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Research Article

Multifractal Early Warning Signals about Sudden Changes in the Stock Exchange States

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Critical phenomena in stock exchange are regularly occurring and difficult to predict events, often leading to disastrous consequences. The presented paper is devoted to the search and research of early warning signals of critical transitions in stock exchange based on the results of a multifractal analysis of a series of transactions in shares of public companies. We have proposed and justified the use of certain features of behavior of multifractal spectrum shape parameters such as signals. As model time series, on which methods of multifractal analysis were tested, we used a series of the number of unstable sites of the sandpile automaton on the random Erdős–Rényi graph, self-organizing into critical and bistable states. It was found that the early warning signals for both cellular automata and stock exchanges are an increase in the magnitude of the maximum position, a decrease in the width, and a decrease, followed by a sharp increase, in the value of the spectrum asymmetry parameter.

1. Introduction

Most complex systems, regardless of their origin, are scale-invariant, have heterogeneity and nonstationary behavior, and contain internal mechanisms of self-organization. Therefore, dynamic processes in such systems are usually nonlinear. In such systems, abrupt changes in states can occur; such changes are often called critical transitions. An example is a phase transition accompanied by a radical change in the properties of the system at the macro level. As a result of a phase transition, the system acquires completely new and unexpected properties that are not reducible to the properties of individual parts.

Ordinary critical phenomena, such as phase transitions of the second kind, are observed only when the control parameter reaches a certain critical value. In other words, a critical state is created artificially by tuning a control parameter to a critical value. For example, if a control parameter such as temperature is adjusted to a critical value, then an order parameter such as magnetization will reach a zero value, and the paramagnetic-ferromagnetic phase transition will occur in the system. The parameters of the system at the critical point are characterized by power laws.

For most complex macroscopic systems and processes of natural origin, it is impossible to adjust the value of a control parameter to a critical value, but, despite this, such systems, while in a critical state, are characterized by power laws. Examples of such systems and processes are financial markets with crashes and crises, seismic activity with catastrophic earthquakes, social networks with information cascading, and other systems (e.g., see papers [1–4]). The answer to the question of how critical transitions occur in such systems was given by Per Bak, Chao Tang, and Kurt Wiesenfeld only in the late 1980s. They discovered the phenomenon of self-organized criticality (SOC) and proposed a theory that explains how such systems reach a critical state without tuning the control parameter (e.g., see papers [5, 6]). It turned out that a critical state, in which even a minor event can lead to a catastrophe, can not only be created artificially (e.g., in laboratory conditions), but also arise as a result of the self-organization of the system. In such a state, the system acquires properties that its elements did not have, demonstrating complex holistic behavior.

The basic model of the SOC theory is a sandpile into which grains of sand fall from time to time (e.g., see papers [2, 7]). At first, the pile simply grows, and in those places where the local slope is greater than the stability threshold, sand grains crumble down the slope to neighboring surface areas. If the average surface slope (z) is small, the set of chaotically directed microcurrents of sand grains is mutually balanced and the macroscopic sand current $J = 0$. If z exceeds some critical value (z_c), then there is a spontaneous sand flow ($J \neq 0$) across the surface of the heap, which increases as z increases. The value of z_c separates the subcritical ($z < z_c$) and supercritical ($z > z_c$) phases, which are resistant to small perturbations. If $z = z_c$, then a single fallen grain of sand can cause avalanches of any size. Thus, the sandpile self-organizes into a critical state at $J = 0^+$, corresponding to a phase transition of the second kind with a control parameter z and a parameter of order J. It should be noted that the sandpile model also allows us to explain the self-organization in the bistable state, corresponding to a phase transition of the first kind. For this purpose, a model of facilitated sandpile was proposed (see the paper [8]), which can demonstrate self-organized bistability (SOB) (e.g., see papers [9, 10]).

There are many studies that substantiate the concept of the similarity of the mechanisms of behavior of economic systems, stock markets, and financial time series with the behavior of the model variables (e.g., see papers [11–17]), as well as studies on the search for early warning signals (EWS) for critical transitions in financial and stock markets (e.g., see papers [18–24]). The determination of the time interval preceding the occurrence of a critical transition in the system not only has important theoretical value, but also has important applied value. The studies we know are mainly focused on the detection of SOC mechanisms in financial systems. Also, there are studies devoted to finding EWS for financial crises using mainly measures of correlation theory (autocorrelation function, skewness, kurtosis, variance, and other measures) according to the results of the analysis of financial series in the selected range. A significant limitation in using such measures for the study of financial series scaling is their applicability only to stationary and fractal time series (e.g., see the paper [25]).

At the moment, we are not aware of any works that present studies of the dynamics of stock exchange self-organization in SOC and SOB states, based on the results of multifractal analysis of the time series of the number of deals made on the shares of companies (volume indicator), as well as multifractal EWS for the corresponding critical transitions. To address this gap, we investigated the possibilities and limitations of multifractal EWS for critical transitions using the results of multifractal stochastic dynamics analysis of volume indicators, using the numbers of unstable cells of the sandpile cellular automaton as the basic (in a sense reference) time series. Research results are presented in this paper.

The paper is structured as follows: Section 2 is devoted to the description of mechanisms of functioning of sandpile cellular automaton and substantiation of similarities in the behavior of such automata and stock markets. Methods for generating time series of the number of unstable automaton nodes and methods for obtaining time series of the number of deals made on shares of companies are also presented. The rationale for the necessity of application and methods of calculation of parameters of a multifractal spectrum of time series as measures of early detection of critical transitions is presented; Section 3 presents and discusses the results of calculations of parameters of multifractal spectra of time series used as measures of early detection of critical transitions; and Section 4 presents the main conclusions, possible practical applications of the obtained results, and the prospects for further research.

2. Data Set and Methods

2.1. Model and Real-Time Series. The self-organized critical sandpile behavior considered in Section 1 can be described using sandpile cellular automata (e.g., see papers [7, 8, 26]). Among the many sandpile cellular automata models, we chose the Manna model (see the paper [27]) on the Erdős–Rényi random graph (e.g., see papers [28, 29]) as the most relevant model of avalanche-like changes in the number of traded shares of companies.

We built three Erdős–Rényi random graphs with a number of sites N equal to 500, 1500, and 2500 by connecting any two sites v_i and v_j with edge e_{ij} with probability p independently of all other pairs of sites.

In this case, the Manna model is a random graph with the number of sites N, the sites of which are assigned integer non-negative numbers $z_i(v_j)$. These numbers are traditionally interpreted as the number of sand grains. If $z_i(v_j)$ is not less than the set threshold z_{cv_j}, then site v_j is unstable and “topples.” This removes z_c sand grains from it, each of which is transferred to one of the randomly chosen neighboring sites. If a site is on the edge of the graph, the sand grains transferred for it are irreversibly lost. Each neighboring site receives a random number of sand grains δ_k. If there are several unstable sites, they “topple” simultaneously—during one time step.

The elementary event that causes the system to move from one steady state to another is initiated by adding a grain of sand to one of the sites. If the addition of a grain of sand causes a site to lose stability, then the grains of sand transferred to neighboring sites during its toppling may violate their stability. The chain reaction of toppling that continues as long as unstable sites remain in the system will be called an avalanche.

Regardless of the initial state of the system, after a certain number of events, the system reaches a critical state (SOC state), in which the processes occurring are scale-invariant, and all characteristics of avalanches correspond to power distributions.

The rules of the standard Manna model on the Erdős–Rényi random graph with the number of sites N, which demonstrates the output of the system in the SOC state, corresponding to the phase transition of the second kind, have the following form:

$$
\begin{aligned}
&z_i\left(v_j\right) \geq z_{cv_j} > 1,\\
&z_{i+1}\left(v_j\right) \longrightarrow z_{i+1}\left(v_j\right) - z_{cv_j},\\
&z_{i+1}(\mathrm{Ne}) \longrightarrow z_{i+1}(\mathrm{Ne}) + \delta_k, \sum_{k=1}^{z_{cv_j}} \delta_k = z_{cv_j}, \quad \delta_k \geq 0,
\end{aligned}
\tag{1}
$$

where Ne denotes the nearest neighboring site to the site v_j.

As noted in Section 1, sandpile cellular automata are also capable of self-organization into a bistable state (SOB state) corresponding to a first-order phase transition. The rules of the automaton allowing to bring it into SOB state are known as "facilitated rules" (see papers [8, 26]). A site v_j of the facilitated automaton is unstable when $z_i(v_j) \geq z_{cv_j}$ and when $f_{i-1}(v_j) \geq 2$ (f_{i-1} is the number of hits to site v_j at the previous iteration). This is the main difference between the facilitated and the standard automaton. Thus, the rules of the facilitated Manna model have the following form:

$$
\begin{aligned}
&z_i\left(v_j\right) \geq z_{cv_j} > 1 \vee f_i\left(v_j\right) \geq 2,\\
&z_i\left(v_j\right) \geq z_{cv_j}: \begin{cases} z_{i+1}\left(v_j\right) \longrightarrow z_{i+1}\left(v_j\right) - z_{cv_j},\\ z_{i+1}(\mathrm{Ne}) \longrightarrow z_{i+1}(\mathrm{Ne}) + \delta_k, \sum_{k=1}^{z_{cv_j}} \delta_k = z_c, \quad \delta_k \geq 0,\\ f_{i+1}(\mathrm{Ne}) \longrightarrow f_{i+1}(\mathrm{Ne}) + \delta_k, \quad \delta_k > 0, \end{cases}\\
&z_i\left(v_j\right) < z_{cv_j}: \begin{cases} z_{i+1}(x,y) \longrightarrow z_{i+1}(x,y) - z_i(x,y),\\ z_{i+1}(\mathrm{Ne}) \longrightarrow z_{i+1}(\mathrm{Ne}) + \delta_k, \sum_{k=1}^{z_{cv_j}} \delta_k = z_i\left(v_j\right), \quad \delta_k \geq 0,\\ f_{i+1}(\mathrm{Ne}) \longrightarrow f_{i+1}(\mathrm{Ne}) + \delta_k, \quad \delta_k > 0. \end{cases}
\end{aligned}
\tag{2}
$$

The avalanche-like propagation of sand grains between sites of the considered sandpile cellular automata in the critical state is a good qualitative econophysical model demonstrating the general regularities of the origin of the avalanche-like change in the number of deals made on the stocks of companies. Indeed, the nodes of the graph can be associated with the agents of the stock market; the edges of the graph, along which the movement of sand grains from unstable sites occurs, can be associated with the deals made between agents; and the random addition of sand grains to the sites can be associated with the market pumping (e.g., information pumping from media, quarterly reports, news feeds, and others). Then, the change in time of the number of unstable sites on the graph corresponds to the change in the number of deals made on the shares of companies. Therefore, we further use the time series of the number of unstable sites with known critical transition times as test series to determine the capabilities and limitations of a particular method of multifractal analysis in the selection and evaluation of EWS for the critical transitions.

We believe it is important to note that besides sandpile cellular automata with Manna model rules, there are other models of self-organized critical cellular automata that cannot be adequate models of stock market transactions. For example, the Bak–Tang–Wiesenfeld model and the Feder–Feder model assume that a nonrandom equal number of sand grains are transferred from an unstable site; the Dhar–Ramaswamy model (e.g., see the paper [30]) and the Pastor-Satorras–Vespignani model (e.g., see the paper [31]) are directional models in which the unstable site has only underlying neighboring sites. In addition, all models can also be realized on square lattices, which implies that there are only four nearest neighbors with an unstable site. Another well-known model that demonstrates self-organized critical behavior is the forest-fire model (e.g., see papers [32, 33]). This model is one of the most popular for simulating sociopolitical and historical processes since it simulates the spread of arousal in some environments.

Time series volume indicators were selected for companies whose shares are listed on any of the stock exchanges. In the stock trading volume data, information is available for 1-day intervals. The exchanges where these companies are traded represent four regions: Asia (Sony Group Corporation, Subaru Corporation); Russia (PJSC Aeroflot—Russian Airlines, Sberbank of Russia); the USA (Apple Inc., Meta Platforms, Inc., and Tesla, Inc.); and Europe (Airbus SE, Allianz SE, Deutsche Lufthansa AG).

2.2. Multifractal Analysis of the Time Series. It is now generally accepted that many financial time series have a complex fractal structure (e.g., see papers [34–37]). In particular, fractal analysis is effectively used to predict market crashes in financial series (e.g., see papers [38–40]). In addition, the universality of multifractal analysis has determined the success of its application to the analysis of time series depicting the dynamics of critical transitions (e.g., see papers [26, 41]).

The features of time series scaling can be studied using different approaches, starting with the classical correlation (or spectral) analysis. Among the obvious drawbacks of such approaches is their applicability only to stationary time series. Since most processes in nature are highly heterogeneous and nonstationary, the attractiveness of the choice of one or another method of analysis is largely determined by its universality and the possibility of its effective application to real processes of any origin.

The most popular methods for analyzing the multifractal structure of nonstationary time series are multifractal detrended fluctuation analysis (MF-DFA) (e.g., see papers [42, 43]); wavelet transform modulus maxima (WTMM), based on continuous wavelet transform (e.g., see papers [44, 45]); and wavelet leaders (WL), based on discrete wavelet transform (e.g., see the paper [46]).

The MF-DFA is a variant of variance analysis of univariate random walks. The method algorithm analyzes the root mean square error of linear approximation ($F^2(s)$) of the generalized random walk model from the size (s) of the approximated area. The analyzed time series is multifractal if the scaling relation is observed for all s:

$$F_q(s) = \left\{ \frac{1}{N_s} \sum_{i=1}^{N_s} \left[F_i^2(s) \right]^{q/2} \right\}^{1/q} \sim s^{H_q}, \quad (3)$$

where N_s is the number of approximated sections, and H_q are generalized Hurst exponents if $q \in (-\infty, +\infty)$.

The multifractal spectrum ($D(H)$) has the following form (see the paper [41]):

$$D_q = \frac{qH_q - 1}{q - 1}, \quad (4)$$

where D_q are the generalized multifractal dimensions.

The WTMM method assumes the existence of the following scaling relation for multifractal time series:

$$Z(q,s) = \sum_{l \in L(s)} \left(\sup_{s' \leq s} |W(s', t_l(s'))| \right)^q \sim s^{\tau_q}. \quad (5)$$

In equation (5), $Z(q,s)$ is the structural function; $L(s)$ is the set of all lines l of maximum modules of wavelet coefficients existing at scale s; $t_l(s')$ characterizes the location of the maximum at scale s, relating to line l; $W(\cdot)$ are the coefficients of the continuous wavelet transform; and τ_q are the scaling exponents.

The multifractal spectrum ($D(h)$) has the following form (see the paper [44]):

$$D_q = qh_q - \tau_q, \quad (6)$$

where D_q are the generalized multifractal dimensions, and h_q are the Hölder exponents.

The WL method assumes the existence of the following scaling relation for a multifractal time series:

$$Z(q,s) = \frac{1}{n_s} \sum_{k=1}^{n_s} L(s,k)^q \sim s^{\tau_q}. \quad (7)$$

In equation (6), $L(k,s) = \sup_{\lambda' \subset 3\lambda_{s,k}} |d(k,s)|$ are the leaders of the wavelet coefficients in which the 2^s scales are translated into the $2^s k$ time positions; $d(k,s)$ are the coefficients of the discrete wavelet transform; k is the time shift; and s is the scale.

The multifractal spectrum ($D(h)$) is defined by the following decomposition:

$$D(h) = d + \frac{c_2}{2!} \left(\frac{h - c_1}{c_2} \right)^2 + \frac{-c_3}{3!} \left(\frac{h - c_1}{c_2} \right)^3 + \cdots, \quad (8)$$

where c_1, c_2, and c_3 are the log-cumulants. c_1 corresponds to the position of the spectrum maximum, c_2 characterizes the width of the spectrum, and c_3 characterizes the asymmetry of the spectrum. The triplet c_1, c_2, c_3 contains the basic information about the multifractal structure of the studied time series.

As will be shown in Section 3, the studied time series are multifractal series, which require an infinite spectrum of fractal dimensions for a complete description. Therefore, as an EWS for critical transitions in the sandpile cellular automata and stock markets, we use the features of changes in the multifractal spectra ($D(H)$ and $D(h)$) of the studied time series as the systems approach critical points.

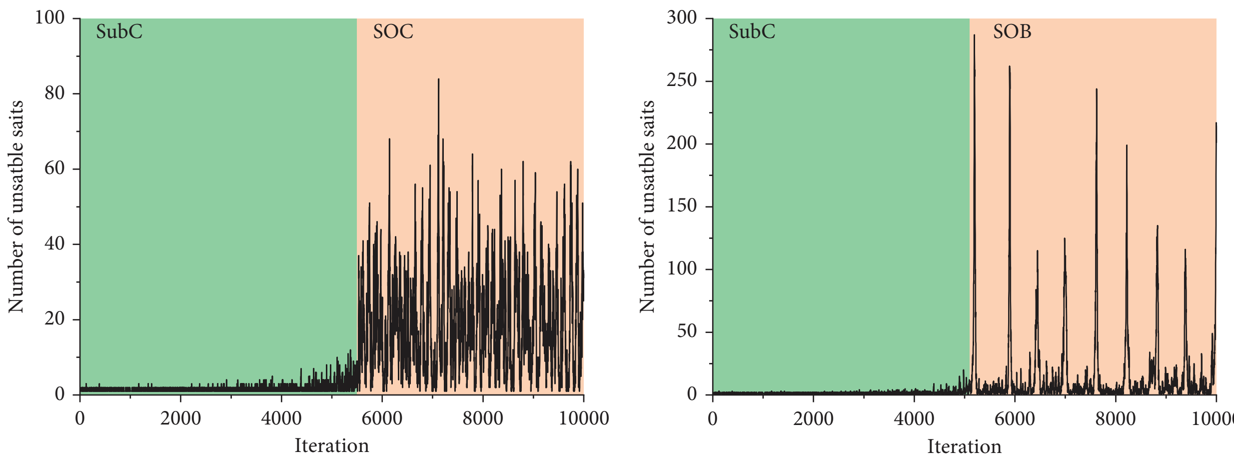
We used three main spectrum shape parameters as early warning measures for critical transitions ($\cdot$): the position of the spectrum maximum (H_0, h_0 and c_1); spectrum width ($W = H_{\max} - H_{\min}$, $W = h_{\max} - h_{\min}$ and c_2); and spectrum asymmetry ($S = H_{\max} - H_0/H_0 - H_{\min}$, $S = h_{\max} - h_0/h_0 - h_{\min}$ and c_3). The spectrum was calculated at $q = \overline{-5, 5}$ in increments of 0.1.

We calculated the time series of early warning measures (m_t) with a fixed left window boundary corresponding to the first value (x_1) of the studied time series $x_t, t = \overline{1, n}$, and a sliding right window boundary (τ) corresponding to some selected value (x_τ) of the studied time series. As a result, we obtained series of early warning measures $m_t, t = \overline{\tau, n}$, with $\tau = 1000$ for the time series of the number of toppled cells of the sandpile cellular automata and $\tau = 50$ for the time series of the volume indicators.

3. Results and Their Discussion

3.1. Time Series of Unstable Sites. The time series of the number of unstable sites of automata with standard rules, which lead to the output of the automaton in SOC state, and with clothed rules, which lead to the output of the automaton in SOB state, of the Manna model are shown in Figure 1. The figure shows time series of automata whose random graphs contain $N = 2500$ sites. Time series for $N = 500$ и $N = 1500$ have similar appearance. The series differ only in the time it takes for the system to enter the critical state (subcritical time) and in the maximum values of the number of unstable sites of cellular automata in the critical state. These values are presented in Table 1.

The time series demonstrate the presence of subcritical phase (SubC phase) and critical state (SOC state and SOB state) of the sandpile cellular automata (see Figure 1). The SubC phase corresponds to the noncatastrophic behavior. The sandpile cellular automaton, being in this chaotic phase, is stable to small perturbations. Only at the critical point (SOC state), catastrophes are possible (see Figure 1(a)) since



(a)

(b)

Figure 1: Time series of the number of unstable sites of sand-cell automata. (a) Standard rules of the Manna model and (b) facilitated rules of the Manna model.

Table 1: Subcritical time (t_{SubC}) and the maximum number of unstable sites (A_{max}) of cellular automata in the critical state.

Number of sites	Standard model		Facilitated model	
	t_{SubC}	A_{max}	t_{SubC}	A_{max}
500	2750	38	4320	196
1500	5050	62	4630	238
2500	5500	80	5100	280

a single added grain of sand in any site of the automaton can cause an avalanche of grains of any size. The SOB state is also characterized by an avalanche of sand grains of any size with the appearance of periodic bursts of activity (see Figure 1(b)). So, Buendia and coauthors state in their paper [9] that "the probability distributions for both avalanche size and duration are bimodal": small avalanches coexist with extremely large ones that span the whole system. These latter "anomalous" outbursts of activity, which are also called "king" avalanches, occur in an almost periodic way. The size of avalanches (the maximum number of unstable sites) increases with the total number of sites of the sandpile cellular automata (see Table 1).

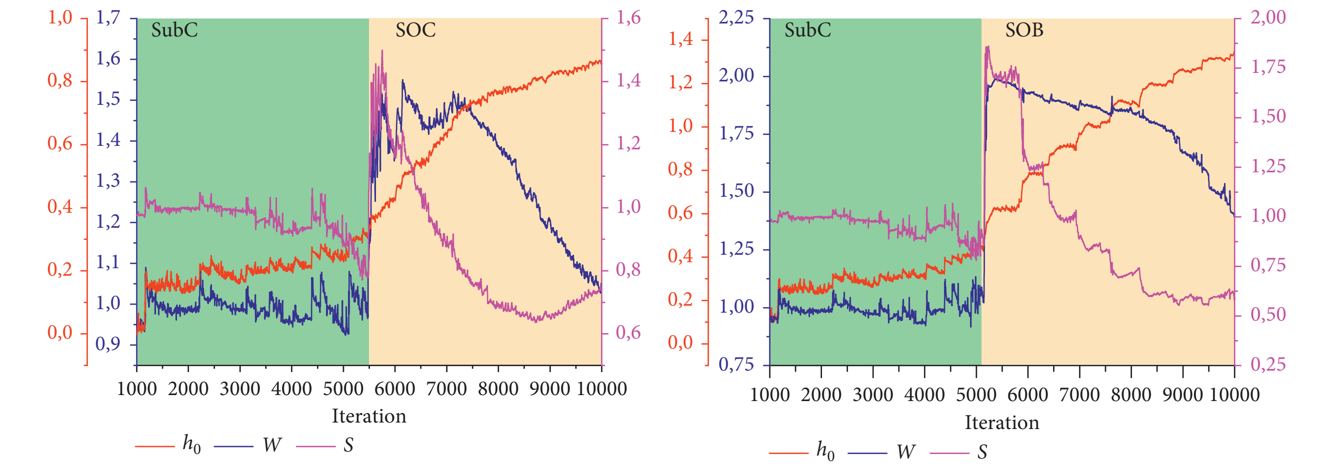
An important characteristic of the process of the system reaching a critical state is the subcritical time (t_{SubC}). It is known (e.g., see papers [2, 5]) that different types of SOC systems have different t_{SubC}. The greatest t_{SubC} is characteristic of the evolution of the Earth's crust and of biological evolution. For many other types of SOC systems, the t_{SubC} is much smaller.

In our opinion, the reason for such large differences in t_{SubC} values is the different levels of complexity of individual SOC systems. Differences in the value of t_{SubC} allow to distinguish different levels of complexity in SOC systems. This extends the applicability of SOC theory, as well as SOB, far beyond the characteristic power laws for the distribution of avalanche size and power spectral density as $1/f$ noise. Given that t_{SubC} increases with the size of the cellular automata (see Table 1), we can use t_{SubC} as a measure of the complexity of the system capable of a critical transition. Note that we previously found a similar change in t_{SubC} with changes in the size of the cellular automata (see the paper [26]). Besides, other things being equal, the value of t_{SubC} for SOB systems is lower than the value of t_{SubC}, characteristic of SOC systems.

Perhaps the formation of a more complex SOC system initially requires a larger value of t_{SubC}, but when such a system is already formed, the corresponding t_{SubC} at the next level is already much smaller.

3.2. Multifractal Measures for Early Detection of Critical Transitions in Sandpile Cellular Automata. The scaling relation (3) of the MF-DFA method is not met for any values of the right boundary x_τ of the sliding window. Therefore, this method cannot be used as a method for calculating measures of early detection of critical transitions in sandpile cellular automata based on the results of multifractal analysis of the number of unstable sites series. The reason why the MF-DFA method does not allow revealing the multifractal structure of model time series is the presence of a large number of repeating values in such series.

In contrast, the scaling relations (see equations (5) and (7)) of wavelet transform-based methods are satisfied for all $\tau \in [1000, 10000]$. This is connected with the fact that these methods do not require the extraction of local trends in repeating values of the time series. The time series of multifractal early warning measures of critical transitions in sandpile cellular automata with a number of 2500 tiles obtained by the WTMM method are presented in Figure 2,



(a)

(b)

Figure 2: Time series of the maximum position (h_0), width (W), and asymmetry (S) of the multifractal spectrum for the sandpile cellular automata. (a) Standard rules of the Manna model and (b) lightweight rules of the Manna model.

and those obtained by the WL method are presented in Figure 3. The conditional time corresponding to the iteration step is used.

The values h_0 (see Figure 2) and c_1 (see Figure 3), which characterize the positions of the maximum of the multifractal spectrum of unstable sandpile cellular automata, increase as the automata approach the critical state. This increase is typical for both standard cellular automata and facilitated cellular automata. Consequently, as the automaton approaches the critical state, the time series of the number of its unstable tiles becomes more "smooth" or less "jagged." Note that a sharp increase in the position of the maximum of the singularity spectrum is also observed in the vicinity of the critical point of the phase transition of the second kind in the Ising model (e.g., see the paper [41]).

The width of the multifractal spectrum (W) of the time series of unstable sites computed by the WTMM method decreases as the standard and facilitated automata approach the critical state (see Figure 2). Also, the value of W calculated by the WL method decreases or is equivalent to the absolute value of the second log-cumulant $|c_2|$ (see Figure 3). The equivalence of W and $|c_2|$ follows from a simple analysis of equation (8). The increasing W value is observed in the SOC state and SOB state. It follows from the decreasing value of W that as automata approach the critical state, the time series of unstable sites become more homogeneous fractal series, with a more uniform distribution of series values. A similar decrease in the width of the spectrum in the vicinity of the critical point is characteristic of a second-order phase transition in the Ising model (e.g., see papers [41, 47]).

The value of the spectrum asymmetry parameter S calculated by the WTMM method first decreases, then sharply increases as the automata approach the critical state (see Figure 2). Consequently, large fluctuations (strong singularities) in the number of unstable tiles of automata as they approach the critical state prevail in the time series. Similar behavior of the parameter S is also observed in the Ising model (e.g., see the paper [41]). The log-cumulant c_3, which characterizes the asymmetry of the spectrum, decreases not only in the vicinity of the time of the critical transition, but also in some noncritical time interval (see Figure 3). In our opinion, such behavior of parameter c_3 contradicts the existence of one of the precursors of the critical transition, known as the critical slowing down (e.g., see papers [19, 21, 24, 48]). Perhaps the incorrect estimation of the asymmetry parameter is one of the drawbacks of the expansion (8) or, moreover, a drawback of the WL method.

The critical slowing down is the phenomenon that when a system approaches a critical point, it relaxes more slowly after small perturbations. It is known (e.g., see papers [48, 49]) that time series showing a critical slowing down are characterized by increases in autocorrelation (or increases in h_0 and decreases in W, as we found), dispersion (or increases in S, as we found), kurtosis and skewness, and the β of the power spectral density $1/f^{\beta}$ (or increases in h_0, as we found). Consequently, of the three multifractal analysis methods, only the WTMM method allows us to obtain correct estimates of the multifractal spectrum shape parameters, at least for the time series of the number of unstable tiles of cellular automata. Recall that the time-varying WL estimation (c_3) for the asymmetry parameter does not explain the critical slowdown.

3.3. Multifractal Measures for Early Detection of Critical Transitions in Stock Exchange. As shown in Subsection 3.2, the multifractal early warning signals are an increase in the magnitude of the maximum position (h_0, c_1) of the multifractal spectrum $D(h)$, a decrease in the spectrum width (W, $|c_2|$), and a decrease followed by a sharp increase in the spectrum asymmetry parameter (S). We also remind that the MF-DFA method did not reveal a multifractal structure in the time series of the number of unstable tiles.

In this subsection, we demonstrate the results of calculations of these three shape parameters of $D(h)$ for the

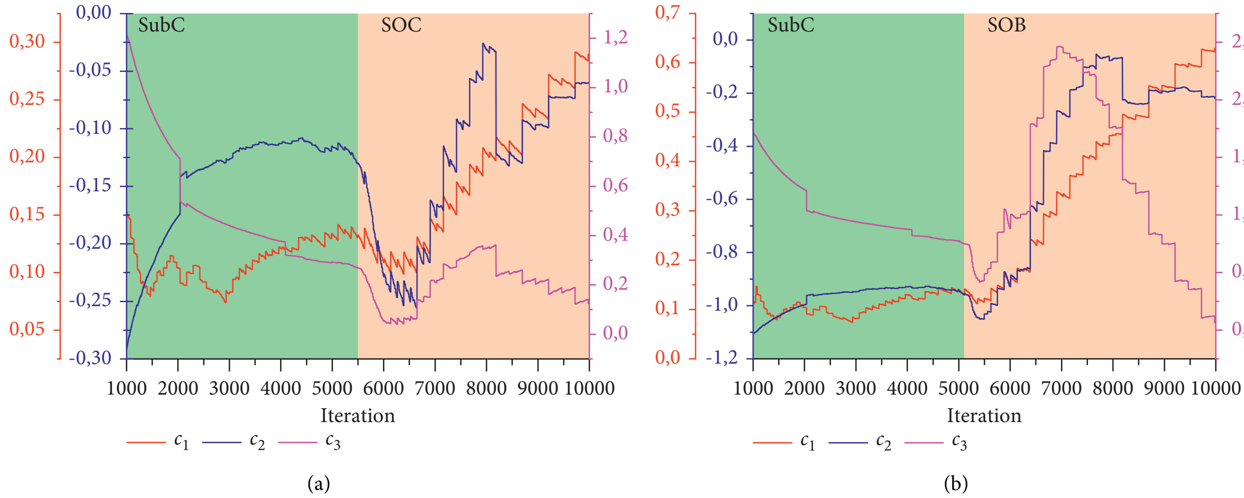


FIGURE 3: Time series of the log-cumulants ($c_i, i = 1, 2, 3$) for the sandpile cellular automata. (a) Standard rules of the Manna model and (b) facilitated rules of the Manna model.

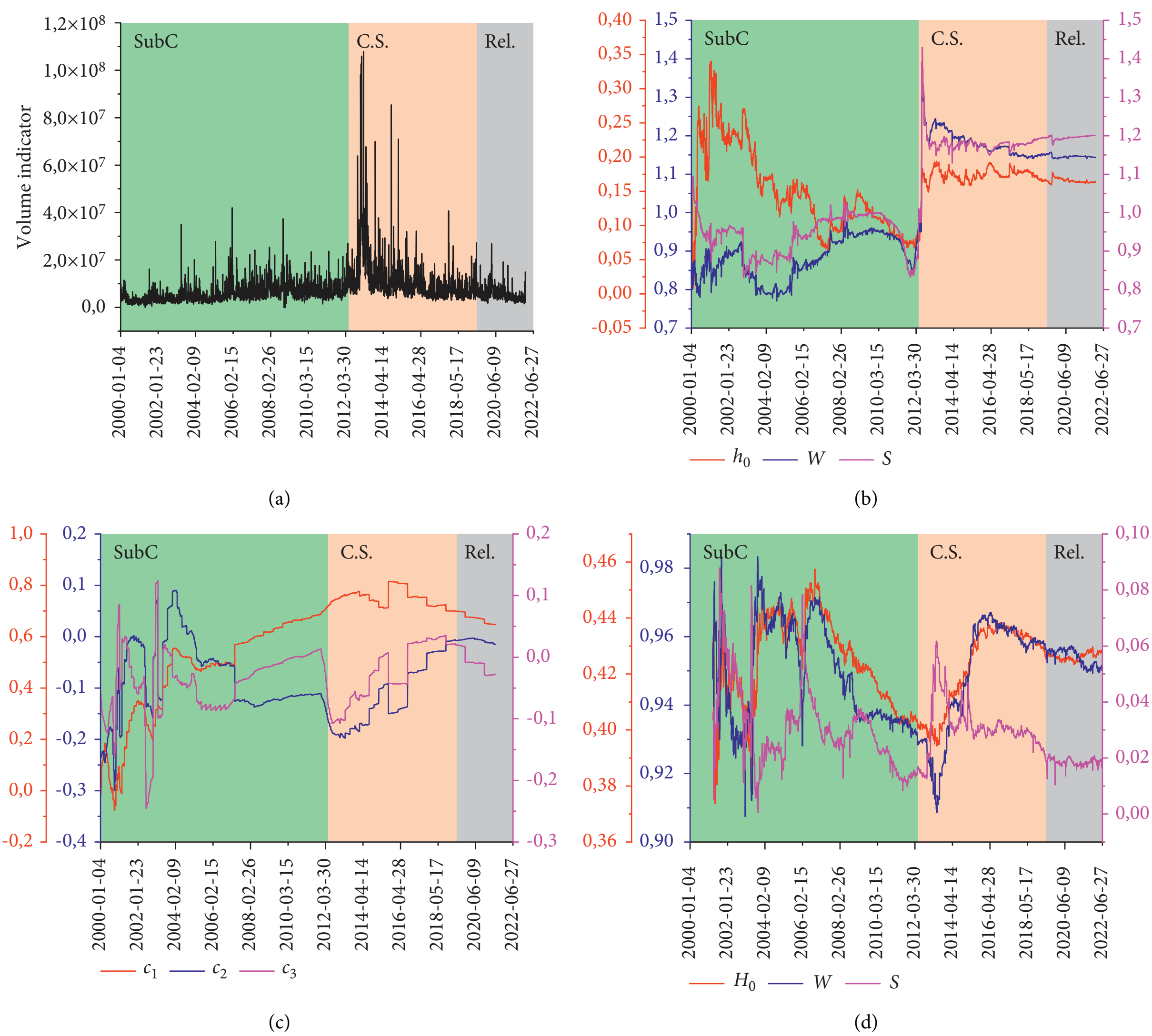


FIGURE 4: Financial time series and time series of multifractal spectrum shape parameters for Sony Group Corporation. The symbol "C.S." denotes the critical state, and the symbol "Rel." denotes the relaxation. (a) Volume indicator, (b) WTMM method, (c) WL method, and (d) MF-DFA method.

Table 2: Multifractal early warning signals and the critical transition date (t_C) of the stock exchange volumes.

Public company	WTMM method			WL method			MF-DFA method			t_C
	h_0	W	S	c_1	c_2	c_3	H_0	W	S	
Sony Group Corporation	+	+	+	+	+	+	−	+	+	May 25, 2012
Subaru Corporation	+	+	+	+	+	+	−	+	+	February 20, 2016
Apple Inc.	+	+	+	+	+	+	−	+	−	September 20, 2014
PJSC Aeroflot—Russian Airlines	+	+	+	+	+	+	−	+	+	March 17, 2002
Airbus SE	+	+	+	+	+	+	+	−	+	July 25, 2007
Allianz SE										April 19, 2004
Meta Platforms, Inc.	+	+	+	+	+	+	−	+	+	December 3, 2021
Deutsche Lufthansa AG	+	+	+	+	+	+	−	+	+	August 11, 2014
Sberbank of Russia	+	+	+	+	+	+	−	+	+	November 25, 2021
Tesla Inc.	+	+	+	+	+	+	−	+	−	January 8, 2020

volume indicator series, using MF-DFA, WTMM, and WL methods. As an illustrative example, Figure 4 shows the time series of the spectrum shape parameters calculated for the time series of the number of transactions on Sony Group Corporation shares. These figures also show the subcritical phase and the subcritical time (t_c), and the possible critical state and relaxation intervals of the segment of the stock exchange whose agents are involved in Sony Group Corporation stock transactions. Hereinafter, the term "stock exchange segment" refers to a stock exchange with agents involved in transactions in the stock of a particular public company, such as Sony Group Corporation.

As shown by the results of the WTMM calculation of changes in the shape parameters (see Figure 4(b)), the corresponding market segment on May 25, 2012, self-organizes into a critical state. Accordingly, the time it takes for a market segment to enter the critical state (subcritical time) is 3035 days.

The volume indicator series shown in this figure demonstrates another interesting phenomenon, which in our opinion cannot be determined by the multifractal early warning signals. This is a market segment relaxation that begins around March 2, 2019 (see Figure 4(a)). In this time interval, the number of transactions on shares decreases, which is probably due to the loss of interest of market players in the corresponding transactions.

The results of the WL method are similar to those of the WTMM method even for the spectrum asymmetry parameter, although the increase in c_3 occurs at the moment of critical transition (see Figure 4(c)). Recall that for time series of the number of unstable tiles of automata, the behavior of $S(t)$ and $c_3(t)$ when approaching the critical point in time is not consistent (see Figures 2 and 2b).

The MF-DFA method made it possible to reveal the multifractal structure in the volume indicator series and, consequently, to give estimates of the spectrum shape parameters (see Figure 4(d)). The behavior of the parameters W and S when approaching the critical point is similarly consistent with the behavior of the same parameters calculated by the WTMM method, and this behavior is characteristic of the critical deceleration. Yet, in spite of this, the parameter H_0 decreases when approaching the critical point. This contradicts with the results of calculations of parameter H_0 by methods based on wavelet transform. Consequently, the critical transition of the exchange segment associated with the trades in Sony Group Corporation shares cannot be detected in advance when calculating the spectrum parameters using the MF-DFA method.

In order not to overload our paper with unnecessary graphical information, for the remaining nine time series studied, we presented the results of calculations of multifractal early warning signals and the most important features of the time series in Table 2. The symbol "+" means that the change in time series values for the corresponding measure when approaching the time of critical transition is similar to the behavior of the time series for sandpile cellular automata. The symbol "–" means that there is no such analogy.

The results presented in Table 2 suggest that all of the considered public companies self-organize into a critical state. At the same time, the time for companies to reach a critical state (t_{SubC}) is different. Methods based on wavelet transform give similar results and allow us to give estimates of the values of the parameters of the shape spectra, acceptable for their application as early warning signals.

4. Conclusion

The time series of the number of unstable tiles of sandpile automata and the time series of the number of transactions in shares of public companies with one-day increments are multifractal. Such series admit decomposition into segments with different local scaling properties, so their quantitative description requires a whole spectrum of fractal dimensions, such as a multifractal spectrum in the form of $D(h)$, sometimes called singularity spectrum.

For early detection of the time moment of the systems reaching a critical state based on the results of analysis of multifractal series generated by such systems, analysis of the change in the shape of the multifractal spectrum as the system approaches the point of critical transition is required. A change in the shape of the spectrum with a good degree of accuracy is determined by a change in its three parameters. These parameters are the position of the spectrum maximum, spectrum width, and spectrum asymmetry.

As the sandpile cellular automata and stock exchange volume approach the time point of critical transition, the value of the maximum position increases, the width decreases, and the value of the spectrum asymmetry parameter decreases, followed by a sharp increase. Such behavior of the spectrum shape in the vicinity of the critical transition point corresponds to the critical slowdown of the system as it approaches the critical transition point. Indeed, in the vicinity of a critical point, the time series becomes more regular and homogeneous, with larger fluctuations prevailing in the value of the number of unstable tiles and the value of the number of stock transactions. Therefore, the indicated behavior of the values of the spectrum shape parameters calculated by the WTMM method is reliable early warning signals for critical transitions.

The sandpile cellular automaton is a very coarse model of the stock exchange, but, despite this, its time series have similar behavior, demonstrating subcritical phase and critical state, and similar behavior of the values of spectrum shape parameters when approaching the time moment of critical transition. Therefore, the series of the number of unstable tiles can be used as reference series for testing various measures of early detection of critical transitions and the method for their calculation. In our opinion, these analogies of time series are caused by multifractality of random graphs with colored (unstable) tiles and stock exchange transaction network-multifractal structures generate multifractal series.

There are different types of self-organized criticality (in particular, self-organized criticality and self-organized bistability), which differ from each other in the level of complexity depending on the characteristic value of subcritical time. The more complex the system of self-organized criticality, in particular, the more tiles the random graph contains, the greater its subcritical time. Also, the subcritical time is different for different public companies. Perhaps this difference is not only due to the different number of market players involved in transactions with the shares of a particular public company, but also due to the different mechanisms of self-organized criticality. In any case, subcritical time can be viewed as one measure of system complexity, along with power laws for the probability density function and autocorrelation function of avalanche size, as well as $1/f$ noise.

In conclusion, we note that not all segments of the stock exchange are capable of self-organization into a critical state; perhaps for some market segments, the time moment of critical transition has not yet arrived. Yet despite this, the possibility of early detection of critical transitions should not be underestimated. In particular, this is due to the irreversibility of a segment of a stock exchange as it approaches a critical point, which can have catastrophic consequences for a company. Multifractal early warning signals will give company managers information about the need to take precritical measures if there is enough time to take such measures.

Data Availability

The time-series data of stock trading volumes used to support the findings of this study are available at https://finance.yahoo.com

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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