Hindawi
Complexity
Volume 2022, Article ID 8155929, 36 pages
https://doi.org/10.1155/2022/8155929

Research Article

Estimation for Parameters of Life of the Marshall-Olkin Generalized-Exponential Distribution Using Progressive Type-II Censored Data

Ahmed Elshahhat,[1] **Abdisalam Hassan Muse**,[2,3] **Omer Mohamed Egeh**,[3] **and Berihan R. Elemary**[4]

[1]*Faculty of Technology and Development, Zagazig University, Zagazig 44519, Egypt*
[2]*Department of Mathematics (Statistics Option), Pan African University, Institute for Basic Sciences, Technology, and Innovation (PAUSTI), Faculty of Commerce, Damietta University, Nairobi 62000-00200, Kenya*
[3]*Department of Mathematics and Statistics, School of Postgraduate Studies and Research, Amoud University, Borama, Somalia*
[4]*Department of Applied, Mathematical and Actuarial Statistics, Faculty of Commerce, Damietta University, Damietta 22052, Egypt*

Correspondence should be addressed to Omer Mohamed Egeh; 1933@amoud.edu.so

Received 13 April 2022; Revised 9 August 2022; Accepted 18 August 2022; Published 3 October 2022

Academic Editor: Yu Zhou

A new three-parameter extension of the generalized-exponential distribution, which has various hazard rates that can be increasing, decreasing, bathtub, or inverted tub, known as the Marshall-Olkin generalized-exponential (MOGE) distribution has been considered. So, this article addresses the problem of estimating the unknown parameters and survival characteristics of the three-parameter MOGE lifetime distribution when the sample is obtained from progressive type-II censoring via maximum likelihood and Bayesian approaches. Making use of the s-normality of classical estimators, two types of approximate confidence intervals are constructed via the observed Fisher information matrix. Using gamma conjugate priors, the Bayes estimators against the squared-error and linear-exponential loss functions are derived. As expected, the Bayes estimates are not explicitly expressed, thus the Markov chain Monte Carlo techniques are implemented to approximate the Bayes point estimates and to construct the associated highest posterior density credible intervals. The performance of proposed estimators is evaluated via some numerical comparisons and some specific recommendations are also made. We further discuss the issue of determining the optimum progressive censoring plan among different competing censoring plans using three optimality criteria. Finally, two real-life datasets are analyzed to demonstrate how the proposed methods can be used in real-life scenarios.

1. Introduction

Several extensions of the exponential distribution such as Weibull, gamma, generalized-exponential, and Nadarajah-Haghighi distributions among others have been proposed in literature. A general technique that allows to adding a new shape parameter to expand a family of distributions is originally proposed by Marshall and Olkin [1]. Applying the Marshall-Olkin technique to generalized-exponential distribution, Ristić and Kundu [2] introduced the three-parameter Marshall-Olkin generalized-exponential (MOGE) distribution. However, suppose that the lifetime random variable X of an individual testing item follows a three-parameter $\mathrm{MOGE}(\alpha, \beta, \theta)$. Hence, its cumulative distribution function (CDF) $F(\cdot)$, probability density function (PDF) $f(\cdot)$, reliability function (RF) $R(\cdot)$, and hazard function (HF) $h(\cdot)$, at distinct time t, are given, respectively, by

$$F(x;\alpha,\beta,\theta)=\frac{\left(1-e^{-\theta x}\right)^{\alpha}}{\beta+\overline{\beta}\left(1-e^{-\theta x}\right)^{\alpha}},\quad x>0, \tag{1}$$

$$f(x;\alpha,\beta,\theta)=\frac{\alpha\beta\theta e^{-\theta x}\left(1-e^{-\theta x}\right)^{\alpha-1}}{\left(\beta+\overline{\beta}\left(1-e^{-\theta x}\right)^{\alpha}\right)^{2}},\quad x>0, \tag{2}$$

$$R(t;\alpha,\beta,\theta)=\frac{\beta\left(1-\left(1-e^{-\theta t}\right)^{\alpha}\right)}{\beta+\overline{\beta}\left(1-e^{-\theta t}\right)^{\alpha}},\quad t>0, \tag{3}$$

$$h(t;\alpha,\beta,\theta)=\frac{\alpha\theta e^{-\theta t}\left(1-e^{-\theta t}\right)^{\alpha-1}}{\left(\beta+\overline{\beta}\left(1-e^{-\theta t}\right)^{\alpha}\right)\left(1-\left(1-e^{-\theta t}\right)^{\alpha}\right)},\quad t>0, \tag{4}$$

where $\overline{\beta}=1-\beta$, $\alpha>0$, and $\beta>0$ are the shape parameters and $\theta>0$ is the scale parameter. Ristić and Kundu [2] derived several properties of the MOGE distribution and estimated its parameters using the likelihood method.

Using some specified values on the range of α, β, and θ, different shapes of the density and failure rate functions of the MOGE distribution are displayed in Figure 1. It shows that the density shapes are decreasing or unimodal while the HF shapes can be increasing, decreasing, bathtub, or inverted tub. Since the bathtub HF shape is quite beneficial in several lifetime data, therefore the MOGE distribution is justly flexible and can be used to provide a good description of different types of censored data, for details, it can be seen in Ristić and Kundu [2]. From (1), three lifetime submodels can be obtained as special cases from the MOGE distribution, namely,

(i) Marshall–Olkin exponential distribution, by Marshall and Olkin [1]; if setting $\alpha=1$.

(ii) Generalized-exponential distribution, by Gupta and Kundu [3]; if setting $\theta=1$.

(iii) Exponential distribution, discussed by Johnson and Kotz [4]; if setting $\alpha=\theta=1$.

Statistical life distributions under censoring plans have received great interest owing to their wide application in numerous fields such as engineering, social sciences, marketing, and medicine. One of the most popular censoring mechanisms, which is of great importance in reliability studies, is known as type-II progressive censoring scheme (PCS-T2). This censoring scheme can be described as follows: suppose that n identical and independent units are put on a life-testing experiment at time zero and the progressive censoring scheme $\mathbf{R}=(R_1,R_2,\ldots,R_m)$ is prefixed, where $m(<n)$ is the number of observed failures that is decided by the experimenter in advance. At the time of the first failure occur (say $x_{1:m:n}$), R_1 of survival items are randomly removed from the remaining live $(n-1)$ units. Again, at the time of the next failure occur (say $x_{2:m:n}$), R_2 of survival items are randomly removed from the remaining live $(n-R_1-2)$ units and continue the test. This procedure continues until all remaining live R_m units are removed from the experiment at the time of the mth failure observed. It is clear that $n-m=\sum_{i=1}^{m}R_i$. For more details, one may refer to an excellent monograph presented by Balakrishnan and Cramer [5].

However, the likelihood function of the PCS-T2 sample $x_{i:m:n}$ for $i=1,2,\ldots,m$, is defined as

$$L(\Theta\mid\mathbf{x})=C\prod_{i=1}^{m}f\left(x_{i:m:n};\Theta\right)\left[1-F\left(x_{i:m:n};\Theta\right)\right]^{R_i}, \tag{5}$$

where $C=n(n-R_1-1)\ldots(n-\sum_{i=1}^{m-1}(R_i+1))$ and Θ is parameter vector.

In literature, several authors including Amin [6], Kim et al. [7], Huang and Wu [8], Dey et al. [9], Panahi [10], Muhammed and Almetwally [11], Elshahhat and Rastogi [12], Maiti and Kayal [13], Dey et al. [14], Maiti et al. [15], and Elshahhat and Abu El Azm [16] studied a wide variety of inferential problems for different lifetime distributions under type-II progressively censored samples.

In the context of censoring mechanisms, to the best of our knowledge, we have not encountered any work related to the estimation of parameters and/or survival characteristics of the MOGE distribution under incomplete (censored) sampling, which is of considerable interest and practical significance in many practical situations. So, in this study, we shall exclusively focus on both classical and Bayesian approaches of estimation to derive the both point and interval estimators of the MOGE parameters α, β, and θ, as well as its time parameters $R(t)$ and $h(t)$ under PCS-T2. Our objectives in this study are as follows: first, deriving the maximum likelihood estimators (MLEs) for the unknown MOGE parameters or any function of them. Correspondingly, using observed Fisher information matrix, two-sided asymptotic confidence intervals are also established. Second, obtain the Bayes estimators of the unknown quantities $\alpha,\beta,\theta,R(t)$, and $h(t)$ using independent gamma priors based on the two popular symmetric and asymmetric loss functions. Markov chain Monte Carlo (MCMC) techniques are also used to carry out the Bayes estimates and associated credible interval estimates from the complex posterior functions. Third, determine the optimum progressive censoring plan from a class of all possible removal patterns that contain a magnificent amount of information regarding the unknown model parameter under consideration. Lastly, the performance of the proposed methods is compared through an extensive simulation study in terms of their simulated root mean squared-error, mean relative absolute bias, average confidence lengths, and coverage percentages. To discuss how the proposed methods can be applied in a real phenomenon, two real-life data sets are analyzed. We also suggest the use of “maxLik” and “coda” statistical packages in R programming software to evaluate the theoretical findings. Finally, some specific recommendations are made.

The rest of the paper is arranged as follows: classical and Bayes estimations of unknown parameters and the reliability characteristics are presented in Sections 2 and 3, respectively. Optimum progressive censoring plans are discussed in Section 4. Monte Carlo simulation is performed in Section

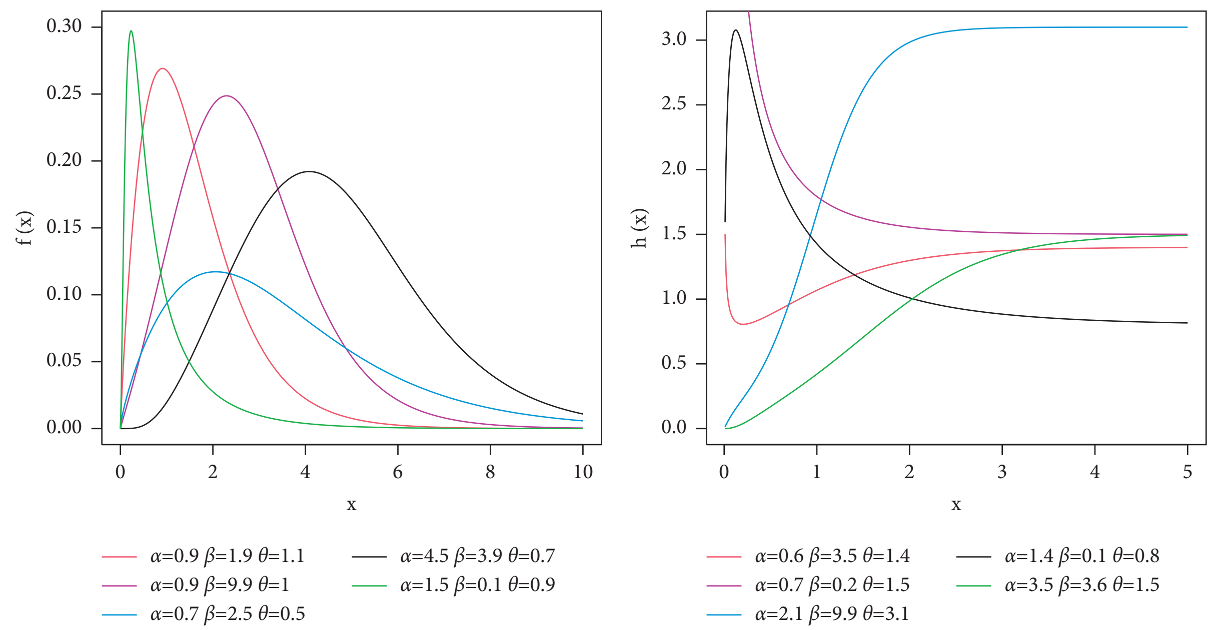


FIGURE 1: Plots of the density and hazard rate functions of the MOGE distribution.

5. Two applications using real data sets are presented in Section 6. Finally, we conclude the paper in Section 7.

2. Likelihood Inference

In this section, the point and interval estimators of the unknown parameters and the reliability characteristics of the MOGE distribution based on PCS-T2 are derived.

2.1. Point Estimators. Suppose $\mathbf{x} = \{(x_{1:m:n}, R_1), (x_{2:m:n}, R_2), \ldots, (x_{m:m:n}, R_m)\}$ is a PCS-T2 data obtained from MOGE (α, β, θ). From (1) and (2), with ignoring any additive constant, the likelihood function (5) becomes

$$L(\alpha, \beta, \theta|\mathbf{x}) \propto \beta^n (\alpha\theta)^m e^{-\theta \sum_{i-1}^{m} x_i} \cdot \prod_{i=1}^{m} \psi_i^{\alpha-1}(\theta)\left(1-\psi_i^{\alpha}(\theta)\right)^{R_i}\left(\beta+\bar{\beta}\psi_i^{\alpha}(\theta)\right)^{-(R_i+2)}, \tag{6}$$

where x_i is used instead of $x_{i:m:n}$ and. $\psi_i(\theta) = 1 - \exp(-\theta x_i), i = 1, 2, \ldots, m$.

From (6), the log-likelihood function, $\ell(\cdot) = \log L(\cdot)$, can be written as

$$\ell(\alpha, \beta, \theta|\mathbf{x}) \propto n \log(\beta) + m \log(\alpha\theta) - \theta m \overline{x} + (\alpha - 1)\sum_{i=1}^{m} \log(\psi_i(\theta)) + \sum_{i=1}^{m} R_i \log\left(1-\psi_i^{\alpha}(\theta)\right) - \sum_{i=1}^{m} (R_i+2)\log\left(\beta + \bar{\beta}\psi_i^{\alpha}(\theta)\right). \tag{7}$$

Setting the first-partial derivatives of (7) with respect to each unknown parameter to zero, the respective MLEs of α, β, and θ are the solutions to the following normal equations as

$$0 = \frac{m}{\widehat{\alpha}} + \sum_{i=1}^{m} \log\left(\widehat{\psi}_i(\widehat{\theta})\right) - \sum_{i=1}^{m} R_i \log\left(\widehat{\psi}_i(\widehat{\theta})\right)\widehat{\psi}_i^{\widehat{\alpha}}(\widehat{\theta})\left(1-\widehat{\psi}_i^{\widehat{\alpha}}(\widehat{\theta})\right)^{-1} - \bar{\widehat{\beta}}\sum_{i=1}^{m} (R_i+2)\log\left(\widehat{\psi}_i(\widehat{\theta})\right)\widehat{\psi}_i^{\widehat{\alpha}}(\widehat{\theta})\left(\widehat{\beta} + \bar{\widehat{\beta}}\widehat{\psi}_i^{\widehat{\alpha}}(\widehat{\theta})\right)^{-1}, \tag{8}$$

$$0 = \frac{n}{\widehat{\beta}} - \sum_{i=1}^{m} (R_i+2)\left(1-\widehat{\psi}_i^{\widehat{\alpha}}(\widehat{\theta})\right)\left(\widehat{\beta} + \bar{\widehat{\beta}}\widehat{\psi}_i^{\widehat{\alpha}}(\widehat{\theta})\right)^{-1}, \tag{9}$$

$$0 = \frac{m}{\widehat{\theta}} - m\overline{x} + (\widehat{\alpha} - 1)\sum_{i=1}^{m} \widehat{\psi}_i'(\widehat{\theta})\widehat{\psi}_i^{-1}(\widehat{\theta}) - \widehat{\alpha}\sum_{i=1}^{m} R_i\widehat{\psi}'_i(\widehat{\theta})\left(\widehat{\psi}_i(\widehat{\theta})\right)^{\widehat{\alpha}-1}\left(1 - \widehat{\psi}_i^{\widehat{\alpha}}(\widehat{\theta})\right)^{-1}$$
$$- \widehat{\alpha}\overline{\widehat{\beta}}\sum_{i=1}^{m}(R_i + 2)\widehat{\psi}_i'(\widehat{\theta})\left(\widehat{\psi}_i(\widehat{\theta})\right)^{\widehat{\alpha}-1}\left(\widehat{\beta} + \overline{\widehat{\beta}}\widehat{\psi}_i^{\widehat{\alpha}}(\widehat{\theta})\right)^{-1}, \tag{10}$$

where $\widehat{\psi}_i'(\widehat{\theta}) = x_i(1 - \widehat{\psi}_i(\widehat{\theta}))$ for $i = 1, \ldots, m$.

Obviously, from (8) and (10), the MLEs $\widehat{\alpha}$, $\widehat{\beta}$, and $\widehat{\theta}$ of the parameters α, β, and θ, respectively, cannot be obtained analytically but can be evaluated using any iterative approximation technique. Once $\widehat{\alpha}$, $\widehat{\beta}$, and $\widehat{\theta}$ calculated, using the invariance property of MLEs, the corresponding MLEs $\widehat{R}(t)$ and $\widehat{h}(t)$ of $R(t)$ and $h(t)$ at any given time $t > 0$ can be easily derived by replacing α, β, and θ with $\widehat{\alpha}, \widehat{\beta}$, and $\widehat{\theta}$ in (3) and (4), respectively.

2.2. Asymptotic Intervals. To construct the $100(1-\tau)\%$ two-sided ACIs for the unknown parameters α, β, and θ, or any life function such as $R(t)$ and $h(t)$ (say φ), the asymptotic variances or covariances (by inverting the Fisher information matrix $\mathbf{I}(\cdot)$) of the MLEs $\widehat{\alpha}$, $\widehat{\beta}$, and $\widehat{\theta}$ must be obtained as

$$\mathbf{I}^{-1}(\widehat{\varphi}) = \begin{bmatrix} \mathscr{L}_{11} & & \\ \mathscr{L}_{12} & \mathscr{L}_{22} & \\ \mathscr{L}_{31} & \mathscr{L}_{32} & \mathscr{L}_{33} \end{bmatrix}^{-1}_{\varphi=\widehat{\varphi}} = \begin{bmatrix} \widehat{\sigma}_{11} & & \\ \widehat{\sigma}_{21} & \widehat{\sigma}_{22} & \\ \widehat{\sigma}_{31} & \widehat{\sigma}_{32} & \widehat{\sigma}_{33} \end{bmatrix}. \tag{11}$$

Under some mild regularity conditions of the MLEs, using the concept of large sample theory, we have the asymptotic normality approximation (NA) of $\widehat{\varphi}$ is approximately multivariate normal with mean vector φ and variance-covariance matrix $\mathbf{I}^{-1}(\varphi)$, i.e., $\widehat{\varphi} \sim N_3[\varphi, \mathbf{I}^{-1}(\widehat{\varphi})]$, where $\varphi = (\alpha, \beta, \theta)^{\top}$ is the parametric vector of α, β, and θ. By differentiating (7) partially with respect to α, β, and θ, locally at their MLEs $\widehat{\alpha}$, $\widehat{\beta}$, and $\widehat{\theta}$, the Fisher's elements $\mathscr{L}_{ij}$ for $i, j = 1, 2, 3$, are obtained and provided in Appendix A.

Similarly, based on NA of the MLEs $\widehat{R}(t)$ and $\widehat{h}(t)$, it is clear that $\widehat{R} \sim N[R(t), \widehat{\sigma}^2_{R(t)}]$ and $\widehat{h}(t) \sim N[h(t), \widehat{\sigma}^2_{h(t)}]$. Thus, ACIs for $R(t)$ and $h(t)$ can be constructed using the corresponding normality. To obtain ACIs of $R(t)$ and $h(t)$, we need to compute their variances. Following Greene [17], the delta method is used to find $\widehat{\sigma}^2_R$ and $\widehat{\sigma}^2_h$ of $R(t)$ and $h(t)$, respectively, as:

$$\widehat{\sigma}^2_R = \left[\nabla_R, \mathbf{I}^{-1}(\varphi), \nabla_R^{\top}\right]|_{\varphi=\widehat{\varphi}},$$
$$\widehat{\sigma}^2_h = \left[\nabla_h, \mathbf{I}^{-1}(\varphi), \nabla_h^{\top}\right]|_{\varphi=\widehat{\varphi}}, \tag{12}$$

where,

$$\nabla_R = \left(\frac{\partial}{\partial\alpha}R(t), \frac{\partial}{\partial\beta}R(t), \frac{\partial}{\partial\theta}R(t)\right),$$
$$\nabla_h = \left(\frac{\partial}{\partial\alpha}h(t), \frac{\partial}{\partial\beta}h(t), \frac{\partial}{\partial\theta}h(t)\right). \tag{13}$$

Then, based on NA, the $100(1-\tau)\%$ two-sided ACIs for the unknown parameter α, β, θ, $R(t)$ or $h(t)$ (say κ) is given by:

$$\widehat{\kappa} \pm z_{\tau/2}\sqrt{\widehat{\sigma}^2_{\widehat{\kappa}}}, \tag{14}$$

where $z_{\tau/2}$ is the upper $(\tau/2)^{\text{th}}$ quantile of the standard normal distribution.

Though, the main problem of asymptotic NA confidence interval is that it may give a negative value in lower bound for a lifetime parameter. To handling this drawback, Meeker and Escobar [18] developed the normal approximation for the log-transformed (NL) of MLE. They also showed that ACI has better CP based on NL compared to NA. However, based on NL, the $100(1-\tau)\%$ two-sided ACI of κ is given by

$$\widehat{\kappa}\exp\left(\mp\frac{z_{\tau/2}\sqrt{\widehat{\sigma}^2_{\widehat{\kappa}}}}{\widehat{\kappa}}\right). \tag{15}$$

3. Bayesian Inference

In this section, we discuss the Bayes inference of the model parameters α, β, and θ, as well as, the reliability parameters $R(t)$ and $h(t)$ using progressively type-II censoring.

3.1. Prior Information and Loss Functions. Since the gamma $G(\cdot)$ density provides various shapes based on parameter values and is flexible in nature, so the utilizing of independent gamma priors are relatively simple which may yield to results with more explicit posterior density expressions Muse et al. [19]; for more details, it can be seen in [14, 20, 21]. Thus, the gamma conjugate priors $G_\alpha(a_1, b_1)$, $G_\beta(a_2, b_2)$, and $G_\theta(a_3, b_3)$ are used to adapt the support of MOGE parameters α, β, and θ, respectively. However, the joint prior PDF of α, β, and θ becomes

$$\pi(\alpha, \beta, \theta) \propto \alpha^{a_1-1}\beta^{a_2-1}\theta^{a_3-1}e^{-\left(b_1\alpha+b_2\beta+b_3\theta\right)}, \quad \alpha, \beta, \theta > 0, \tag{16}$$

where the hyperparameters a_i, b_i, $i = 1, 2, 3$, are chosen to reflect prior knowledge about the unknown parameters α, β, and θ, and they assumed to be known as non-negative.

The choice of (symmetric and/or symmetric) loss function is an important issue in Bayesian analysis. So, to develop the objective estimates, two well-known loss functions namely, squared-error loss (SEL) and linear-exponential loss (LL), are considered. However, any other loss function can be easily incorporated. The SEL function is defined as

$$\mathfrak{I}_S(\phi,\widetilde{\phi}) = (\widetilde{\phi}-\phi)^2, \tag{17}$$

where $\widetilde{\phi}$ being an estimate of ϕ. Under (13), the Bayes estimate $\widetilde{\phi}$ is the posterior mean of ϕ.

On the other hand, the LL function, $\mathfrak{I}_L(\cdot)$, is defined as

$$\mathfrak{I}_L(\phi,\widetilde{\phi}) = \exp(v(\widetilde{\phi}-\phi)) - v(\widetilde{\phi}-\phi) - 1, \quad v \neq 0. \tag{18}$$

The sign and magnitude of v represent the direction and degree of symmetry, respectively (if $v>0$, overestimation is more serious than underestimation and $v<0$ means the opposite). Using (14), the Bayes estimator $\widetilde{\phi}_L$ of ϕ is given by:

$$\widetilde{\phi}_L = -v^{-1}\ln\left(E_\phi\left[e^{-v\phi}|\mathbf{x}\right]\right), \quad v \neq 0, \tag{19}$$

provided the above exception exists, and is finite. When v close to 0, the LL function is approximately the SEL function and it can become almost symmetric, as can be seen in Pandey and Rai [22].

3.2. Posterior Analysis. Using (6) and (12), the joint posterior PDF of α, β, and θ can be written as

$$\pi(\alpha,\beta,\theta|\mathbf{x}) = K^{-1}\alpha^{a_1+m-1}\beta^{a_2+n-1}\theta^{a_3+m-1}e^{-\left(\alpha b_1+\beta b_2+\theta\left(b_3+m\overline{x}\right)\right)} \times \prod_{i=1}^{m}\psi_i^{\alpha-1}(\theta)\left(1-\psi_i^{\alpha}(\theta)\right)^{R_i}\left(\beta+\overline{\beta}\psi_i^{\alpha}(\theta)\right)^{-(R_i+2)}, \tag{20}$$

where K is the normalizing constant of (15).

Due to nonlinear expression in (6), the closed-form of the marginal PDF for each unknown parameter is not possible. Thus, we purpose to apply some simulation algorithms such as the MCMC methods to approximate the Bayesian estimates and to find associated credible intervals. So the MCMC methods have been widely used in Bayesian computational analysis. At first, from (15), we derive the conditional posterior distributions of α, β, and θ, respectively, as

$$\pi_\alpha^*(\alpha|\beta,\theta,\mathbf{x}) \propto \alpha^{a_1+m-1}e^{-\alpha b_1^*(\theta)}\prod_{i=1}^{m}\left(1-\psi_i^{\alpha}(\theta)\right)^{R_i}\left(\beta+\overline{\beta}\psi_i^{\alpha}(\theta)\right)^{-(R_i+2)}, \tag{21}$$

$$\pi_\beta^*(\beta|\alpha,\theta,\mathbf{x}) \propto \beta^{a_2+n-1}e^{-\beta b_2}\prod_{i=1}^{m}\left(\beta+\overline{\beta}\psi_i^{\alpha}(\theta)\right)^{-(R_i+2)}, \tag{22}$$

$$\pi_\theta^*(\theta|\alpha,\beta,\mathbf{x}) \propto \theta^{a_3+m-1}e^{-\theta\left(b_3+m\overline{x}\right)}\prod_{i=1}^{m}\psi_i^{\alpha-1}(\theta)\left(1-\psi_i^{\alpha}(\theta)\right)^{R_i}\left(\beta+\overline{\beta}\psi_i^{\alpha}(\theta)\right)^{-(R_i+2)}, \tag{23}$$

where $b_1^*(\theta) = b_1 - \sum_{i=1}^{m}\log(\psi_i(\theta))$.

It can be seen, from (16)–(18), that the conditional distributions of α, β, and θ cannot be reduced to any standard distribution so that the exact inferences from their marginal densities cannot be easily obtained. Thus, to generate samples from posterior distribution whenever the posterior PDF cannot be reduced to any familiar distribution, the Metropolis–Hastings algorithm can be easily used for this purpose (see Gelman et al. [23] for detail).

3.3. M-H Algorithm. The Metropolis-Hastings (M-H) algorithm is a general member of the MCMC family of simulation techniques. In this subsection, the M-H algorithm with normal proposal distributions is conducted to generate samples of α, β, and θ from (16), (17), and (18), respectively. Precisely, Figure 2 indicates that the distribution of MOGE parameters behaves similar to the normal distribution. Next, to obtain the Bayes estimates and the associated credible intervals, do the following generation steps:

Step 1: Put $g=1$.

Step 2: Set initial values $(\alpha^{(0)},\beta^{(0)},\theta^{(0)}) = (\widehat{\alpha},\widehat{\beta},\widehat{\theta})$.

Step 3: Generate α^*, β^*, and θ^* from (16)–(18) with normal distributions $N(\widehat{\alpha},\widehat{\sigma}_\alpha^2)$, $N(\widehat{\beta},\widehat{\sigma}_\beta^2)$, and $N(\widehat{\theta},\widehat{\sigma}_\theta^2)$, respectively, as follows:

(a) Calculate $Q_\alpha = \pi_\alpha^*(\alpha^*|\ \beta^{(g-1)},\theta^{(g-1)},\mathbf{x})/\pi_\alpha^*(\alpha^{(g-1)}|\beta^{(g-1)},\theta^{(g-1)},\mathbf{x})$, $Q_\beta = \pi_\beta^*(\beta^*|\alpha^{(g)},\theta^{(g-1)},\mathbf{x})/\pi_\beta^*(\beta^{(g-1)}|\alpha^{(g)},\theta^{(g-1)},\mathbf{x})$ and $Q_\theta = \pi_\theta^*(\theta^*|\alpha^{(g)},\beta^{(g)},\mathbf{x})/\pi_\theta^*(\theta^{(g-1)}|\alpha^{(g)},\beta^{(g)},\mathbf{x})$.

(b) Generate sample variates u_1, u_2, and u_3 from the uniform $U(0,1)$ distribution.

(c) If $u_1 \leqslant \min\{1,Q_\alpha\}$, set $\alpha^{(g)}=\alpha^*$, else set $\alpha^{(g)}=\alpha^{(g-1)}$.

(d) If $u_2 \leqslant \min\{1,Q_\beta\}$, set $\beta^{(g)}=\beta^*$, else set $\beta^{(g)}=\beta^{(g-1)}$.

(e) If $u_3 \leqslant \min\{1,Q_\theta\}$, set $\theta^{(g)}=\theta^*$, else set $\theta^{(g)}=\theta^{(g-1)}$.

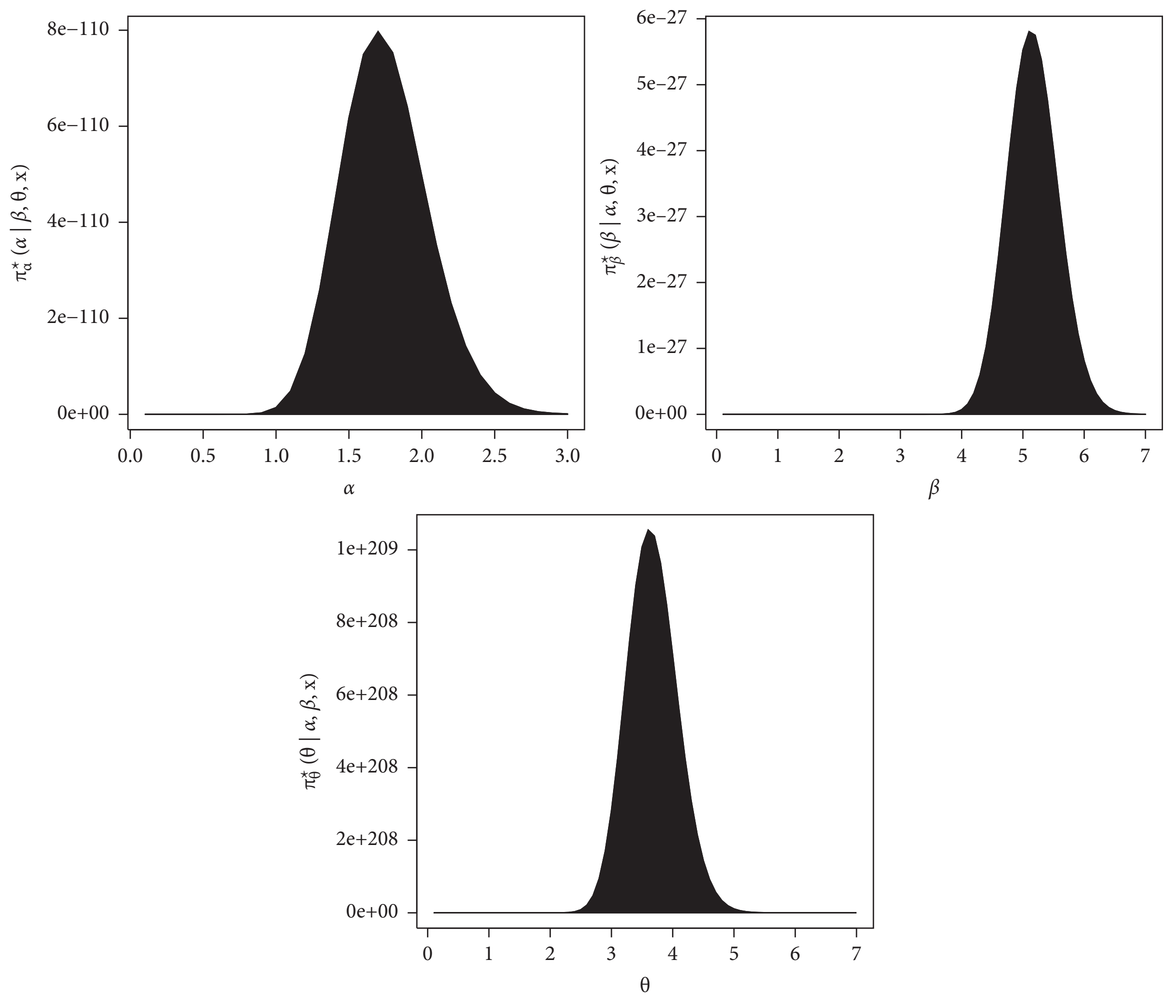


Figure 2: Plots of the conditional distributions of α, β, and θ.

Step 4: Obtain $R^{(g)}(t)$ and $h^{(g)}(t)$, for a specified time $t > 0$ by replacing α, β, and θ with their $\alpha^{(g)}$, $\beta^{(g)}$, and $\theta^{(g)}$, respectively.

Step 5: Put $g = g + 1$.

Step 6: Redo Steps 3–5 G times to obtain the MCMC simulated variates of α, β, θ, $R(t)$, or $h(t)$ (say κ) as $\kappa^{(g)} = (\alpha^{(g)}, \beta^{(g)}, \theta^{(g)}, R^{(g)}(t), h^{(g)}(t))$ for $g = 1, 2, \ldots, G$.

Step 7: Use outputs in Step 6, compute the Bayesian estimates of κ under SEL and LL functions given in (13) and (14), respectively, as:

$$\widetilde{\kappa}_S = \frac{\sum_{g=G_0+1}^{G} \kappa^{(g)}}{(G - G_0)}, \tag{24}$$

$$\widetilde{\kappa}_L = -\frac{1}{\upsilon} \log\left[\frac{\sum_{g=G_0+1}^{G} \exp\left(-\upsilon\kappa^{(g)}\right)}{(G - G_0)}\right], \quad \upsilon \neq 0, \tag{25}$$

where G_0 is the burn-in period.

3.4. Credible Intervals. To construct the Bayes credible interval (BCI) and highest posterior density (HPD) credible interval for the parameter of interest, the MCMC simulated variates of the target parameter are used. Under HPD credible interval, for every point inside the interval, the study distribution must be greater than that for every point outside the interval. Therefore, it has the shortest length among all possible credible intervals. According to the procedure proposed by Chen and Shao [24], the HPD credible interval can be constructed. Now, to establish the BCIs (or HPD credible intervals) of the unknown parameters α, β, and θ or the survival characteristics $R(t)$ and $h(t)$, for short say κ, one can be easily perform the following process:

Step 1: Order the simulated MCMC variates of $\kappa^{(g)}$ for $g = G_0 + 1, \ldots, G$, as $\kappa_{(G_0+1)}, \kappa_{(G_0+2)}, \ldots, \kappa_{(G)}$.

Step 2: Set the level of significance τ.

Step 3: Obtain the $100(1 - \tau)$ BCI of κ as $(\kappa_{(G-G_0)(\tau/2)}, \kappa_{(G-G_0)(1-\tau/2)})$.

Step 4: Obtain the $100(1 - \tau)\%$ HPD credible interval κ is given by,

$$\kappa_{(g^*)}, \kappa_{(g^*+(1-\tau)(G-G_0))}, \tag{26}$$

where $g^* = G_0 + 1, \ldots, G$ is chosen, such that,

$$\begin{aligned}&\kappa_{(g^*+[(1-\tau)(G-G_0)])} - \kappa_{(g^*)}\\ &\quad = \min_{1\leqslant g\leqslant \tau(G-G_0)}\left(\kappa_{(g+[(1-\tau)(G-G_0)])} - \kappa_{(g)}\right).\end{aligned} \tag{27}$$

Here, $[x]$ denotes the largest integer less than or equal to x.

4. Optimum Progressive Censoring Plans

In reliability field, the problem to determine the "optimal" censoring scheme from a set of available censoring schemes is main issue. So, to great the information included in the unknown parameters under consideration, Balakrishnan and Aggarwala [25] first discussed the optimal censoring plan via several setups. Following Ng et al. [26], the values of n (total test units) and m (effective sample) must be fixed in advance, then one can be easily determine the optimal censoring design $(R_1, R_2, \ldots, R_m$ where $\sum_{i=1}^{m} R_i = n - m)$ under type-II progressive censoring by considering the following optimum criteria:

(i) Criterion-A: Minimize $\longrightarrow$ trace$(\mathbf{I}^{-1}(\cdot))$.
(ii) Criterion-B: Minimize $\longrightarrow$ det$(\mathbf{I}^{-1}(\cdot))$.
(iii) Criterion-C: Maximize $\longrightarrow$ trace$(\mathbf{I}(\cdot))$.

Regarding to criteria A and B, our goal is to minimize the trace and determinant of $\mathbf{I}^{-1}(\cdot)$ while regarding to the criterion C our goal is to maximize the main diagonal elements of $\mathbf{I}(\cdot)$, with respect to MLE $\widehat{\varphi}$ of φ. The optimized PCS-T2 plan that provides more information corresponds to the smallest value of criteria (A and B) with the highest value of C-criterion.

5. Monte Carlo Simulation

To show the performance of the proposed estimators of the unknown parameters α, β, and θ as well as the survival characteristics $R(t)$ and $h(t)$, a Monte Carlo simulation study is carried out. Using various combinations of n (total test units), m (effective sample size), and $\mathbf{R}$ (progressive censoring), a large 1,000 PCS-T2 samples are generated from the MOGE distribution when the true value of parameters α, β, θ is taken as $(0.8, 0.5, 0.2)$. Thus, for time $t = 0.1$, the actual value of the reliability and hazard functions is used as 0.91683 and 0.68856, respectively. In this study, some specified values of n, m, and $\mathbf{R}$ are also taken in account such as $n = 30$ (small), 60 (moderate), and 90 (large). As soon as the number of failed subjects achieves (or exceeds) m, where the failure proportion (FP), $(m/n)100\%$, is taken as 30, 60, and 90%, the experiment is terminated.

Moreover, for given n and m, different censoring schemes $\mathbf{R}$ to remove survival items during the test are also considered as:

$$\begin{aligned}&\text{CS}-1\colon R_1 = n-m,\\ &\qquad R_i = 0 \text{ for } i \neq 1,\\ &\text{CS}-2\colon \begin{cases} R_{m/2} = n-m, R_i = 0 \text{ for } \dfrac{i\neq m}{2}, \text{ if } m \text{ is even},\\ R_{(m+1)/2} = n-m, R_i = 0 \text{ for } \dfrac{i\neq (m+1)}{2}, \text{ if } m \text{ is odd},\\ \text{CS}-3\colon R_m = n-m, R_i = 0 \text{ for } i\neq m.\end{cases}\end{aligned} \tag{28}$$

To examine the effects of the prior parameters a_i and b_i for $i = 1, 2, 3$ on the Bayesian analysis, two informative sets of the hyperparameters are used; called Prior-1 (say P1): $(a_1, a_2, a_3) = (1.6, 1, 0.4)$ and $b_i = 2, i = 1, 2, 3$, and Prior-II (say P2): $(a_1, a_2, a_3) = (4, 2.5, 1)$ and $b_i = 5, i = 1, 2, 3$. These values are specified in such a way that the expected value of prior became the expected value of the corresponding parameter. If one setting $a_i, b_i = 0$, $i = 1, 2, 3$, the posterior distribution (20) is reduced in proportion to the corresponding likelihood function (6). So, if one does not have prior information on the unknown parameter, we recommend to use the frequentist estimates instead of the Bayes estimates because the latter are more computationally expensive.

To develop the Bayesian calculations, following the M–H algorithm proposed in Section 3.3, we generate $G = 12,000$ MCMC samples and then the first $G_0 = 2,000$ iterations have been discarded as burn-in. Hence, using 10,000 MCMC samples, the average Bayes point estimates and associated BCI/HPD credible intervals of the unknown parameters α, β, θ, $R(t)$, and $h(t)$ are computed. Here, the Bayesian estimates relative to SEL and LL $(\upsilon(= -3, -0.03, +3))$ are developed.

For each considered setting, the average point estimates (APEs) using the proposed frequentist and Bayes approaches of α, β, θ, $R(t)$, or $h(t)$ (say η) are computed by

$$\mathrm{APE}\left(\widehat{\eta}_q\right)=\frac{1}{\mathscr{B}}\sum_{j=1}^{\mathscr{B}}\widehat{\eta}_\tau^{(j)}, q=1,2,3,4,5, \tag{29}$$

where $\widehat{\eta}$ is the desired estimate of the parametric function η, $\widehat{\eta}_q^{(j)}$ denotes the obtained estimate at the $j-th$ sample of the unknown parameter η_q, $\mathscr{B}$ is number of generated sequence data, $\eta_1=\alpha$, $\eta_2=\beta$, $\eta_3=\theta$, $\eta_4=R(t)$, and $\eta_5=h(t)$.

Comparison between different point estimates of α, β, θ, $R(t)$, or $h(t)$ (say η) is made based on their root mean squared-error (RMSE) and mean relative absolute bias (MRAB) values using the following formulae, respectively, as

$$\mathrm{RMSE}\left(\widehat{\eta}_q\right)=\sqrt{\frac{1}{\mathscr{B}}\sum_{j=1}^{\mathscr{B}}\left(\widehat{\eta}_q^{(j)}-\eta_q\right)^2}, \quad q=1,2,3,4,5, \tag{30}$$

$$\mathrm{MRAB}\left(\widehat{\eta}_q\right)=\frac{1}{\mathscr{B}}\sum_{j=1}^{\mathscr{B}}\frac{\left|\widehat{\eta}_q^{(j)}-\eta_q\right|}{\eta_q}, \quad q=1,2,3,4,5. \tag{31}$$

Further, the performances of different type of the $(1-\tau)\%$ two-sided asymptotic/credible intervals estimates are compared using their average confidence lengths (ACLs) and the average coverage percentages (CPs) as:

$$\mathrm{ACL}\left(\eta_q\right)=\frac{1}{\mathscr{B}}\sum_{j=1}^{\mathscr{B}}\left(\mathscr{U}\left(\widehat{\eta}_q^{(j)}\right)-\mathscr{L}\left(\widehat{\eta}_q^{(j)}\right)\right), \quad q=1,2,3,4,5, \tag{32}$$

$$\mathrm{CP}\left(\eta_q\right)=\frac{1}{\mathscr{B}}\sum_{j=1}^{\mathscr{B}}1_{\left(\mathscr{L}\left(\widehat{\eta}_q^{(j)}\right);\mathscr{U}\left(\widehat{\eta}_q^{(j)}\right)\right)}\left(\eta_q\right), \quad q=1,2,3,4,5, \tag{33}$$

where $\mathscr{L}(\cdot)$ and $\mathscr{U}(\cdot)$ denote the lower and upper bounds, respectively, of $(1-\tau)\%$ asymptotic (or credible) interval of the unknown parameter η_q.

All numerical computations were performed using R 4.0.4 software with three useful packages, namely: (i) "coda" package proposed by Plummer et al. [27]; (ii) "maxLik" package proposed by Henningsen and Toomet [28]; (iii) "GoFKernel" package by Pavia [29]. Recently, the same packages are also recommended by Elshahhat and Nassar [30].

The simulation results of α, β, θ, $R(t)$, and $h(t)$ are presented in Tables 1–10 and are also displayed with heatmaps in Figures 3–7. In each heatmap plot, the "x-lab" display the proposed estimation point (or interval) methods while the "y-lab" represents the different choices of n, FP%, and censoring schemes. For specification, using P1 for example, we have used the notation "SEL-P1" for the Bayes (SEL) estimates; "LL1-P1" for the Bayes (LL) estimates when $\upsilon=-3$; "LL2-P1" for the Bayes (LL) estimates when $\upsilon=-0.03$; "LL3-P1" for the Bayes (LL) estimates when $\upsilon=+3$; "BCI-P1" and "HPD-P1" denote to the BCI and HPD credible intervals, respectively. Also, "ACI-NA" and "ACI-NL" denote to ACI based on NA and NL methods, respectively. All simulation results are presented in Appendix B. Now, we can make the following observations:

(i) In general, all estimates of the unknown parameters α, β, and θ or the time parameters $R(t)$ and $h(t)$ are very good in term of minimum RMSEs and MRABs.

(ii) As n (or m) increases, the proposed estimate become more better. This pattern is also observed when $\sum_{i=1}^{m}R_i$ decreases.

(iii) Due to gamma prior information, the Bayesian estimates perform better than the competing frequentist estimates.

(iv) Bayes estimates using LL function of all unknown parameters are or may be overestimates (or underestimates) when $\upsilon<0$ (or $\upsilon>0$).

(v) RMSEs and MRABs of the Bayes (LL) estimates are smaller than the Bayes (SEL) estimates, in most cases, when υ close to a positive value.

(vi) On the basis of smallest ACLs and highest CPs, it is noted that the ACI-NA approach is the best compared to those obtained based on the ACI-NL approach, similarly, the interval estimates based on the HPD method are the best compared to those obtained based on the BCI method.

(vii) Comparing CSs 1 and 3, it is observed that RMSEs, MRABs, and ACLs of α, β, and θ are greater based on CS-3 than CS-1, thus the associated CPs of the same unknown parameters are greater based on CS-1 than CS-3.

(viii) It is also a predictable fact because of the expected duration of the test items using CS-1 (where $n-m$ subjects at the time of 1^{st} failure observed are removed) is greater than the CS-3 (where $n-m$ subjects are withdrawn at the time of m^{th} failure observed).

(ix) Estimates of $R(t)$ and $h(t)$ have the lowest RMSEs, MRABs, and ACLs as well as the highest CPs based on CS-3 than CS-1.

(x) Since the corresponding variance of P2 is less than the variance of P1, one can be seen that the Bayes (point/interval) estimates performed satisfactory based on P2 than P1 as expected.

Table 1: The APEs (first-line), RMSEs (second-line), and MRABs (third-line) of α.

t				SEL		LL					
Prior $\longrightarrow$	FP	Scheme	MLE	I	II	I			II		
$\upsilon \longrightarrow$						−3	−0.03	+3	−3	−0.03	+3
	30%	1	0.8465	0.7242	0.7955	0.7305	0.7243	0.7100	0.8022	0.7957	0.7759
			0.3001	0.1508	0.0898	0.1081	0.1190	0.1361	0.0306	0.0307	0.0557
			0.2790	0.1484	0.0512	0.1347	0.1482	0.1681	0.0344	0.0357	0.0394
		2	0.9038	0.8281	0.8056	0.8360	0.8282	0.8050	0.8124	0.8057	0.7892
			0.3662	0.1113	0.0949	0.0632	0.0719	0.0906	0.0350	0.0436	0.0659
			0.3144	0.0924	0.0729	0.0785	0.0888	0.1077	0.0427	0.0524	0.0720
		3	0.9373	0.7476	0.8241	0.7547	0.7477	0.7268	0.8312	0.8242	0.8026
			0.5291	0.1148	0.0943	0.0599	0.0695	0.0919	0.0321	0.0306	0.0536
			0.3934	0.0861	0.0579	0.0744	0.0858	0.1085	0.0391	0.0360	0.0563
	60%	1	0.8669	0.7651	0.8201	0.7764	0.7653	0.7396	0.8271	0.8202	0.8023
			0.2799	0.1105	0.0879	0.0698	0.0766	0.0956	0.0078	0.0186	0.0503
			0.2573	0.0948	0.0312	0.0869	0.0947	0.1125	0.0087	0.0157	0.0531
		2	0.8742	0.8039	0.7980	0.8136	0.8040	0.7771	0.8050	0.7981	0.7827
30			0.2755	0.1029	0.0847	0.0371	0.0351	0.0633	0.0162	0.0205	0.0482
			0.2568	0.0640	0.0494	0.0450	0.0411	0.0621	0.0190	0.0231	0.0404
		3	0.8447	0.7963	0.7312	0.8034	0.7964	0.7837	0.7405	0.7314	0.7132
			0.2701	0.1045	0.0819	0.0462	0.0559	0.0907	0.0199	0.0278	0.0452
			0.2573	0.0700	0.0535	0.0567	0.0653	0.0914	0.0199	0.0327	0.0466
	90%	1	0.8641	0.7906	0.6813	0.7961	0.7907	0.7767	0.6922	0.6814	0.6655
			0.2482	0.1094	0.0771	0.0245	0.0377	0.0769	0.0077	0.0176	0.0442
			0.2382	0.0797	0.0271	0.0295	0.0434	0.0755	0.0065	0.0129	0.0301
		2	0.8580	0.7288	0.7580	0.7372	0.7290	0.7138	0.7659	0.7581	0.7424
			0.2566	0.0994	0.0848	0.0165	0.0231	0.0585	0.0153	0.0193	0.0463
			0.2383	0.0602	0.0485	0.0176	0.0224	0.0484	0.0177	0.0208	0.0380
		3	0.8714	0.7785	0.8176	0.7841	0.7786	0.7656	0.8261	0.8177	0.8001
			0.2576	0.0926	0.0796	0.0289	0.0290	0.0521	0.0198	0.0248	0.0426
			0.2412	0.0658	0.0434	0.0332	0.0268	0.0430	0.0207	0.0238	0.0346
	30%	1	0.8904	0.8470	0.8063	0.8543	0.8472	0.8280	0.8141	0.8064	0.7824
			0.2091	0.1016	0.0956	0.0563	0.0532	0.0609	0.0160	0.0203	0.0602
			0.2003	0.0842	0.0436	0.0678	0.0607	0.0718	0.0186	0.0228	0.0494
		2	0.8819	0.7575	0.7643	0.7637	0.7576	0.7406	0.7704	0.7644	0.7491
			0.2231	0.0962	0.0891	0.0493	0.0561	0.0760	0.0302	0.0383	0.0631
			0.2098	0.0693	0.0542	0.0608	0.0671	0.0830	0.0370	0.0445	0.0636
		3	0.8770	0.7792	0.8481	0.7847	0.7793	0.7661	0.8551	0.8482	0.8291
			0.2592	0.1015	0.0843	0.0654	0.0617	0.0622	0.0206	0.0303	0.0602
			0.2262	0.0956	0.0341	0.0805	0.0728	0.0737	0.0235	0.0306	0.0507
	60%	1	0.8690	0.8154	0.7935	0.8219	0.8155	0.7958	0.7994	0.7936	0.7761
			0.1940	0.0890	0.0835	0.0238	0.0255	0.0528	0.0090	0.0200	0.0532
			0.1836	0.0466	0.0248	0.0277	0.0302	0.0519	0.0088	0.0158	0.0374
		2	0.8592	0.7462	0.8143	0.7513	0.7463	0.7336	0.8200	0.8144	0.7997
60			0.1919	0.0933	0.0877	0.0419	0.0455	0.0740	0.0299	0.0298	0.0513
			0.1829	0.0672	0.0475	0.0511	0.0530	0.0743	0.0355	0.0352	0.0543
		3	0.8461	0.7755	0.8581	0.7812	0.7756	0.7594	0.8644	0.8582	0.8422
			0.1838	0.1013	0.0803	0.0569	0.0539	0.0615	0.0191	0.0253	0.0509
			0.1773	0.0864	0.0340	0.0689	0.0618	0.0727	0.0222	0.0259	0.0475
	90%	1	0.8621	0.7934	0.7912	0.7986	0.7935	0.7810	0.7959	0.7912	0.7796
			0.1801	0.0743	0.0715	0.0098	0.0186	0.0418	0.0082	0.0171	0.0395
			0.1721	0.0317	0.0230	0.0090	0.0144	0.0291	0.0059	0.0116	0.0260
		2	0.8568	0.8221	0.8339	0.8284	0.8222	0.8032	0.8408	0.8341	0.8132
			0.1848	0.0914	0.0801	0.0369	0.0397	0.0560	0.0219	0.0228	0.0412
			0.1767	0.0532	0.0443	0.0454	0.0460	0.0633	0.0253	0.0270	0.0421
		3	0.8436	0.8102	0.7770	0.8164	0.8103	0.7920	0.7824	0.7771	0.7632
			0.1808	0.0858	0.0782	0.0201	0.0292	0.0550	0.0166	0.0226	0.0492
			0.1726	0.0394	0.0310	0.0220	0.0286	0.0460	0.0192	0.0265	0.0424

Table 1: Continued.

t				SEL		LL					
Prior ⟶	FP	Scheme	MLE	I	II	I			II		
v ⟶						−3	−0.03	+3	−3	−0.03	+3
	30%	1	0.8978	0.7893	0.7682	0.7945	0.7894	0.7753	0.7737	0.7683	0.7535
			0.1915	0.0840	0.0783	0.0272	0.0353	0.0613	0.0174	0.0259	0.0512
			0.1805	0.0397	0.0259	0.0329	0.0396	0.0581	0.0193	0.0252	0.0416
		2	0.8705	0.7742	0.7940	0.7794	0.7743	0.7607	0.7994	0.7941	0.7791
			0.1889	0.0825	0.0809	0.0233	0.0317	0.0576	0.0136	0.0197	0.0475
			0.1816	0.0357	0.0302	0.0280	0.0348	0.0531	0.0159	0.0216	0.0409
		3	0.8751	0.8020	0.8011	0.8076	0.8021	0.7866	0.8067	0.8012	0.7856
			0.1910	0.0830	0.0748	0.0181	0.0276	0.0551	0.0117	0.0193	0.0470
			0.1772	0.0326	0.0247	0.0199	0.0265	0.0447	0.0138	0.0197	0.0385
	60%	1	0.8762	0.7918	0.7805	0.7978	0.7919	0.7746	0.7857	0.7806	0.7667
			0.1686	0.0835	0.0770	0.0250	0.0332	0.0588	0.0094	0.0203	0.0496
			0.1598	0.0370	0.0253	0.0301	0.0367	0.0548	0.0074	0.0145	0.0410
		2	0.8516	0.7721	0.8036	0.7776	0.7722	0.7575	0.8093	0.8037	0.7876
90			0.1590	0.0797	0.0784	0.0221	0.0305	0.0555	0.0086	0.0186	0.0473
			0.1534	0.0322	0.0201	0.0258	0.0321	0.0492	0.0082	0.0147	0.0332
		3	0.8507	0.7787	0.7884	0.7841	0.7788	0.7642	0.7932	0.7885	0.7764
			0.1505	0.0804	0.0729	0.0135	0.0209	0.0501	0.0102	0.0189	0.0429
			0.1493	0.0271	0.0207	0.0157	0.0217	0.0414	0.0120	0.0173	0.0296
	90%	1	0.8489	0.7797	0.7705	0.7846	0.7798	0.7672	0.7759	0.7706	0.7562
			0.1504	0.0825	0.0757	0.0160	0.0251	0.0536	0.0089	0.0195	0.0469
			0.1460	0.0226	0.0173	0.0179	0.0243	0.0360	0.0068	0.0132	0.0309
		2	0.8454	0.7981	0.7920	0.8035	0.7982	0.7835	0.7970	0.7921	0.7791
			0.1413	0.0780	0.0745	0.0105	0.0189	0.0459	0.0083	0.0179	0.0431
			0.1373	0.0231	0.0193	0.0120	0.0180	0.0359	0.0060	0.0120	0.0281
		3	0.8471	0.8010	0.8007	0.8071	0.8011	0.7845	0.8058	0.8008	0.7875
			0.1474	0.0800	0.0800	0.0127	0.0195	0.0468	0.0101	0.0168	0.0410
			0.1432	0.0291	0.0281	0.0149	0.0207	0.0393	0.0085	0.0144	0.0334

Table 2: The APEs (first-line), RMSEs (second-line), and MRABs (third-line) of β.

n				SEL		LL					
Prior ⟶	FP	Scheme	MLE	I	II	I			II		
v ⟶						−3	−0.03	+3	−3	−0.03	+3
	30%	1	1.9985	0.5629	0.5427	0.5691	0.5630	0.5571	0.5471	0.5428	0.5390
			1.9726	0.0894	0.0749	0.0692	0.0630	0.0571	0.0480	0.0432	0.0402
			2.4722	0.1327	0.1067	0.1383	0.1260	0.1142	0.0941	0.0855	0.0788
		2	1.8432	0.5834	0.5514	0.5870	0.5834	0.5804	0.5586	0.5515	0.5449
			1.5105	0.0957	0.0859	0.0874	0.0836	0.0805	0.0741	0.0680	0.0643
			2.1497	0.1668	0.1355	0.1739	0.1668	0.1608	0.1454	0.1350	0.1282
		3	6.0528	0.5304	0.4064	0.5336	0.5304	0.5279	0.4145	0.4065	0.4009
			1.6522	0.1195	0.1161	0.0899	0.0904	0.0961	0.0835	0.0954	0.1001
			2.9469	0.2079	0.2070	0.1710	0.1777	0.1914	0.1539	0.1870	0.1983
	60%	1	1.0393	0.5479	0.4952	0.5490	0.5480	0.5470	0.5055	0.4953	0.4887
			1.7948	0.0672	0.0546	0.0493	0.0482	0.0472	0.0381	0.0392	0.0393
			1.5525	0.0973	0.0961	0.0980	0.0959	0.0940	0.0729	0.0762	0.0780
		2	0.9105	0.5565	0.5675	0.5583	0.5565	0.5548	0.5727	0.5675	0.5641
30			1.4539	0.0852	0.0660	0.0592	0.0566	0.0549	0.0586	0.0522	0.0458
			1.3115	0.1350	0.1129	0.1171	0.1129	0.1097	0.1167	0.1030	0.0898
		3	1.6928	0.4110	0.5678	0.4243	0.4111	0.4043	0.5715	0.5678	0.5641
			2.1120	0.0841	0.0612	0.0715	0.0679	0.0642	0.0353	0.0316	0.0286
			2.1506	0.1420	0.0803	0.1429	0.1356	0.1282	0.0671	0.0609	0.0557
	90%	1	0.8250	0.5188	0.4619	0.5214	0.5188	0.5169	0.4635	0.4619	0.4606
			1.1076	0.0501	0.0429	0.0238	0.0201	0.0177	0.0282	0.0168	0.0155
			1.1141	0.0873	0.0490	0.0427	0.0376	0.0339	0.0408	0.0305	0.0289
		2	0.8093	0.4851	0.5093	0.4869	0.4851	0.4837	0.5117	0.5093	0.5071
			0.9211	0.0409	0.0363	0.0158	0.0163	0.0171	0.0154	0.0125	0.0100
			1.0725	0.0623	0.0484	0.0295	0.0302	0.0326	0.0233	0.0186	0.0142
		3	0.7959	0.4810	0.5145	0.4866	0.4811	0.4770	0.5180	0.5145	0.5122
			0.8940	0.0536	0.0461	0.0261	0.0254	0.0270	0.0229	0.0170	0.0136
			1.1052	0.0696	0.0413	0.0486	0.0460	0.0478	0.0359	0.0290	0.0245

Table 2: Continued.

n				SEL		LL					
Prior $\longrightarrow$	FP	Scheme	MLE	I	II	I			II		
$v \longrightarrow$						-3	-0.03	$+3$	-3	-0.03	$+3$
	30%	1	0.8289	0.4609	0.5271	0.4696	0.4609	0.4563	0.5292	0.5271	0.5254
			1.2792	0.0775	0.0648	0.0484	0.0458	0.0457	0.0302	0.0312	0.0347
			1.1895	0.1102	0.0973	0.0955	0.0910	0.0874	0.0584	0.0606	0.0688
		2	0.8221	0.5196	0.5524	0.5226	0.5197	0.5177	0.5534	0.5524	0.5515
			1.0056	0.0674	0.0582	0.0536	0.0525	0.0522	0.0246	0.0206	0.0181
			1.1491	0.1150	0.1069	0.1048	0.1049	0.1031	0.0451	0.0393	0.0353
		3	2.3964	0.4963	0.4363	0.4996	0.4964	0.4943	0.4451	0.4363	0.4319
			1.2605	0.0918	0.0505	0.0275	0.0279	0.0289	0.0632	0.0661	0.0691
			1.7870	0.1550	0.0725	0.0508	0.0512	0.0551	0.1159	0.1273	0.1361
	60%	1	0.6404	0.5408	0.4696	0.5440	0.5408	0.5387	0.4756	0.4697	0.4656
			0.6240	0.0585	0.0582	0.0452	0.0435	0.0440	0.0280	0.0277	0.0258
			0.7339	0.0910	0.0817	0.0880	0.0817	0.0876	0.0519	0.0543	0.0509
		2	0.6507	0.4515	0.5102	0.4552	0.4515	0.4490	0.5145	0.5102	0.5072
60			0.6429	0.0503	0.0485	0.0475	0.0504	0.0516	0.0203	0.0136	0.0093
			0.7673	0.0564	0.0897	0.0477	0.0970	0.1020	0.0290	0.0205	0.0145
		3	0.8705	0.4746	0.5283	0.4780	0.4747	0.4725	0.5325	0.5283	0.5257
			1.2345	0.0552	0.0500	0.0212	0.0207	0.0217	0.0355	0.0294	0.0293
			1.5016	0.0726	0.0595	0.0404	0.0385	0.0421	0.0651	0.0566	0.0562
	90%	1	0.5737	0.5100	0.5455	0.5139	0.5100	0.5078	0.5478	0.5455	0.5438
			0.4576	0.0458	0.0447	0.0438	0.0414	0.0390	0.0197	0.0129	0.0091
			0.6049	0.0639	0.0317	0.0820	0.0794	0.0773	0.0277	0.0201	0.0155
		2	0.5792	0.4880	0.5030	0.4920	0.4881	0.4857	0.5072	0.5030	0.5004
			0.4528	0.0478	0.0448	0.0182	0.0156	0.0157	0.0179	0.0107	0.0068
			0.6110	0.0440	0.0462	0.0342	0.0294	0.0291	0.0208	0.0140	0.0105
		3	0.6153	0.4814	0.4741	0.4855	0.4814	0.4790	0.4780	0.4742	0.4719
			0.5390	0.0523	0.0421	0.0143	0.0100	0.0085	0.0284	0.0284	0.0261
			0.7138	0.0596	0.0348	0.0235	0.0184	0.0159	0.0531	0.0525	0.0514
	30%	1	0.6918	0.4963	0.4965	0.4988	0.4963	0.4947	0.4989	0.4966	0.4950
			1.0070	0.0400	0.0370	0.0184	0.0138	0.0110	0.0143	0.0097	0.0070
			0.8784	0.0299	0.0305	0.0271	0.0221	0.0190	0.0186	0.0150	0.0132
		2	0.6603	0.5138	0.4904	0.5163	0.5138	0.5122	0.4936	0.4904	0.4885
			0.7114	0.0442	0.0435	0.0283	0.0245	0.0223	0.0222	0.0163	0.0131
			0.8485	0.0469	0.0285	0.0518	0.0467	0.0435	0.0343	0.0276	0.0245
		3	1.4694	0.4988	0.5048	0.5026	0.4988	0.4967	0.5082	0.5049	0.5029
			1.2001	0.0448	0.0425	0.0181	0.0154	0.0149	0.0176	0.0112	0.0075
			1.0183	0.0425	0.0250	0.0334	0.0288	0.0272	0.0190	0.0136	0.0106
	60%	1	0.5322	0.5110	0.5059	0.5136	0.5111	0.5095	0.5083	0.5059	0.5044
			0.4071	0.0387	0.0365	0.0159	0.0107	0.0083	0.0120	0.0082	0.0067
			0.5394	0.0277	0.0198	0.0225	0.0181	0.0158	0.0166	0.0125	0.0099
		2	0.5573	0.5193	0.5138	0.5212	0.5194	0.5181	0.5171	0.5138	0.5118
90			0.3903	0.0440	0.0391	0.0234	0.0205	0.0187	0.0171	0.0143	0.0130
			0.5467	0.0399	0.0388	0.0424	0.0387	0.0362	0.0317	0.0270	0.0237
		3	0.6249	0.4900	0.5005	0.4935	0.4901	0.4879	0.5033	0.5005	0.4988
			0.8033	0.0442	0.0403	0.0175	0.0104	0.0076	0.0155	0.0096	0.0061
			1.0769	0.0251	0.0204	0.0247	0.0178	0.0141	0.0187	0.0126	0.0088
	90%	1	0.5475	0.5075	0.5008	0.5103	0.5075	0.5057	0.5029	0.5008	0.4994
			0.3430	0.0377	0.0345	0.0133	0.0098	0.0075	0.0110	0.0080	0.0059
			0.4828	0.0228	0.0171	0.0206	0.0150	0.0114	0.0165	0.0118	0.0087
		2	0.5332	0.4969	0.5234	0.5004	0.4969	0.4948	0.5259	0.5234	0.5217
			0.3258	0.0426	0.0378	0.0198	0.0156	0.0138	0.0165	0.0110	0.0088
			0.4729	0.0296	0.0281	0.0325	0.0276	0.0252	0.0264	0.0199	0.0166
		3	0.5489	0.5036	0.5049	0.5073	0.5036	0.5015	0.5079	0.5049	0.5031
			0.4135	0.0438	0.0390	0.0168	0.0101	0.0064	0.0137	0.0088	0.0060
			0.5669	0.0203	0.0198	0.0198	0.0126	0.0085	0.0162	0.0102	0.0067

Table 3: The APEs (first-line), RMSEs (second-line), and MRABs (third-line) of θ.

n				SEL		LL					
Prior ⟶	FP	Scheme	MLE	I	II	I			II		
v ⟶						−3	−0.03	+3	−3	−0.03	+3
	30%	1	0.3717	0.1597	0.1297	0.1647	0.1597	0.1562	0.1365	0.1297	0.1250
			0.3998	0.0943	0.0671	0.0398	0.0421	0.0445	0.0676	0.0719	0.0758
			1.1666	0.4028	0.2552	0.1830	0.2015	0.2188	0.3174	0.3513	0.3750
		2	0.5022	0.0924	0.2262	0.1026	0.0925	0.0868	0.2341	0.2262	0.2212
			0.6359	0.1299	0.0712	0.1031	0.1093	0.1139	0.0462	0.0384	0.0339
			1.7861	0.6028	0.2124	0.4870	0.5376	0.5658	0.2090	0.1817	0.1645
		3	1.1230	0.1331	0.1982	0.1378	0.1332	0.1298	0.2000	0.1982	0.1968
			1.8096	0.0854	0.0654	0.0655	0.0686	0.0713	0.0354	0.0362	0.0381
			1.6811	0.3790	0.2378	0.3108	0.3342	0.3509	0.1651	0.1677	0.1849
	60%	1	0.2717	0.1672	0.2235	0.1730	0.1673	0.1636	0.2256	0.2235	0.2218
			0.1825	0.0653	0.0430	0.0346	0.0356	0.0377	0.0273	0.0246	0.0225
			0.6336	0.2301	0.1441	0.1605	0.1641	0.1818	0.1281	0.1177	0.1092
		2	0.2958	0.1643	0.2363	0.1719	0.1643	0.1591	0.2418	0.2363	0.2329
30			0.2429	0.0753	0.0656	0.0376	0.0396	0.0430	0.0427	0.0311	0.0245
			0.7644	0.2776	0.2008	0.1738	0.1798	0.2045	0.1705	0.1312	0.1058
		3	0.3477	0.1967	0.1665	0.2017	0.1967	0.1922	0.1724	0.1666	0.1630
			0.4242	0.0683	0.0401	0.0074	0.0074	0.0099	0.0104	0.0089	0.0082
			1.4417	0.3104	0.1088	0.0313	0.0316	0.0440	0.0427	0.0386	0.0363
	90%	1	0.2443	0.2157	0.1969	0.2191	0.2157	0.2133	0.1986	0.1970	0.1957
			0.1392	0.0467	0.0316	0.0234	0.0180	0.0146	0.0090	0.0074	0.0067
			0.4836	0.1247	0.0788	0.0955	0.0785	0.0667	0.0394	0.0344	0.0318
		2	0.2546	0.2167	0.2185	0.2196	0.2168	0.2148	0.2215	0.2185	0.2165
			0.1539	0.0433	0.0450	0.0218	0.0178	0.0152	0.0254	0.0208	0.0179
			0.5352	0.1202	0.1101	0.0978	0.0838	0.0738	0.1073	0.0926	0.0827
		3	0.2506	0.2621	0.1967	0.2633	0.2621	0.2609	0.1995	0.1967	0.1947
			0.1698	0.0564	0.0335	0.0635	0.0622	0.0610	0.0095	0.0067	0.0064
			0.6328	0.2147	0.1031	0.3167	0.3105	0.3047	0.0377	0.0307	0.0297
	30%	1	0.2633	0.1889	0.1408	0.1903	0.1889	0.1878	0.1469	0.1408	0.1372
			0.1908	0.0827	0.0574	0.0586	0.0614	0.0639	0.0243	0.0149	0.0132
			0.6170	0.3539	0.1095	0.2683	0.2960	0.3142	0.0965	0.0685	0.0614
		2	0.3201	0.1452	0.2157	0.1526	0.1452	0.1406	0.2207	0.2158	0.2129
			0.2967	0.0842	0.0538	0.0541	0.0572	0.0606	0.0292	0.0204	0.0157
			0.8839	0.3342	0.1129	0.2474	0.2738	0.2972	0.1037	0.0789	0.0645
		3	0.6700	0.2126	0.1977	0.2147	0.2126	0.2112	0.1994	0.1977	0.1965
			1.1081	0.0626	0.0543	0.0408	0.0431	0.0452	0.0218	0.0133	0.0083
			1.2887	0.2553	0.1497	0.1921	0.2103	0.2237	0.0771	0.0508	0.0336
	60%	1	0.2305	0.1945	0.1990	0.2011	0.1946	0.1912	0.2039	0.1990	0.1958
			0.1106	0.0551	0.0477	0.0398	0.0347	0.0319	0.0193	0.0127	0.0101
			0.4023	0.1675	0.1050	0.1853	0.1676	0.1566	0.0698	0.0538	0.0468
		2	0.2409	0.2121	0.2267	0.2135	0.2121	0.2109	0.2298	0.2267	0.2245
60			0.1442	0.0640	0.0428	0.0467	0.0484	0.0500	0.0182	0.0167	0.0168
			0.4870	0.2808	0.1113	0.2199	0.2366	0.2472	0.0855	0.0776	0.0760
		3	0.2676	0.1579	0.2101	0.1616	0.1579	0.1553	0.2154	0.2102	0.2067
			0.2763	0.0509	0.0333	0.0219	0.0207	0.0213	0.0093	0.0070	0.0060
			1.0459	0.1464	0.0734	0.1029	0.0950	0.0967	0.0390	0.0322	0.0287
	90%	1	0.2121	0.2098	0.2335	0.2140	0.2098	0.2073	0.2371	0.2335	0.2313
			0.0846	0.0526	0.0315	0.0197	0.0128	0.0123	0.0128	0.0127	0.0087
			0.3163	0.1479	0.0971	0.0698	0.0593	0.0579	0.0601	0.0492	0.0366
		2	0.2148	0.1527	0.1875	0.1560	0.1527	0.1506	0.1904	0.1875	0.1856
			0.0909	0.0502	0.0319	0.0324	0.0281	0.0252	0.0148	0.0128	0.0113
			0.3345	0.1478	0.0931	0.1492	0.1336	0.1225	0.0676	0.0604	0.0546
		3	0.2157	0.1836	0.1973	0.1877	0.1837	0.1810	0.1992	0.1973	0.1960
			0.1115	0.0368	0.0314	0.0183	0.0147	0.0124	0.0089	0.0068	0.0058
			0.4229	0.0721	0.0691	0.0737	0.0632	0.0561	0.0375	0.0311	0.0274

Table 3: Continued.

n				SEL		LL					
Prior ⟶	FP	Scheme	MLE	I	II		I			II	
v ⟶						−3	−0.03	+3	−3	−0.03	+3
	30%	1	0.2435	0.2101	0.1928	0.2126	0.2101	0.2085	0.1957	0.1928	0.1911
			0.1384	0.0424	0.0398	0.0175	0.0129	0.0101	0.0165	0.0124	0.0115
			0.4771	0.0896	0.0543	0.0763	0.0640	0.0584	0.0443	0.0588	0.0538
		2	0.2616	0.2103	0.1986	0.2134	0.2104	0.2085	0.2009	0.1986	0.1972
			0.1991	0.0451	0.0413	0.0243	0.0190	0.0160	0.0153	0.0118	0.0108
			0.6324	0.1075	0.0840	0.1025	0.0863	0.0764	0.0685	0.0560	0.0506
		3	0.4731	0.2010	0.2066	0.2035	0.2010	0.1995	0.2090	0.2066	0.2052
			0.7870	0.0494	0.0422	0.0367	0.0333	0.0313	0.0194	0.0140	0.0107
			0.9913	0.1631	0.0824	0.1755	0.1630	0.1549	0.0615	0.0465	0.0371
	60%	1	0.2107	0.1925	0.2057	0.1957	0.1926	0.1906	0.2086	0.2058	0.2040
			0.0837	0.0410	0.0382	0.0169	0.0137	0.0127	0.0162	0.0110	0.0077
			0.3132	0.0853	0.0525	0.0628	0.0504	0.0426	0.0432	0.0291	0.0204
		2	0.2203	0.1892	0.1933	0.1929	0.1892	0.1871	0.1964	0.1933	0.1915
90			0.0973	0.0420	0.0357	0.0198	0.0146	0.0144	0.0125	0.0090	0.0071
			0.3522	0.0948	0.0549	0.0813	0.0690	0.0666	0.0486	0.0388	0.0330
		3	0.2094	0.1940	0.2326	0.1962	0.1940	0.1926	0.2351	0.2326	0.2310
			0.2080	0.0442	0.0366	0.0263	0.0218	0.0193	0.0126	0.0104	0.0096
			0.8395	0.1030	0.0737	0.1167	0.1024	0.0936	0.0577	0.0492	0.0451
	90%	1	0.2093	0.2057	0.2060	0.2088	0.2057	0.2039	0.2086	0.2060	0.2044
			0.0672	0.0406	0.0381	0.0161	0.0112	0.0081	0.0154	0.0102	0.0066
			0.2584	0.0505	0.0493	0.0448	0.0322	0.0244	0.0401	0.0289	0.0196
		2	0.2087	0.2021	0.2172	0.2044	0.2022	0.2007	0.2205	0.2173	0.2153
			0.0718	0.0410	0.0354	0.0174	0.0140	0.0106	0.0116	0.0073	0.0047
			0.2694	0.0607	0.0510	0.0672	0.0518	0.0425	0.0331	0.0226	0.0162
		3	0.2053	0.2093	0.2205	0.2123	0.2093	0.2074	0.2233	0.2205	0.2187
			0.0925	0.0369	0.0355	0.0156	0.0111	0.0082	0.0129	0.0084	0.0058
			0.3563	0.0453	0.0434	0.0454	0.0337	0.0263	0.0429	0.0312	0.0241

Table 4: The APEs (first-line), RMSEs (second-line), and MRABs (third-line) of $R(t)$.

n				SEL		LL					
Prior ⟶	FP	Scheme	MLE	I	II		I			II	
v ⟶						−3	−0.03	+3	−3	−0.03	+3
	30%	1	0.9212	0.9094	0.9365	0.9151	0.9095	0.8942	0.9430	0.9366	0.9177
			0.1454	0.0984	0.0797	0.0564	0.0625	0.0761	0.0095	0.0203	0.0696
			0.1394	0.0673	0.0236	0.0612	0.0672	0.0788	0.0099	0.0166	0.0444
		2	0.9191	0.9565	0.9112	0.9636	0.9566	0.9349	0.9164	0.9112	0.8991
			0.1461	0.1013	0.0742	0.0480	0.0456	0.0717	0.0209	0.0280	0.0473
			0.1377	0.0671	0.0325	0.0510	0.0461	0.0612	0.0218	0.0269	0.0390
		3	0.9240	0.9206	0.9016	0.9285	0.9207	0.8929	0.9070	0.9017	0.8869
			0.1425	0.1006	0.0854	0.0324	0.0367	0.0736	0.0124	0.0224	0.0513
			0.1365	0.0380	0.0340	0.0346	0.0380	0.0524	0.0107	0.0166	0.0326
	60%	1	0.9237	0.9146	0.9096	0.9226	0.9148	0.8882	0.9135	0.9096	0.9007
			0.1385	0.0942	0.0658	0.0280	0.0284	0.0510	0.0090	0.0179	0.0477
			0.1342	0.0450	0.0176	0.0286	0.0293	0.0461	0.0060	0.0108	0.0271
		2	0.9200	0.9284	0.9097	0.9363	0.9285	0.9004	0.9138	0.9097	0.9007
30			0.1364	0.1002	0.0656	0.0362	0.0425	0.0615	0.0113	0.0196	0.0419
			0.1315	0.0478	0.0298	0.0390	0.0446	0.0566	0.0097	0.0143	0.0264
		3	0.9246	0.8979	0.9074	0.9001	0.8979	0.8942	0.9141	0.9075	0.8898
			0.1371	0.0793	0.0651	0.0179	0.0244	0.0466	0.0105	0.0184	0.0502
			0.1328	0.0319	0.0236	0.0183	0.0231	0.0449	0.0108	0.0140	0.0297
	90%	1	0.9242	0.9072	0.8551	0.9113	0.9073	0.8978	0.8607	0.8552	0.8446
			0.1346	0.0889	0.0643	0.0093	0.0168	0.0353	0.0074	0.0166	0.0337
			0.1303	0.0314	0.0171	0.0065	0.0104	0.0207	0.0052	0.0106	0.0202
		2	0.9216	0.8758	0.8921	0.8811	0.8759	0.8649	0.8968	0.8921	0.8810
			0.1348	0.0819	0.0542	0.0220	0.0262	0.0590	0.0106	0.0178	0.0348
			0.1305	0.0401	0.0246	0.0221	0.0268	0.0555	0.0079	0.0117	0.0210
		3	0.9240	0.8820	0.9195	0.8851	0.8820	0.8757	0.9235	0.9196	0.9108
			0.1354	0.0636	0.0484	0.0161	0.0213	0.0275	0.0069	0.0147	0.0294
			0.1316	0.0228	0.0213	0.0163	0.0206	0.0247	0.0045	0.0108	0.0228

Table 4: Continued.

n				SEL		LL					
Prior $\longrightarrow$	FP	Scheme	MLE	I	II	I			II		
$v \longrightarrow$						−3	−0.03	+3	−3	−0.03	+3
	30%	1	0.9268	0.9228	0.9327	0.9266	0.9229	0.9143	0.9403	0.9328	0.9073
			0.1321	0.0985	0.0778	0.0255	0.0278	0.0656	0.0129	0.0199	0.0482
			0.1285	0.0436	0.0237	0.0258	0.0288	0.0537	0.0132	0.0160	0.0289
		2	0.9206	0.9200	0.9031	0.9261	0.9201	0.9024	0.9082	0.9031	0.8891
			0.1291	0.0844	0.0777	0.0341	0.0396	0.0545	0.0158	0.0216	0.0486
			0.1248	0.0409	0.0295	0.0365	0.0408	0.0509	0.0160	0.0199	0.0344
		3	0.9256	0.9011	0.9167	0.9047	0.9011	0.8933	0.9205	0.9168	0.9079
			0.1295	0.0765	0.0658	0.0180	0.0210	0.0435	0.0109	0.0183	0.0384
			0.1258	0.0299	0.0206	0.0182	0.0194	0.0306	0.0111	0.0160	0.0268
	60%	1	0.9248	0.9223	0.9042	0.9281	0.9224	0.9056	0.9095	0.9043	0.8903
			0.1285	0.0822	0.0629	0.0147	0.0222	0.0487	0.0123	0.0159	0.0291
			0.1254	0.0285	0.0232	0.0150	0.0196	0.0370	0.0080	0.0137	0.0246
		2	0.9216	0.8793	0.9102	0.8834	0.8794	0.8701	0.9138	0.9103	0.9022
60			0.1268	0.0774	0.0770	0.0209	0.0223	0.0516	0.0118	0.0196	0.0402
			0.1235	0.0356	0.0177	0.0211	0.0231	0.0378	0.0094	0.0149	0.0302
		3	0.9219	0.9144	0.9296	0.9196	0.9145	0.9005	0.9331	0.9297	0.9222
			0.1271	0.0762	0.0622	0.0140	0.0203	0.0430	0.0092	0.0169	0.0309
			0.1238	0.0222	0.0175	0.0133	0.0171	0.0284	0.0093	0.0135	0.0258
	90%	1	0.9240	0.9096	0.9071	0.9129	0.9096	0.9024	0.9104	0.9071	0.9000
			0.1270	0.0586	0.0584	0.0097	0.0161	0.0301	0.0083	0.0148	0.0288
			0.1242	0.0125	0.0114	0.0070	0.0106	0.0184	0.0047	0.0082	0.0160
		2	0.9226	0.9307	0.9244	0.9358	0.9307	0.9170	0.9297	0.9245	0.9101
			0.1261	0.0745	0.0613	0.0137	0.0204	0.0426	0.0084	0.0154	0.0309
			0.1229	0.0272	0.0162	0.0141	0.0193	0.0360	0.0056	0.0093	0.0179
		3	0.9207	0.9165	0.9013	0.9217	0.9166	0.9027	0.9052	0.9014	0.8922
			0.1263	0.0633	0.0606	0.0139	0.0187	0.0353	0.0085	0.0148	0.0267
			0.1230	0.0182	0.0170	0.0127	0.0169	0.0257	0.0085	0.0130	0.0224
	30%	1	0.9295	0.9050	0.9032	0.9091	0.9051	0.8953	0.9079	0.9033	0.8911
			0.1275	0.0769	0.0677	0.0142	0.0224	0.0455	0.0107	0.0185	0.0385
			0.1247	0.0178	0.0133	0.0130	0.0178	0.0302	0.0085	0.0138	0.0238
		2	0.9207	0.9028	0.9090	0.9070	0.9028	0.8926	0.9132	0.9091	0.8991
			0.1228	0.0720	0.0696	0.0127	0.0206	0.0405	0.0088	0.0169	0.0365
			0.1201	0.0194	0.0154	0.0108	0.0153	0.0264	0.0054	0.0099	0.0233
		3	0.9246	0.9120	0.9108	0.9162	0.9121	0.9021	0.9151	0.9109	0.9006
			0.1243	0.0712	0.0676	0.0159	0.0217	0.0406	0.0106	0.0190	0.0351
			0.1217	0.0184	0.0196	0.0158	0.0194	0.0242	0.0078	0.0124	0.0273
	60%	1	0.9257	0.9125	0.9050	0.9178	0.9126	0.8984	0.9092	0.9051	0.8950
			0.1239	0.0732	0.0666	0.0120	0.0210	0.0447	0.0107	0.0184	0.0374
			0.1210	0.0179	0.0130	0.0097	0.0148	0.0292	0.0083	0.0128	0.0235
		2	0.9208	0.9094	0.9168	0.9141	0.9095	0.8974	0.9213	0.9169	0.9058
90			0.1226	0.0684	0.0670	0.0098	0.0171	0.0403	0.0083	0.0162	0.0361
			0.1201	0.0144	0.0131	0.0100	0.0143	0.0261	0.0052	0.0102	0.0208
		3	0.9218	0.9054	0.8990	0.9097	0.9054	0.8949	0.9024	0.8991	0.8918
			0.1214	0.0689	0.0614	0.0098	0.0185	0.0395	0.0093	0.0163	0.0342
			0.1190	0.0144	0.0141	0.0071	0.0119	0.0240	0.0067	0.0110	0.0221
	90%	1	0.9232	0.9055	0.9004	0.9092	0.9055	0.8969	0.9049	0.9005	0.8891
			0.1230	0.0717	0.0635	0.0109	0.0189	0.0440	0.0091	0.0179	0.0347
			0.1203	0.0153	0.0126	0.0083	0.0128	0.0281	0.0084	0.0123	0.0217
		2	0.9212	0.9104	0.9084	0.9143	0.9105	0.9013	0.9121	0.9084	0.9000
			0.1214	0.0645	0.0625	0.0086	0.0159	0.0336	0.0081	0.0157	0.0324
			0.1188	0.0121	0.0117	0.0056	0.0098	0.0198	0.0052	0.0092	0.0184
		3	0.9221	0.9094	0.9074	0.9140	0.9094	0.8980	0.9110	0.9074	0.8994
			0.1213	0.0667	0.0609	0.0085	0.0166	0.0366	0.0083	0.0160	0.0320
			0.1187	0.0140	0.0116	0.0070	0.0113	0.0222	0.0064	0.0103	0.0190

Table 5: The APEs (first-line), RMSEs (second-line), and MRABs (third-line) of $h(t)$.

n				SEL		LL					
Prior ⟶	FP	Scheme	MLE	I	II	I			I		
v ⟶						−3	−0.03	+3	−3	−0.03	+3
	30%	1	0.6786	0.6490	0.4730	1.0081	0.6503	0.6053	0.8772	0.4746	0.4135
			0.4189	0.3983	0.3037	0.3506	0.2297	0.2758	0.2550	0.0978	0.1235
			0.4643	0.4215	0.1969	0.3822	0.3112	0.3995	0.2773	0.1351	0.1792
		2	0.7198	0.3130	0.7049	0.7986	0.3149	0.2535	1.0613	0.7064	0.6347
			0.4300	0.4258	0.3264	0.4134	0.3863	0.3559	0.3609	0.1400	0.0874
			0.4225	0.3955	0.2592	0.4186	0.3267	0.3182	0.3559	0.1918	0.1243
		3	0.7028	0.5604	0.7902	0.9726	0.5619	0.5121	1.0398	0.7911	0.7488
			0.4035	0.3519	0.2761	0.3228	0.2499	0.2226	0.2378	0.1138	0.0916
			0.3973	0.2968	0.2013	0.3362	0.2596	0.2563	0.2216	0.1490	0.1329
	60%	1	0.6373	0.6189	0.7311	1.0722	0.6206	0.5652	0.9240	0.7318	0.7019
			0.2817	0.2489	0.2273	0.2967	0.2206	0.2524	0.2053	0.0696	0.0235
			0.3326	0.2614	0.1357	0.2540	0.2630	0.2644	0.2421	0.0655	0.0281
		2	0.6791	0.5351	0.7158	1.0068	0.5368	0.4875	0.9794	0.7168	0.6678
30			0.2723	0.2760	0.2196	0.2166	0.2310	0.2026	0.2084	0.0909	0.0682
			0.3064	0.2625	0.2374	0.2433	0.2363	0.2921	0.2290	0.0960	0.0863
		3	0.6295	0.8216	0.6625	0.9191	0.8221	0.7905	1.0846	0.6642	0.5971
			0.2569	0.2416	0.2293	0.2519	0.1547	0.1796	0.2307	0.0712	0.0825
			0.3064	0.2601	0.1867	0.2792	0.2070	0.2231	0.2102	0.0982	0.1188
	90%	1	0.6354	0.7311	1.0063	0.9878	0.7320	0.6968	1.2238	1.0074	0.9395
			0.2435	0.2181	0.2073	0.2160	0.0735	0.0835	0.2031	0.0685	0.0108
			0.2830	0.2400	0.1280	0.2346	0.1007	0.1209	0.2316	0.0631	0.0119
		2	0.6478	0.9153	0.8195	1.1731	0.9164	0.8494	1.0931	0.8206	0.7742
			0.2392	0.2236	0.2048	0.2042	0.1767	0.1625	0.1868	0.0784	0.0412
			0.2783	0.2421	0.1951	0.2037	0.2210	0.1594	0.1876	0.0743	0.0504
		3	0.6329	0.9357	0.6420	1.0617	0.9362	0.9110	0.8771	0.6429	0.6068
			0.2520	0.2101	0.1811	0.2309	0.1368	0.1021	0.2238	0.0674	0.0605
			0.2906	0.2589	0.1722	0.2419	0.1940	0.1481	0.2047	0.0899	0.0874
	30%	1	0.6572	0.6334	0.5008	0.8137	0.6341	0.6085	0.9592	0.5025	0.4503
			0.2770	0.2216	0.1936	0.2428	0.2062	0.2392	0.2269	0.0965	0.0813
			0.3112	0.2877	0.1898	0.2408	0.2760	0.2461	0.2334	0.1031	0.1162
		2	0.7069	0.5881	0.7386	0.9764	0.5896	0.5409	1.0929	0.7398	0.7037
			0.2509	0.2335	0.1953	0.2291	0.1600	0.1669	0.2032	0.1093	0.1201
			0.2686	0.2560	0.1108	0.2125	0.1755	0.2144	0.1871	0.1484	0.1724
		3	0.6664	0.7734	0.6881	0.9706	0.7741	0.7482	0.8778	0.6888	0.6640
			0.2240	0.2168	0.2107	0.2125	0.1148	0.1314	0.1838	0.0797	0.0778
			0.2495	0.1789	0.1540	0.2096	0.1546	0.1901	0.2033	0.1071	0.1123
	60%	1	0.6459	0.6104	0.7490	1.0076	0.6118	0.5727	1.0645	0.7502	0.7022
			0.2024	0.1942	0.1316	0.1860	0.1100	0.1175	0.1846	0.0753	0.0318
			0.2390	0.2236	0.1566	0.1953	0.1509	0.1682	0.1622	0.0895	0.0338
		2	0.6694	0.9138	0.7268	1.0909	0.9145	0.8849	0.9477	0.7276	0.6962
60			0.1812	0.1645	0.1403	0.1761	0.1295	0.1656	0.1558	0.0855	0.0192
			0.2099	0.1719	0.1248	0.1844	0.1522	0.2052	0.1763	0.0744	0.0220
		3	0.6636	0.6473	0.5887	0.9432	0.6483	0.6112	0.8412	0.5896	0.5577
			0.1903	0.1728	0.1686	0.1814	0.0976	0.0758	0.1754	0.0712	0.0536
			0.2194	0.1673	0.1129	0.1995	0.1243	0.1041	0.1723	0.0982	0.0770
	90%	1	0.6474	0.7180	0.7401	0.9405	0.7188	0.6931	0.9683	0.7409	0.7137
			0.1790	0.1405	0.1203	0.1800	0.0752	0.0272	0.1548	0.0597	0.0099
			0.2122	0.0968	0.0691	0.1662	0.0760	0.0365	0.1597	0.0445	0.0082
		2	0.6556	0.5421	0.6030	0.8506	0.5432	0.5095	0.9075	0.6041	0.5698
			0.1668	0.1608	0.1393	0.1537	0.1288	0.1496	0.1505	0.0675	0.0146
			0.1916	0.1562	0.1220	0.1589	0.1328	0.1600	0.1442	0.0567	0.0136
		3	0.6640	0.6565	0.7657	0.9680	0.6576	0.6169	0.9671	0.7664	0.7416
			0.1719	0.1650	0.1341	0.1667	0.0918	0.0600	0.1616	0.0520	0.0279
			0.1975	0.1260	0.0966	0.1957	0.1130	0.0866	0.1522	0.0610	0.0367

Table 5: Continued.

n	FP	Scheme	MLE	SEL		LL					
Prior ⟶				I	II		I			I	
v ⟶						−3	−0.03	+3	−3	−0.03	+3
	30%	1	0.6373	0.7468	0.7389	0.9890	0.7476	0.7205	1.0140	0.7398	0.7106
			0.2188	0.1862	0.1647	0.2079	0.0963	0.0513	0.1662	0.0791	0.0461
			0.2541	0.1226	0.0864	0.1992	0.1135	0.0733	0.1727	0.0876	0.0618
		2	0.6982	0.7529	0.7146	1.0105	0.7538	0.7262	0.9498	0.7154	0.6888
			0.1849	0.1549	0.1386	0.1543	0.0876	0.0640	0.1480	0.0622	0.0268
			0.2073	0.1416	0.1010	0.1675	0.0975	0.0902	0.1481	0.0641	0.0373
		3	0.6728	0.6961	0.7050	0.9358	0.6969	0.6698	0.9488	0.7059	0.6785
			0.1762	0.1599	0.1423	0.1481	0.1236	0.0880	0.1320	0.0724	0.0236
			0.2017	0.1664	0.0849	0.1652	0.1668	0.1274	0.1401	0.0650	0.0304
	60%	1	0.6504	0.6771	0.7389	0.9836	0.6782	0.6460	0.9926	0.7398	0.7121
			0.1648	0.1605	0.1503	0.1567	0.0791	0.0329	0.1555	0.0735	0.0257
			0.1910	0.1122	0.0848	0.1524	0.0857	0.0464	0.1492	0.0774	0.0342
		2	0.6759	0.6928	0.6549	0.9933	0.6938	0.6629	0.9278	0.6559	0.6264
90			0.1509	0.1448	0.1258	0.1360	0.0715	0.0395	0.1291	0.0585	0.0118
			0.1721	0.0948	0.0710	0.1494	0.0947	0.0547	0.1402	0.0499	0.0205
		3	0.6722	0.7314	0.8027	0.9686	0.7322	0.7050	1.0055	0.8034	0.7763
			0.1462	0.1419	0.1124	0.1359	0.0734	0.0281	0.1249	0.0696	0.0228
			0.1706	0.1277	0.0841	0.1440	0.0768	0.0393	0.1268	0.0634	0.0296
	90%	1	0.6526	0.7377	0.7658	0.9747	0.7386	0.7119	1.0271	0.7667	0.7390
			0.1459	0.1422	0.1259	0.1526	0.0764	0.0251	0.1394	0.0696	0.0249
			0.1686	0.0889	0.0783	0.1441	0.0744	0.0339	0.1416	0.0726	0.0320
		2	0.6702	0.7082	0.7249	0.9323	0.7089	0.6833	0.9652	0.7258	0.6980
			0.1391	0.1232	0.1142	0.1305	0.0639	0.0121	0.1202	0.0562	0.0105
			0.1616	0.0880	0.0665	0.1471	0.0540	0.0144	0.1388	0.0487	0.0137
		3	0.6626	0.7148	0.7406	0.9811	0.7157	0.6822	0.9649	0.7414	0.7156
			0.1380	0.1259	0.1109	0.1211	0.0611	0.0190	0.1201	0.0583	0.0169
			0.1637	0.0882	0.0833	0.1407	0.0626	0.0239	0.1091	0.0587	0.0226

Table 6: The ACLs/CPs of asymptotic/credible interval estimates of α.

n	FP	Scheme	ACI-NA		BCI			
			ACI-NL		HPD			
Prior ⟶					I	II	I	II
			ACL	CP	ACL	CP	ACL	CP
	30%	1	1.5136	0.963	0.3198	0.988	0.2507	0.999
			2.0153	0.935	0.2705	0.959	0.1942	0.972
		2	1.6241	0.963	0.3085	0.988	0.2710	0.999
			2.0466	0.933	0.2496	0.957	0.2316	0.970
		3	1.8642	0.930	0.2966	0.954	0.2804	0.967
			2.0752	0.931	0.2569	0.955	0.2393	0.968
	60%	1	1.2520	0.966	0.2967	0.991	0.2257	0.999
			1.4132	0.942	0.2644	0.966	0.1706	0.979
30		2	1.2288	0.970	0.2835	0.995	0.2668	0.999
			1.3672	0.947	0.2401	0.972	0.2202	0.984
		3	1.1503	0.948	0.2776	0.973	0.2344	0.985
			1.6574	0.935	0.2320	0.959	0.1790	0.972
	90%	1	1.1156	0.982	0.2389	0.997	0.2069	0.999
			1.2204	0.958	0.1992	0.983	0.1595	0.996
		2	1.1042	0.978	0.2630	0.994	0.2420	0.999
			1.2040	0.949	0.2074	0.974	0.1959	0.986
		3	1.1252	0.966	0.2347	0.991	0.2147	0.999
			1.2621	0.941	0.1799	0.965	0.1697	0.978

Table 6: Continued.

n	FP	Scheme	ACI-NA / ACI-NL		BCI / HPD			
Prior ⟶					I	II	I	II
			ACL	CP	ACL	CP	ACL	CP
60	30%	1	1.0578	0.971	0.2607	0.997	0.2495	0.999
			1.1375	0.939	0.2003	0.964	0.1968	0.977
		2	1.0423	0.965	0.2327	0.991	0.2083	0.999
			1.1156	0.943	0.1841	0.968	0.1577	0.982
		3	1.3312	0.953	0.2442	0.979	0.2028	0.992
			1.6050	0.930	0.1917	0.955	0.1597	0.968
	60%	1	0.8704	0.979	0.2009	0.995	0.1897	0.999
			0.9127	0.947	0.1452	0.973	0.1349	0.986
		2	0.8333	0.971	0.2249	0.997	0.1962	0.999
			0.8713	0.949	0.1687	0.973	0.1460	0.985
		3	0.9457	0.956	0.2241	0.982	0.1908	0.995
			1.0345	0.938	0.1770	0.963	0.1386	0.976
	90%	1	0.7658	0.990	0.1903	0.997	0.1730	0.999
			0.7945	0.961	0.1457	0.987	0.1294	0.999
		2	0.7514	0.987	0.2059	0.996	0.1925	0.999
			0.7789	0.949	0.1600	0.974	0.1404	0.988
		3	0.7766	0.969	0.1927	0.995	0.1848	0.999
			0.8087	0.942	0.1419	0.967	0.1334	0.980
90	30%	1	0.8749	0.977	0.1865	0.997	0.1730	0.999
			0.9156	0.945	0.1338	0.973	0.1243	0.988
		2	0.8202	0.968	0.1867	0.997	0.1801	0.999
			0.8548	0.945	0.1376	0.973	0.1283	0.988
		3	1.0393	0.958	0.2030	0.987	0.1799	0.999
			1.1490	0.938	0.1517	0.966	0.1292	0.981
	60%	1	0.7025	0.981	0.1818	0.999	0.1674	0.999
			0.7232	0.949	0.1322	0.977	0.1177	0.992
		2	0.6679	0.985	0.1783	0.999	0.1702	0.999
			0.6866	0.950	0.1281	0.979	0.1193	0.993
		3	0.7937	0.974	0.1869	0.999	0.1658	0.999
			0.8015	0.940	0.1358	0.968	0.1183	0.983
	90%	1	0.6217	0.996	0.1817	0.999	0.1575	0.999
			0.6369	0.969	0.1318	0.998	0.1104	0.999
		2	0.6081	0.998	0.1731	0.998	0.1664	0.999
			0.6224	0.961	0.1247	0.990	0.1188	0.999
		3	0.6364	0.981	0.1730	0.999	0.1645	0.999
			0.6531	0.947	0.1230	0.975	0.1179	0.990

(xi) In summary, the Bayesian inference via M-H algorithm of the unknown life parameters α, β, θ, $R(t)$, and $h(t)$ of the Marshall-Olkin generalized-exponential lifetime model is recommended.

6. Real-Life Applications

To show the potentiality and flexibility of the proposed methodologies to real phenomenon, in this section we shall provide two real data sets obtained from different fields. First data set (denoted by Data-I) represents the failure times of $n = 18$ electronic devices, as can be seen in Wang [31]. Recently, this data set has also been discussed by Elshahhat and Abu El Azm [16]. Other data set (denoted by data-II), reported by Hinkley [32], consists of the amount of precipitation (inches) having $n = 30$ successive values in March at Minneapolis/St Paul. The lifetime points (in order) of both data sets I and II are presented in Table 11.

Using Table 11, plots of the empirical/estimated scaled TTT-transform of the MOGE distribution are provided in Figure 8. It shows that the scaled TTT-transform is (a) convex and then concave for data-I while is concave for data-II. It also suggests that increasing and bathtub shaped failure rates are suitable for the fitting MOGE lifetime model.

Before discussing our proposed estimators, the MOGE distribution is compared with seven popular Marshall-Olkin

Table 7: The ACLs/CPs of asymptotic/credible interval estimates of β.

n	FP	Scheme	ACI-NA		BCI			
			ACI-NL		HPD			
Prior $\longrightarrow$					I	II	I	II
			ACL	CP	ACL	CP	ACL	CP
30	30%	1	2.1029	0.935	0.2470	0.959	0.1885	0.970
			2.8978	0.946	0.2156	0.965	0.1737	0.976
		2	2.5156	0.933	0.2201	0.968	0.1847	0.979
			2.9665	0.947	0.2080	0.970	0.1693	0.981
		3	2.6728	0.930	0.2628	0.941	0.1667	0.951
			3.2831	0.931	0.2238	0.945	0.1612	0.955
	60%	1	2.0646	0.947	0.1941	0.968	0.1159	0.975
			2.8450	0.959	0.1883	0.968	0.0998	0.979
		2	2.1457	0.949	0.1811	0.969	0.1210	0.980
			2.9067	0.953	0.1541	0.975	0.1088	0.986
		3	2.2594	0.935	0.2288	0.949	0.1598	0.959
			3.1135	0.935	0.2030	0.967	0.1436	0.978
	90%	1	2.0477	0.961	0.1289	0.968	0.1022	0.979
			2.4218	0.967	0.1099	0.970	0.0925	0.981
		2	2.0345	0.959	0.1401	0.975	0.1055	0.986
			2.7836	0.968	0.1266	0.982	0.0899	0.993
		3	2.0779	0.942	0.1986	0.956	0.1369	0.967
			2.8034	0.952	0.1746	0.977	0.1134	0.988
60	30%	1	1.2260	0.942	0.2037	0.966	0.1402	0.975
			1.3398	0.960	0.1696	0.966	0.1155	0.981
		2	1.5492	0.943	0.1570	0.968	0.1501	0.984
			1.4873	0.976	0.1351	0.973	0.1247	0.986
		3	1.6245	0.936	0.1887	0.946	0.1353	0.957
			1.8239	0.946	0.1531	0.965	0.1128	0.961
	60%	1	1.1900	0.949	0.1913	0.969	0.1296	0.984
			1.1409	0.968	0.1652	0.975	0.1129	0.984
		2	1.2970	0.959	0.1435	0.971	0.1233	0.985
			1.2537	0.981	0.1177	0.983	0.0968	0.992
		3	1.1520	0.941	0.1408	0.955	0.1259	0.965
			1.3875	0.965	0.1152	0.971	0.1005	0.983
	90%	1	0.8932	0.964	0.1273	0.978	0.1136	0.984
			1.0309	0.973	0.1007	0.979	0.0948	0.986
		2	0.9701	0.970	0.1225	0.976	0.0947	0.992
			1.1368	0.988	0.1010	0.985	0.0809	0.999
		3	1.0142	0.947	0.1339	0.966	0.1202	0.972
			1.2175	0.978	0.1074	0.980	0.0927	0.994
90	30%	1	0.6642	0.945	0.1061	0.979	0.0905	0.985
			0.7152	0.979	0.0833	0.977	0.0676	0.985
		2	0.7568	0.945	0.1033	0.975	0.0900	0.987
			0.9429	0.979	0.0799	0.978	0.0671	0.992
		3	0.8374	0.940	0.1188	0.977	0.1022	0.965
			1.1004	0.948	0.0938	0.980	0.0753	0.984
	60%	1	0.5277	0.958	0.1021	0.980	0.0874	0.988
			0.6272	0.981	0.0811	0.979	0.0643	0.995
		2	0.5950	0.967	0.1031	0.975	0.0817	0.990
			0.6820	0.982	0.0775	0.985	0.0612	0.998
		3	0.8045	0.947	0.1046	0.985	0.0979	0.974
			0.8527	0.979	0.0771	0.984	0.0737	0.990
	90%	1	0.3216	0.969	0.0882	0.983	0.0800	0.998
			0.4431	0.986	0.0658	0.980	0.0589	0.999
		2	0.4885	0.981	0.1019	0.988	0.0996	0.996
			0.6032	0.989	0.0770	0.987	0.0716	0.998
		3	0.5997	0.958	0.1037	0.986	0.0973	0.985
			0.6263	0.980	0.0760	0.986	0.0724	0.998

Table 8: The ACLs/CPs of asymptotic/credible interval estimates of θ.

n / Prior ⟶	FP	Scheme	ACI-NA		BCI			
			ACI-NL		HPD			
					I	II	I	II
			ACL	CP	ACL	CP	ACL	CP
30	30%	1	1.0555	0.956	0.2068	0.967	0.1862	0.973
			1.6742	0.947	0.1822	0.960	0.1589	0.970
		2	1.5785	0.951	0.2301	0.964	0.2296	0.972
			1.9251	0.943	0.2042	0.956	0.2004	0.969
		3	1.8426	0.950	0.2061	0.962	0.1821	0.973
			2.1218	0.947	0.1950	0.960	0.1556	0.972
	60%	1	0.6223	0.959	0.1748	0.972	0.1369	0.978
			0.9152	0.952	0.1504	0.969	0.1103	0.978
		2	0.7489	0.959	0.2242	0.966	0.1769	0.975
			1.2099	0.950	0.1894	0.963	0.1497	0.973
		3	1.4160	0.955	0.1805	0.969	0.1349	0.978
			1.8741	0.951	0.1577	0.964	0.1156	0.973
	90%	1	0.4788	0.965	0.1491	0.978	0.1036	0.985
			0.6136	0.956	0.1224	0.971	0.0871	0.974
		2	0.5126	0.967	0.1378	0.972	0.1291	0.978
			0.6598	0.951	0.1168	0.964	0.1065	0.976
		3	1.2041	0.957	0.1161	0.972	0.1029	0.984
			1.5174	0.962	0.0979	0.974	0.0951	0.975
60	30%	1	0.6254	0.960	0.1750	0.969	0.1545	0.974
			1.5286	0.948	0.1455	0.965	0.1210	0.977
		2	0.8538	0.958	0.1975	0.971	0.1460	0.978
			1.4900	0.962	0.1668	0.974	0.1171	0.978
		3	1.6051	0.959	0.1680	0.963	0.1561	0.975
			1.6117	0.958	0.1409	0.965	0.1338	0.976
	60%	1	0.4083	0.973	0.1679	0.976	0.1254	0.984
			0.4828	0.960	0.1421	0.973	0.1088	0.981
		2	0.4731	0.960	0.1305	0.974	0.1234	0.980
			0.5953	0.965	0.1100	0.978	0.0998	0.983
		3	1.1866	0.958	0.1392	0.972	0.0940	0.980
			1.4610	0.960	0.1143	0.969	0.0759	0.977
	90%	1	0.3181	0.979	0.1205	0.985	0.0916	0.986
			0.3563	0.962	0.0956	0.975	0.0771	0.981
		2	0.3324	0.968	0.1193	0.980	0.0992	0.982
			0.3783	0.968	0.0965	0.982	0.0855	0.984
		3	0.4527	0.962	0.0971	0.975	0.0902	0.987
			1.0251	0.964	0.0773	0.975	0.0726	0.982
90	30%	1	0.4902	0.971	0.1126	0.984	0.1041	0.986
			0.9675	0.961	0.0877	0.974	0.0799	0.983
		2	0.6296	0.962	0.1087	0.975	0.1066	0.979
			0.9274	0.967	0.0839	0.980	0.0784	0.989
		3	1.2145	0.965	0.1153	0.978	0.1017	0.982
			1.4497	0.967	0.0912	0.980	0.0793	0.984
	60%	1	0.3222	0.976	0.0979	0.986	0.0959	0.987
			0.3627	0.967	0.0722	0.980	0.0707	0.984
		2	0.3707	0.976	0.0954	0.989	0.0945	0.991
			0.4293	0.972	0.0713	0.985	0.0709	0.986
		3	0.9149	0.968	0.1003	0.981	0.0863	0.990
			0.9867	0.969	0.0756	0.982	0.0643	0.992
	90%	1	0.2581	0.980	0.0936	0.989	0.0885	0.990
			0.2776	0.968	0.0704	0.983	0.0656	0.986
		2	0.2686	0.979	0.0927	0.990	0.0920	0.992
			0.2911	0.975	0.0694	0.988	0.0695	0.995
		3	0.3640	0.978	0.0880	0.988	0.0825	0.991
			0.5262	0.970	0.0650	0.983	0.0596	0.990

Table 9: The ACLs/CPs of asymptotic/credible interval estimates of $R(t)$.

n	FP	Scheme	ACI-NA		BCI			
			ACI-NL		HPD			
Prior $\longrightarrow$					I	II	I	II
			ACL	CP	ACL	CP	ACL	CP
30	30%	1	0.2452	0.947	0.2138	0.961	0.1930	0.970
			0.1823	0.948	0.1936	0.962	0.1731	0.971
		2	0.2263	0.952	0.1981	0.966	0.1601	0.975
			0.1658	0.950	0.1605	0.964	0.1536	0.973
		3	0.2355	0.941	0.2297	0.955	0.2101	0.964
			0.2134	0.946	0.1770	0.960	0.1699	0.969
	60%	1	0.1968	0.954	0.1637	0.968	0.1555	0.977
			0.1639	0.950	0.1593	0.964	0.1449	0.973
		2	0.2215	0.944	0.1865	0.958	0.1522	0.967
			0.1621	0.947	0.1524	0.961	0.1427	0.970
		3	0.1848	0.949	0.1804	0.963	0.1386	0.972
			0.1857	0.948	0.1349	0.962	0.1033	0.971
	90%	1	0.1571	0.955	0.1504	0.969	0.1112	0.979
			0.1506	0.952	0.1150	0.966	0.1107	0.975
		2	0.2060	0.942	0.1761	0.956	0.1494	0.965
			0.1608	0.945	0.1496	0.959	0.1365	0.968
		3	0.1571	0.956	0.1609	0.970	0.1291	0.980
			0.1467	0.953	0.1163	0.967	0.1012	0.976
60	30%	1	0.2225	0.968	0.1797	0.981	0.1412	0.989
			0.1599	0.961	0.1413	0.973	0.1267	0.982
		2	0.1952	0.962	0.1693	0.975	0.1144	0.983
			0.1419	0.968	0.1200	0.981	0.1145	0.989
		3	0.1764	0.965	0.1425	0.978	0.1417	0.986
			0.1272	0.968	0.1028	0.981	0.1009	0.989
	60%	1	0.1868	0.969	0.1423	0.982	0.1210	0.990
			0.1356	0.965	0.1211	0.978	0.1023	0.986
		2	0.1694	0.965	0.1603	0.978	0.1092	0.986
			0.1203	0.970	0.1156	0.983	0.1078	0.991
		3	0.1738	0.967	0.1450	0.980	0.1316	0.988
			0.1246	0.971	0.1104	0.984	0.0996	0.992
	90%	1	0.1271	0.972	0.1304	0.985	0.1096	0.993
			0.1097	0.974	0.0911	0.987	0.0896	0.995
		2	0.1664	0.969	0.1441	0.982	0.1077	0.990
			0.1182	0.973	0.1092	0.986	0.1048	0.994
		3	0.1408	0.968	0.1465	0.981	0.1283	0.989
			0.1130	0.976	0.1151	0.989	0.0961	0.997
90	30%	1	0.1641	0.964	0.1391	0.977	0.1165	0.988
			0.1166	0.960	0.1032	0.972	0.0964	0.984
		2	0.1542	0.970	0.1476	0.983	0.0933	0.994
			0.1068	0.968	0.1018	0.981	0.0934	0.992
		3	0.0969	0.968	0.1398	0.981	0.1129	0.992
			0.1153	0.974	0.0992	0.987	0.0979	0.998
	60%	1	0.1502	0.972	0.1389	0.985	0.1002	0.996
			0.1144	0.976	0.1003	0.989	0.0956	0.998
		2	0.1411	0.974	0.1398	0.987	0.0904	0.998
			0.0980	0.973	0.0961	0.986	0.0904	0.997
		3	0.0932	0.977	0.1397	0.990	0.1103	0.998
			0.0933	0.975	0.1007	0.988	0.0940	0.999
	90%	1	0.1459	0.965	0.1256	0.978	0.0911	0.989
			0.0986	0.961	0.0896	0.973	0.0883	0.985
		2	0.1344	0.975	0.1244	0.988	0.0899	0.999
			0.0940	0.976	0.0922	0.989	0.0899	0.998
		3	0.0895	0.978	0.1376	0.991	0.1091	0.998
			0.0895	0.976	0.1091	0.989	0.0888	0.999

Table 10: The ACLs/CPs of asymptotic/credible interval estimates of $h(t)$.

n	FP	Scheme	ACI-NA		BCI			
			ACI-NL		HPD			
Prior ⟶			ACL	CP	I ACL	II CP	I ACL	II CP
30	30%	1	1.5013	0.949	0.9965	0.962	0.9859	0.973
			1.8475	0.958	0.8069	0.966	0.8042	0.974
		2	1.3225	0.946	1.0254	0.961	1.0064	0.970
			1.5260	0.951	0.8197	0.962	0.8468	0.972
		3	1.1862	0.960	1.0191	0.965	0.9190	0.972
			1.3247	0.948	0.8363	0.960	0.7316	0.971
	60%	1	1.0750	0.960	0.8859	0.964	0.8399	0.974
			1.2229	0.962	0.7755	0.968	0.6627	0.975
		2	0.9821	0.966	0.9481	0.967	0.8518	0.973
			1.0785	0.961	0.8087	0.964	0.7109	0.975
		3	0.9753	0.950	0.7517	0.962	0.6660	0.973
			0.9963	0.958	0.6133	0.965	0.5898	0.972
	90%	1	0.9178	0.955	0.7219	0.966	0.6399	0.977
			1.0102	0.965	0.5738	0.974	0.4976	0.982
		2	0.8950	0.968	0.9327	0.970	0.8243	0.982
			0.9778	0.962	0.7221	0.975	0.6718	0.984
		3	0.8953	0.954	0.7328	0.968	0.5662	0.980
			0.9828	0.962	0.5670	0.971	0.4578	0.983
60	30%	1	1.0510	0.959	0.9113	0.974	0.7610	0.986
			1.1716	0.961	0.7126	0.972	0.5693	0.983
		2	0.8889	0.959	0.8581	0.972	0.7133	0.983
			0.9495	0.959	0.6782	0.974	0.5331	0.985
		3	0.8095	0.964	0.7332	0.975	0.6713	0.982
			0.8654	0.957	0.5011	0.969	0.5254	0.980
	60%	1	0.7787	0.966	0.8358	0.975	0.5622	0.985
			0.8301	0.964	0.6715	0.973	0.4359	0.984
		2	0.6770	0.960	0.6730	0.974	0.6690	0.985
			0.7076	0.960	0.5248	0.975	0.5102	0.986
		3	0.7082	0.966	0.6931	0.976	0.5625	0.986
			0.7527	0.960	0.5689	0.970	0.4243	0.981
	90%	1	0.6606	0.969	0.5723	0.976	0.5362	0.986
			0.6917	0.965	0.4253	0.974	0.3939	0.985
		2	0.6405	0.967	0.6526	0.978	0.6484	0.989
			0.6680	0.961	0.5093	0.973	0.4960	0.985
		3	0.6488	0.968	0.6590	0.975	0.5191	0.987
			0.6765	0.967	0.5528	0.978	0.3816	0.984
90	30%	1	0.8486	0.966	0.6336	0.977	0.5739	0.986
			0.9153	0.964	0.4651	0.973	0.4120	0.984
		2	0.6994	0.966	0.6088	0.977	0.5780	0.988
			0.7293	0.971	0.4337	0.982	0.4182	0.990
		3	0.6569	0.969	0.5669	0.979	0.5424	0.988
			0.6857	0.968	0.4324	0.976	0.4062	0.990
	60%	1	0.6434	0.967	0.5452	0.978	0.5434	0.989
			0.6713	0.971	0.3947	0.978	0.3889	0.990
		2	0.5641	0.969	0.5525	0.982	0.5402	0.990
			0.5685	0.970	0.4125	0.981	0.3821	0.991
		3	0.6081	0.972	0.5508	0.983	0.5379	0.992
			0.6571	0.969	0.4015	0.979	0.3904	0.990
	90%	1	0.5399	0.968	0.5339	0.982	0.5300	0.993
			0.5562	0.973	0.3789	0.984	0.3728	0.995
		2	0.5448	0.970	0.5272	0.982	0.5153	0.992
			0.5414	0.972	0.3959	0.983	0.3681	0.993
		3	0.5299	0.977	0.5475	0.984	0.5206	0.994
			0.5447	0.971	0.3964	0.980	0.3738	0.989

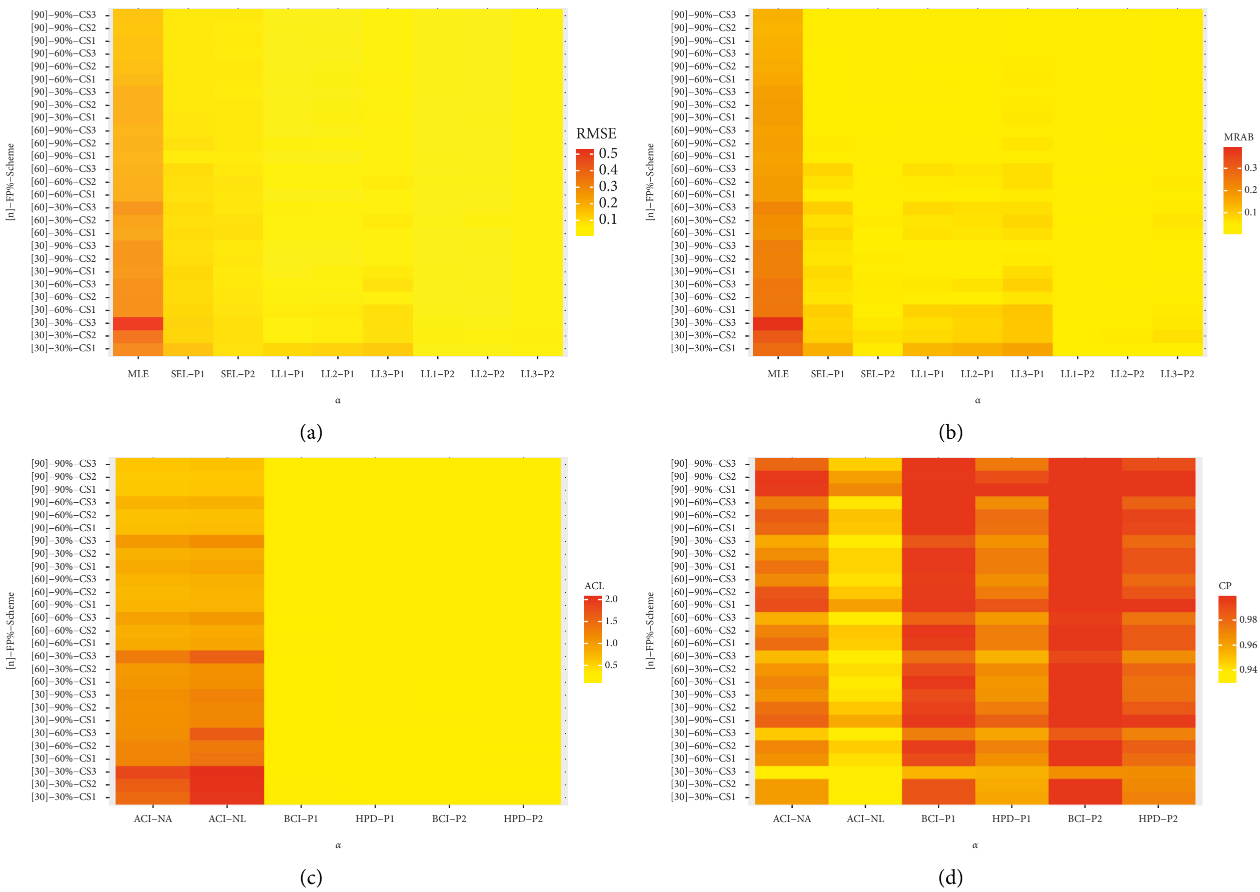


Figure 3: Heatmaps for the simulated results. (a) RMSE, (b) RMAB, (c) ACL, and (d) CP of α.

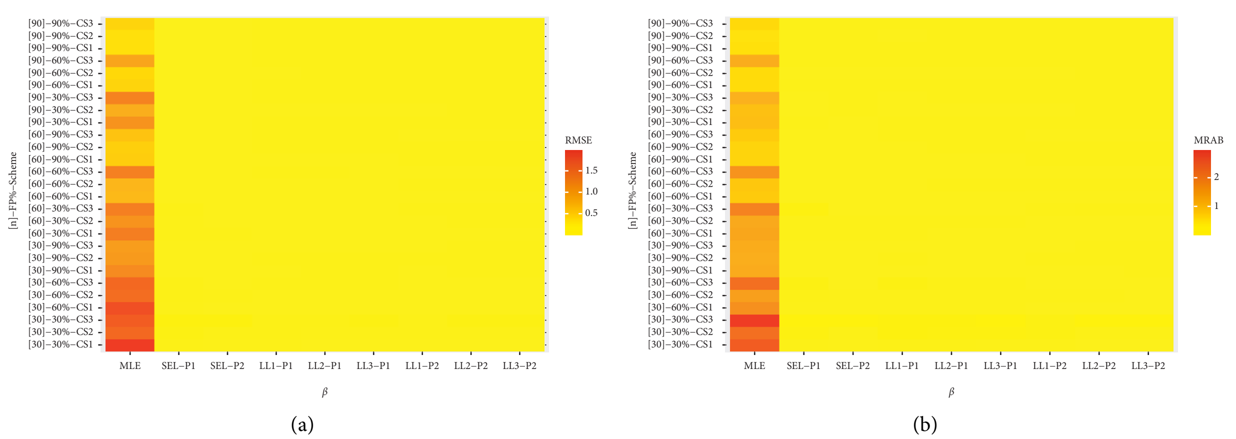
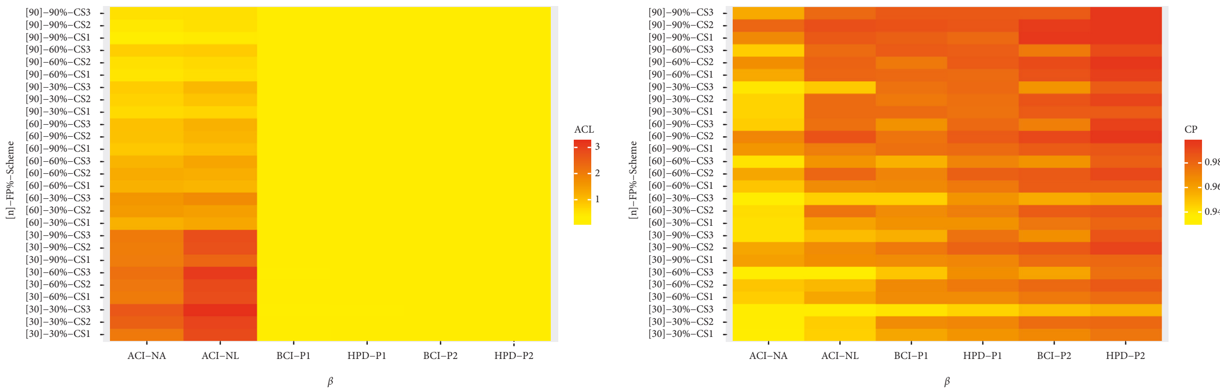
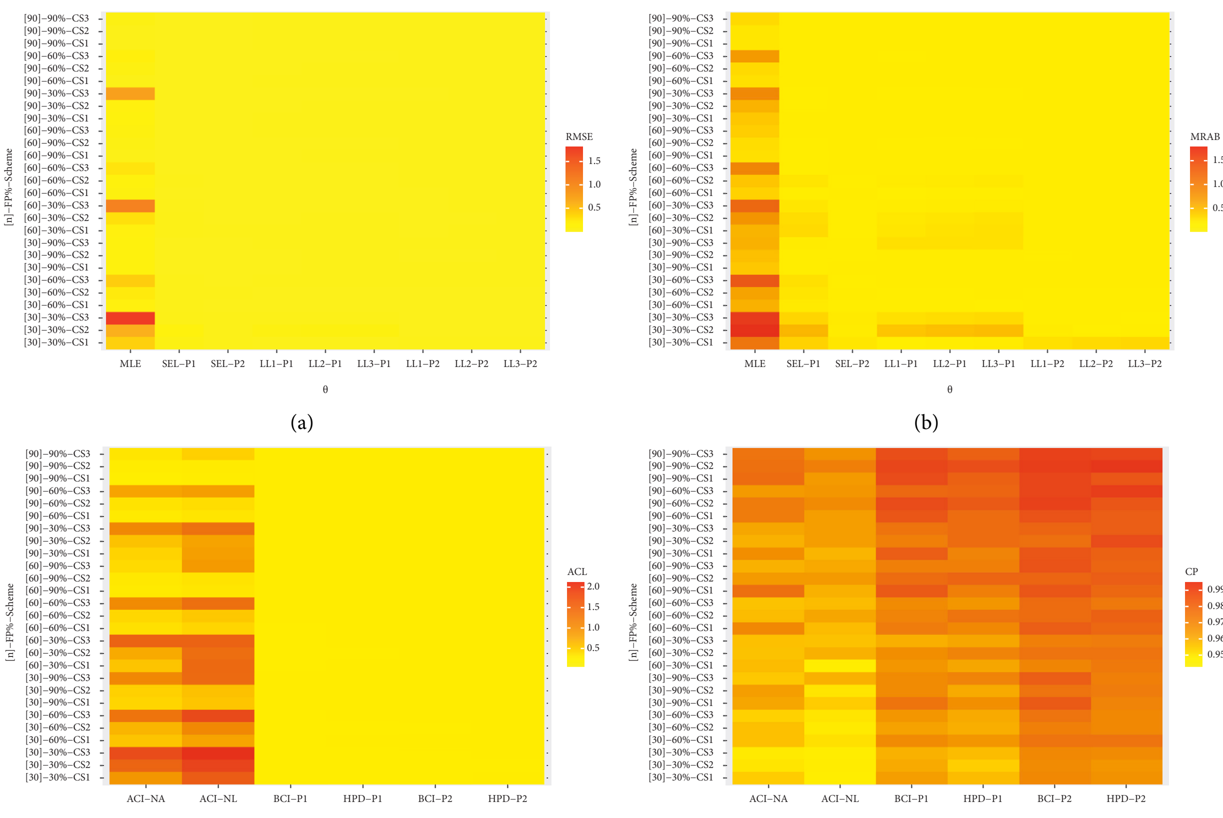


Figure 4: Continued.



(c) (d)

Figure 4: Heatmaps for the simulated results. (a) RMSE, (b) RMAB, (c) ACL, and (d) CP of β.



(c) (d)

Figure 5: Heatmaps for the simulated results. (a) RMSE, (b) RMAB, (c) ACL, and (d) CP of θ.

extended distributions as competitors, namely: Marshall-Olkin extended Weibull (MOEW), Marshall-Olkin logistic exponential (MOLE), Marshall-Olkin alpha power exponential (MOAPE), Marshall-Olkin Nadarajah-Haghighi (MONH), Marshall-Olkin Gompertz (MOG), Marshall-Olkin Lomax (MOL), and Marshall-Olkin exponential (MOE) distributions. The corresponding probability densities of the competing models (for $x > 0$ and $\alpha, \beta, \theta > 0$) are tabulated in Table 12.

To judge the performance of the considered models, several goodness-of-fit criteria, namely, estimated negative log-likelihood (ENL), Akaike information (AI), consistent

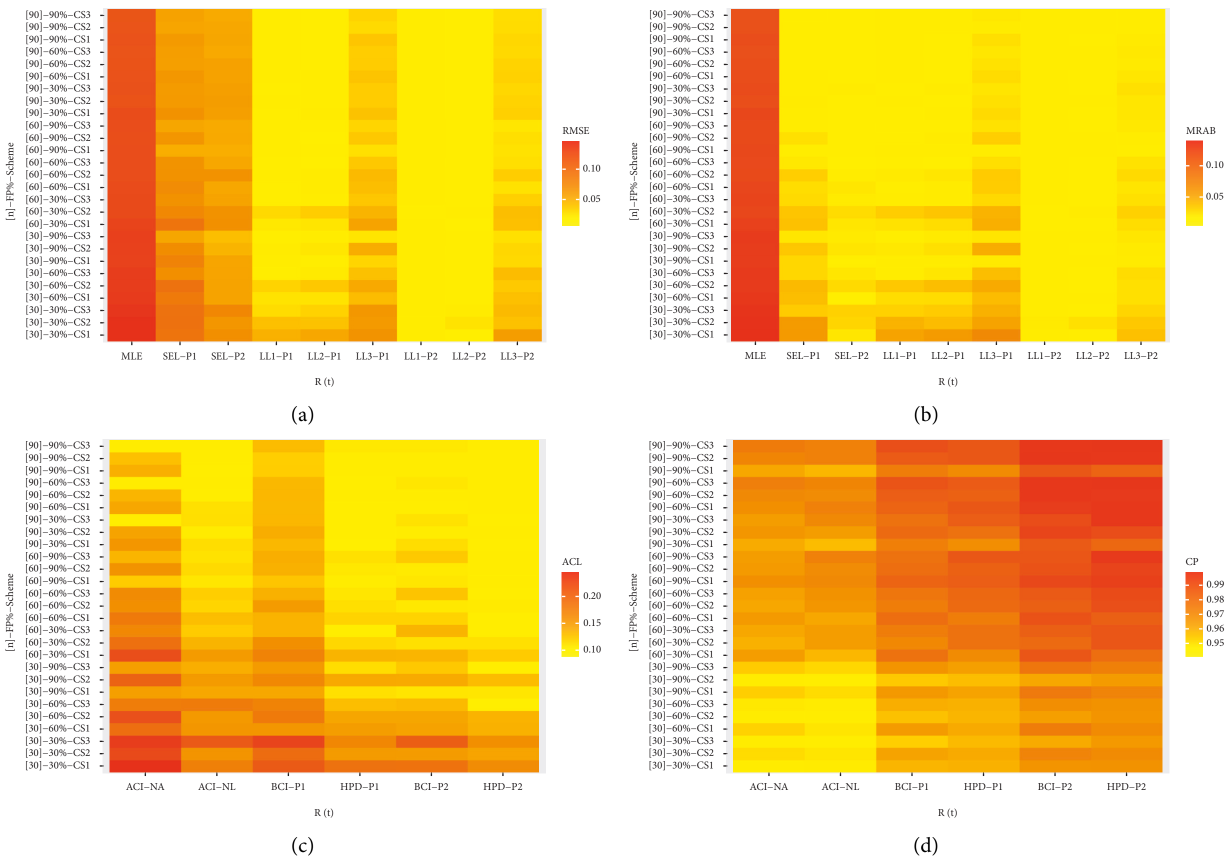


Figure 6: Heatmaps for the simulated results. (a) RMSE, (b) RMAB, (c) ACL, and (d) CP of $R(t)$.

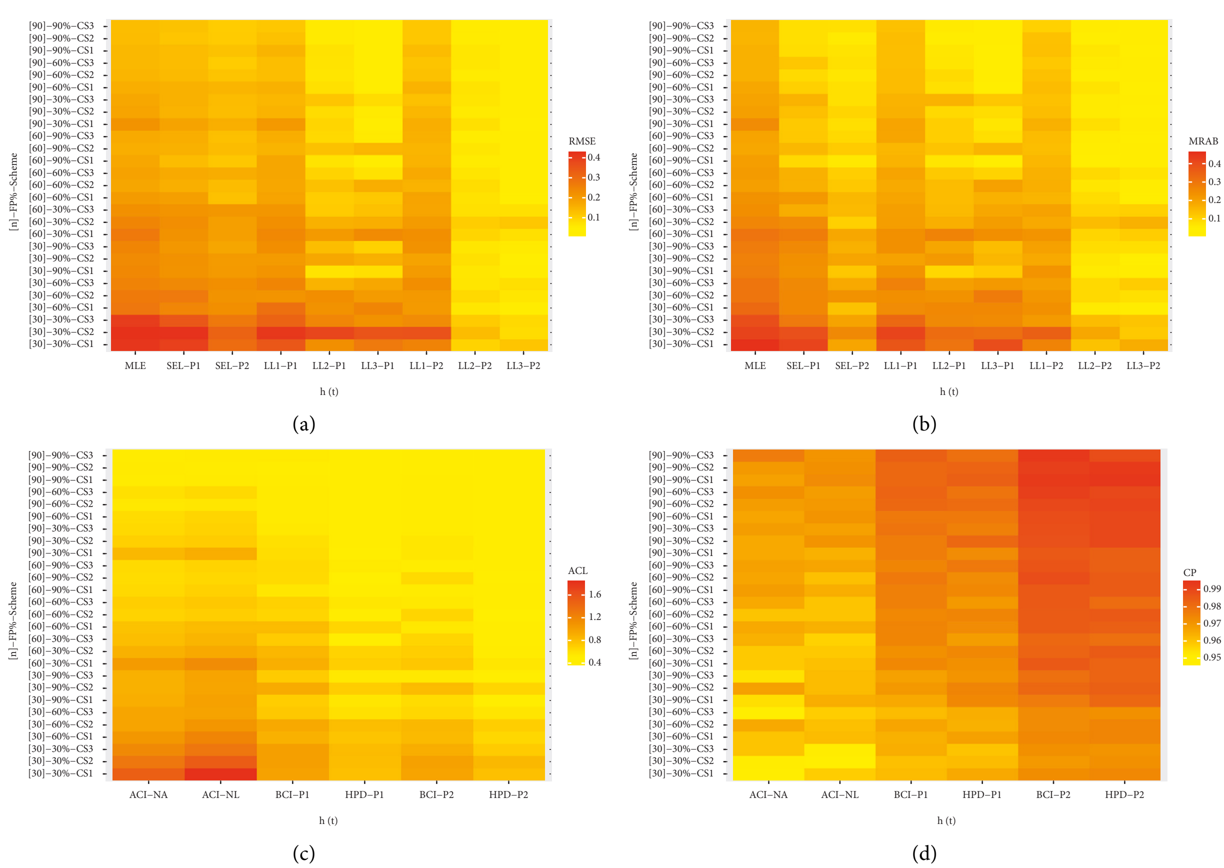
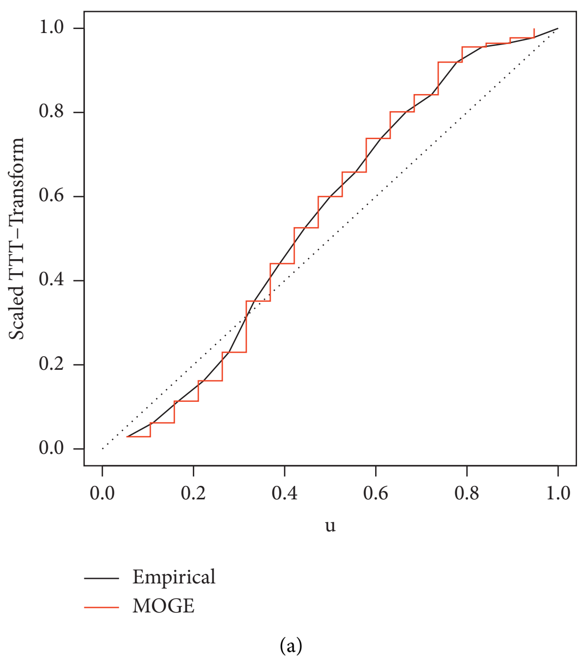


Figure 7: Heatmaps for the simulated results. (a) RMSE, (b) RMAB, (c) ACL, and (d) CP of $h(t)$.

Table 11: Failure times of data sets I and II.

Data	Times									
I	5	11	21	31	46	75	98	122	145	165
	196	224	245	293	321	330	350	420		
II	0.32	0.47	0.52	0.59	0.77	0.81	0.81	0.90	0.96	1.18
	1.20	1.20	1.31	1.35	1.43	1.51	1.62	1.74	1.87	1.89
	1.95	2.05	2.10	2.20	2.48	2.81	3.00	3.09	3.37	4.75



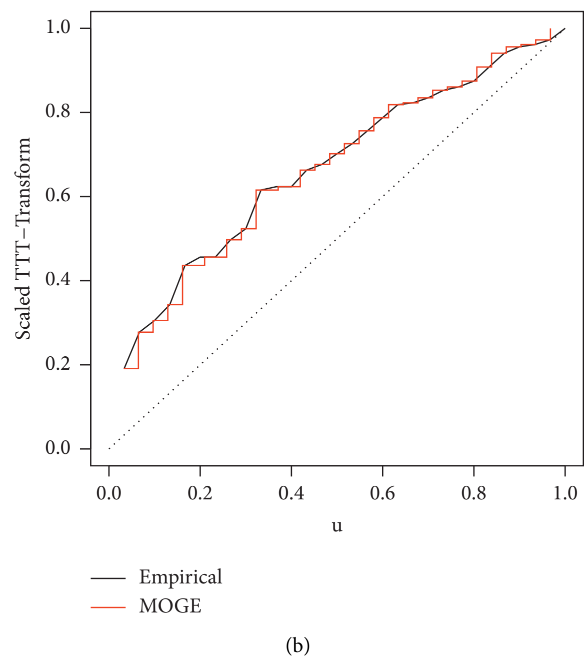


Figure 8: Empirical and estimated scaled TTT-transform plot of the MOGE distribution. (a) Data-I. (b) Data-II.

Table 12: Some useful Marshall-Olkin life models.

Model	$f(x)$	Author(s)
MOEW	$\alpha\beta\theta x^{\alpha-1}\exp(-\theta x^{\alpha})/(1-\overline{\beta}\exp(-\theta x^{\alpha}))^2$	Cordeiro and Lemonte [33]
MOLE	$\alpha\beta\theta\exp(\theta x)(\exp(\theta x)-1)^{-\alpha-1}/(1+\beta(\exp(\theta x)-1)^{-\alpha})^2$	Mansoor et al. [34]
MOAPE	$\theta\beta\log(\alpha)\exp(-\theta x)\alpha^{1-\exp(-\theta x)}/(\alpha-1)(\beta+\overline{\beta}(1/(\alpha-1))(\alpha(^{1-\exp}(-\theta x))-1))^2$	Nassar et al. [35]
MONH	$\theta\beta\alpha(1+\theta x)^{\alpha-1}\exp(1-(1+\theta x)^{\alpha})/(1-\overline{\beta}\exp(1-(1+\theta x)^{\alpha}))^2$	Lemonte et al. [36]
MOG	$\alpha\beta\exp(\theta x)\exp(-(\alpha/\theta)(\exp(\theta x)-1))/(1-\overline{\beta}\exp(-(\alpha/\theta)(\exp(\theta x)-1)))^2$	Eghwerido et al. [37]
MOL	$\alpha\beta\theta(1+\theta x)^{\alpha-1}/((1+\theta x)^{\alpha}-\overline{\beta})^2$	Ghitany et al. [38]
MOE	$\theta\beta\exp(-\theta x)/(\beta+\overline{\beta}(1-\exp(-\theta x)))^2$	Marshall and Olkin [1]

Akaike information (CAI), Bayesian information (BI), Hannan-Quinn information (HQI), Anderson-Darling (AD), Cramer von Mises (CvM) and Kolmogorov-Smirnov (KS) distance with its P-value are used. For each model, we estimate the parameters with their standard errors (SEs) by utilizing the method of maximum likelihood, see Table 13. Generally, the lowest values of these statistics (except the highest P-value) represent the best fit to the given data. The calculated statistics based on data sets I and II are reported in Table 14. It indicates that the MOGE model provides the best fit than other competing models for the given data sets. Plots of the estimated densities and reliability functions of all fitted models are displayed in Figure 9. It shows that the MOGE distribution for both data sets I and II captures the general pattern of the histograms and also supports the numerical findings established here. Also, quantile-quantile plots of the

Table 13: MLEs with their (SEs) of the competing distributions.

Model	MLE (SE)		
	α	β	θ
Data-I			
MOGE	0.6490 (0.5526)	4.5237 (6.4116)	0.0090 (0.0030)
MOEW	0.6927 (0.1490)	0.2761 (0.2874)	0.0114 (0.0157)
MOLE	0.8497 (0.2631)	1.9676 (1.5960)	0.0086 (0.0048)
MOAPE	1.0462 (2.6066)	2.3850 (3.5866)	0.0086 (0.0029)
MONH	1.0053 (0.5547)	2.6755 (2.7688)	0.0088 (0.0106)
MOG	0.0139 (0.0148)	4.8900 (7.7044)	0.0014 (0.0045)
MOL	2.4924 (1.0329)	2.6491 (2.4099)	0.0055 (0.0035)
MOE	—	9.8556 (3.5453)	0.0132 (0.0051)
Data-II			
MOGE	3.3527 (1.2959)	1.2211 (1.5027)	1.2271 (0.4267)
MOEW	2.4651 (0.5673)	0.1748 (0.2675)	0.0624 (0.0951)
MOLE	1.9930 (1.0694)	0.9614 (4.5808)	0.4598 (0.7649)
MOAPE	17.022 (34.498)	2.4018 (2.9264)	1.4241 (0.3697)
MONH	0.7013 (0.4966)	24.895 (59.086)	4.5347 (11.584)
MOG	1.9963 (0.9514)	13.623 (12.192)	0.1311 (0.2301)
MOL	12.687 (10.343)	12.129 (8.4767)	0.1508 (0.1575)
MOE	—	9.8857 (6.2279)	1.5511 (0.3205)

Table 14: Summary fit for the competing models.

Model	ENL	AI	CAI	BI	HQI	AD	CvM	KS (P-value)
Data-I								
MOGE	109.9511	225.9023	227.6165	228.5734	226.2706	0.2602	0.0349	0.0994 (0.9863)
MOEW	115.9579	237.9373	239.6516	240.6084	238.3056	0.6553	0.1059	0.2019 (0.4019)
MOLE	110.0574	226.1147	227.8290	228.7858	226.4830	0.2956	0.0419	0.0857 (0.9977)
MOAPE	110.1670	226.3339	228.0482	229.0051	226.7022	0.2966	0.0417	0.1122 (0.9580)
MONH	110.1754	226.3507	228.0650	229.0218	226.7190	0.2880	0.0401	0.1199 (0.9309)
MOG	110.9341	227.8682	229.5825	230.5393	228.2365	0.3399	0.0481	0.1288 (0.8898)
MOL	111.6748	229.3495	231.0638	232.0206	229.7178	0.4579	0.0673	0.1349 (0.8565)
MOE	111.7332	227.4664	228.2664	229.2471	227.7119	0.2395	0.0310	0.1993 (0.4172)
Data-II								
MOGE	38.0814	82.1628	83.0858	86.3664	83.5075	0.1030	0.0141	0.0601 (0.9999)
MOEW	38.1435	82.2871	83.2102	86.4907	83.6318	0.1107	0.0149	0.0602 (0.9999)
MOLE	38.3690	82.7381	83.6611	86.9417	84.0828	0.1163	0.0157	0.0585 (0.9999)
MOAPE	38.9348	83.8696	84.7927	88.0732	85.2144	0.1744	0.0228	0.0538 (0.9999)
MONH	39.0908	84.1817	85.1047	88.3852	85.5264	0.1860	0.0246	0.0549 (0.999)
MOG	39.1869	84.3738	85.2968	88.5773	85.7185	0.1740	0.0230	0.0661 (0.9994)
MOL	39.2394	84.4788	85.4019	88.6824	85.8236	0.1039	0.0140	0.0603 (0.9999)
MOE	39.3771	82.7542	83.1986	85.5566	83.6507	0.2401	0.0328	0.0646 (0.9996)

fitted models are also investigated in Figure 10. It yields that the MOGE model fits the given data sets more closely and supports the numerical findings.

Now, using some specified choices of m and R, different PCS-T2 samples are generated from data sets I and II, see Table 15. For brevity, the PCS $(2, 0, 0, 0, 2)$ is abbreviated by $(2, 0^3, 2)$. Using the M-H algorithm described

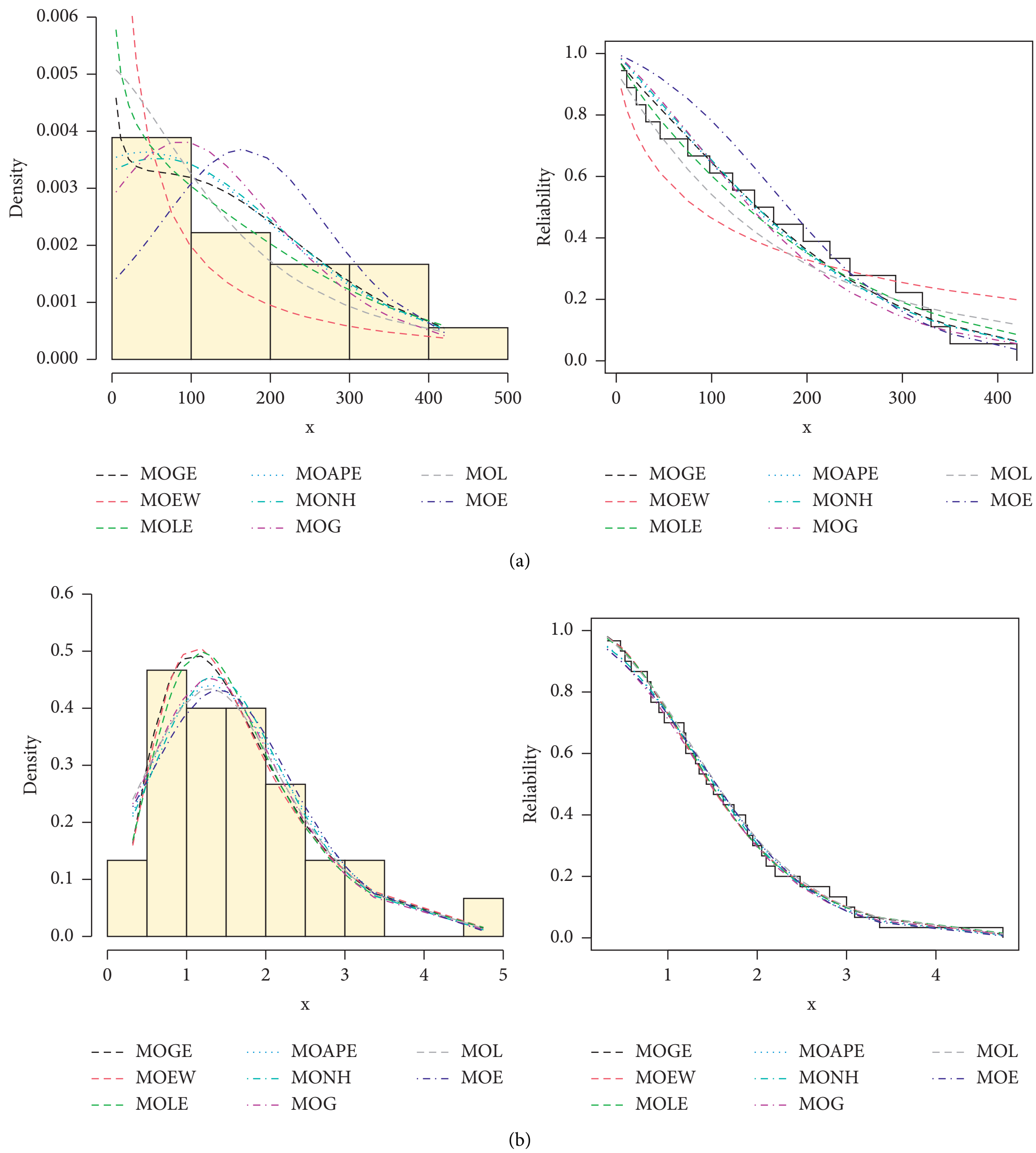
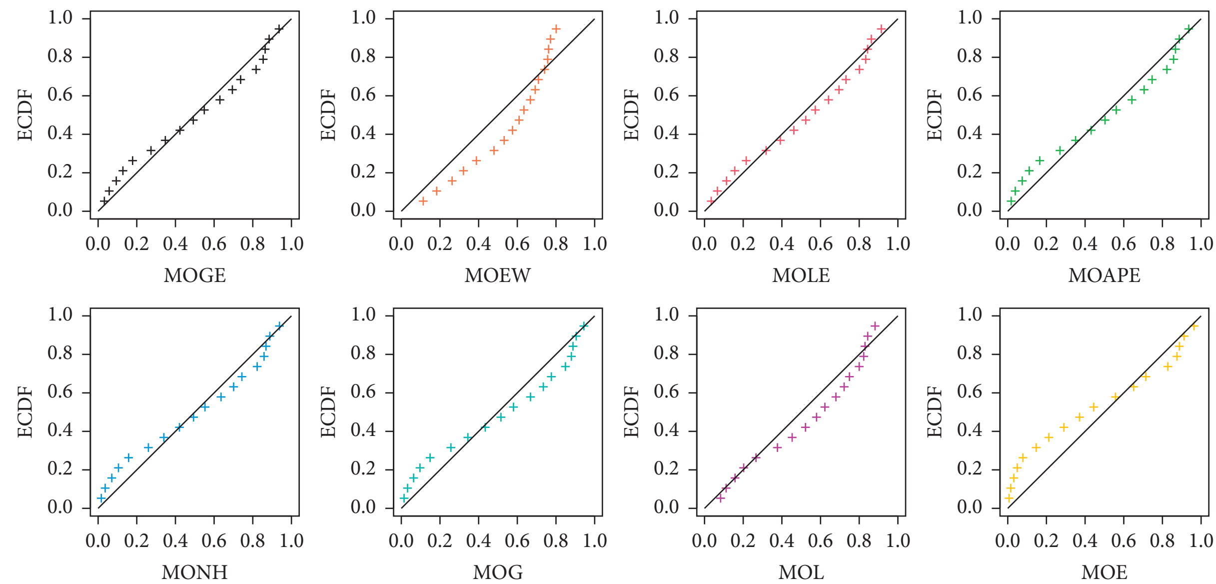
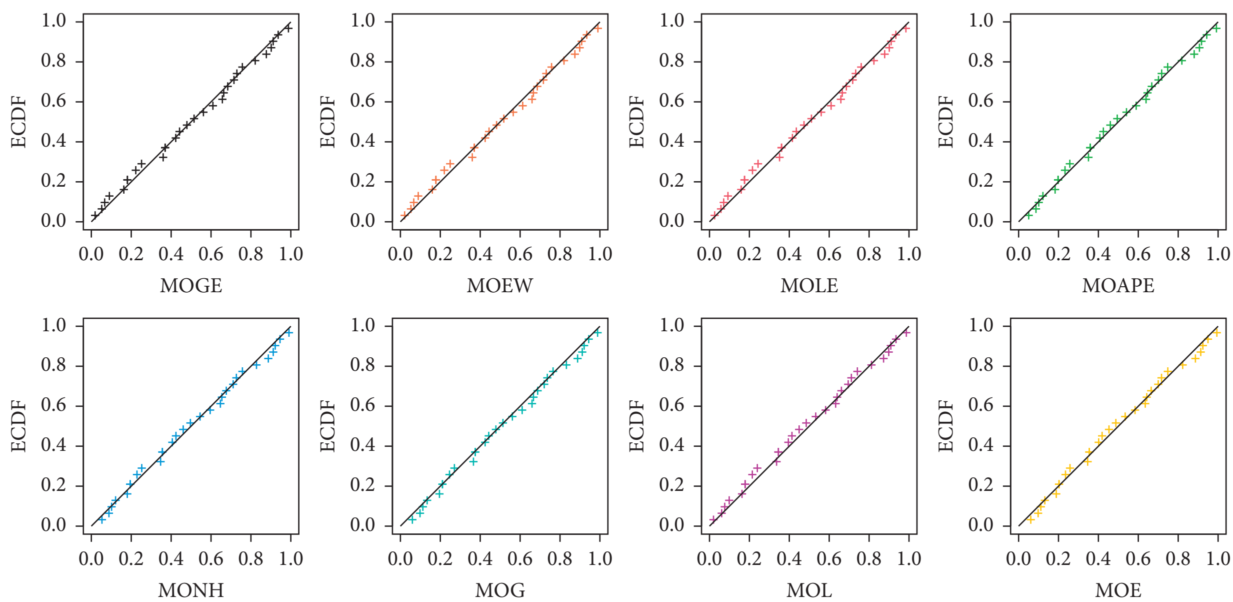


Figure 9: Plots of histogram and fitted PDFs (left-panel) fitted/empirical RFs (right-panel) of MOGE and competing models under data sets I and II. (a) Data-I. (b) Data-II.



(a)



(b)

FIGURE 10: The Q-Q plots of MOGE and competing models under data sets I and II. (a) Data-I. (b) Data-II.

TABLE 15: Three generated PCS-T2 samples from data sets I and II.

Data	$n(m)$	Scheme	Censored data
I	18 (9)	$\mathbf{R}_1 = (9, 0^8)$	5, 196, 224, 245, 293, 321, 330, 350, 420
		$\mathbf{R}_2 = (0^4, 9, 0^4)$	5, 11, 21, 31, 46, 321, 330, 350, 420
		$\mathbf{R}_3 = (0^8, 9)$	5, 11, 21, 31, 46, 75, 98, 122, 145
II	30 (15)	$\mathbf{R}_1 = (15, 0^{14})$	0.77, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05
		$\mathbf{R}_2 = (0^7, 15, 0^7)$	0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05
		$\mathbf{R}_3 = (0^{14}, 15)$	0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52

Table 16: The frequentist and bayes estimates (with their SEs) under data sets I and II.

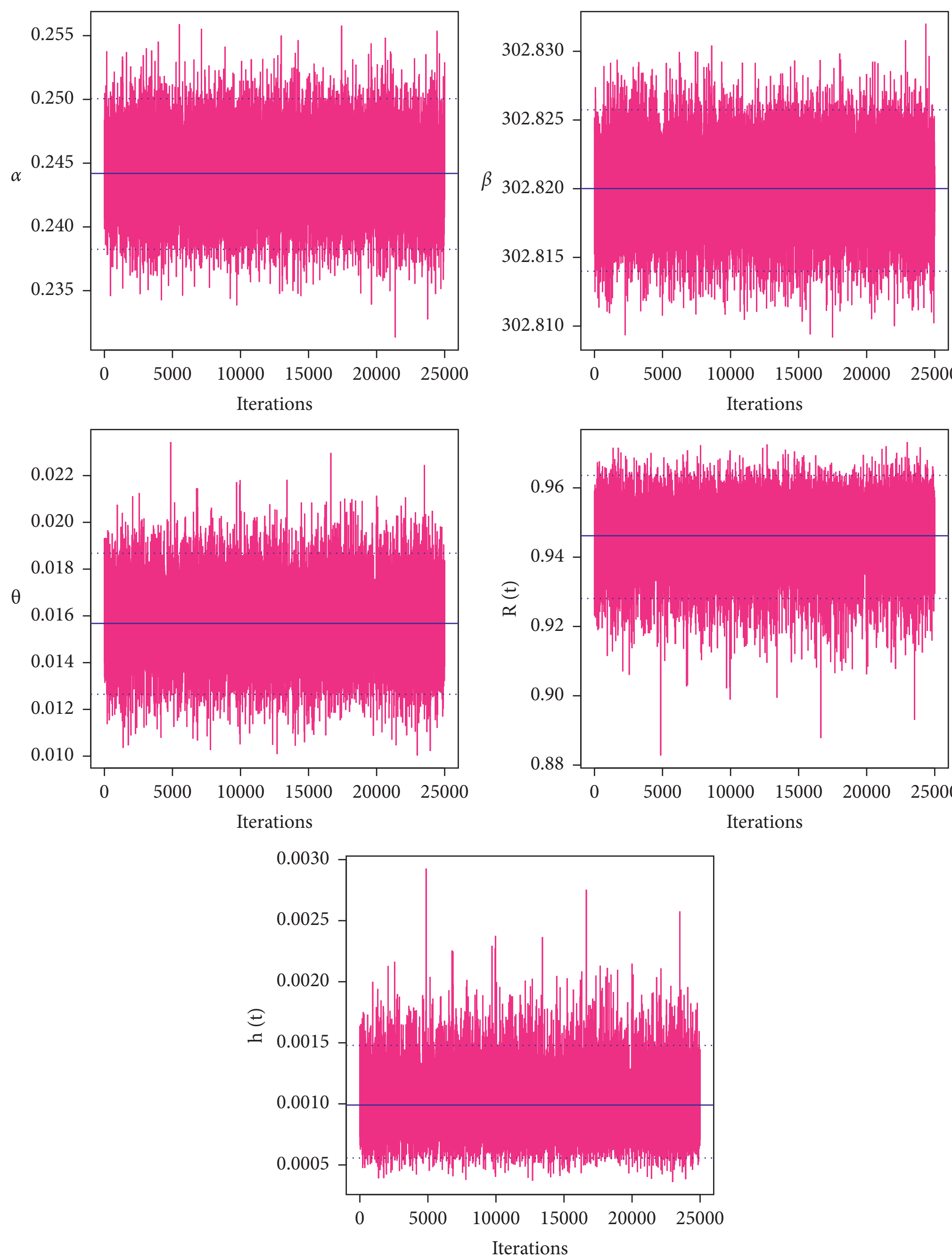
Scheme	Parameter	MLE	SEL	LL		
$\upsilon \longrightarrow$				-2	-0.02	2
Data-I						
	α	0.2442 (0.3133)	0.2441 (1.10×10^{-5})	0.2442 (2.38×10^{-8})	0.2442 (3.33×10^{-8})	0.2442 (9.16×10^{-8})
	β	302.82 (2.9810)	302.82 (1.09×10^{-5})	302.82 (1.14×10^{-7})	302.82 (5.71×10^{-8})	302.82 (1.19×10^{-9})
$\mathbf{R}_1$	θ	0.0156 (0.0042)	0.0157 (9.77×10^{-6})	0.0157 (5.05×10^{-7})	0.0157 (4.90×10^{-7})	0.0157 (4.75×10^{-7})
	$R(100)$	0.9471 (0.0451)	0.9462 (5.84×10^{-5})	0.9463 (6.21×10^{-6})	0.9462 (6.75×10^{-6})	0.9461 (7.29×10^{-6})
	$h(100)$	0.0010 (0.0006)	0.0009 (1.54×10^{-5})	0.0010 (2.24×10^{-7})	0.0010 (2.23×10^{-7})	0.0010 (2.23×10^{-7})
	α	0.6520 (0.4854)	0.6520 (1.90×10^{-5})	0.6520 (3.68×10^{-8})	0.6519 (1.94×10^{-8})	0.6520 (7.68×10^{-8})
	β	2.7522 (4.3540)	2.7521 (1.89×10^{-5})	2.7521 (4.54×10^{-8})	2.7522 (1.02×10^{-7})	2.7521 (1.59×10^{-7})
$\mathbf{R}_2$	θ	0.0057 (0.0031)	0.0057 (5.44×10^{-6})	0.0057 (2.06×10^{-7})	0.0057 (2.02×10^{-7})	0.0057 (1.97×10^{-7})
	$R(100)$	0.6659 (0.1229)	0.6646 (2.45×10^{-4})	0.6661 (5.21×10^{-6})	0.6646 (4.17×10^{-6})	0.6661 (1.37×10^{-5})
	$h(100)$	0.0039 (0.0016)	0.0039 (4.16×10^{-6})	0.0039 (2.52×10^{-7})	0.0039 (2.49×10^{-7})	0.0039 (2.47×10^{-7})
	α	0.9315(0.5133)	0.9315 (6.31×10^{-6})	0.9315 (1.10×10^{-7})	0.9315 (1.16×10^{-7})	0.9315 (1.23×10^{-7})
	β	0.5257 (2.5912)	0.5257 (6.29×10^{-6})	0.5257 (1.36×10^{-7})	0.5257 (1.42×10^{-7})	0.5257 (1.48×10^{-7})
$\mathbf{R}_3$	θ	0.0026 (0.0098)	0.0027 (2.81×10^{-6})	0.0027 (8.15×10^{-7})	0.0027 (8.13×10^{-7})	0.0027 (8.12×10^{-7})
	$R(100)$	0.6057 (0.1072)	0.5974 (2.75×10^{-4})	0.5993 (5.39×10^{-5})	0.5974 (6.59×10^{-5})	0.5955 (7.79×10^{-5})
	$h(100)$	0.0043(0.0020)	0.0044 (3.69×10^{-6})	0.0044 (9.90×10^{-7})	0.0044 (9.88×10^{-7})	0.0044 (9.86×10^{-7})
Data-II						
	α	4.2551 (1.4664)	4.2535 (5.09×10^{-4})	4.2600 (3.12×10^{-5})	4.2536 (9.42×10^{-6})	4.2471 (5.09×10^{-5})
	β	0.3270 (0.8587)	0.3455(4.39×10^{-4})	0.3504 (1.48×10^{-4})	0.3456 (1.17×10^{-4})	0.3407 (8.69×10^{-5})
$\mathbf{R}_1$	θ	0.8510 (0.8031)	0.8586 (4.28×10^{-4})	0.8632 (7.69×10^{-5})	0.8586 (4.82×10^{-5})	0.8540 (1.90×10^{-5})
	$R(1)$	0.7602 (0.0747)	0.7600 (3.06×10^{-4})	0.7623 (1.39×10^{-5})	0.7600 (6.12×10^{-7})	0.7576 (1.58×10^{-5})
	$h(1)$	0.7139 (0.2696)	0.7119 (8.57×10^{-4})	0.7310 (1.07×10^{-4})	0.7121 (1.18×10^{-5})	0.6942 (1.25×10^{-4})
	α	4.9887 (2.1453)	4.9941 (1.25×10^{-3})	5.0334 (2.83×10^{-4})	4.9945 (3.63×10^{-5})	4.9548 (2.14×10^{-4})
	β	0.9138 (2.3699)	0.9500 (1.15×10^{-3})	0.9835 (4.41×10^{-4})	0.9503 (2.31×10^{-4})	0.9173 (2.20×10^{-5})
$\mathbf{R}_2$	θ	1.0688 (0.9528)	1.0867 (7.30×10^{-4})	1.1002 (1.98×10^{-4})	1.0868 (1.14×10^{-4})	1.0735 (2.98×10^{-5})
	$R(1)$	0.8674 (0.0514)	0.8619 (2.42×10^{-4})	0.8633 (2.57×10^{-5})	0.8619 (3.47×10^{-5})	0.8604 (4.42×10^{-5})
	$h(1)$	0.4216 (0.1402)	0.4359 (6.97×10^{-4})	0.4485 (1.70×10^{-4})	0.4360 (9.14×10^{-5})	0.4242 (1.64×10^{-5})
	α	4.3478 (1.9903)	4.3514 (1.92×10^{-3})	4.4437 (6.07×10^{-4})	4.3523 (2.83×10^{-5})	4.2601 (5.55×10^{-4})
	β	1.6147 (2.4821)	1.6355 (1.82×10^{-3})	1.7195 (6.63×10^{-4})	1.6363 (1.37×10^{-4})	1.5536 (3.86×10^{-4})
$\mathbf{R}_3$	θ	1.4146 (0.5795)	1.4307 (1.04×10^{-3})	1.4583 (2.77×10^{-4})	1.4310 (1.04×10^{-4})	1.4042 (6.55×10^{-5})
	$R(1)$	0.7918 (0.0788)	0.7835 (3.61×10^{-4})	0.7867 (3.23×10^{-5})	0.7835 (5.24×10^{-5})	0.7802 (7.35×10^{-5})
	$h(1)$	0.5857 (0.2394)	0.6097 (9.78×10^{-4})	0.6348 (3.11×10^{-4})	0.6099 (1.53×10^{-4})	0.5869 (7.61×10^{-6})

Table 17: The 95% asymptotic/credible interval estimates [with their CLs] under data sets I and II.

Scheme	Parameter	ACI-NA	ACI-NL	BCI	HPD
Data-I					
	α	(0.0000, 0.8595) [0.8582]	(0.0198, 3.0176) [2.9978]	(0.2382, 0.2501) [0.0118]	(0.2382, 0.2501) [0.0118]
	β	(296.14, 307.82) [11.680]	(296.19, 307.88) [11.691]	(302.81, 302.83) [0.0117]	(302.81, 302.83) [0.0117]
$\mathbf{R}_1$	θ	(0.0073, 0.0239) [0.0166]	(0.0092, 0.0265) [0.0173]	(0.0127, 0.0188) [0.0061]	(0.0126, 0.0187) [0.0060]
	$R(100)$	(0.8588, 0.9999) [0.1411]	(0.8628, 0.9999) [0.1371]	(0.9260, 0.9620) [0.0360]	(0.9280, 0.9636) [0.0356]
	$h(100)$	(0.0000, 0.0022) [0.0022]	(0.0003, 0.0036) [0.0033]	(0.0006, 0.0015) [0.0009]	(0.0006, 0.0015) [0.0009]
	α	(0.0000, 1.6033) [1.6033]	(0.1515, 2.8051) [2.6536]	(0.6460, 0.6578) [0.0118]	(0.6461, 0.6579) [0.0118]
	β	(0.0000, 11.286) [11.286]	(0.1239, 61.137) [61.137]	(2.7463, 2.7581) [0.0118]	(2.7465, 2.7583) [0.0118]
$\mathbf{R}_2$	θ	(0.0000, 0.0118) [0.0118]	(0.0019, 0.0166) [0.0147]	(0.0041, 0.0075) [0.0034]	(0.0041, 0.0075) [0.0034]
	$R(100)$	(0.4251, 0.9067) [0.4816]	(0.4638, 0.9560) [0.4922]	(0.5889, 0.7407) [0.1518]	(0.5900, 0.7417) [0.1516]
	$h(100)$	(0.0007, 0.0070) [0.0063]	(0.0017, 0.0088) [0.0071]	(0.0027, 0.0053) [0.0026]	(0.0026, 0.0052) [0.0026]
	α	(0.0000, 1.9375) [1.9375]	(0.3163, 2.7428) [2.4264]	(0.9296, 0.9335) [0.0039]	(0.9295, 0.9334) [0.0039]
	β	(0.0000, 5.6043) [5.6043]	(0.0001, 8244.3) [8244.3]	(0.5237, 0.5276) [0.0039]	(0.5237, 0.5276) [0.0039]
$\mathbf{R}_3$	θ	(0.0000, 0.0219) [0.0219]	(0.0001, 4.0602) [4.0601]	(0.0019, 0.0036) [0.0017]	(0.0019, 0.0036) [0.0017]
	$R(100)$	(0.3956, 0.8159) [0.4203]	(0.4282, 0.8569) [0.4287]	(0.5170, 0.6866) [0.1696]	(0.5164, 0.6854) [0.1690]
	$h(100)$	(0.0004, 0.0082) [0.0078]	(0.0017, 0.0107) [0.0090]	(0.0033, 0.0056) [0.0023]	(0.0033, 0.0056) [0.0023]
Data-II					
	α	(1.3811, 7.1292) [5.7481]	(2.1656, 8.3608) [6.1952]	(4.0952, 4.4105) [0.3153]	(4.0955, 4.4107) [0.3152]
	β	(0.0000, 2.0101) [2.0101]	(0.0019, 56.228) [56.226]	(0.2136, 0.4867) [0.2731]	(0.2147, 0.4873) [0.2726]
$\mathbf{R}_1$	θ	(0.0000, 2.4250) [2.4250]	(0.1338, 5.4104) [5.2765]	(0.7269, 0.9931) [0.2662]	(0.7294, 0.9952) [0.2659]
	$R(1)$	(0.6137, 0.9066) [0.2929]	(0.6270, 0.9216) [0.1371]	(0.6563, 0.8470) [0.1906]	(0.6664, 0.8539) [0.1875]
	$h(1)$	(0.1854, 1.2424) [1.0570]	(0.3405, 1.4967) [1.1562]	(0.4684, 1.0015) [0.5331]	(0.4530, 0.9793) [0.5263]

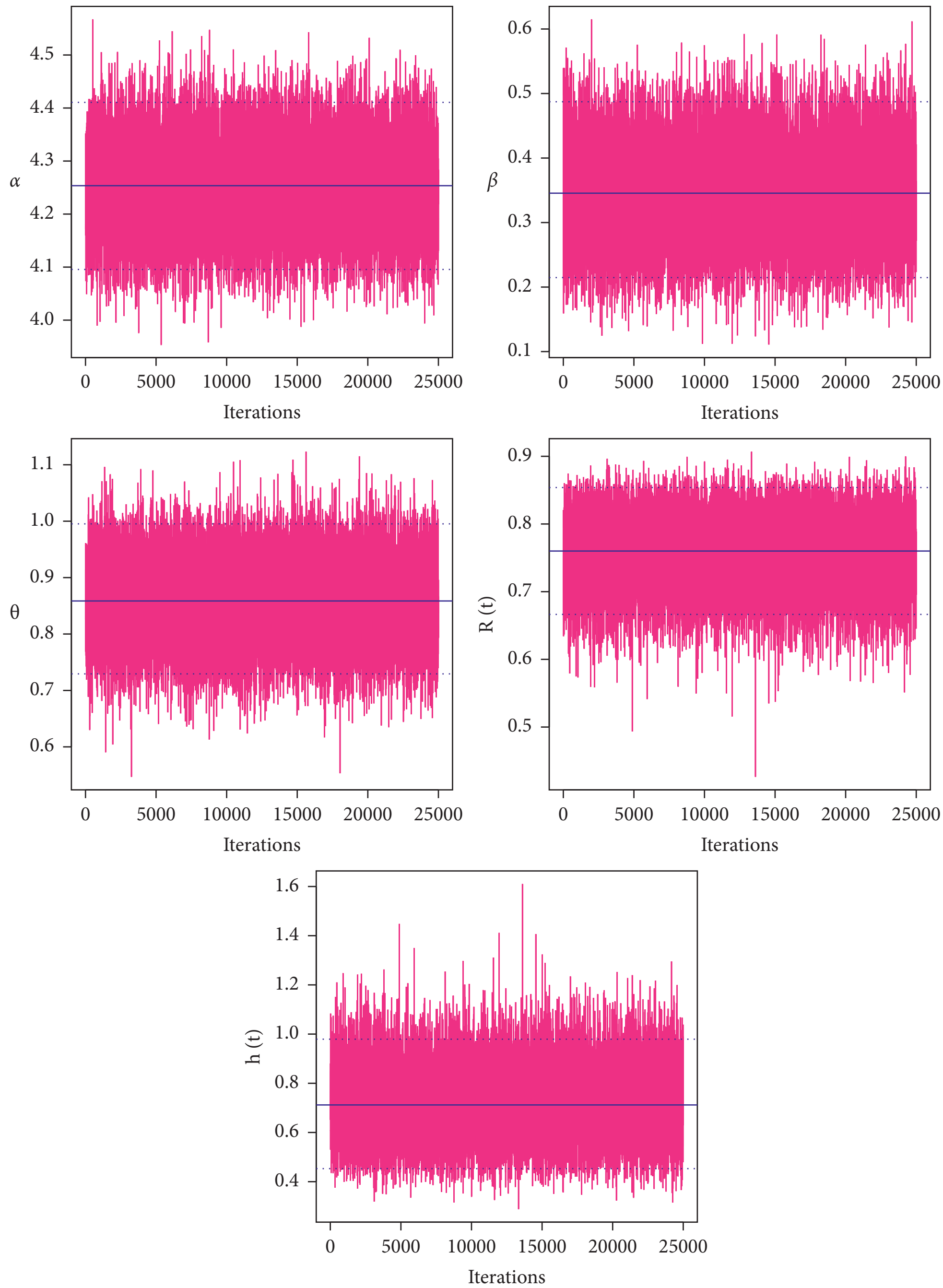
Table 17: Continued.

Scheme	Parameter	ACI-NA	ACI-NL	BCI	HPD
	α	(0.7840, 9.1933) [8.4093]	(2.1475, 11.588) [9.4409]	(4.6080, 5.3856) [0.7776]	(4.6064, 5.3816) [0.7752]
	β	(0.0000, 5.5587) [5.5587]	(0.0057, 147.40) [147.39]	(0.6005, 1.3118) [0.7113]	(0.5975, 1.3079) [0.7103]
$\mathbf{R}_2$	θ	(0.0000, 2.9362) [2.9362]	(0.1862, 6.1331) [5.9469]	(0.8663, 1.3248) [0.4585]	(0.8644, 1.3219) [0.4574]
	$R(1)$	(0.7667, 0.9680) [0.2013]	(0.7723, 0.9741) [0.2018]	(0.7776, 0.9269) [0.1494]	(0.7851, 0.9325) [0.1474]
	$h(1)$	(0.1468, 0.6964) [0.5496]	(0.2197, 0.8090) [0.5894]	(0.2458, 0.6759) [0.4301]	(0.2370, 0.6642) [0.4272]
	α	(0.4469, 8.2488) [7.8018]	(1.7726, 10.664) [8.8915]	(3.7588, 4.9459) [1.1871]	(3.7527, 4.9383) [1.1856]
	β	(0.0000, 6.4795) [6.4795]	(0.0794, 32.851) [32.772]	(1.0759, 2.2085) [1.1326]	(1.0858, 2.2128) [1.1270]
$\mathbf{R}_3$	θ	(0.2788, 2.5505) [2.2717]	(0.6338, 3.1576) [2.5238]	(1.1212, 1.7705) [0.6493]	(1.1070, 1.7542) [0.6472]
	$R(1)$	(0.6373, 0.9463) [0.3090]	(0.6515, 0.9624) [0.3109]	(0.6612, 0.8832) [0.2220]	(0.6740, 0.8934) [0.2194]
	$h(1)$	(0.1165, 1.0548) [0.9383]	(0.2629, 1.3048) [1.0419]	(0.3420, 0.9462) [0.6042]	(0.3258, 0.9194) [0.5936]



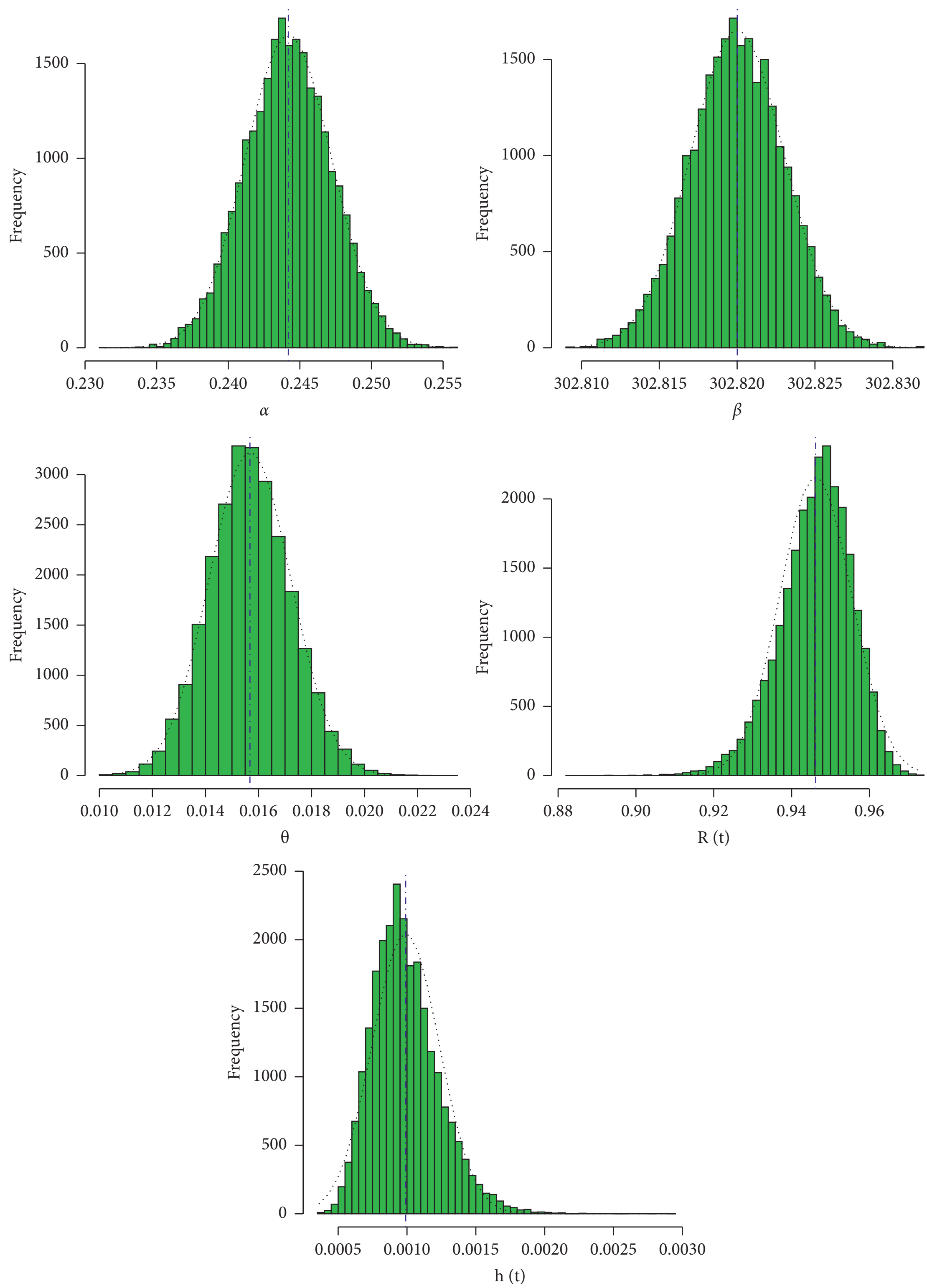
(a)

Figure 11: Continued.



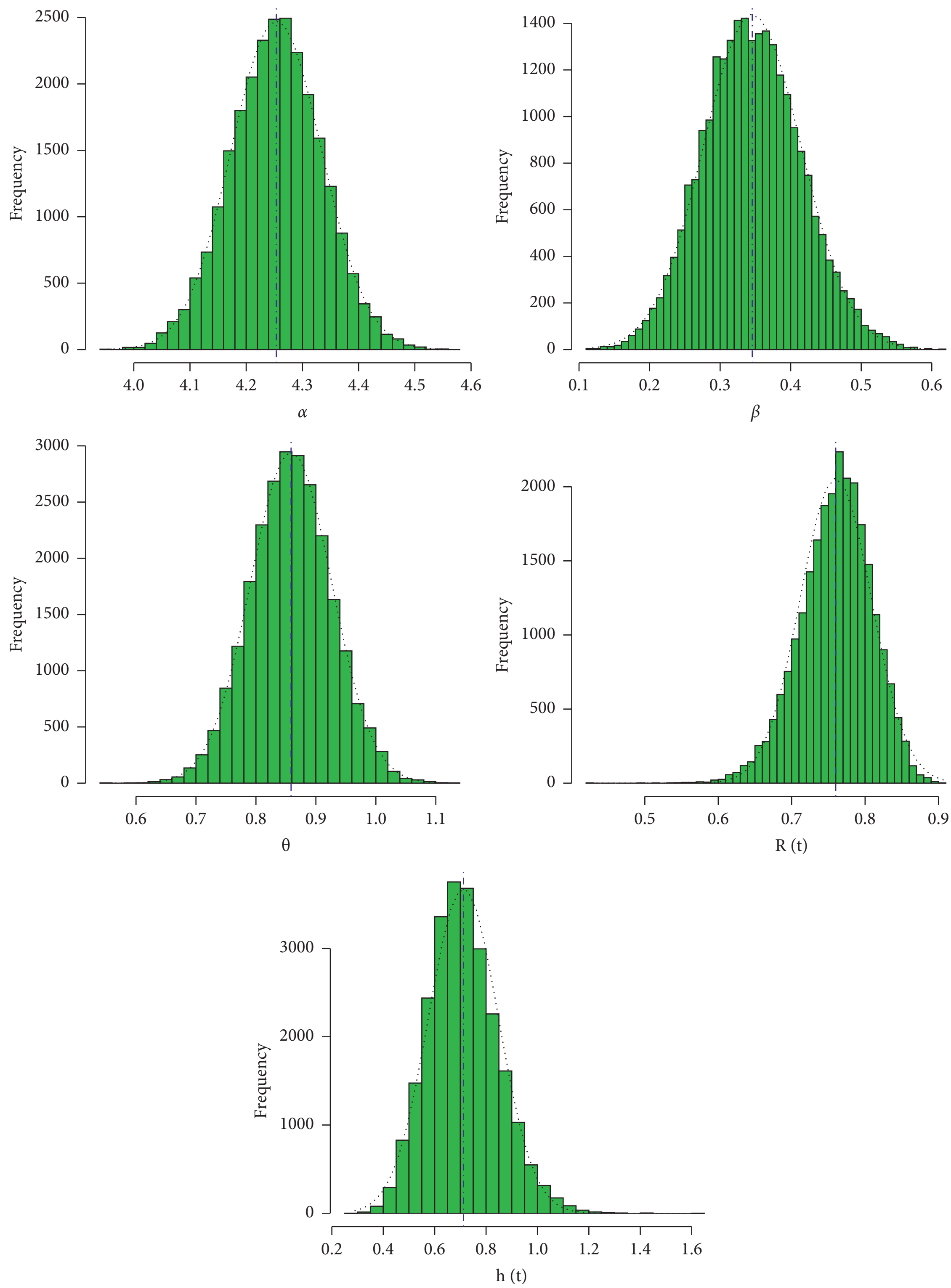
(b)

FIGURE 11: Trace plots for MCMC draws of the unknown parameters under data sets I and II. (a) $\mathbf{R}_1$ of data-I. (b) $\mathbf{R}_1$ of data-II.



(a)

FIGURE 12: Continued.



(b)

FIGURE 12: Histogram plots of the unknown parameters under data sets I and II. (a) $\mathbf{R}_1$ from data-I. (b) $\mathbf{R}_1$ from data-II.

Table 18: Optimum censoring scheme under data sets I and II.

Data	Scheme	Criterion		
		A	B	C
I	$\mathbf{R}_1$	8.9824	2.822×10^{-6}	306091.5
	$\mathbf{R}_2$	19.193	4.203×10^{-6}	330194.6
	$\mathbf{R}_3$	6.9776	7.775×10^{-7}	943744.4
II	$\mathbf{R}_1$	3.5326	1.988×10^{-2}	138.9751
	$\mathbf{R}_2$	11.127	3.890×10^{-1}	76.48990
	$\mathbf{R}_3$	10.458	6.801×10^{-1}	31.96376

in Section 3.3, we generate 30,000 MCMC samples and discard the first 5,000 samples as burn-in. Under SEL and LL (with $v(= -2, -0.02, +2)$), the Bayesian MCMC estimates and associated credible intervals of α, β, $\theta R(t)$, and $h(t)$ using noninformative priors are developed. Also, MLEs and two-sided 95% asymptotic (NA and NL) interval estimates of the unknown parameters are computed. The estimates of RF and HF are calculated when the mission time t is taken as 100 and 1 for data sets I and II, respectively. The point estimates (with their SEs) and interval estimates with their confidence lengths (CLs) and reported in Tables 16 and 17, respectively. It is noted that the proposed point and interval estimates, obtained based on data sets I and II, of α, β, $\theta R(t)$, and $h(t)$ are very similar to each other, as expected.

To examine the convergence of M-H algorithm, trace plot based on 25,000 MCMC simulated variates of each unknown parameter (for $\mathbf{R}_1$ as an example) using data sets I and II, is plotted in Figure 11. For each trace plot, the sample mean and two bounds of 95% HPD credible intervals are mentioned by solid (—) and dashed (- - -) lines, respectively. It indicates that the M-H algorithm converges very well. Using the Gaussian kernel density, the corresponding histogram plot of each unknown parameter (for $\mathbf{R}_1$ as an example) is also plotted and represented in Figure 12. In each histogram, the sample mean is depicted as vertical dash-dotted line (:). It represents that the simulated posteriors of α, β, and θ are fairly symmetric while the simulated posteriors of $R(t)$ and $h(t)$ are close to negative and positive skewed, respectively.

Using the optimum criteria given in Section 4, using the observed Fisher information (11) obtained from both data sets I and II, the concept of optimal censoring is illustrated, see Table 18. It shows that the scheme $\mathbf{R}_3$ is the optimal censoring (for data set I) while $\mathbf{R}_1$ is the optimal censoring (for data set II) than other competing scheme plans. However, the parameter and optimal estimates examined here support the same findings reported in the simulation study.

7. Conclusion Remarks

This article investigates the problem of estimating the unknown parameters and reliability characteristics of the MOGE distribution when the data are collected under type-II progressive censoring. Besides the likelihood inference, the Bayes MCMC estimates of the unknown parameters, reliability, and hazard functions are derived. As expected the joint posterior density cannot be obtained explicitly. Therefore, using gamma informative priors under different loss functions, the Bayes point and interval estimates of the unknown parametric functions are approximated utilizing the Metropolis–Hasting algorithm. From the simulation outputs, the Bayes MCMC estimation method for any unknown function of the MOGE parameters is recommended. To show the practical utility of the proposed methods in real-life phenomenon and to determine the optimum censoring plan, two data sets from engineering and physics are analyzed for this purpose. Additionally, based on the given data sets, we showed that the proposed model provides an adequate fit with respect to its competing Marshall–Olkin extended models. We hope that the results and methodologies discussed in this paper will be beneficial to data analyst and reliability practitioners. [39, 40].

Appendix

A. Fisher's Elements

Differentiating (7) partially with respect to α, β, and θ, the Fisher elements $\mathscr{L}_{ij}, i, j = 1, 2, 3$, are expressed as

$$\mathscr{L}_{11} = -\frac{m}{\alpha^2} - \sum_{i=1}^{m} R_i \log^2\psi_i(\theta)\psi_i^{\alpha}(\theta)\left(1-\psi_i^{\alpha}(\theta)\right)^{-1}\left[1+\psi_i^{\alpha}(\theta)\left(1-\psi_i^{\alpha}(\theta)\right)^{-1}\right]$$
$$-\overline{\beta}\sum_{i=1}^{m}(R_i+2)\log^2\psi_i(\theta)\psi_i^{\alpha}(\theta)\left(\beta+\overline{\beta}\psi_i^{\alpha}(\theta)\right)^{-1}\left[1-\overline{\beta}\psi_i^{\alpha}(\theta)\left(\beta+\overline{\beta}\psi_i^{\alpha}(\theta)\right)^{-1}\right],$$
$$\mathscr{L}_{22} = -\frac{n}{\beta^2} + \sum_{i=1}^{m}(R_i+2)\left(1-\psi_i^{\alpha}(\theta)\right)^2\left(\beta+\overline{\beta}\psi_i^{\alpha}(\theta)\right)^{-2},$$
$$\mathscr{L}_{33} = -\frac{m}{\theta^2} + (\alpha-1)\sum_{i=1}^{m}\psi_i^{-1}(\theta)\left[\psi_i''(\theta)-\psi_i'^2(\theta)\psi_i^{-1}(\theta)\right]$$
$$-\alpha\sum_{i=1}^{m}R_i\psi_i^{\alpha-1}(\theta)\left(1-\psi_i^{\alpha}(\theta)\right)^{-1}\left[\psi_i''(\theta)-\psi_i'^2(\theta)\psi_i^{-1}(\theta)\left\{1-\alpha-\alpha\psi_i^{\alpha}(\theta)\left(1-\psi_i^{\alpha}(\theta)\right)^{-1}\right\}\right]$$
$$-\alpha\overline{\beta}\sum_{i=1}^{m}(R_i+2)\psi_i^{\alpha-1}(\theta)\left(\beta+\overline{\beta}\psi_i^{\alpha}(\theta)\right)^{-1}\left[\psi_i''(\theta)-\psi_i'^2(\theta)\psi_i^{-1}(\theta)\left\{1-\alpha+\alpha\overline{\beta}\psi_i^{\alpha}(\theta)\left(\beta+\overline{\beta}\psi_i^{\alpha}(\theta)\right)^{-1}\right\}\right], \tag{A.1}$$
$$\mathscr{L}_{12} = \sum_{i=1}^{m}(R_i+2)\log\left(\psi_i(\theta)\right)\psi_i^{\alpha}(\theta)\left(\beta+\overline{\beta}\left(\psi_i(\theta)\right)^{\alpha}\right)^{-1}\left[1+\overline{\beta}\left(1-\psi_i^{\alpha}(\theta)\right)\left(\beta+\overline{\beta}\psi_i^{\alpha}(\theta)\right)^{-1}\right],$$
$$\mathscr{L}_{23} = \alpha\sum_{i=1}^{m}(R_i+2)\psi_i'(\theta)\psi_i^{\alpha-1}(\theta)\left(\beta+\overline{\beta}\psi_i^{\alpha}(\theta)\right)^{-1}\left[1+\overline{\beta}\left(1-\psi_i^{\alpha}(\theta)\right)\left(\beta+\overline{\beta}\psi_i^{\alpha}(\theta)\right)^{-1}\right],$$

$$\mathscr{L}_{13} = \sum_{i=1}^{m}\psi_i'(\theta)\psi_i^{-1}(\theta) - \sum_{i=1}^{m}R_i\psi_i'(\theta)\left(\psi_i(\theta)\right)^{\alpha-1}\left(1-\left(\psi_i(\theta)\right)^{\alpha}\right)^{-1}\left[1+\alpha\log\left(\psi_i(\theta)\right)\left\{1+\psi_i^{\alpha}(\theta)\left(1-\psi_i^{\alpha}(\theta)\right)^{-1}\right\}\right]$$
$$-\overline{\beta}\sum_{i=1}^{m}(R_i+2)\psi_i'(\theta)\psi_i^{\alpha-1}(\theta)\left(\beta+\overline{\beta}\psi_i^{\alpha}(\theta)\right)^{-1}\left[1+\alpha\log\left(\psi_i(\theta)\right)\left\{1-\overline{\beta}\psi_i^{\alpha}(\theta)\left(\beta+\overline{\beta}\psi_i^{\alpha}(\theta)\right)^{-1}\right\}\right], \tag{A.2}$$

where $\psi_i''(\theta) = -x_i^2(1-\psi_i(\theta)), i = 1, 2, \ldots, m$.

Appendix B. Simulation Outputs

All simulation outputs including APEs, RMSEs, MRABs, ACLs, and CPs of α, β, θ, $R(t)$, and $h(t)$ based on different choices of progressive censoring schemes (also different complete and effective sample sizes) are reported here.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] A. W. Marshall and I. Olkin, "A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families," *Biometrika*, vol. 84, no. 3, pp. 641–652, 1997.

[2] M. M. Ristić and D. Kundu, "Marshall-Olkin generalized exponential distribution," *Metron*, vol. 73, no. 3, pp. 317–333, 2015.

[3] R. D. Gupta and D. Kundu, "Generalized exponential distribution: different method of estimations," *Journal of Statistical Computation and Simulation*, vol. 69, no. 4, pp. 315–337, 2001.

[4] N. L. Johnson and S. Kotz, *Continuous Univariate Distribution*, Wiley, New York, NY, USA, 1970.

[5] N. Balakrishnan and E. Cramer, *The Art of Progressive Censoring*, Springer, Birkhäuser, New York, NY, USA, 2014.

[6] Z. H. Amin, "Bayesian inference for the Pareto lifetime model under progressive censoring with binomial removals," *Journal of Applied Statistics*, vol. 35, no. 11, pp. 1203–1217, 2008.

[7] C. Kim, J. Jung, and Y. Chung, "Bayesian estimation for the exponentiated Weibull model under Type II progressive censoring," *Statistical Papers*, vol. 52, no. 1, pp. 53–70, 2011.

[8] S. R. Huang and S. J. Wu, "Bayesian estimation and prediction for Weibull model with progressive censoring," *Journal of Statistical Computation and Simulation*, vol. 82, no. 11, pp. 1607–1620, 2012.

[9] S. Dey, M. Nassar, R. K. Maurya, and Y. M. Tripathi, "Estimation and prediction of Marshall–Olkin extended exponential distribution under progressively type-II censored data," *Journal of Statistical Computation and Simulation*, vol. 88, no. 12, pp. 2287–2308, 2018.

[10] H. Panahi, "Interval estimation of Kumaraswamy parameters based on progressively Type-II censored sample and record values," *Miskolc Mathematical Notes*, vol. 21, no. 1, pp. 319–334, 2020.

[11] H. Z. Muhammed and E. M. Almetwally, "Bayesian and non-Bayesian estimation for the bivariate inverse Weibull distribution under progressive Type-II censoring," *Annals of Data Science*, pp. 1–32, 2020.

[12] A. Elshahhat and M. K. Rastogi, "Estimation of parameters of life for an inverted Nadarajah–Haghighi distribution from Type-II progressively censored samples," *Journal of the Indian Society for Probability and Statistics*, vol. 22, no. 1, pp. 113–154, 2021.

[13] K. Maiti and S. Kayal, "Estimation of parameters and reliability characteristics for a generalized Rayleigh distribution under progressive type-II censored sample," *Communications in Statistics - Simulation and Computation*, vol. 50, no. 11, pp. 3669–3698, 2021.

[14] S. Dey, A. Elshahhat, and M. Nassar, "Analysis of progressive Type-II censored gamma distribution," *Computational Statistics*, 2022.

[15] K. Maiti, S. Kayal, and D. Kundu, "Statistical inference on the Shannon and Rényi entropy measures of generalized exponential distribution under the progressive censoring," *SN Computer Science*, vol. 3, no. 4, pp. 317–321, 2022.

[16] A. Elshahhat and W. S. Abu El Azm, "Statistical reliability analysis of electronic devices using generalized progressively hybrid censoring plan," *Quality and Reliability Engineering International*, vol. 38, no. 2, pp. 1112–1130, 2022.

[17] W. H. Greene, *Econometric Analysis*, Prentice-Hall, New York, NY, USA, 2000.

[18] W. Q. Meeker and L. A. Escobar, *Statistical Methods for Reliability Data*, John Wiley & Sons, New York, NY, USA, 1998.

[19] A. H. Muse, S. Mwalili, O. Ngesa, H. M. Alshanbari, S. K. Khosa, and E. Hussam, "Bayesian and frequentist approach for the generalized log-logistic accelerated failure time model with applications to larynx-cancer patients," *Alexandria Engineering Journal*, vol. 61, no. 10, pp. 7953–7978, 2022.

[20] A. H. Muse, S. Mwalili, O. Ngesa, S. J. Almalki, and G. Abd-Elmougod, "Bayesian and classical inference for the generalized log-logistic distribution with applications to survival data," *Computational Intelligence and Neuroscience*, vol. 2021, Article ID 5820435, 24 pages, 2021.

[21] A. H. Muse, O. Ngesa, S. Mwalili, H. M. Alshanbari, and A. A. H. El-Bagoury, "A flexible Bayesian parametric proportional hazard model: simulation and applications to right-censored healthcare data," *Journal of Healthcare Engineering*, pp. 1–28, 2022.

[22] B. N. Pandey and O. Rai, "Bayesian estimation of mean and square of mean of normal distribution using LINEX loss function," *Communications in Statistics - Theory and Methods*, vol. 21, no. 12, pp. 3369–3391, 1992.

[23] A. Gelman, J. B. Carlin, H. S. Stern, and D. B. Rubin, *Bayesian Data Analysis*, Chapman and Hall/CRC, New York, NY, USA, 2004.

[24] M. H. Chen and Q. M. Shao, "Monte Carlo estimation of Bayesian credible and HPD intervals," *Journal of Computational & Graphical Statistics*, vol. 8, no. 1, pp. 69–92, 1999.

[25] N. Balakrishnan and R. Aggarwala, *Progressive Censoring Theory, Methods and Applications*, Birkhäuser, Boston, MA, USA, 2000.

[26] H. K. T. Ng, P. S. Chan, and N. Balakrishnan, "Optimal progressive censoring plans for the Weibull distribution," *Technometrics*, vol. 46, no. 4, pp. 470–481, 2004.

[27] M. Plummer, N. Best, K. Cowles, and K. Vines, "coda: convergence diagnosis and output analysis for MCMC," *R News*, vol. 6, pp. 7–11, 2006.

[28] A. Henningsen and O. Toomet, "maxLik: a package for maximum likelihood estimation in R," *Computational Statistics*, vol. 26, no. 3, pp. 443–458, 2011.

[29] J. M. Pavia, "Testing goodness-of-fit with the kernel density estimator: GoFKernel," *Journal of Statistical Software*, vol. 66, pp. 1–27, 2015.

[30] A. Elshahhat and M. Nassar, "Bayesian survival analysis for adaptive Type-II progressive hybrid censored Hjorth data," *Computational Statistics*, vol. 36, no. 3, pp. 1965–1990, 2021.

[31] F. K. Wang, "A new model with bathtub-shaped failure rate using an additive Burr XII distribution," *Reliability Engineering & System Safety*, vol. 70, no. 3, pp. 305–312, 2000.

[32] D. Hinkley, "On quick choice of power transformation," *Applied Statistics*, vol. 26, no. 1, pp. 67–69, 1977.

[33] G. M. Cordeiro and A. J. Lemonte, "On the Marshall–Olkin extended Weibull distribution," *Statistical Papers*, vol. 54, no. 2, pp. 333–353, 2013.

[34] M. Mansoor, M. H. Tahir, G. M. Cordeiro, S. B. Provost, and A. Alzaatreh, "The Marshall-Olkin logistic-exponential distribution," *Communications in Statistics - Theory and Methods*, vol. 48, no. 2, pp. 220–234, 2019.

[35] M. Nassar, D. Kumar, S. Dey, G. M. Cordeiro, and A. Z. Afify, "The Marshall–Olkin alpha power family of distributions with applications," *Journal of Computational and Applied Mathematics*, vol. 351, pp. 41–53, 2019.

[36] A. J. Lemonte, G. M. Cordeiro, and G. Moreno–Arenas, "A new useful three-parameter extension of the exponential distribution," *Statistics*, vol. 50, no. 2, pp. 1–26, 2015.

[37] J. T. Eghwerido, J. O. Ogbo, and A. E. Omotoye, "The Marshall-Olkin Gompertz distribution: properties and applications," *Statistica*, vol. 81, no. 2, pp. 183–215, 2021.

[38] M. E. Ghitany, F. A. Al-Awadhi, and L. A. Alkhalfan, "Marshall–Olkin extended Lomax distribution and its application to censored data," *Communications in Statistics - Theory and Methods*, vol. 36, no. 10, pp. 1855–1866, 2007.

[39] N. Balakrishnan and R. A. Sandhu, "A simple simulational algorithm for generating progressive Type-II censored samples," *The American Statistician*, vol. 49, no. 2, pp. 229-230, 1995.

[40] S. Dey and A. Elshahhat, "Analysis of Wilson-Hilferty distribution under progressive Type-II censoring," *Quality and Reliability Engineering International*, 2022.