of the path is defined to be $\sum_{i=1}^{n} \omega(v_{i-1}, v_i)$. If there exists a path between the vertex $v_i$ and vertex $v_j$, the length of the shortest possible path is defined as the distance between them, denoted as $\rho(v_i, v_j)$. If there is no path between vertices $v_i$ and $v_j$, we define the distance $\rho(v_i, v_j) = \infty$. A graph is said to be connected if any two distinct vertices of $\mathscr{G}$ are connected [6, 37].

Let $\mathscr{G} = (V, E, \omega)$ and $\mathscr{G}_c = (V_c, E_c, \omega_c)$ be two graphs; we say that $\mathscr{G}_c$ is the coarse-grained graph of $\mathscr{G}$ if $V_c$ is a partition of $V$. In this case, there exists subsets $V_1, \ldots, V_k$ of $V$ for some $k \in \mathbb{N}$ such that

$$\begin{gathered} V_c = \{V_1, \ldots, V_k\}, \\ V_1 \cup \cdots \cup V_k = V, \\ V_i \cap V_j = \emptyset,\ 1 \leq i < j \leq k. \end{gathered} \tag{14}$$

That is, each vertex $V_j$ of $\mathscr{G}_c$ is called a cluster for $\mathscr{G}$. The edges of $\mathscr{G}_c$ are the links between clusters of $\mathscr{G}$. We define that two vertices $v_i$ and $v_j$ are equivalent, if $v_i$ and $v_j$ are in the same cluster, denoted by $v_i \sim v_j$ [6]. Generally, we use $[v]$ to denote a cluster in $\mathscr{G}$ with respect to a vertex in the coarse-grained graph $\mathscr{G}_c$.

Let $J, J_0, J \geq J_0$ be two integers; a coarse-grained chain $\mathscr{G}_{J\longrightarrow J_0} := (\mathscr{G}_J, \mathscr{G}_{J-1}, \ldots, \mathscr{G}_{J_0})$ of $\mathscr{G}$ is a sequence of graphs with $\mathscr{G}_J \equiv \mathscr{G}$. Each $\mathscr{G}_j = (V_j, E_j, \omega_j)$ is a coarse-grained graph of $\mathscr{G}$ for all $J_0 \leqslant j \leqslant J$, and $[v]_{\mathscr{G}_j} \subseteq [v]_{\mathscr{G}_{j-1}}$ for all $j = J_0 + 1, \ldots, J$ and all $v \in \mathscr{G}$. The graph $\mathscr{G}_j$ is the level $j$ graph of the chain $\mathscr{G}_{J\longrightarrow J_0}$, and the $\mathscr{G}_{j-1}$ can be viewed as a coarse-grained graph of $\mathscr{G}_j$ for $j = J_0 + 1, \ldots, J$. If $\mathscr{G}_j \equiv \mathscr{G}$ for all $j = J_0 + 1, \ldots, J$, the chain $\mathscr{G}_{J\longrightarrow J_0}$ is called as an undecimated chain of $\mathscr{G}$. Otherwise, $\mathscr{G}_{J\longrightarrow J_0}$ is called an decimated chain of $\mathscr{G}$. For convenience of discussion, it is usually assumed each vertex $v$ of the finest level graph $\mathscr{G}_J \equiv \mathscr{G}$ as a cluster of singleton. When there is only one vertex in the coarsest graph $\mathscr{G}_{J_0}$, we call $\mathscr{G}_{J\longrightarrow J_0}$ a tree [6].

*2.3. Orthonormal Bases on Graphs.* In this subsection, we introduce the orthonormal bases for the coarse-grained chain on a graph [6, 35–37].

Let $L_2(\mathscr{G}) := L_2(\mathscr{G}, \langle\cdot,\cdot\rangle_{\mathscr{G}})$ be the Hilbert space of vectors $f\colon V \longrightarrow \mathbb{C}$ on the graph $\mathscr{G}$ with the inner product $\langle\cdot,\cdot\rangle_{\mathscr{G}}$, which is defined as follows:

$$\langle f, g\rangle_{\mathscr{G}} := \sum_{v\in V} f(v)\overline{g(v)}, \quad f,\ g \in L_2(\mathscr{G}), \tag{15}$$

where $\overline{g}$ is the complex conjugate to $g$. The norm $\|\cdot\|_{\mathscr{G}}$ is given by $\|f\|_{\mathscr{G}} := \sqrt{\langle f, f\rangle_{\mathscr{G}}}$ for $f \in L_2(\mathscr{G})$. Let $\delta_{l,l'}$ be the Kronecker delta satisfying $\delta_{l,l'} = 1$ if $l = l'$ and 0 otherwise, and $N := |V|$ is the number of vertices. A set $\{\mu_l\}_{l=1}^N$ of vectors in $f \in L_2(\mathscr{G})$ is an orthonormal basis for $f \in L_2(\mathscr{G})$ if

$$\langle \mu_l, \mu_{l'}\rangle = \delta_{l,l'}, \quad 1 \leq l,\ l' \leq N. \tag{16}$$

The generalized Fourier coefficient of degree $l$ for $f \in L_2(\mathscr{G})$ with respect to $\mu_l$ is defined to be $\widehat{f}_l := \langle f, \mu_l\rangle$. Then, for all $f \in L_2(\mathscr{G})$, we have $f = \sum_{l=1}^N \widehat{f}_l \mu_l$. Let $\{\lambda_l\}_{l=1}^N \subseteq \mathbb{R}$ be a nondecreasing sequence of non-negative numbers satisfying $0 = \lambda_1 \leq \cdots \leq \lambda_N$, if $\{\mu_l\}_{l=1}^N$ is an orthonormal basis for $f \in L_2(\mathscr{G})$ with $\mu_1 \equiv (1/\sqrt{N})$; we say $\{(\mu_l, \lambda_l)\}_{l=1}^N$ is an orthonormal eigenpair for $f \in L_2(\mathscr{G})$ [35]. Meanwhile, Hammond [35] also gives the definition of the graph Laplacian $\mathscr{L}\colon L_2(\mathscr{G}) \longrightarrow L_2(\mathscr{G})$

$$\begin{gathered} [\mathscr{L}f](p) := d(p)f(p) - \sum_{v\in V} \omega(p, v) f(v), \\ p \in V,\ f \in L_2(\mathscr{G}). \end{gathered} \tag{17}$$

If $\langle f, \mathscr{L}f\rangle \geq 0$, the equation $\mathscr{L}\mu_l = \lambda_l \mu_l$ corresponding to the eigenvalues $\lambda_l, l = 1, \ldots, N$ and eigenvectors $\mu_l$ is non-negative, and satisfies $0 = \lambda_1 \leq \cdots \leq \lambda_N$ with $\mu_1 \equiv (1/\sqrt{N})$ [37].

Based on the properties of the eigenvalues and eigenvectors, Wang and Zhuang in [37] gave an orthonormal basis for the chain. Let $\mathscr{G}_{J\longrightarrow J_0} := (\mathscr{G}_J, \mathscr{G}_{J-1}, \ldots, \mathscr{G}_{J_0})$ be a chain on the graph $\mathscr{G}$ with $N$ vertices; $L_2(\mathscr{G}_{J\longrightarrow J_0})$ be the set of all vectors $f$ defined on the union of vertices on all levels $V_J \cup \cdots \cup V_{J_0}$. If the restriction $\{\mu_l, \lambda_l\}_{l=1}^N$ on the $j$ th-level graphs $\mathscr{G}_j$ is an orthonormal basis for $L_2(\mathscr{G}_j)$ at each level $j = J_0, \ldots, J$, a set pairs of vectors and complex numbers $\{\mu_l, \lambda_l\}_{l=1}^N$ in $L_2(\mathscr{G}_{J\longrightarrow J_0})$ are called an orthonormal basis for the chain $\mathscr{G}_{J\longrightarrow J_0}$.

*2.4. Decimated Tight Framelets on Graphs.* In this subsection, we introduce the construction methods of a decimated tight framelet on a graph. The conclusion of this part mainly comes from reference [6]. Let $\mathscr{G} = (V, E, \omega)$ be a graph and $\mathscr{G}_{J\longrightarrow J_0} := (\mathscr{G}_J, \mathscr{G}_{J-1}, \ldots, \mathscr{G}_{J_0})$ be a chain on the graph $\mathscr{G}$. For each vertex $[p]$ in $\mathscr{G}_j = (V_j, E_j, \omega_j)$, a weight $\omega_{j,[p]} \in \mathbb{R}$ is defined. For the bottom level with $j = J$, let $\omega_{J,[p]_{\mathscr{G}_J}} \equiv 1$. Let $\mathcal{Q}_j := \{\omega_{j,[p]}\colon [p] \in V_j\}$ be the set of weights on $\mathscr{G}_j$, and $\mathcal{Q}_{J\longrightarrow J_0} := (\mathcal{Q}_J, \ldots, \mathcal{Q}_{J_0})$ be the sequence of weights for the coarse-grained chain $\mathscr{G}_{J\longrightarrow J_0}$. Zheng et al. in [6] gave the construction method of the decimated framelets on the graph, and the details are as follows.

*Definition 1.* Let $\Psi_j = \{\phi_j; \psi_j^1, \ldots, \psi_j^r\}$ be a tight frame in $L_2(\mathbb{R})$ at scale $j$ for $j = J_0, \ldots, J$. The decimated framelets $\phi_{j,[p]}(v)$ and $\psi_{j,[p]}^n(v), p, v \in V$, at scale $j = J_0, \ldots, J$ for the chain $\mathscr{G}_{J\longrightarrow J_0}$ on the graph $\mathscr{G}_j$ are defined by

$$\begin{gathered} \phi_{j,[p]}(v) = \sqrt{\omega_{j,[p]}} \sum_{l=1}^N \widehat{\phi}_j\left(\frac{\lambda_l}{\Lambda_j}\right)\overline{\mu_l([p])}\mu_l(v), \\ [p] \in V_j, \\ \psi_{j,[p]}^n(v) = \sqrt{\omega_{j+1,[p]}} \sum_{l=1}^N \widehat{\psi_j^{(n)}}\left(\frac{\lambda_l}{\Lambda_j}\right)\overline{\mu_l([p])}\mu_l(v), \\ [p] \in V_{j+1},\ n = 1, \ldots, r, \end{gathered} \tag{18}$$

where for $j = J$, we let $V_{J+1} = V_J$ and $\omega_{J+1,[p]} = \omega_{J,[p]}$. We call $\phi_{j,[p]}$ and $\psi_{j,[p]}^n$ low-pass and high-pass framelets at scale $j$.

The decimated tight framelets in Definition 1 are constructed based on framelet generators in $L_2(\mathbb{R})$ and the orthonormal basis associated with the chain $\mathscr{G}_{J\longrightarrow J_0}$. The function $\mu_l([p])$ can be defined by $\mu_l([p]) = \min_{v\in[p]} \mu_l(v)$. In order to obtain the decimated tight framelets on the graph $\mathscr{G}$, first, we study the construction methods of tight frames on $\mathbb{R}$. Then, in next section, we focus on the specific construction process of the graph wavelet frame.

## 3. Construction of Tight Wavelet Frames on the Graph

In this section, we introduce the construction process of decimated tight framelets on the graph $\mathcal{G}$ in detail. First, we start from the construction of the tight wavelet frames on $\mathbb{R}$ by a given scaling function.

*3.1. Construction of Tight Wavelet Frames on* $\mathbb{R}$. For the construction of classical tight wavelet frames on $\mathbb{R}$, there have been many creative results. These works are generally based on the multiresolution analysis viewpoint. Chui and He in [11] demonstrated that a tight wavelet frame with 3 symmetric generators can be derived from the B-spline functions $B_m (m \in \mathbb{N})$. However, this construction only has single-order vanishing moments. It is desirable to construct symmetric tight wavelet frames with high vanishing moments in applications. In order to achieve high order of vanishing moments, Daubechies et al. [5] considered a tight wavelet frame with 2 compactly supported generators from the B-spline functions $B_m (m \in \mathbb{N})$, which have $m$ order vanishing moments. Unfortunately, this tight wavelet frames are not symmetry. For studying symmetric tight wavelet frames, based on symmetric compactly supported refinable functions, Petukhov [66] discussed the symmetry of tight wavelet frames using the unitary extension principle and obtained the existence criterion of the symmetric or antisymmetric compactly supported framelets. For any compactly supported symmetric real-valued refinable function, Bin and Mo [67] showed that symmetric tight wavelet frames with 3 generators and high vanishing moments can be derived.

The tight wavelet frames with desired approximation orders are very critical in practical applications. Dong and Shen in [68] proved that the tight frame system derived from a pseudospline normally have better approximation order than that derived from B-splines. Later, Dong showed that the shifts of an arbitrarily given pseudosplines are linearly independent [69]. The pseudosplines are considered an important family of refinable functions and provide a wide variety of choices of refinable functions. By selecting different parameters, pseudosplines with various orders fill in the gaps between the B-splines and orthogonal refinable functions for the first type and between B-splines and interpolatory refinable functions for the second type [69]. Hence, pseudosplines have large flexibilities in wavelet and framelet construction. In this subsection, we focus on the construction of tight wavelet frames on $\mathbb{R}$ based on the pseudosplines.

In many applications, such as computational cost and storage concern [70–75], we hope a symmetric tight wavelet frame with as small as possible number of generators; that is, the symmetric tight wavelet frame is generated by a single wavelet function. Yet, except the tight frames generated by the discontinuous Haar wavelet function or its dilated version, it is impossible to exist an MRA compactly supported real-valued symmetric tight wavelet frame with one continuous generator [25]. Therefore, one is interested in considering a symmetric tight wavelet frame with two generators. They have been extensively studied in [13, 68, 76–80]. As shown in [68], a necessary and sufficient condition has been derived for the existence of a symmetric tight wavelet frame with two generators, and details are as follows: $2\pi$, $\Theta$

(1) There exists a $2\pi$-periodic trigonometric polynomial $\Theta$ with real coefficients such that $\Theta(0) = 1$ and $\Theta(\xi) \geq 0$ for all $\xi \in \mathbb{R}$;

(2) There exists a real-valued sequence $b$ on $\mathbb{Z}$ with symmetry such that $\det M_{\Theta}(\xi) = |\widehat{b}(2\xi)|^2$, where the matrix $M_{\Theta}$ is defined in (10);

(3) The greatest common factor of all the entries of the matrix $M_{\Theta}$ satisfies a technical "gcd" condition. The technical "gcd" condition is shown in [68].

However, it is difficult to obtain a $2\pi$-periodic trigonometric polynomial $\Theta$ satisfying all above the conditions, since there exist nonlinear equations in these conditions. Therefore, the symmetric tight wavelet frame with three generators is usually considered. Here, we will derive a new construction method of a symmetric tight wavelet frame with three generators based on pseudosplines.

Pseudosplines are defined in terms of their refinement masks [5, 68]. The refinement mask of the first type of pseudosplines with order $(m, l)$ is given by

$$\left|{}_1\widehat{a}(\xi)\right|^2 = \left|{}_1\widehat{a}_{(m,l)}(\xi)\right|^2 = \cos^{2m}\left(\frac{\xi}{2}\right)\sum_{j=0}^{l}\begin{pmatrix} m+l \\ j \end{pmatrix}\sin^{2j}\left(\frac{\xi}{2}\right)\cos^{2(l-j)}\left(\frac{\xi}{2}\right), \tag{19}$$

and the refinement mask of the second type of pseudosplines with order $(m, l)$ is given by

$${}_2\widehat{a}(\xi) = {}_2\widehat{a}_{(m,l)}(\xi) = \cos^{2m}\left(\frac{\xi}{2}\right)\sum_{j=0}^{l}\begin{pmatrix} m+l \\ j \end{pmatrix}\sin^{2j}\left(\frac{\xi}{2}\right)\cos^{2(l-j)}\left(\frac{\xi}{2}\right), \tag{20}$$

where $0 \leq l \leq m-1$.

In this subsection, we only consider the construction of tight wavelet frame from the second type of pseudosplines using the oblique extension principle. In order to state the main results, we review the following results in [67].

**Theorem 2** (see [67]). *Let a be a finitely supported real-valued mask on* $\mathbb{Z}$ *such that* $|\widehat{a}(2\pi/3)| > 1$ *and* $|\widehat{a}(2\pi/3)| \notin \{2^j\colon\ j \in \mathbb{N}\}$ *. Then, there does not exist a*$2\pi$*-periodic trigonometric rational polynomial* $\Theta$ *with real coefficient such that*

*(i)* $\Theta(0) = 1$ *and* $\Theta(\xi) \geq 0$, *a.e.*$\xi \in \mathbb{R}$*;*

*(ii)* $\Theta(\xi) - \Theta(2\xi)|\widehat{a}(\xi)|^2$ *can be regarded as a* $2\pi$*-periodic trigonometric polynomial and* $\Theta(\xi) - \Theta(2\xi)$ $|\widehat{a}(\xi)|^2 \geq 0$ *for all* $\xi \in \mathbb{R}$.

Consequently, for any positive integer $r$, there do not exist finitely supported real-valued sequences $\psi^1, \ldots, \psi^r$ and a $2\pi$-periodic trigonometric rational polynomial $\Theta$ with real coefficient such that all the conditions in Theorem 2 are satisfied.

**Theorem 3** (see [67]). *Let $a$ be a finitely supported real-valued mask on $\mathbb{Z}$ such that $\widehat{a}(\xi) = (1+e^{-i\xi})^m \widehat{b}(\xi)$ for some positive integer $m$ and some finitely supported sequence $b$ on $\mathbb{Z}$ . Let $\phi$ be the compactly supported real-valued refinable function associated with mask $a$ . Suppose that $\phi \in L_2(\mathbb{R})$ and the shifts of $\phi$ are stable. Then, there exists a$2\pi$-periodic trigonometric polynomial$\theta$such that*

$$\theta(0) = 1,\ \theta(\xi) > 0, \quad \xi \in \mathbb{R}. \tag{21}$$

**Theorem 4** (see [67]). *Let $a$ be a finitely supported mask in $\mathbb{Z}$ such that $|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 \geq 1$ for all $\xi \in \mathbb{R}$ and $|\widehat{a}(\xi_0)|^2 + |\widehat{a}(\xi_0+\pi)|^2 > 1$ for some $\xi_0 \in \mathbb{R}$ . Then, there does not exist a$2\pi$-periodic trigonometric polynomial $\theta_0$ such that $\theta_0(0) = 1$ and*

$$\theta_0(\xi) - \theta_0(2\xi)|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 \geq 0 \quad \forall \xi \in [-\pi, \pi]. \tag{22}$$

Symmetry is an important property in various purposes. For a Laurent polynomial $p(z)$ with real coefficients, we say that $p(z)$ is symmetric (or antisymmetric) about $(k/2)$ for some $k \in \mathbb{Z}$ if $p(z) = z^k\ p(1/z)$ (or $p(z) = -z^k p(1/z)$). For a nonzero Laurent polynomial $p$, we introduce an operator $S$ to be

$$[Sp](z) = \frac{p(z)}{p(1/z)}, \quad z \in \mathbb{Z}\setminus\{0\}. \tag{23}$$

The following result can be given, as shown in [76].

**Theorem 5** (see [76]). *Let $p$ and $q$ be two Laurent polynomials with real coefficients. Then,*

*(1) $p$ is (anti)symmetric about $(k/2)$ for some $k \in \mathbb{Z}$ if and only if $[Sp](z) = \pm z^k$ .*

*(2) $[S(p(1/\cdot))](z) = [Sp](1/z) = 1/[Sp](z)$ .*

*(3) $[S(pq)](z) = [Sp](z)[Sq](z)$ and $[S((\cdot)^k)](z) = z^{2k}$ for $k \in \mathbb{Z}$ .*

*(4) If $p$ and $q$ are (anti)symmetric such that $Sq = Sp$ , then $p \pm q$ is (anti)symmetric and $S(p \pm q) = Sp = Sq$ .*

Next, we give the construction of symmetric tight wavelet frame. For a given refinable function with mask $a$, the key is to find a $2\pi$-periodic $\Theta$, such that OEP condition is satisfied. We have the following the main result.

**Theorem 6.** *Let ${}_2\phi$ denote the second type of pseudosplines of order $(m,l)$ with a finitely supported mask $a$ , which is defined in (20). Suppose that there is a$2\pi$-periodic trigonometric polynomial $\theta$ with real coefficients such that $\theta(0) = 1$, $\theta$ is symmetric, and $\theta(\xi) > 0$ for all $\xi \in \mathbb{R}$. In addition, assume the following:*

$$\theta_0(\xi) = \Theta(\xi) - \Theta(2\xi)\left(|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2\right) \geq 0, \quad \forall \xi \in \mathbb{R}, \tag{24}$$

*where$\Theta(\xi) = |\theta(\xi)|^2$. By the Fejér–Rieszelemma, there exists a$2\pi$-periodic trigonometric polynomial $\theta_1$with real coefficients such that $|\theta_1(\xi)|^2 = \theta_0(\xi)$, defined by*

$$\widehat{b}_1(\xi) = e^{-i\xi}\overline{\widehat{a}(\xi+\pi)}\theta(2\xi), \tag{25}$$

$$\widehat{b}_2(\xi) = \frac{1}{2}\left[\theta_1(\xi) + e^{-i\xi}\overline{\theta_1(\xi)}\right], \tag{26}$$

$$\widehat{b}_3(\xi) = \frac{1}{2}\left[-\theta_1(\xi) + e^{-i\xi}\overline{\theta_1(\xi)}\right]. \tag{27}$$

The wavelet functions $\psi^1, \psi^2$, and $\psi^3$ are defined by

$$\begin{aligned} \widehat{\psi}^1(\xi) &= \widehat{b}_1\left(\frac{\xi}{2}\right)_2\widehat{\phi}\left(\frac{\xi}{2}\right), \\ \widehat{\psi}^2(\xi) &= \widehat{b}_2\left(\frac{\xi}{2}\right)_2\widehat{\phi}\left(\frac{\xi}{2}\right), \\ \widehat{\psi}^3(\xi) &= \widehat{b}_3\left(\frac{\xi}{2}\right)_2\widehat{\phi}\left(\frac{\xi}{2}\right). \end{aligned} \tag{28}$$

Then, $\{\psi^1, \psi^2, \psi^3\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$ and each of the wavelet functions $\psi^1, \psi^2$, and $\psi^3$ is either symmetric or antisymmetric.

*Proof 1.* By the oblique extension principle, in order to prove $\{\psi^1, \psi^2, \psi^3\}$ generates a tight wavelet frame, we need to check the condition (9). From (26) and (27), we deduce the following:

$$\begin{aligned} \left|\widehat{b}_2(\xi)\right|^2 &= \widehat{b}_2(\xi)\overline{\widehat{b}_2(\xi)} \\ &= \frac{1}{4}\left[\theta_1(\xi) + e^{-i\xi}\overline{\theta_1(\xi)}\right]\left[\overline{\theta_1(\xi)} + e^{i\xi}\theta_1(\xi)\right] \\ &= \frac{1}{4}\left[\left|\theta_1(\xi)\right|^2 + e^{i\xi}\left|\theta_1(\xi)\right|^2 + e^{-i\xi}\left|\overline{\theta_1(\xi)}\right|^2 + \left|\theta_1(\xi)\right|^2\right], \end{aligned} \tag{29}$$

and

$$\begin{aligned} \left|\widehat{b}_3(\xi)\right|^2 &= \widehat{b}_3(\xi)\overline{\widehat{b}_3(\xi)} \\ &= \frac{1}{4}\left[-\theta_1(\xi) + e^{-i\xi}\overline{\theta_1(\xi)}\right]\left[-\overline{\theta_1(\xi)} + e^{i\xi}\theta_1(\xi)\right] \\ &= \frac{1}{4}\left[\left|\theta_1(\xi)\right|^2 - e^{i\xi}\left|\theta_1(\xi)\right|^2 - e^{-i\xi}\left|\overline{\theta_1(\xi)}\right|^2 + \left|\theta_1(\xi)\right|^2\right]. \end{aligned} \tag{30}$$

Then,

$$\left|\widehat{b}_2(\xi)\right|^2 + \left|\widehat{b}_3(\xi)\right|^2 = \left|\theta_1(\xi)\right|^2 = \theta_0(\xi). \tag{31}$$

Hence,

$$\begin{aligned} &|\widehat{a}(\xi)|^2\Theta(2\xi) + \left|\widehat{b}_1(\xi)\right|^2 + \left|\widehat{b}_2(\xi)\right|^2 + \left|\widehat{b}_3(\xi)\right|^2 \\ &= |\widehat{a}(\xi)|^2\Theta(2\xi) + \Theta(2\xi)|\widehat{a}(\xi+\pi)|^2 \\ &\quad + \Theta(\xi) - \Theta(2\xi)\left(|a(\xi)|^2 + |a(\xi+\pi)|^2\right) \\ &= \Theta(\xi). \end{aligned} \tag{32}$$

Since $\theta_1$ is a $2\pi$-periodic trigonometric polynomial with real coefficients, we have the following:

$$\widehat{b_2}(\xi)\overline{\widehat{b_2}(\xi+\pi)} + \widehat{b_3}(\xi)\overline{\widehat{b_3}(\xi+\pi)} = 0, \tag{33}$$

which implies the following:

$$\begin{aligned} &\widehat{b_1}(\xi)\overline{\widehat{b_1}(\xi+\pi)} + \widehat{b_2}(\xi)\overline{\widehat{b_2}(\xi+\pi)} + \widehat{b_3}(\xi)\overline{\widehat{b_3}(\xi+\pi)} \\ &= -\widehat{a}(\xi)\overline{\widehat{a}(\xi+\pi)}\Theta(2\xi). \end{aligned} \tag{34}$$

Therefore, all the conditions in (9) are satisfied. By Theorem 1, $\{\psi^1, \psi^2, \psi^3\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$.

Now, we show the symmetry of the wavelet functions $\psi^1, \psi^2$, and $\psi^3$. ${}_2\phi$ is the second type of pseudospline of order $(m, l)$, which is symmetric. Then, according to the definition of symmetry and Theorem 5, we can obtain the wavelet functions $\psi^1, \psi^2$, and $\psi^3$ that are symmetric (or antisymmetric). □

*Remark 1.* It is easy to see that $2\pi$-periodic trigonometric polynomial $\theta$ is existent. ${}_2\phi$ is the second type of pseudospline of order $(m, l)$ with a finitely supported mask $a$, which is defined in (20), and the shifts of ${}_2\phi$ are stable. So, the conditions of Theorem 3 are satisfied. That is, there exists a $2\pi$-periodic trigonometric polynomial $\theta$ such that $\theta(0) = 1, \theta(\xi) > 0$ for all $\xi \in \mathbb{R}$.

*Remark 2.* Pseudospline's definition starts with the simple identity $1 = (\cos^2(\xi/2) + \sin^2(\xi/2))^{m+l}$ for given non-negative integers $l$ and $m$ with $l \le m-1$. By the summation of the first $l+1$ terms of the binomial expansion of this identity, we can define the refinement mask of pseudosplines in (20). So, we have $|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 \le 1$. That is, $\theta_0$ of Theorem 6 can be found.

According to the above results, we give the following theorem.

**Theorem 7.** *Let ${}_2\phi$ denote the second type of pseudospline of order $(m, l)$ with a finitely supported mask $a$ . Suppose that there are the wavelet functions $\psi^1, \psi^2, \psi^3$ being defined in (28). Then,*

*(1) $\{\psi^1, \psi^2, \psi^3\}$ has $l$ vanishing moments.*

*(2) The approximation order of the framelet system, which be obtained by $\{\psi^1, \psi^2, \psi^3\}$, is* $\min\{m, 2l+2\}$.

*Proof 2.* Since ${}_2\phi$ is the second type of pseudosplines of order $(m, l)$ and $\Theta$ is a $2\pi$-periodic trigonometric polynomial, we have the following:

$$\theta_0(\xi) = O\left(|\xi|^{2l}\right) \quad \text{as } \xi \longrightarrow 0, \tag{35}$$

where $\theta_0$ is defined in (24). It is straightforward to see that $\widehat{b}_l(\xi) = O(|\xi|^l)$ as $\xi \longrightarrow 0$ for all $l = 1, 2, 3$, where $\widehat{b}_1, \widehat{b}_2, \widehat{b}_3$ are defined in (3.5) – (3.7). Therefore, $\{\psi^1, \psi^2, \psi^3\}$ has $l$ vanishing moments. □

Next, we give an example to illustrate our constructed tight wavelet frame by Theorem 6.

Example: Let ${}_2\phi$ denote the second type of pseudosplines of order $(3, 1)$ with a finitely supported mask,

$$\widehat{a}(\xi) = \cos^6\left(\frac{\xi}{2}\right)\left(1 + 3\sin^2\left(\frac{\xi}{2}\right)\right), \tag{36}$$

in which there exists a $2\pi$-periodic trigonometric polynomial $\theta(\xi)$, which is expressed as follows:

$$\theta(\xi) = \frac{437}{320} - \frac{97}{240}\cos(2\xi) + \frac{37}{960}\cos(4\xi), \tag{37}$$

and satisfies

$$\theta(0) = 1, \quad \theta(\xi) > 0, \forall \xi \in \mathbb{R}. \tag{38}$$

According to Theorem 6, we have the following:

$$\begin{aligned} &\theta_0(\xi) = \Theta(\xi) - \Theta(2\xi)\left(|a(\xi)|^2 + |a(\xi+\pi)|^2\right) \ge 0, \\ &\left|\theta_1(\xi)\right|^2 = \theta_0(\xi), \quad \forall \xi \in \mathbb{R}. \end{aligned} \tag{39}$$

The wavelet filters $b_1, b_2, b_3$ are defined as follows:

$$\begin{aligned} \widehat{b}_1(\xi) &= e^{-i\xi}\overline{\widehat{a}(\xi+\pi)}\theta(2\xi), \\ \widehat{b}_2(\xi) &= \frac{1}{2}\left[\theta_1(\xi) + e^{-i\xi}\overline{\theta_1(\xi)}\right], \\ \widehat{b}_3(\xi) &= \frac{1}{2}\left[-\theta_1(\xi) + e^{-i\xi}\overline{\theta_1(\xi)}\right]. \end{aligned} \tag{40}$$

Then, the wavelet functions $\psi^1, \psi^2$, and $\psi^3$ are defined as follows:

$$\begin{aligned} \widehat{\psi}^1(\xi) &= \widehat{b}_1\left(\frac{\xi}{2}\right){}_2\widehat{\phi}\left(\frac{\xi}{2}\right), \\ \widehat{\psi}^2(\xi) &= \widehat{b}_2\left(\frac{\xi}{2}\right){}_2\widehat{\phi}\left(\frac{\xi}{2}\right), \\ \widehat{\psi}^3(\xi) &= \widehat{b}_3\left(\frac{\xi}{2}\right){}_2\widehat{\phi}\left(\frac{\xi}{2}\right). \end{aligned} \tag{41}$$

According to Theorem 6, $\{\psi^1, \psi^2, \psi^3\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$. Moreover, all the wavelet functions $\psi^1, \psi^2, \psi^3$ are symmetric or antisymmetric. Figure 1 shows the tight filter bank $\{a; b_1, b_2, b_3\}$ with symmetric functions constructed by Theorem 6. $(a)$ is the graph of the second type of pseudosplines of order $(3, 1)$. $(b)$ – $(d)$ are the graphs of the framelet functions $\psi^1, \psi^2, \psi^3$, respectively. The set $\{\psi^1, \psi^2, \psi^3\}$ generates a symmetric tight wavelet frame in $L_2(\mathbb{R})$.

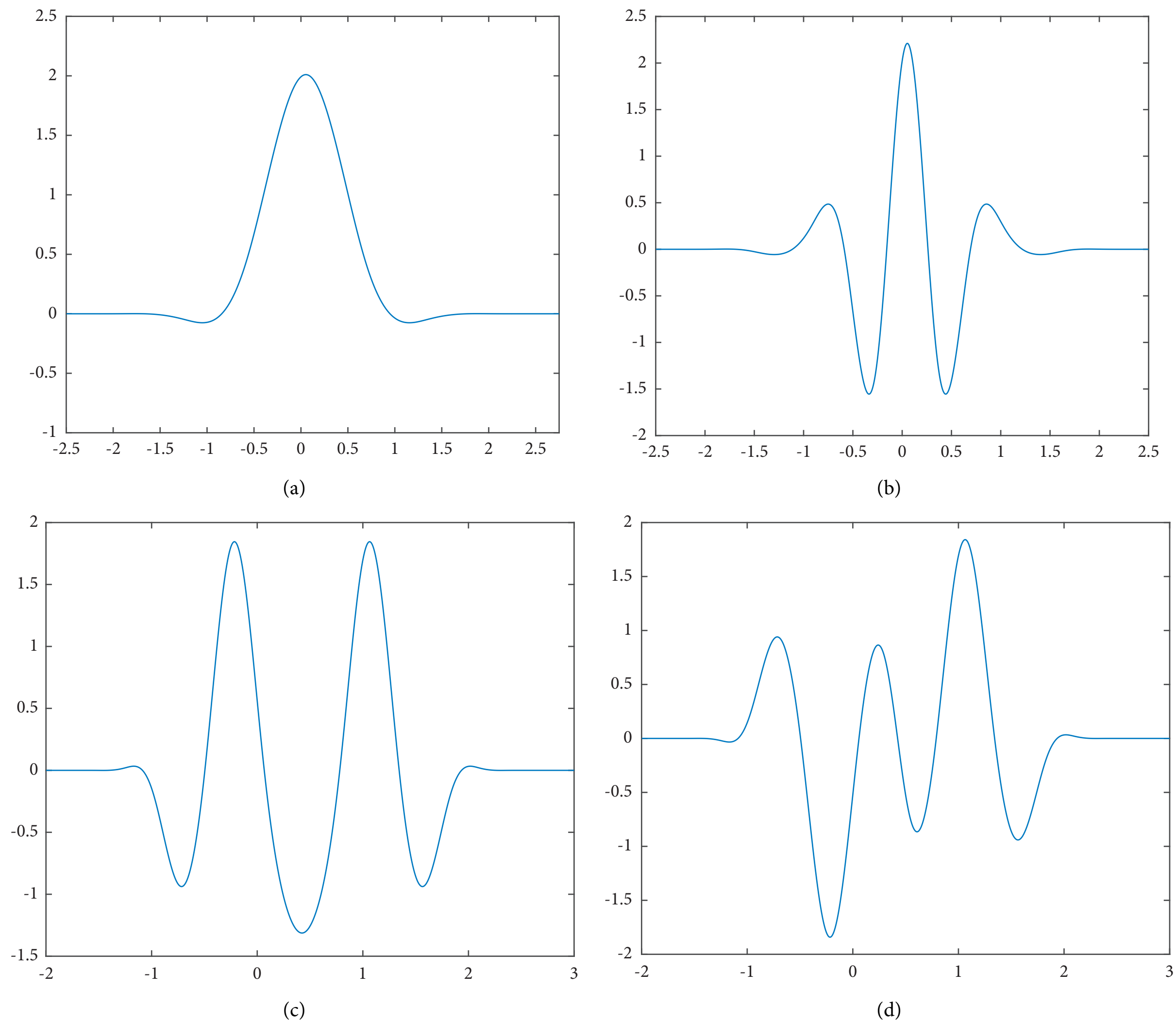


Figure 1: Graphs of the scaling function and the corresponding wavelets.

*3.2. Construction of Orthogonal Bases on $\mathcal{G}$.* In above subsection, we constructed the tight wavelet framelets on $\mathbb{R}$. In order to achieve the construction of decimated tight framelets on $\mathcal{G}$, according to Definition 1, we need to give the orthonormal basis associated with the chain $\mathcal{G}_{J\longrightarrow J_0}$. We only consider the Haar wavelet basis for the chain $\mathcal{G}_{J\longrightarrow J_0}$ in the paper, which is a particular case of Daubechies wavelets, and is developed onto a graph by Chui, Filbir, and Mhaskar in [23]. The Haar basis $\{\mu_l^{(j)}\}_{l=1}^{N_j}$, $j = J_0, \ldots, J$ is a sequence of collections of vectors. Each Haar basis is associated with a single layer of the chain $\mathcal{G}_{J\longrightarrow J_0}$ on a graph $\mathcal{G}$. The detail discussed about the Haar basis is based on the coarse-grained chain on a graph in [6, 25, 37, 81].

We first give the construction of the Haar basis for a chain with two levels. For the construction of the Haar basis for a chain with more levels, one can use this method recursively. Let $\mathcal{G}_c = (V_c, E_c, \omega_c)$ be a coarse-grained graph of $\mathcal{G} = (V, E, \omega)$ with $N_c = |V_c|$. We sequence the vertices of $\mathcal{G}_c$ by their degrees as

$$V_c = \left\{[p_j]_{\mathcal{G}_c}: j = 1, \ldots, N_c\right\},\ d\left([p_j]_{\mathcal{G}_c}\right) \geq d\left([p_{j+1}]_{\mathcal{G}_c}\right). \tag{42}$$

$N_c$ vectors $\mu_l^c$ on $\mathcal{G}_c$ are defined by

$$\mu_1^c(v^c) = \frac{1}{\sqrt{N_c}}1, \quad v^c \in V_c;$$

$$\mu_l^c = \sqrt{\frac{N_c - l + 1}{N_c - l + 2}}\left(\chi_{l-1}^c - \frac{\sum_{j=1}^{N_c}\chi_j^c}{N_c - l + 1}\right), \quad l = 2, \ldots, N_c, \tag{43}$$

where $\chi_j^c$ is the indicator function for the $j$ th vertex $[p_j]_{\mathcal{G}_c}$ on $\mathcal{G}_c$, which is given by

$$\chi_j^c([v]) = \begin{cases} 1, & [v] = [p_j]_{\mathcal{G}_c}, \\ 0, & \text{otherwise}. \end{cases} \tag{44}$$

Then, the set of function $\{\mu_l^c\}_{l=1}^{N_c}$ forms an orthonormal basis for $L_2(\mathcal{G}_c)$ [6, 37].

Now, we extend the orthonormal basis $\{\mu_l^c\}_{l=1}^{N_c}$ for $\mathcal{G}_c$ on the $\mathcal{G}$. For each element of $\{\mu_l^c: l = 1, \ldots, N_c\}$ with the vertex $[p_l]_{\mathcal{G}_c}$ on $\mathcal{G}_c$, we define

$$\mu_{l,1}(v) = \frac{\mu_l^c([v])}{\sqrt{N_c}}, \quad v \in V, l = 1, \ldots, N_c. \tag{45}$$

Let $k_l = |[p_l]_{\mathcal{G}_c}|$; we order the cluster $[p_l]_{\mathcal{G}_c}$ according to their degrees,

$$[p_l]_{\mathcal{G}_c} = \{v_{l,1}, \ldots, v_{l,k_l}\} \subseteq V, \quad d(v_{l,j}) \geq d(v_{l,j+1}). \tag{46}$$

For $k = 2, \ldots, k_l$, we define

$$\mu_{l,k} = \sqrt{\frac{k_l - k + 1}{k_l - k + 2}} \left( \chi_{l,k-1} - \frac{\sum_{j=k}^{k_l} \chi_{l,j}}{k_l - k + 1} \right), \quad j = 1, \ldots, k_l, \tag{47}$$

where $\chi_{l,j}$ is given by

$$\chi_{l,j}(v) = \begin{cases} 1, & v = v_{l,j}, \\ 0, & \text{otherwise}. \end{cases} \tag{48}$$

Then, the resulting $\{\mu_{l,k}: l = 1, \ldots, N_c, k = 1, \ldots, k_l\}$ is an orthonormal basis for $L_2(\mathcal{G})$ [6].

Next, we can give the Haar basis for the coarse-grained chain on a graph by repeating the above process. Starting from $\mathcal{G}_{J_0}$, an orthonormal basis $\{\mu_l^{J_0}: l = 1, \ldots, N_{J_0}\}$ for $L_2(\mathcal{G}_{J_0})$ is generated as the above definition. By the chain relation of $\mathcal{G}_{J_0}$ and $\mathcal{G}_{J_1}$, we can obtain an orthonormal basis $\{\mu_l^{J_1}: l = 1, \ldots, N_{J_1}\}$ for $L_2(\mathcal{G}_{J_1})$. Continuing carrying out this process on each $\mathcal{G}_j$, $j = 1, \ldots, J$, for $L_2(\mathcal{G}_j)$, we can obtain orthonormal basis $\{\mu_l^{(j)}\}_{l=1}^{N_j}$, $j = 1, \ldots, J$. Then, the resulting orthonormal basis $\{\mu_l\}_{l=1}^{N}$ forms a Haar global orthonormal basis for a coarse-grained chain $\mathcal{G}_{J \longrightarrow J_0}$ on the graph $\mathcal{G}$ [6].

In the following, we give a new orthogonal basis on the graph $\mathcal{G}$.

**Theorem 8.** *Let $\mathcal{G}' = (V', E', \omega')$ be a coarse-grained graph of $\mathcal{G} = (V, E, \omega)$ with $N' = |V'|$ . We sequence the vertices of $\mathcal{G}'$ by their degrees as*

$$V' = \{[p_j]_{\mathcal{G}'}: j = 1, \ldots, N'\}, \quad d([p_j]_{\mathcal{G}'}) \geq d([p_{j+1}]_{\mathcal{G}'}). \tag{49}$$

*$N'$ vectors $\mu_l'$ on $\mathcal{G}'$* are defined *by*

$$\begin{aligned} \mu_1'(v') &= \frac{1}{\sqrt{N'}} 1, \quad v' \in V'; \\ \mu_l' &= \chi_{l-1}' - \frac{\sum_{j=1}^{N'} \chi_j^c}{N' - l + 1}, \quad l = 2, \ldots, N', \end{aligned} \tag{50}$$

where $\chi_j'$ is the indicator function for the $j$ th vertex $[p_j]_{\mathcal{G}'}$ on $\mathcal{G}'$, which is given by

$$\chi_j'([v]) = \begin{cases} 1, & [v] = [p_j]_{\mathcal{G}'}, \\ 0, & \text{otherwise}. \end{cases} \tag{51}$$

Then, the set of function $\{\mu_l'\}_{l=1}^{N'}$ forms an orthogonal basis for $L_2(\mathcal{G}')$.

*Proof 3.* For $l = 2, \ldots, N'$,

$$\langle \mu_1', \mu_l' \rangle = \frac{1}{\sqrt{N'}} \langle 1, \chi_{l-1}' - \frac{\sum_{j=l}^{N'} \chi_j'}{N' - l + 1} \rangle. \tag{52}$$

And for $2 \leq k \leq l \leq N'$,

$$\begin{aligned} \langle \mu_l', \mu_k' \rangle &= \langle \chi_{l-1}' - \frac{\sum_{j=l}^{N'} \chi_j'}{N' - l + 1}, \chi_{k-1}' - \frac{\sum_{j=k}^{N'} \chi_j'}{N' - k + 1} \rangle \\ &= \left( -\frac{\sum_{j=k}^{N'} \langle \chi_{l-1}', \chi_j' \rangle}{N' - k + 1} + \frac{\langle \sum_{j=l}^{N'} \chi_j', \sum_{j=k}^{N'} \chi_j' \rangle}{(N' - k + 1)(N' - l + 1)} \right) \\ &= \left( -\frac{1}{N' - k + 1} + \frac{N' - l + 1}{(N' - k + 1)(N' - l + 1)} \right) \\ &= 0. \end{aligned} \tag{53}$$

Thus, the set of function $\{\mu_l'\}_{l=1}^{N'}$ is an orthogonal basis for $L_2(\mathcal{G}')$. □

Analogous to the above construction of the Haar basis for a chain, we can obtain the resulting orthonormal basis$\{\mu_l'\}_{l=1}^{N}$ that forms a global orthogonal basis for a coarse-grained chain $\mathcal{G}_{J \longrightarrow J_0}$ on the graph $\mathcal{G}$. Once we obtain the set of function that forms an orthonormal basis for $L_2(\mathcal{G}_{J \longrightarrow J_0})$, the decimated tight framelet on chain $\mathcal{G}_{J \longrightarrow J_0}$ can be constructed by Definition 1. In general, the weight function $\omega_c$ on $V_c \times V_c$ is defined as follows:

$$\omega_c([p], [v]) = \sum_{p \in [p]} \sum_{v \in [v]} \frac{\omega(p, v)}{\text{vol}(\mathcal{G})}, \quad [p], [v] \in V_c. \tag{54}$$

## 4. Conclusion

In this paper, we surveyed the construction methods of decimated tight framelets on the graphs. The related theory of the graph wavelet frame was analyzed, including wavelet frame on $L_2(\mathbb{R})$, graph, and chain, orthonormal bases on graphs, and the specific construction of decimated tight framelets. Because the wavelet frames on $L_2(\mathbb{R})$ are the basis for the construction of the graph wavelet frame, and the filter bank of decimated tight framelets is closely related to the classical wavelet framelets, then based on the second type of pseudosplines, we presented a symmetric tight wavelet frame with 3 generators in $L_2(\mathbb{R})$ using the oblique extension principle. By considering the general fundamental function $\Theta$ instead of the case $\Theta = 1$, we obtained a tight wavelet frame with good vanishing moments. Moreover, we analyzed the construction of the Haar basis for the coarse-grained chain on a graph $\mathcal{G}$ and obtained the new orthogonal basis on a graph $\mathcal{G}$.

The study of decimated tight framelets on a graph is only limited to the Haar basis. The specific construction method and practical application are only completed on the transformation of the graph Haar wavelet frame. In the future, we can consider more generalized graph wavelet frame research with a non-Haar basis. In this paper, we mainly introduced related theory of the graph wavelet frame and the detailed construction methods of the graph Haar frame, and provided theoretical basis for the later non-Haar wavelet frame construction on a graph.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

This work was carried out in collaboration between the two authors. Z.Z. Zhang analyzed and interpreted the classical wavelet framelets on $L_2(\mathbb{R})$. J. Zhou introduced a specific construction method and the detailed construction process of wavelet frame on a graph and was a major contributor in writing the manuscript. All authors read and approved the final manuscript.

## Acknowledgments

This research was supported by the Natural Science Foundation of Shaanxi Provincial Department of Education of China (Program No. 21JK0654).

## References

[1] I. Daubechies, "Orthonormal bases of compactly supported wavelets," *Communications on Pure and Applied Mathematics*, vol. 41, no. 7, pp. 909–996, 1988.

[2] Z. Shen, "Wavelet frames and image restorations," in *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) (In 4 Volumes) Vol. I: Plenary Lectures and Ceremonies Vols. II–IV: Invited Lectures*, pp. 2834–2863, World Scientific, Singapore, 2010.

[3] A. Ron and Z. Shen, "Affine systems in $l2(r\ d)$: the analysis of the analysis operator," *Journal of Functional Analysis*, vol. 148, no. 2, pp. 408–447, 1997.

[4] A. Chai and Z. Shen, "Deconvolution: a wavelet frame approach," *Numerische Mathematik*, vol. 106, no. 4, pp. 529–587, 2007.

[5] I. Daubechies, B. Han, A. Ron, and Z. Shen, "Framelets: mra-based constructions of wavelet frames," *Applied and Computational Harmonic Analysis*, vol. 14, no. 1, pp. 1–46, 2003.

[6] X. Zheng, B. Zhou, Y. Wang, and X. Zhuang, "Decimated framelet system on graphs and fast g-framelet transforms," *Journal of Machine Learning Research*, vol. 23, no. 18–1, 2022.

[7] A. Petukhov, "Explicit construction of framelets," *Applied and Computational Harmonic Analysis*, vol. 11, no. 2, pp. 313–327, 2001.

[8] Q. Jiang, "Hexagonal tight frame filter banks with idealized high-pass filters," *Advances in Computational Mathematics*, vol. 31, no. 1-3, pp. 215–236, 2009.

[9] C. K. Chui, W. He, and J. Stöckler, "Compactly supported tight and sibling frames with maximum vanishing moments," *Applied and Computational Harmonic Analysis*, vol. 13, no. 3, pp. 224–262, 2002.

[10] I. Daubechies, *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics, Philadelphia, 1992.

[11] C. K. Chui and W. He, "Compactly supported tight frames associated with refinable functions," *Applied and Computational Harmonic Analysis*, vol. 8, no. 3, pp. 293–319, 2000.

[12] J. Zhang and J. M. F. Moura, "Diffusion in social networks as sis epidemics: beyond full mixing and complete graphs," *IEEE Journal of Selected Topics in Signal Processing*, vol. 8, no. 4, pp. 537–551, 2014.

[13] W. Hu, G. Cheung, A. Ortega, and O. C. Au, "Multiresolution graph fourier transform for compression of piecewise smooth images," *IEEE Transactions on Image Processing*, vol. 24, no. 1, pp. 419–433, 2015.

[14] B. A. Miller, M. S. Beard, P. J. Wolfe, and N. T. Bliss, "A spectral framework for anomalous subgraph detection," *IEEE Transactions on Signal Processing*, vol. 63, no. 16, pp. 4191–4206, 2015.

[15] G. Cheung, E. Magli, Y. Tanaka, and M. K. Ng, "Graph spectral image processing," *Proceedings of the IEEE*, vol. 106, no. 5, pp. 907–930, 2018.

[16] K. Yamamoto, M. Onuki, and Y. Tanaka, "Deblurring of point cloud attributes in graph spectral domain," in *Proceedings of the 2016 IEEE International Conference on Image Processing (ICIP)*, pp. 1559–1563, IEEE, Phoenix, AZ, USA, September 2016.

[17] F. Cotter and N. Kingsbury, "Deep Learning in the Wavelet Domain," 2018, https://arxiv.org/abs/1811.06115.

[18] S. Mallat, "Understanding deep convolutional networks," *Philosophical Transactions of the Royal Society A: Mathematical, Physical & Engineering Sciences*, vol. 374, Article ID 20150203, 2016.

[19] O. Ronneberger, P. Fischer, and T. Brox, "U-net: Convolutional networks for biomedical image segmentation," in *International Conference on Medical Image Computing and Computer-Assisted Intervention*Springer, New York, NY, USA, 2015.

[20] J. C. Ye, Y. Han, and E. Cha, "Deep convolutional framelets: a general deep learning framework for inverse problems," *SIAM Journal on Imaging Sciences*, vol. 11, no. 2, pp. 991–1048, 2018.

[21] P. Yu, B. Yeo, P. Grant, B. Fischl, and P. Golland, "Cortical folding development study based on over-complete spherical wavelets," in *Proceedings of the 2007 IEEE 11th International Conference on Computer Vision*, October 2007.

[22] K. Zhang, W. Zuo, Y. Chen, D. Meng, and L. Zhang, "Beyond a Gaussian denoiser: residual learning of deep cnn for image denoising," *IEEE Transactions on Image Processing*, vol. 26, no. 7, pp. 3142–3155, 2017.

[23] C. Chui, F. Filbir, and H. Mhaskar, "Representation of functions on big data: graphs and trees," *Applied and Computational Harmonic Analysis*, vol. 38, no. 3, pp. 489–509, 2015.

[24] C. K. Chui, H. Mhaskar, and X. Zhuang, "Representation of functions on big data associated with directed graphs," *Applied and Computational Harmonic Analysis*, vol. 44, no. 1, pp. 165–188, 2018.

[25] M. Li, Z. Ma, Y. G. Wang, and X. Zhuang, "Fast haar transforms for graph neural networks," *Neural Networks*, vol. 128, pp. 188–198, 2020.

[26] M. M. Bronstein, J. Bruna, Y. LeCun, A. Szlam, and P. Vandergheynst, "Geometric deep learning: going beyond euclidean data," *IEEE Signal Processing Magazine*, vol. 34, no. 4, pp. 18–42, 2017.

[27] H. Mhaskar, "Eignets for function approximation on manifolds," *Applied and Computational Harmonic Analysis*, vol. 29, no. 1, pp. 63–87, 2010.

[28] P. D'Urso and E. A. Maharaj, "Wavelets-based clustering of multivariate time series," *Fuzzy Sets and Systems*, vol. 193, pp. 33–61, 2012.

[29] E. Ann Maharaj, P. D'Urso, and D. U. A. Galagedera, "Wavelet-based fuzzy clustering of time series," *Journal of Classification*, vol. 27, no. 2, pp. 231–275, 2010.

[30] Y. Kakizawa, R. H. Shumway, and M. Taniguchi, "Discrimination and clustering for multivariate time series," *Journal of the American Statistical Association*, vol. 93, no. 441, pp. 328–340, 1998.

[31] R. H. Chan, T. F. Chan, L. Shen, and Z. Shen, "Wavelet algorithms for high-resolution image reconstruction," *SIAM Journal on Scientific Computing*, vol. 24, no. 4, pp. 1408–1432, 2003.

[32] R. H. Chan, T. F. Chan, L. Shen, and Z. Shen, "Wavelet deblurring algorithms for spatially varying blur from high-resolution image reconstruction," *Linear Algebra and Its Applications*, vol. 366, pp. 139–155, 2003.

[33] R. R. Coifman and M. Maggioni, "Diffusion wavelets," *Applied and Computational Harmonic Analysis*, vol. 21, no. 1, pp. 53–94, 2006.

[34] M. Maggioni and H. Mhaskar, "Diffusion polynomial frames on metric measure spaces," *Applied and Computational Harmonic Analysis*, vol. 24, no. 3, pp. 329–353, 2008.

[35] D. K. Hammond, P. Vandergheynst, and R. Gribonval, "Wavelets on graphs via spectral graph theory," *Applied and Computational Harmonic Analysis*, vol. 30, no. 2, pp. 129–150, 2011.

[36] B. Dong, "Sparse representation on graphs by tight wavelet frames and applications," *Applied and Computational Harmonic Analysis*, vol. 42, no. 3, pp. 452–479, 2017.

[37] Y. Wang and X. Zhuang, "Tight framelets on graphs for multiscale data analysis,"vol. 11138, pp. 100–111, in *Proceedings of the Wavelets and Sparsity XVIII*, vol. 11138, SPIE, San Diego, CA, USA, 2019.

[38] A. Singer, "From graph to manifold laplacian: the convergence rate," *Applied and Computational Harmonic Analysis*, vol. 21, no. 1, pp. 128–134, 2006.

[39] M. Cheung, J. Shi, O. Wright, L. Y. Jiang, X. Liu, and J. M. F. Moura, "Graph signal processing and deep learning: convolution, pooling, and topology," *IEEE Signal Processing Magazine*, vol. 37, no. 6, pp. 139–149, 2020.

[40] M. Gavish, B. Nadler, and R. Coifman, "Multiscale Wavelets on Trees, Graphs and High Dimensional Data: Theory and Applications to Semi Supervised Learning," in *Proceedings of the 27th International Conference on International Conference on Machine Learning*, June 2010.

[41] W. Hamilton, Z. Ying, and J. Leskovec, "Inductive representation learning on large graphs," *Advances in Neural Information Processing Systems*, vol. 30, 2017.

[42] B. Han and X. Zhuang, "Smooth affine shear tight frames with mra structure," *Applied and Computational Harmonic Analysis*, vol. 39, no. 2, pp. 300–338, 2015.

[43] Y. Ma, X. Liu, T. Zhao, Y. Liu, J. Tang, and N. Shah, "A unified view on graph neural networks as graph signal denoising," in *Proceedings of the 30th ACM International Conference on Information & Knowledge Management*, pp. 1202–1211, Virtual Event, Queensland, Australia, October 2021.

[44] F. Monti, M. Bronstein, and X. Bresson, "Geometric matrix completion with recurrent multi-graph neural networks," *Advances in Neural Information Processing Systems*, vol. 30, 2017.

[45] C. Aggarwal and H. Wang, "A survey of clustering algorithms for graph data," in *Managing and Mining Graph Data*Springer, New York, NY, USA, 2010.

[46] M. Filippone, F. Camastra, F. Masulli, and S. Rovetta, "A survey of kernel and spectral methods for clustering," *Pattern Recognition*, vol. 41, no. 1, pp. 176–190, 2008.

[47] M. C. Nascimento and A. C. P. L. F. De Carvalho, "Spectral methods for graph clustering–a survey," *European Journal of Operational Research*, vol. 211, no. 2, pp. 221–231, 2011.

[48] S. E. Schaeffer, "Graph clustering," *Computer Science Review*, vol. 1, no. 1, pp. 27–64, 2007.

[49] D. Xu and Y. Tian, "A comprehensive survey of clustering algorithms," *Annals of Data Science*, vol. 2, no. 2, pp. 165–193, 2015.

[50] P. Qian, Y. Jiang, S. Wang et al., "Affinity and penalty jointly constrained spectral clustering with all-compatibility, flexibility, and robustness," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 28, no. 5, pp. 1123–1138, 2017.

[51] C. K. Chui, W. He, J. Stöckler, and Q. Sun, "Compactly supported tight affine frames with integer dilations and maximum vanishing moments," *Advances in Computational Mathematics*, vol. 18, no. 2/4, pp. 159–187, 2003.

[52] Y. Li and S. Yang, "Dual multiwavelet frames with symmetry from two-direction re nable functions," *Bulletin of the Iranian Mathematical Society*, vol. 37, no. 1, pp. 199–214, 2011.

[53] N. D. Atreas, M. Papadakis, and T. Stavropoulos, "Extension principles for dual multiwavelet frames of $l_2(\mathbb{R})$ constructed from multirefinable generators," *Journal of Fourier Analysis and Applications*, vol. 22, no. 4, pp. 854–877, 2016.

[54] B. Han, "Compactly supported tight wavelet frames and orthonormal wavelets of exponential decay with a general dilation matrix," *Journal of Computational and Applied Mathematics*, vol. 155, no. 1, pp. 43–67, 2003.

[55] Y. G. Cen, L. H. Cen, X. F. Chen, and Z. J. Miao, "Explicit construction of symmetric compactly supported biorthogonal multiwavelets via group transformations," *Journal of Computational and Applied Mathematics*, vol. 244, pp. 49–66, 2013.

[56] B. Li and L. Peng, "Biorthogonal multiwavelets with sampling property and application in image compression," *Circuits, Systems, and Signal Processing*, vol. 35, no. 3, pp. 933–951, 2016.

[57] S. S. Goh, T. N. Goodman, and S. L. Lee, "Constructing tight frames of multivariate functions," *Journal of Approximation Theory*, vol. 158, no. 1, pp. 49–68, 2009.

[58] F. Abdullah, "Construction of ultivariate tight framelet packets associated with dilation matrix," *Analysis in Theory and Applications*, vol. 2, 2015.

[59] F. Shah and N. Sheikh, "Construction of vector-valued multivariate wavelet frame packets," *Thai Journal of Mathematics*, vol. 10, no. 2, pp. 401–414, 2012.

[60] M. Charina, M. Putinar, C. Scheiderer, and J. Stöckler, "An algebraic perspective on multivariate tight wavelet frames," *Constructive Approximation*, vol. 38, no. 2, pp. 253–276, 2013.

[61] Y. Hur, Z. Lubberts, and K. Okoudjou, "An Algebraic Perspective on Multivariate Tight Wavelet Frames with Rational Masks," 2019, https://arxiv.org/abs/1403.2184.
[62] M. Salvatori and P. M. Soardi, "Multivariate tight affine frames with a small number of generators," *Journal of Approximation Theory*, vol. 127, no. 1, pp. 61–73, 2004.
[63] M. Ehler and B. Han, "Wavelet bi-frames with few generators from multivariate refinable functions," *Applied and Computational Harmonic Analysis*, vol. 25, no. 3, pp. 407–414, 2008.
[64] A. V. Krivoshein, "Multivariate symmetric refinable functions and function vectors," *International Journal of Wavelets, Multiresolution and Information Processing*, vol. 14, no. 05, Article ID 1650034, 2016.
[65] J. Hrivnak and L. Motlochová, "Discrete transforms and orthogonal polynomials of (anti) symmetric multivariate cosine functions," *SIAM Journal on Numerical Analysis*, vol. 52, no. 6, pp. 3021–3055, 2014.
[66] A. Petukhov, "Symmetric framelets," *Constructive Approximation*, vol. 19, no. 2, pp. 309–328, 2003.
[67] B. Han and Q. Mo, "Symmetric mra tight wavelet frames with three generators and high vanishing moments," *Applied and Computational Harmonic Analysis*, vol. 18, no. 1, pp. 67–93, 2005.
[68] B. Dong and Z. Shen, "Construction of biorthogonal wavelets from pseudo-splines," *Journal of Approximation Theory*, vol. 138, no. 2, pp. 211–231, 2006.
[69] B. Dong and Z. Shen, "Linear independence of pseudo-splines," *Proceedings of the American Mathematical Society*, vol. 134, no. 9, pp. 2685–2694, 2006.
[70] R. Chan, S. D Riemenschneider, L. Shen, and Z. Shen, "Tight frame: an efficient way for high-resolution image reconstruction," *Applied and Computational Harmonic Analysis*, vol. 17, no. 1, pp. 91–115, 2004.
[71] R. Chan, L. Shen, and Z. Shen, "A framelet-based approach for image inpainting," *Preprint*, vol. 4, p. 325, 2005.
[72] E. J. Candès, J. Romberg, and T. Tao, "Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information," *IEEE Transactions on Information Theory*, vol. 52, no. 2, pp. 489–509, 2006.
[73] Q. Li, L. Shen, and L. Yang, "Split-bregman iteration for framelet based image inpainting," *Applied and Computational Harmonic Analysis*, vol. 32, no. 1, pp. 145–154, 2012.
[74] T. F. Chan, J. Shen, and H. M. Zhou, "Total variation wavelet inpainting," *Journal of Mathematical Imaging and Vision*, vol. 25, no. 1, pp. 107–125, 2006.
[75] L. Borup, R. Gribonval, and M. Nielsen, "Bi-framelet systems with few vanishing moments characterize besov spaces," *Applied and Computational Harmonic Analysis*, vol. 17, no. 1, pp. 3–28, 2004.
[76] B. Han and Q. Mo, "Splitting a matrix of laurent polynomials with symmetry and its application to symmetric framelet filter banks," *SIAM Journal on Matrix Analysis and Applications*, vol. 26, no. 1, pp. 97–124, 2004.
[77] J. Sun, Y. Huang, S. Sun, and L. Cui, "Parameterizations of masks for 3-band tight wavelet frames by symmetric extension of polyphase matrix," *Applied Mathematics and Computation*, vol. 225, pp. 461–474, 2013.
[78] N. Emirov, C. Cheng, J. Jiang, and Q. Sun, "Polynomial graph filters of multiple shifts and distributed implementation of inverse filtering," *Sampling Theory, Signal Processing, and Data Analysis*, vol. 20, no. 1, pp. 2–39, 2022.
[79] X. Han, Y. Chen, J. Shi, and Z. He, "An extended cell transmission model based on digraph for urban traffic road network," in *Proceedings of the 2012 15th International IEEE Conference on Intelligent Transportation Systems*, pp. 558–563, IEEE, Anchorage, AK, USA, September 2012.
[80] D. Mohan, M. Asif, N. Mitrovic, J. Dauwels, and P. Jaillet, "Wavelets on graphs with application to transportation networks," in *Proceedings of the 17th International IEEE Conference on Intelligent Transportation Systems (ITSC)*, pp. 1707–1712, IEEE, Qingdao, China, October 2014.
[81] Y. G. Wang and X. Zhuang, "Tight framelets and fast framelet filter bank transforms on manifolds," *Applied and Computational Harmonic Analysis*, vol. 48, no. 1, pp. 64–95, 2020.