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Research Article

Fixed Point Results of Dynamic Process $\check{D}(\Upsilon, \mu_0)$ through F_I^C-Contractions with Applications

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This article constitutes the new fixed point results of dynamic process D(ϒ, μ0) through FIC-integral contractions of the Ciric kind and investigates the said contraction to iterate a fixed point of set-valued mappings in the module of metric space. To do so, we use the dynamic process instead of the conventional Picard sequence. The main results are examined by tangible nontrivial examples which display the motivation for such investigation. The work is completed by giving an application to Liouville-Caputo fractional differential equations.

1. Introduction and Preliminaries

In the recent past, the study of metric fixed point theory untied a portal to a new area of pure and applied mathematics, the fixed point theory and its application. This concept was prolonged by either extending metric space into its extensions or by modifying the structure of the contractions (see [1–7]). The most classical structure known as Banach contraction principle (or contraction) theorem was introduced by a Polish mathematician Banach in 1922 [8]. The applications of fixed points of Banach contraction mappings defined for different kinds of spaces is the guarantee of the existence and uniqueness of solutions of differential and integral type equations. The variety of these nonlinear problems imposes the search for more and better tools, which are recently very remarkable in the literature. One of such tools was recently conveyed by Wardowski [9], where the author originated a new class of contractive mapping called F-contraction.

Nadler [10] using the idea of Pompeiu–Hausdorff metric discussed the Banach contraction mappings for set-valued functions rather than single-valued functions. Let (Δ, δ) be a metric space. For $\mu_1, \mu_2 \in \Delta$ and $A, B \subseteq \Delta$, define the Pompeiu–Hausdorff metric $\widehat{H}$ induced by δ on $CB(\Delta)$ as follows:

$$\widehat{H}(A, B) = \max\left\{\sup_{\mu_1 \in A} \check{D}(\mu_1, B), \sup_{\mu_2 \in B} \check{D}(\mu_2, A)\right\}, \tag{1}$$

for each $A, B \in CB(\Delta)$, where $CB(\Delta)$ denotes the set of all nonempty closed bounded subsets of Δ and $\check{D}(\mu_1, B) = \inf_{\mu_2 \in B} \delta(\mu_1, \mu_2)$. An element $\mu \in \Delta$ is called a fixed point of a set-valued mapping, i.e., $\Upsilon\colon \Delta \longrightarrow CB(\Delta)$, then $\mu \in \Upsilon(\mu)$. Also, denote the family of nonempty compact subsets of Δ by $K(\Delta)$.

Some well-known results are related to this section and hereafter.

Lemma 1. *Let A and B be nonempty proximal subsets of a metric space (Δ, δ). If $\alpha \in A$, then*

$$\delta(\alpha, B) \leq H(A, B). \tag{2}$$

Lemma 2 (see [11]). *Let (Δ, δ) be a metric space and a sequence $(\mu_i)_{i \in N}$ in (Δ, δ) such that*

$$\lim_{i \longrightarrow \infty} \delta(\mu_i, \mu_{i+1}) = 0 \tag{3}$$

is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ and subsequences of positive integers (μ_{i_j}) and (μ_{l_j}), $\mu_{i_j} > \mu_{l_j} > j$ such that

$$\left[\delta\left(\mu_{i_j},\mu_{l_j}\right),\delta\left(\mu_{i_j+1},\mu_{i_j}\right),\delta\left(\mu_{i_j},\mu_{l_j-1}\right),\delta\left(\mu_{i_j+1},\mu_{l_j-1}\right),\delta\left(\mu_{i_j+1},\mu_{l_j+1}\right)\right]\longrightarrow\varepsilon^{+},\text{as } j\longrightarrow +\infty. \tag{4}$$

Definition 1 (see [12]). Let $\Upsilon\colon \Delta \longrightarrow N(\Delta)$ be a multivalued mapping and $\mu_0 \in \Delta$ be arbitrary and fixed. Define

$$\check{D}(\Upsilon,\mu_0)=\left\{\left(\mu_j\right)_{j\in N\cup\{0\}}\colon \mu_j\in\Upsilon\left(\mu_{j-1}\right),\text{for all } j\in N\right\}. \tag{5}$$

Each element of $\check{D}(\Upsilon,\mu_0)$ is called a dynamic process of Υ starting point μ_0. The dynamic process $(\mu_j)_{j\in N\cup\{0\}}$ onward be written as (μ_j).

Example 1 (see [12]). Let $\Delta = C([0,1])$ be a Banach space with a norm $\|\mu\| = \sup_{r\in[0,1]}|\mu(r)|$, $\mu\in\Delta$. Let $\Upsilon\colon \Delta\longrightarrow 2^{\Delta}$ be such that, for every $\mu\in\Delta$, $\Upsilon(\mu)$ is a collection of the functions

$$r\mapsto k\int_0^r \mu(t)\mathrm{d}t,\quad k\in[0,1], \tag{6}$$

that is,

$$(\Upsilon(\mu))(r)=\left\{k\int_0^r \mu(t)\mathrm{d}t\colon k\in[0,1]\right\},\mu\in\Delta, \tag{7}$$

and let $\mu_0(r) = r$, $r\in[0,1]$, then the sequence $(1/(j!(j+1)!)r^{j+1})$ is a dynamic process of Υ with starting point μ_0.

A mapping $\Upsilon\colon \Delta\longrightarrow R$ is said to be $\check{D}(\Upsilon,\mu_0)$-dynamic lower semicontinuous at $\mu\in\Delta$, if for every dynamic process $(\mu_j)\in\check{D}(\Upsilon,\mu_0)$ and for every subsequence $(\mu_{j(i)})$ of (μ_j) convergent to μ, we get $\Upsilon(\mu)\le\liminf_{i\longrightarrow\infty}\Upsilon((\mu_{j(i)}))$. If Υ is $\check{D}(\Upsilon,\mu_0)$-dynamic lower semicontinuous at each $\mu\in\Delta$, then Υ is said to be $\check{D}(\Upsilon,\mu_0)$-dynamic lower semicontinuous. If for every sequence $(\mu_j)\subset\Delta$ and $\mu\in\Delta$ such that $\mu_j\longrightarrow\mu$, we have $\Upsilon(\mu)\le\liminf_{j\longrightarrow\infty}\Upsilon(\mu_j)$, then Υ is known as lower semicontinuous.

As of now, Branciari [5] generalized the second well-known contraction of Banach contraction mappings is determined, i.e., let (Δ,δ) be a metric space and a mapping $\Upsilon\colon \Delta\longrightarrow\Delta$ such that

$$\int_0^{\delta(\Upsilon\mu_1,\Upsilon\mu_2)}\varphi(s)ds\ \le\beta\int_0^{\delta(\mu_1,\mu_2)}\varphi(s)ds \tag{8}$$

for all $\mu_1,\mu_2\in\Delta$, where $\beta\in(0,1)$, $\varphi\in\Phi$, and Φ is the class of all functions $\varphi\colon [0,+\infty)\longrightarrow[0,+\infty)$ which is Lebesgue integrable, summable on each compact subset of $[0,+\infty)$ and $\int_0^{\varepsilon}\varphi(s)\mathrm{d}s>0$ for all $\varepsilon>0$. Then, Υ has a fixed point.

The following lemmas are helpful for our main results. We shall also suppose that $\varphi\in\Phi$.

Lemma 3 (see [6]). *Let $(\mu_i)_{i\in N}$ be a nonnegative sequence in such a way that $\lim_{i\longrightarrow+\infty}\mu_i=\mu$. Then,*

$$\lim_{i\longrightarrow+\infty}\int_0^{\mu_i}\varphi(s)\mathrm{d}s=\int_0^{\mu}\varphi(s)\mathrm{d}s. \tag{9}$$

Lemma 4 (see [6]). *Let $(\mu_i)_{i\in N}$ be a nonnegative sequence. Then,*

$$\lim_{i\longrightarrow+\infty}\int_0^{\mu_i}\varphi(s)\mathrm{d}s=0\Leftrightarrow\lim_{i\longrightarrow+\infty}\mu_i=0. \tag{10}$$

In 2012, Wardowski [9] initiated the term of F-contraction and implemented on fixed point theorem related with F-contraction. So, with the intent that, he generalizes contraction theorem which is a purely altered from many past results in the literature frame.

Definition 2 (see [9]). Let $\Upsilon\colon \Delta\longrightarrow\Delta$ is called an $\mathscr{F}$-contraction on a metric space (Δ,δ), if there exist $\mathscr{F}\in\nabla_{\mathrm{F}}$ and $\tau\in R_+$ in such a way that, $\delta(\Upsilon\mu_1,\Upsilon\mu_2)>0$ implies

$$\tau+\mathscr{F}(\delta(\Upsilon\mu_1,\Upsilon\mu_2))\le\mathscr{F}(\delta(\mu_1,\mu_2)). \tag{11}$$

For each $\mu_1,\mu_2\in\Delta$, where ∇_{F} is the class of all functions $\mathscr{F}\colon R_+\longrightarrow R$ such that

($\mathscr{F}_i$) $\mu_1<\mu_2$ implies $\mathscr{F}(\mu_1)<\mathscr{F}(\mu_2)$ for all $\mu_1,\mu_2\in R_+$.

($\mathscr{F}_{ii}$) For each sequence $\{\mu_j\}$ of positive real numbers,

$$\lim_{j\longrightarrow\infty}\mu_j=0\text{ iff }\lim_{j\longrightarrow\infty}\mathscr{F}\left(\mu_j\right)=-\infty. \tag{12}$$

($\mathscr{F}_{\text{iii}}$) There is $k\in(0,1)$ in such a way that $\lim_{c\longrightarrow0^+}c^k\mathscr{F}(c)=0$.

From now, we present some well-defined examples of $\mathscr{F}$-contraction that are listed as follows:

($\mathscr{F}a$): $\mathscr{F}(\mu)=\ln\mu$

($\mathscr{F}b$): $\mathscr{F}(\mu)=\ln\mu+\mu$

($\mathscr{F}c$): $\mathscr{F}(\mu)=-1/\sqrt{\mu}$

($\mathscr{F}\ d$): $\mathscr{F}(\mu)=\ln(\mu^2+\mu)$

Owing to ($\mathscr{F}_i$) and (11), clearly, we conclude that every F-contraction Υ is a contractive mapping. Consequently, every F-contraction is a continuous mapping (see more [13]).

The main purpose of this manuscript is to introduce the new concept of dynamic iterative process $\check{D}(\Upsilon,\mu_0)$ based on F_I^C-integral contractions and prove some related multivalued fixed point results in the class of metric space. To approximate our main results by tangible examples are also determined. At the end, the work is completed by giving an application to Liouville–Caputo fractional differential equations.

2. Main Result

First, we give our main definition.

Definition 3. Let (Δ,δ) be a metric space, $\mu_0\in\Delta$, $\mathscr{F}\in\nabla_{\mathrm{F}}$ and $\varphi\in\Phi$. A set-valued map $\Upsilon\colon \Delta\longrightarrow CB(\Delta)$ is said to be F_I^C-integral contraction with respect to a dynamic process $(\mu_i)\in\check{D}(\Upsilon,\mu_0)$, if there exists $\tau\colon R_+\longrightarrow R_+$ such that

$$\widehat{H}(\Upsilon\mu_i, \Upsilon\mu_{i+1}) > 0 \Rightarrow \tau(U(\mu_{i-1}, \mu_i)) + \mathscr{F}\left(\int_0^{\widehat{H}(\Upsilon\mu_i, \Upsilon\mu_{i+1})} \varphi(s)\mathrm{d}s\right) \leq \mathscr{F} \cdot \left(\int_0^{U(\mu_{i-1}, \mu_i)} \varphi(s)\mathrm{d}s\right), \quad (13)$$

for all $i \in N$, where

$$U(\mu_{i-1}, \mu_i) = \max\left\{\delta(\mu_{i-1}, \mu_i), \check{D}(\mu_{i-1}, \Upsilon\mu_{i-1}), \check{D}(\mu_i, \Upsilon\mu_i), \cdot \frac{\check{D}(\mu_{i-1}, \Upsilon\mu_i) + \check{D}(\mu_i, \Upsilon\mu_{i-1})}{2}\right\}. \quad (14)$$

Remark 1. For the act of continuing our results, we consider only the dynamic processes $(\mu_i) \in \check{D}(\Upsilon, \mu_0)$ satisfying the following structure:

$$\delta(\mu_i, \mu_{i+1}) > 0 \Rightarrow \delta(\mu_{i-1}, \mu_i) > 0 \text{ for each } i \in N. \quad (15)$$

If the investigated process does not satisfy (15), then there is $i_0 \in N$ such that

$$\delta\left(\mu_{i_0}, \mu_{i_{0+1}}\right) > 0 \quad (16)$$

and

$$\delta\left(\mu_{i_{0-1}}, \mu_{i_0}\right) = 0. \quad (17)$$

Then, we get $\mu_{i_{0-1}} = \mu_{i_0} \in \Upsilon\mu_{i_{0-1}}$ which implies the existence of fixed point due to this consideration of dynamic process that satisfying (15) does not depreciate a generality of our approach.

Example 2. Let $\mathscr{F}: R_+ \longrightarrow R$ be defined by $\mathscr{F}(\mu) = \ln\mu$. Each set-valued F_I^C-integral contraction Υ on a metric space (Δ, δ) with respect to dynamic process $\check{D}(\Upsilon, \mu_0)$ assures that

$$\tau(U(\mu_{i-1}, \mu_i)) + \mathscr{F}\left(\int_0^{\widehat{H}(\Upsilon\mu_i, \Upsilon\mu_{i+1})} \varphi(s)\mathrm{d}s\right) \leq \mathscr{F}\left(\int_0^{U(\mu_{i-1}, \mu_i)} \varphi(s)\mathrm{d}s\right). \quad (18)$$

Upon setting, we have

$$\int_0^{\widehat{H}(\Upsilon\mu_i, \Upsilon\mu_{i+1})} \varphi(s)\mathrm{d}s \leq e^{-\tau(U(\mu_{i-1}, \mu_i))} \int_0^{U(\mu_{i-1}, \mu_i)} \varphi(s)\mathrm{d}s, \quad (19)$$

for all $i \in N$, $(\mu_i) \in \check{D}(\Upsilon, \mu_0)$, and $\Upsilon\mu_{i-1} \neq \Upsilon\mu_i$. In view of the above observations, clearly, for $(\mu_{i_0-1}), (\mu_{i_0}) \in \check{D}(\Upsilon, \mu_0)$ such that $\Upsilon\mu_{i_{0-1}} = \Upsilon\mu_{i_0}$, the following inequality also holds through $\check{D}(\Upsilon, \mu_0)$

$$\int_0^{\widehat{H}\left(\Upsilon\mu_{i_0-1}, \Upsilon\mu_{i_0}\right)} \varphi(s)\mathrm{d}s \leq e^{-\tau\left(U\left(\mu_{i_0-1}, \mu_{i_0}\right)\right)} \int_0^{U\left(\mu_{i_0-1}, \mu_{i_0}\right)} \varphi(s)\mathrm{d}s, \quad (20)$$

that is, Υ is a contraction.

Theorem 1. *Let (Δ, δ) be a complete metric space, $\mu_0 \in \Delta$ and $\Upsilon: \Delta \longrightarrow K(\Delta)$ be a set-valued F_I^C-integral contraction with respect to the dynamic process $(\mu_i) \in \check{D}(\Upsilon, \mu_0)$. Assume that*

Proof. In view of $(\mu_i) \in \check{D}(\Upsilon, \mu_0)$, if there exists $i_0 \in N$ such that $\mu_{i_0} = \mu_{i_0+1}$, then the existence of a fixed point is obvious. Therefore, if we let $\mu_i \notin \Upsilon\mu_i$, then $\check{D}(\mu_i, \Upsilon\mu_i) > 0$ for every $i \in N$. Using (15) and by Lemma 1, one writes

$$\mathscr{F}\left(\int_0^{\check{D}(\mu_i, \Upsilon\mu_i)} \varphi(s)\mathrm{d}s\right) \leq \mathscr{F}\left(\int_0^{\widehat{H}(\Upsilon\mu_i, \Upsilon\mu_{i+1})} \varphi(s)\mathrm{d}s\right), \quad (21)$$

$$\leq \mathscr{F}\left(\int_0^{U(\mu_{i-1}, \mu_i)} \varphi(s)\mathrm{d}s\right) - \tau(U(\mu_{i-1}, \mu_i)) = \mathscr{F}\left(\int_0^{\max}\left\{\begin{array}{c}\delta(\mu_{i-1}, \mu_i), \check{D}(\mu_{i-1}, \Upsilon\mu_{i-1}), \check{D}(\mu_i, \Upsilon\mu_i), \\ \dfrac{\check{D}(\mu_{i-1}, \Upsilon\mu_i) + \check{D}(\mu_i, \Upsilon\mu_{i-1})}{2}\end{array}\right\}\right) \quad (22)$$

Moreover, since $\Upsilon\mu_i$ is compact, we obtain $\mu_{i+1} \in \Upsilon\mu_i$ such that $\delta(\mu_i, \mu_{i+1}) = \check{D}(\mu_i, \Upsilon\mu_i)$. Using (21), we have

$$\mathscr{F}\left(\int_0^{\delta(\mu_i, \mu_{i+1})} \varphi(s)\mathrm{d}s\right) \leq \mathscr{F}\left(\int_0^{\widehat{H}(\Upsilon\mu_{i-1}, \Upsilon\mu_i)} \varphi(s)\mathrm{d}s\right) \leq \mathscr{F}\left(\int_0^{\delta(\mu_{i-1}, \mu_i)} \varphi(s)\mathrm{d}s\right) - \tau(\delta(\mu_{i-1}, \mu_i)) < \mathscr{F}\left(\int_0^{\delta(\mu_{i-1}, \mu_i)} \varphi(s)\mathrm{d}s\right). \quad (23)$$

In view of the above observations, $\{\delta(\mu_i, \mu_{i+1})\}$ is decreasing and hence convergent. We now show that $\lim_{i\longrightarrow\infty}\delta(\mu_i, \mu_{i+1}) = 0$. In the light of (D1), there exist $\sigma > 0$ and $i_0 \in N$ such that $\tau(\delta(\mu_{i-1}, \mu_i)) > \sigma$ for all $i > i_0$. So, we have

$$\begin{aligned}
\mathscr{F}\left(\int_0^{\delta(\mu_i,\mu_{i+1})} \varphi(s)\mathrm{d}s\right) &\le \mathscr{F}\left(\int_0^{\delta(\mu_{i-1},\mu_i)} \varphi(s)\mathrm{d}s\right) - \tau(\delta(\mu_{i-1}, \mu_i)) \\
&\le \mathscr{F}\left(\int_0^{\delta(\mu_{i-2},\mu_{i-1})} \varphi(s)\mathrm{d}s\right) - \tau(\delta(\mu_{i-2}, \mu_{i-1})) - \tau(\delta(\mu_{i-1}, \mu_i)) \\
&\vdots \\
&\le \mathscr{F}\left(\int_0^{\delta(\mu_0,\mu_1)} \varphi(s)\mathrm{d}s\right) - \tau(\delta(\mu_0, \mu_1)) - \cdots - \tau(\delta(\mu_{i-1}, \mu_i)) \\
&= \mathscr{F}\left(\int_0^{\delta(\mu_0,\mu_1)} \varphi(s)\mathrm{d}s\right) - \left(\tau(\delta(\mu_0, \mu_1)) + \cdots + \tau\left(\delta\left(\mu_{i_{0-1}}, \mu_{i_0}\right)\right)\right) \\
&\quad - \tau\left(\delta\left(\mu_{i_0}, \mu_{i_{0+1}}\right)\right)) + \cdots + \tau(\delta(\mu_{i-1}, \mu_i)) \\
&\le \mathscr{F}\left(\int_0^{\delta(\mu_0,\mu_1)} \varphi(s)\mathrm{d}s\right) - (i - i_0)\sigma.
\end{aligned} \tag{24}$$

Let us set $\lambda_i = \int_0^{\delta(\mu_i,\mu_{i+1})} \varphi(s)\mathrm{d}s > 0$ for $i = 0, 1, 2, \ldots$ and from (24), we see that $\lim_{i\longrightarrow\infty}\mathscr{F}(\lambda_i) = -\infty$. By means of $(\mathscr{F}_{\text{ii}})$, we have

$$\lim_{i\longrightarrow\infty} \lambda_i = 0. \tag{25}$$

Also, in the light of $(\mathscr{F}_{\text{iii}})$, there is $\alpha \in (0, 1)$ such that

$$\lim_{i\longrightarrow\infty} [\lambda_i]^\alpha \mathscr{F}[\lambda_i] = 0. \tag{26}$$

Furthermore, from (24), we can write for all $i > i_0$

$$\begin{aligned}
[\lambda_i]^\alpha \mathscr{F}[\lambda_i] - [\lambda_i]^\alpha \mathscr{F}[\lambda_0] &\le [\lambda_i]^\alpha (\mathscr{F}(\lambda_0) - (i - i_0)\sigma) \\
&\quad - [\lambda_i]^\alpha \mathscr{F}[\lambda_0] \\
&= -[\lambda_i]^\alpha (i - i_0)\sigma \le 0.
\end{aligned} \tag{27}$$

Taking limit as $i \longrightarrow \infty$ in (27) and using (26), we have

$$\lim_{i\longrightarrow\infty} i[\lambda_i]^\alpha = 0. \tag{28}$$

Let us perceive that, from (28), there is $i_1 \in \mathbb{N}$ such that $i[\lambda_i]^\alpha \le 1$ for all $i \ge i_1$. We have

$$\lambda_i \le \frac{1}{i^{1/\alpha}}. \tag{29}$$

Now, in order to show that $\{\mu_i\}$ is a Cauchy sequence, we consider $j_1, j_2 \in \mathbb{N}$ such that $j_1 > j_2 \ge i_1$. From (29) and by virtue of metric condition, we have

$$\begin{aligned}
&\int_0^{\delta\left(\mu_{j_1},\mu_{j_2}\right)} \varphi(s)\mathrm{d}s \\
&\quad \le \int_0^{\delta\left(\mu_{j_1},\mu_{j_1+1}\right)} \varphi(s)\mathrm{d}s \\
&\qquad + \int_0^{\delta\left(\mu_{j_1+1},\mu_{j_1+2}\right)} \varphi(s)\mathrm{d}s + \cdots + \int_0^{\delta\left(\mu_{j_2-1},\mu_{j_2}\right)} \varphi(s)\mathrm{d}s \\
&\quad = \lambda_{j_1} + \lambda_{j_1+1} + \cdots + \lambda_{j_2-1} \\
&\quad = \sum_{l=j_1}^{j_2-1} \lambda_l \le \sum_{l=j_1}^{\infty} \lambda_l \le \sum_{l=j_1}^{\infty} \frac{1}{l^{1/\alpha}}.
\end{aligned} \tag{30}$$

In the light of (30) and view of convergence of series $\sum_{l=j_1}^{\infty} 1/l^{1/\alpha}$, we see that $\int_0^{\delta(\mu_{j_1},\mu_{j_2})} \varphi(s)\mathrm{d}s \longrightarrow 0$. Hence, $\{\mu_i\}$ is Cauchy sequence in (Δ, δ). Furthermore, for the completeness of Δ, there is $\mu^* \in \Delta$ such that $\lim_{i\longrightarrow\infty}\mu_i = \mu^*$. Since Υ is compact, then we have $\Upsilon\mu_i \longrightarrow \Upsilon\mu^*$. By Lemma 1, one writes

$$\check{D}(\mu_i, \Upsilon\mu^*) \le \widehat{H}(\Upsilon\mu_{i-1}, \Upsilon\mu^*). \tag{31}$$

So, $\check{D}(\mu^*, \Upsilon\mu^*) = 0$ and $\mu^* \in \Upsilon\mu^*$. Suppose, on the contrary, $\mu^* \notin \Upsilon\mu^*$. Then, there exist $i_0 \in \mathbb{N}$ and subsequence $\{\mu_{i_k}\}$ of $\{\mu_i\}$ such that $\check{D}(\mu_{i_k+1}, \Upsilon\mu^*) > 0$ for each $i_k \ge i_0$ (otherwise, there is $i_1 \in \mathbb{N}$ such that $\mu_i \in \Upsilon\mu^*$ for every $i \ge i_1$, which yields that $\mu^* \in \Upsilon\mu^*$). By contractive condition, one writes

$$\mathcal{F}\left(\int_0^{\check{D}\left(\mu_{i_k+1},\Upsilon\mu^*\right)}\varphi(s)\mathrm{d}s\right) \leq \mathcal{F}\left(\int_0^{\widehat{H}\left(\Upsilon\mu_{i_k},\Upsilon\mu^*\right)}\varphi(s)\mathrm{d}s\right) \leq \mathcal{F}\left(\int_0^{U\left(\mu_{i_k},\mu^*\right)}\varphi(s)\mathrm{d}s\right) - \tau\left(U\left(\mu_{i_k},\mu^*\right)\right). \tag{32}$$

Upon letting $k \longrightarrow \infty$ in (32),

$$\mathcal{F}\left(\int_0^{\check{D}(\mu^*,\Upsilon\mu^*)}\varphi(s)\mathrm{d}s\right) \leq \mathcal{F}\left(\int_0^{\check{D}(\mu^*,\Upsilon\mu^*)}\varphi(s)\mathrm{d}s\right) - \tau\left(\check{D}(\mu^*,\mu^*)\right) < \mathcal{F}\left(\int_0^{\check{D}(\mu^*,\Upsilon\mu^*)}\varphi(s)\mathrm{d}s\right). \tag{33}$$

which is a contradiction. On the other hand, we see that the mapping $\Delta \ni \mu_i \mapsto \delta(\mu_i, \Upsilon\mu_i)$ is $\check{D}(\Upsilon,\mu_0)$-dynamic lower semicontinuous, we have

$$\int_0^{\check{D}(\mu^*,\Upsilon\mu^*)}\varphi(s)\mathrm{d}s \leq \liminf_{n\longrightarrow\infty}\int_0^{\check{D}\left(\mu_{i_k},\Upsilon\mu_{i_k}\right)}\varphi(s)\mathrm{d}s \leq \liminf_{n\longrightarrow\infty}\int_0^{\check{D}(\mu_i,\Upsilon\mu_i)}\varphi(s)\mathrm{d}s = 0 \tag{34}$$

and by virtue of closedness of $\Upsilon\mu^*$ implies that $\mu^* \in \Upsilon\mu^*$ which means that μ^* is a fixed point of Υ. □

Remark 2. If in Theorem 1, instead of the contractive condition (13), we assume the following condition

$$\widehat{H}(\Upsilon\mu_i,\Upsilon\mu_{i+1}) > 0 \Rightarrow \tau\left(U_j(\mu_{i-1},\mu_i)\right) + \mathcal{F}\left(\int_0^{\widehat{H}(\Upsilon\mu_i,\Upsilon\mu_{i+1})}\varphi(s)\mathrm{d}s\right) \leq \mathcal{F}\left(\int_0^{U_j(\mu_{i-1},\mu_i)}\varphi(s)\mathrm{d}s\right), \tag{35}$$

where $j \in \{1,2,3\}$ and

$$\begin{aligned} U_1(\mu_{i-1},\mu_i) &= \delta(\mu_{i-1},\mu_i),\\ U_2(\mu_{i-1},\mu_i) &= \max\{\delta(\mu_{i-1},\mu_i), \check{D}(\mu_{i-1},\Upsilon\mu_{i-1}), \check{D}(\mu_i,\Upsilon\mu_i)\},\\ U_3(\mu_{i-1},\mu_i) &= \max\left\{\delta(\mu_{i-1},\mu_i), \frac{\check{D}(\mu_{i-1},\Upsilon\mu_{i-1}) + \check{D}(\mu_i,\Upsilon\mu_i)}{2}, \cdot\frac{\check{D}(\mu_{i-1},\Upsilon\mu_i) + \check{D}(\mu_i,\Upsilon\mu_{i-1})}{2}\right\}, \end{aligned} \tag{36}$$

for all $i \in N$, $(\mu_i) \in \check{D}(\Upsilon,\mu_0)$, then there exists a fixed point of the mapping Υ with the assumptions (D1) and (D2) on Theorem 1.

Corollary 1. *Let (Δ,δ) be a complete metric space, $\mu_0 \in \Delta$, $\mathcal{F} \in \nabla_F$, $\varphi \in \Phi$, and $\Upsilon\colon \Delta \longrightarrow K(\Delta)$. Assume that there exists $\tau\colon R_+ \longrightarrow R_+$ such that*

$$\widehat{H}(\Upsilon\mu_i,\Upsilon\mu_{i+1}) > 0 \Rightarrow \tau(U(\mu_{i-1},\mu_i)) - \frac{1}{\int_0^{\widehat{H}(\Upsilon\mu_i,\Upsilon\mu_{i+1})}\varphi(s)\mathrm{d}s} \leq -\frac{1}{\int_0^{U(\mu_{i-1},\mu_i)}\varphi(s)\mathrm{d}s}, \tag{37}$$

for all $i \in N$, $\mu_i \in \check{D}(\Upsilon,\mu_0)$, where

$$U(\mu_{i-1},\mu_i) = \max\left\{\delta(\mu_{i-1},\mu_i), \check{D}(\mu_{i-1},\Upsilon\mu_{i-1}), \check{D}(\mu_i,\Upsilon\mu_i), \cdot\frac{\check{D}(\mu_{i-1},\Upsilon\mu_i) + \check{D}(\mu_i,\Upsilon\mu_{i-1})}{2}\right\}. \tag{38}$$

Then, there exists a fixed point of the mapping Υ with the assumptions (D1) and (D2) on Theorem 1.

Proof. If we choose $\mathcal{F}(\mu) = -1/\mu$, the proof follows from Theorem 1. □

Corollary 2. *Let (Δ,δ) be a complete metric space, $\mu_0 \in \Delta$, $\mathcal{F} \in \nabla_F$, $\varphi \in \Phi$, and $\Upsilon\colon \Delta \longrightarrow K(\Delta)$. Assume that there exists $\tau\colon R_+ \longrightarrow R_+$ such that*

$$\widehat{H}(\Upsilon\mu_i,\Upsilon\mu_{i+1}) > 0 \Rightarrow \tau(U(\mu_{i-1},\mu_i)) + \frac{1}{1-\exp\int_0^{\widehat{H}(\Upsilon\mu_i,\Upsilon\mu_{i+1})}\varphi(s)\mathrm{d}s} \leq \frac{1}{1-\exp\int_0^{U(\mu_{i-1},\mu_i)}\varphi(s)\mathrm{d}s}, \tag{39}$$

for all $i \in N$, $\mu_i \in \check{D}(\Upsilon,\mu_0)$, where

$$U(\mu_{i-1},\mu_i) = \max\left\{\delta(\mu_{i-1},\mu_i), \check{D}(\mu_{i-1},\Upsilon\mu_{i-1}), \check{D}(\mu_i,\Upsilon\mu_i), \frac{\check{D}(\mu_{i-1},\Upsilon\mu_i) + \check{D}(\mu_i,\Upsilon\mu_{i-1})}{2}\right\}. \tag{40}$$

Then, there exists a fixed point of the mapping Υ with the assumptions (D1) and (D2) on Theorem 1.

Proof. If we choose $\mathcal{F}(\mu) = 1/(1-\exp(\mu))$, the proof follows from Theorem 1.

The direct consequence of Theorem 1 for single-valued maps is the following. □

Corollary 3. *Let (Δ,δ) be a complete metric space, $\mu_0 \in \Delta$, $\mathcal{F} \in \nabla_F$, $\varphi \in \Phi$, and $\Upsilon\colon \Delta \longrightarrow \Delta$. Assume that there exists $\tau\colon R_+ \longrightarrow R_+$ such that $\delta(\Upsilon^i\mu_0, \Upsilon^{i+1}\mu_0) > 0$ implies*

$$\tau\left(\delta\left(\Upsilon^{i-1}\mu_0, \Upsilon^{i}\mu_0\right)\right) + \mathscr{F}\left(\int_0^{\delta\left(\Upsilon^{i}\mu_0, \Upsilon^{i+1}\mu_0\right)} \varphi(s)ds\right) \leq \mathscr{F}\left(\int_0^{\delta\left(\Upsilon^{i-1}\mu_0, \Upsilon^{i}\mu_0\right)} \varphi(s)ds\right), \quad (41)$$

for all $i \in N$ *and* $\liminf_{k\longrightarrow l^+}\tau(k) > 0$ *for each* $l \geq 0$. *Suppose also that a mapping* $\Delta \ni \mu \mapsto \delta(\mu, \Upsilon\mu)$ *is* $\check{D}(\Upsilon, \mu_0)$*-dynamic lower semicontinuous. Then,* Υ *has a fixed point.*

Corollary 4. *Let* (Δ, δ) *be a complete metric space,* $\mathscr{F} \in \nabla_F$, $\varphi \in \Phi$, *and* $\Upsilon: \Delta \longrightarrow \Delta$. *Assume that there exists* $\tau: R_+ \longrightarrow R_+$ *such that* $\delta(\Upsilon\mu, \Upsilon^2\mu) > 0$ *implies*

$$\tau(\delta(\mu, \Upsilon\mu)) + \mathscr{F}\left(\int_0^{\delta(\Upsilon\mu, \Upsilon^2\mu)} \varphi(s)ds\right) \leq \mathscr{F}\left(\int_0^{\delta(\mu, \Upsilon\mu)} \varphi(s)ds\right), \quad (42)$$

for all $\mu \in \Delta$ *and* $\liminf_{k\longrightarrow l^+}\tau(k) > 0$ *for each* $l \geq 0$. *Suppose also that a mapping* $\Delta \ni \mu \mapsto \delta(\mu, \Upsilon\mu)$ *is lower semi-continuous. Then,* Υ *has a fixed point.*

Example 3. Let $\Delta = [0, +\infty)$ with the usual metric, Δ constitutes a complete metric space. Consider a mapping $\Upsilon: \Delta \longrightarrow K(\Delta)$ by $\Upsilon(\mu) = [0, \mu/2]$, $\mu > 0$ and $\tau: R_+ \longrightarrow R_+$ by

$$\tau(\mu) = \begin{cases} -\ln\mu, \ \mu \in \left(0, \dfrac{1}{2}\right) \\ \ln 2, \ \mu \in \left[\dfrac{1}{2}, \infty\right) \end{cases} \quad (43)$$

Define dynamic iterative process $\check{D}(\Upsilon, \mu_0)$: a sequence $\{\mu_i\}$ is given by $\mu_i = \mu_0 g^{i-1}$ for all $i \in N$ with initial point $\mu_0 = 2$ and $g = 1/2$ such that

$i \geq 2$	$\mu_i = i_0 g^{i-1}$	$\in$	$\Upsilon\mu_{i-1} = [0, \mu/2]$
$\mu_{i=2}$	1	–	$\Upsilon\mu_{i=1} = [0, 1]$
$\mu_{i=3}$	1/2	–	$\Upsilon\mu_{i=2} = [0, 1/2]$
$\mu_{i=4}$	1/4	–	$\Upsilon\mu_{i=3} = [0, 1/4]$
$\mu_{i=5}$	1/8	–	$\Upsilon\mu_{i=4} = [0,1/8]$

Continuing the above iterative process, we see that

$$\check{D}(\Upsilon, \mu_0) = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\} \quad (44)$$

is a dynamic iterative process of Υ starting from the point $\mu_0 = 2$. Setting $\varphi(s) = 1$ for all $s \in R$ and $\mathscr{F}(s) = \ln(s)$. For $\mu_i \in \check{D}(\Upsilon, \mu_0)$ and $\widehat{H}(\Upsilon\mu_i, \Upsilon\mu_{i+1}) > 0$, we have

$$\begin{cases} e^{\tau\left(\left|\mu_{i-1}-\mu_i\right|\right) + \mathscr{F}\left(\int_0^{\frac{\left|\mu_{i-1}-\mu_i\right|}{2}} \varphi(s)ds\right)} \leq e^{\mathscr{F}\left(\int_0^{\left|\mu_{i-1}-\mu_i\right|} \varphi(s)ds\right)} \\ e^{\tau\left(\left|\mu_{i-1}-\mu_i\right|\right) + \ln\left(\int_0^{\frac{\left|\mu_{i-1}-\mu_i\right|}{2}} \varphi(s)ds\right)} \leq e^{\ln\left(\int_0^{\left|\mu_{i-1}-\mu_i\right|} \varphi(s)ds\right)} \\ e^{\tau\left(\left|\mu_{i-1}-\mu_i\right|\right)} e^{\ln\left(\int_0^{\frac{\left|\mu_{i-1}-\mu_i\right|}{2}} \varphi(s)ds\right)} \leq e^{\ln\left(\int_0^{\left|\mu_{i-1}-\mu_i\right|} \varphi(s)ds\right)} \\ e^{\tau\left(\left|\mu_{i-1}-\mu_i\right|\right)} \int_0^{\frac{\left|\mu_{i-1}-\mu_i\right|}{2}} \varphi(s)ds \leq \int_0^{\left|\mu_{i-1}-\mu_i\right|} \varphi(s)ds \\ \dfrac{\left|\mu_{i-1}-\mu_i\right|}{2} \leq e^{-\tau\left(\left|\mu_{i-1}-\mu_i\right|\right)}\left|\mu_{i-1}-\mu_i\right| \end{cases}, \quad (45)$$

and so

$$\tau\left(U\left(\mu_{i-1},\mu_i\right)\right)+\mathscr{F}\left(\int_0^{\widehat{H}\left(\Upsilon\mu_i,\Upsilon\mu_{i+1}\right)}\varphi(s)\mathrm{d}s\right)\leq\mathscr{F}\left(\int_0^{U\left(\mu_{i-1},\mu_i\right)}\varphi(s)\mathrm{d}s\right). \tag{46}$$

Hence, all the required hypotheses of Theorem 1 are satisfied and hence 0 is a fixed point of Υ.

3. An Application

In this frame of study, we deal with some new aspects of Liouville–Caputo fractional differential equations in module of complete metric space. Several earlier developments on fixed point theory and its applications involving fractional calculus can be found in [14].

Define the Liouville–Caputo fractional differential equations based on order κ ($\check{D}_{(c,\kappa)}$) by

$$\check{D}_{(c,\kappa)}(\alpha(g))=\frac{1}{\Gamma(i-\kappa)}\int_0^g(g-t)^{i-\kappa-1}\alpha^{(i)}(t)\mathrm{d}t, \tag{47}$$

where $i-1<\kappa<i$, $i=[\kappa]+1$, $\alpha\in C^i([0,+\infty])$, and the collection $[\kappa]$ represents positive real number and Γ represents the Gamma function. Let $\Delta{:}\ =C(I,R)$ be the space of all continuous real-valued functions on I. And, complete metric space $\delta_\varsigma{:}\ \Delta\times\Delta\longrightarrow[0,+\infty)$ be given by

$$\delta_\varsigma(\varepsilon_1,\varepsilon_2)=\sup_{a\in I}\left|\varepsilon_1(a)-\varepsilon_2(a)\right|. \tag{48}$$

Now, consider the following fractional differential equations and its integral boundary valued problem:

$$\check{D}_{(c,\kappa)}(\beta(g))=L(g,\beta(g)), \tag{49}$$

where $g\in(0,1)$, $\kappa\in(1,2]$ and

$$\begin{cases}\beta(0)=0,\\ \beta(1)=\int_0^\vartheta\beta(g)\mathrm{d}g,\ \vartheta\in(0,1),\end{cases} \tag{50}$$

where $I=[0,1]$, $\beta\in C(I,R)$ and $L{:}\ I\times R\longrightarrow R$ be a continuous function. Let $P{:}\ \Delta\longrightarrow\Delta$ be defined by

$$\mathrm{P}v(r)=\begin{cases}\dfrac{1}{\Gamma(\kappa)}\displaystyle\int_0^g(g-t)^{\kappa-1}L(t,v(t))\mathrm{d}t\\ -\dfrac{2g}{\left(2-\vartheta^2\right)\Gamma(\kappa)}\displaystyle\int_0^1(1-t)^{\kappa-1}L(t,v(t))\mathrm{d}t\\ +\dfrac{2g}{\left(2-\vartheta^2\right)\Gamma(\kappa)}\displaystyle\int_0^\vartheta\left(\int_0^{g_1}(g_1-t_1)^{\kappa-1}L(t_1,v(t_1))\mathrm{d}t_1\right)\mathrm{d}t\end{cases}, \tag{51}$$

for $v\in\Delta$ and $g\in[0,1]$. Now, we start the main result of this section.

Theorem 2. *Let $L{:}\ I\times R\longrightarrow R$ be a continuous function, nondecreasing on second variable and there is a nonconstant function τ such that $\varepsilon_i\in\check{D}_\varsigma(\Upsilon,\varepsilon_0)$ and $g\in[0,1]$ implies*

$$\left|P\varepsilon_{i-1}(r)-P\varepsilon_i(r)\right|\leq\Omega\frac{U\left(\varepsilon_{i-1},\varepsilon_i\right)(r)}{\left(1+\tau\sqrt{\max_{g\in I}U\left(\varepsilon_{i-1},\varepsilon_i\right)(r)}\right)^2}, \tag{52}$$

where $\Omega=(2\kappa-1)(\Gamma(\kappa+1))/2(5\kappa+2)$ and

$$U\left(\varepsilon_{i-1},\varepsilon_i\right)(r)=\max\left\{\begin{array}{c}\left|\varepsilon_{i-1}(r)-\varepsilon_i(r)\right|,\left|\varepsilon_{i-1}(r)-\Upsilon\varepsilon_{i-1}(r)\right|,\left|\varepsilon_i(r)-\Upsilon\varepsilon_i(r)\right|,\\ \dfrac{\left|\varepsilon_{i-1}(r)-\Upsilon\varepsilon_i(r)\right|+\left|\varepsilon_i(r)-\Upsilon\varepsilon_{i-1}(r)\right|}{2}\end{array}\right\}. \tag{53}$$

Then, equations (49) and (50) has at least one solution on Δ.

Proof. For every $g\in I$ and owing to operator $P{:}\ \Delta\longrightarrow\Delta$, one writes

Upon setting, we see that

In the light of above observation, we have

which implies that

$$\left|P\varepsilon_{i-1}(r)-P\varepsilon_i(r)\right|\leq\frac{U\left(\varepsilon_{i-1},\varepsilon_i\right)(r)}{\left(1+\tau\sqrt{\max_{g\in I}U\left(\varepsilon_{i-1},\varepsilon_i\right)(r)}\right)^2}. \tag{54}$$

By above virtue, we have

$$\delta_\varsigma\left(P\varepsilon_{i-1}(r)-P\varepsilon_i(r)\right)=\sup_{a\in I}\left|P\varepsilon_{i-1}(r)-P\varepsilon_i(r)\right|\leq\frac{U\left(\varepsilon_{i-1},\varepsilon_i\right)(r)}{\left(1+\tau\sqrt{\max_{g\in I}U\left(\varepsilon_{i-1},\varepsilon_i\right)(r)}\right)^2}. \tag{55}$$

Furthermore, by contractive condition (13) upon setting of $\varphi(s)=1$ for all $s\in R$ and $\mathscr{F}(s)=-1/\sqrt{s}$, we have

$$\widehat{H}\left(\Upsilon\varepsilon_i,\Upsilon\varepsilon_{i+1}\right)>0\Rightarrow\tau\left(U\left(\varepsilon_{i-1},\varepsilon_i\right)\right)+\mathscr{F}\left(\int_0^{\widehat{H}\left(\Upsilon\varepsilon_i,\Upsilon\varepsilon_{i+1}\right)}\varphi(s)\mathrm{d}s\right)\leq\mathscr{F}\left(\int_0^{U\left(\varepsilon_{i-1},\varepsilon_i\right)}\varphi(s)\mathrm{d}s\right), \tag{56}$$

for all $i \in N$, $\varepsilon_i \in \check{D}_{\varsigma}(\Upsilon, \varepsilon_0)$ and for each given $\epsilon > 0$ such that $\int_0^{\epsilon} \varphi(s)ds > 0$. Thus, all the required hypotheses of Theorem 1 are satisfied, and hence equations (49) and (50) has at least one solution on Δ. □

Example 4. Based upon the Liouville–Caputo fractional differential equations based on order $\kappa(\check{D}_{(c,\kappa)})$. Let us consider the following integral boundary-value problem:

$$\check{D}_{\left(c,\frac{3}{2}\right)}(\beta(g)) = \frac{1}{(g+3)^2}\frac{|\beta(g)|}{1+|\beta(g)|} \tag{57}$$

and

$$\begin{cases} \beta(0) = 0, \\ \beta(1) = \int_0^{3/4} \beta(g)dg, \ \vartheta \in (0,1), \end{cases} \tag{58}$$

where $\kappa = 3/2$, $\vartheta = 34$, and $L(t, v(t)) = 1/(g+3)^2|\alpha(g)|/1+|\alpha(g)|$. So, the above setting is an example of equations (49) and (50). Hence, here is clearly the pair of equations (57) and (58) has at least one solution.

4. Conclusions

In this paper, we have investigated the preexisting results of fixed point for set-valued mappings rather than the conventional mappings. Based upon a Wardowski integral and with a nonnegative Lebesque integrable mapping, we have transformed the conventional theorems of fixed point into the module of F_I^C. Instead of the traditional Picard sequence, the dynamic process $\check{D}(\Upsilon, \mu_0)$ is adopted to iterate the fixed point. Afterwards, the results have been explained by rendering concrete examples, and some foremost corollaries have been deduced from the prime results. Also, we provide illustrative applications to Liouville–Caputo fractional differential equations.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All authors contributed equally in writing this article. All authors read and approved the final manuscript.

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