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Research Article

Multiple Quasisynchronization of Uncertain Fractional-Order Delayed Neural Networks by Impulsive Control Mechanism

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We study the dynamical behavior of multiple quasi-synchronization of a type of fractional-order coupled neural networks (FCNNs) with delay and uncertain parameters. By utilizing the pinned pulse control strategy technique, we establish a new pulse controller, which realizes the multiple quasisynchronization of the system. Furthermore, we derive some new criteria of multiple quasisynchronization by using the comparison principle and mathematical analysis. Eventually, simulations are carried out with two examples to explicate the effectiveness of the conclusions.

1. Introduction

Fractional-order calculus is related to model memory, complexity, and heritability, so it has advantages over integer calculus (see [1, 2]). In many actual questions, we generally first consider the fractional dynamic system because it can better descript the actual problem than the integral order dynamic system. There have been many reports on fractional-order system dynamics. It plays an extremely significant effect in the modeling of engineering system, power system, and physical system (see [3, 4]). In Reference [3], Bao and Cao combined Caputo derivatives and fractional calculus inequalities and sufficiency condition for projection synchronization of fractional memristor-based neural networks (FMNNs) was theoretically derived. Xu et al. in Reference [5] studied a new fractional Hopfield neural network chaotic system and its application in image encryption. Li et al. in Reference [4] studied the application of neural network fractional-order PID in the control of piezoelectric stacks.

Parameter uncertainty is caused by incomplete understanding of some knowledge of mathematical model, such as constitutive law and empirical quantity (see [6]). In various engineering discipline systems, the model parameters studied are often uncertain, so the parameter uncertainty needs to be considered when facing the actual system. Fortunately, in recent years, many scholars have considered the parameter uncertainty in the model. In Reference [7], the author designs an appropriate event triggering mechanism and controller to ensure the stability of randomly nonlinearity system of time lag with uncertain parameters. However, with the continuous maturity of technology, the task of designing a good controller for an uncertain fractional-order neural network so that the network can achieve the desired effect is still very arduous, and many problems need to be further studied.

Synchronization is an extremely important dynamic behavior in complex dynamic networks, which has a wide range of applications in many fields. Therefore, many researchers have studied it (see [8]). Moreover, the synchronization behavior of fractional-order coupled neural networks (FCNNs) is also discussed. For example, Xu et al. designed a suitable controller in Reference [9], so that FCNNs with time-variable delays could realize the synchronization behavior in a finite time. Chen et al. [10] investigated the synchronization of FMNNs with time lag. It is a well-known fact that time delay often exists in plenty of complex networks; hence, it is very significant to premeditate time delay when studying FCNNs.

In general, the coupled neural network is not synchronized without external force interference, so it is necessary to develop a controller to make it synchronize. Some scholars also use various control technologies, for example, impulse control, adaptive control, pinning control, and feedback control to achieve synchronization. However, if there are too many nodes in the network, the cost of applying controller to each node is too high and difficult to implement. Therefore, it is possible to try to control the network by only controlling the fixed part of the time and some nodes, so as to arrive the purpose of reducing the control cost. But, few authors have applied the pinning impulse control project to NNs (see Reference [11]). In Reference [11], Wang et al. theoretically derived a few sufficient conditions for pinning synchronization and robust synchronization of FCNNs through the pinning control strategy. Also, some networks can only achieve quasi-synchronization due to external and internal interference, and there are few studies on quasisynchronization of NNs with couple (see [12, 13]). In Reference [12], Feng et al., based on the matrix-related knowledge theory and Lyapunov functional method, derived several simple sufficient optimality conditions for quasisynchronization of coupled memristor NNs theoretically, and a suitable controller is constructed to ensure the quasisynchronization of such networks. In Reference [13], Lv et al. introduced a type of activation function and a few sufficient conditions to guarantee that each subnetwork in the time-delay coupled neural network has multiple equilibrium states and made the network achieve dynamic and static multisynchronization by constructing an appropriate impulse controller and Lyapunov function. Based on the abovementioned phenomenon, this paper will research the multiple quasi-synchronization issue of FCNNs with uncertain parameters and delay by pinning pulse control method.

For as much as the above discussion, the major dedications of this article involve the following: (1) The multiple quasisynchronization problem of FCNNs with uncertain parameters and time delays is studied. (2) The concept of multiple quasisynchronization is proposed. (3) Aiming at the problem of multiple quasisynchronization, a new method combining pinning and pulse control is proposed.

The rest of the main content of this article is as follows: Section 2 mainly describes the prerequisites and models required in this article. The primary contribution is in Section 3. Section 4 gives two examples that demonstrate the validity of the conclusion. Finally, Section 5 gives the main conclusions of this article.

2. Preliminary Knowledge and Model Description

2.1. Fraction-Order Calculus. Firstly, existing definitions of fraction-order calculus are given, which can be seen in Reference [14], that are needed later.

Define the Gamma function $\Gamma(\cdot)$ as below:

$$\Gamma(p) = \int_{t_0}^{+\infty} t^{p-1}\exp\{-t\}\mathrm{d}t, \tag{1}$$

where $p > 0$.

Define the Caputo fractional derivative ${}^{c}D_{t_0,t}^{p} g(\cdot)$ of the function $g(t)$ as below:

$${}^{c}D_{t_0,t}^{p} g(t) = \frac{1}{\Gamma(n-p)}\int_{t_0}^{t}\frac{g^{(n)}(s)}{(t-s)^{p-n+1}}\mathrm{d}s, \tag{2}$$

where t_0 is the initial time, $t \geq t_0$, p is the order, $n-1 < p < n, n \in Z^{+}$.

Define the fractional integral $I_{t_0,t}^{p} g(\cdot)$ of the function $g(t)$ as below:

$$I_{t_0,t}^{p} g(t) = \frac{1}{\Gamma(p)}\int_{t_0}^{t}(t-s)^{p-1}g(s)\mathrm{d}s, \tag{3}$$

where t_0 is the initial time, $t \geq t_0$.

Define the Mittag-Leffler function with single parameter $E_p(\cdot)$ as below:

$$E_p(s) = \sum_{k=0}^{+\infty}\frac{s^k}{\Gamma(kp+1)}, \tag{4}$$

where $p > 0, s$ is a complex number.

Define the Mittag-Leffler function with double parameters $E_{p,\overline{p}}(\cdot)$ as below:

$$E_{p,\overline{p}}(s) = \sum_{k=0}^{+\infty}\frac{s^k}{\Gamma(kp+\overline{q})}, \tag{5}$$

where $p > 0, \overline{p} > 0, s$ is a complex number.

2.2. Model Description. A collection that makes Z^{+} a positive integer. The superscript T represents the transpose, and $\#\mathfrak{J}$ is an element in the finite collection $\mathfrak{J}$. Denote R^n is the set of n-dimensional real-valued vectors. R^{+} is the group of fixed non-negative numbers. For an arbitrary vector $a \in R^n$ and the existence of a constant $\sigma_0 > 0$, we record $\mathscr{M}(a,\sigma_0) = \{x\|x-a\| < \sigma_0\}$ as a set of vectors, where the distance between x and a is less than σ_0. The set of $n\times n$ real matrices is written as $R^{n\times n}$. If a real matrix $X > 0$, then X is a positive definite matrix. $A\otimes B$ represents the Kronecker product of matrices A and B. For any matrix A, $\lambda_{\min}(A), \lambda_{\max}(A)$ denotes its minimum eigenvalue and maximum eigenvalue, respectively, and the norm of A is defined as $\|A\| = (\lambda_{\max}(A^TA))^{(1/2)}$. In this article, we regard the following FCNNs with N same nodes, uncertainties, and time delays.

$$\begin{aligned}{}^{c}D_{t_0,t}^{\alpha} x_i(t) = &-(\widehat{P}+\Delta\mathscr{P}(t))x_i(t) + (\widehat{Q}+\Delta\mathscr{Q}(t))f_i(x_i(t)) \\ &+ (\widehat{R}+\Delta\mathscr{R}(t))f_i(x_i(t-\tau)) \\ &+ \sum_{j=1}^{N}\widehat{G}_{ij}\Gamma x_j(t) + J,\end{aligned} \tag{6}$$

where $i = 1, 2, \ldots, N$, and $N \geq 2$ represents the quantity of subnetworks; τ represents the time delay in transmission; $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T$ is the state vector of i-th neuron; $\Delta\mathscr{P}(t), \Delta\mathscr{Q}(t), \Delta\mathscr{R}(t)$ are the norm-bounded parametric uncertainties; $\widehat{P}$ is a diagonal matrix that expresses the self-feedback item of the j-th network, in which the diagonal elements are $p_1, p_2, \ldots, p_n, p_i > 0$. $\widehat{Q} = (Q_{ij})_{n\times n}$ is the connection weight matrices and $\widehat{R} = (r_{ij})_{n\times n}$ is the time lag join matrices, where $i, j = 1, 2, \ldots, n$. $f_i(x_i(t))$ is the activation function; $\widehat{G} = (\widehat{G}_{ij})_{N\times N}$ is the coupled matrix, when there exists a connection among the i-th node with the j-th node, $i \neq j, \widehat{G}_{ij} \neq 0$, if not, $\widehat{G}_{ij} = 0$, in which the diagonal elements are defined by $\widehat{G}_{ii} = -\sum_{j=1,j\neq i}^{N} \widehat{G}_{ij}$; $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is the internal coupling matrix; $J = (J_1, J_2, \ldots, J_N)$ is the input vector.

Next, we give several basic assumptions.

(A1) The activation function $f_i(\cdot)$ is continuous, for any vector x, y, exists $\mathscr{L}_i > 0$, and the following formula holds:

$$\left|f_i(x) - f_i(y)\right| \leq \mathscr{L}_i |x - y|. \tag{7}$$

(A2)

$$\% \ \Delta\mathscr{P} = \widehat{\mathscr{A}}_1 \widehat{\mathscr{C}}_1(t) \widehat{\mathscr{B}}_1, \Delta\mathscr{Q} = \widehat{\mathscr{A}}_2 \widehat{\mathscr{C}}_2(t) \widehat{\mathscr{B}}_2, \Delta\mathscr{R} = \widehat{\mathscr{A}}_3 \widehat{\mathscr{C}}_3(t) \widehat{\mathscr{B}}_3, \tag{8}$$

where $\widehat{\mathscr{A}}_i, \widehat{\mathscr{B}}_i (i = 1, 2, 3)$ are constant matrices with the corresponding matching dimensions and $\widehat{\mathscr{C}}_i(t) (i = 1, 2, 3)$ is an indeterminate matrix, where $\widehat{\mathscr{C}}_i^T(t)\widehat{\mathscr{C}}_i(t) \leq I$ (I is the unity matrix with the corresponding matching dimensions).

Remark 1. $\#D_k = \kappa_k$ means that the group D_k has κ_k nodes and $\kappa_k \neq 0$.

For any initial state $x(t) = (x_1^T(t), x_2^T(t), \ldots, x_N^T(t))^T$ where $x_i(t) \in C([-\tau, 0], R), i = 1, 2, \ldots, N$, for any given initial value condition, there is a solution $s(t)$, if all the node trajectories in the network satisfy the formula $\lim_{t\longrightarrow\infty}\left\|x_i(t) - s(t)\right\| = 0, i = 1, 2, \ldots, N$, then this network is called complete synchronization. Furthermore, if the margin of error $\sigma > 0$, exists $T > 0$, for all $x(t)$ and $\forall t > T$, $\|x_i(t) - s(t)\| < \sigma$ holds, then this network is called uniformly quasisynchronized.

Definition 1 (see [15]). For an arbitrary complex network with N nodes, $\{D_1, D_2, \ldots, D_m\}$ is a set of disjoint nodes, that is, $\sum_{k=1}^{m} D_k = \{1, 2, \ldots, N\}$, $D_k = \{l_{k1}, l_{k2}, \ldots\}$, $D_k \cap D_u = \varnothing$ for $k \neq u$. The network is called multiple quasisynchronization with the error vector $\delta = \{\delta_1, \delta_2, \ldots, \delta_m\}^T > 0$ under any initial value conditions, if there exist a series of reference solutions $\{s_1(t), s_2(t), \ldots, s_m(t)\}$ and for any constant $\kappa > 0$ small enough, T exists, for $\forall t > T$, the nodes $x_i(t) \in \mathscr{M}(s_k(t), \sigma_k), i \in D_k$ holds, in which $\mathscr{M}(s_k(t), \sigma_k) \neq \mathscr{M}(s_u(t), \overline{\sigma}_u), u \neq k$.

Remark 2. It can be seen from Definition 1 that $s_k(t)$ is the reference trajectory for all nodes in set $\#D_k$.

The target trajectory $s_{\bar{i}}(t)$ satisfies the following formula:

$$\begin{aligned} {}^{c}D_{t_0,t}^{\alpha} s_{\bar{i}}(t) = {} & -(\widehat{P} + \Delta\mathscr{P}(t)) s_{\bar{i}}(t) + (\widehat{Q} + \Delta\mathscr{Q}(t)) f_{\bar{i}}\left(s_{\bar{i}}(t)\right) \\ & + (\widehat{R} + \Delta\mathscr{R}(t)) f_{\bar{i}}\left(s_{\bar{i}}(t - \tau)\right) + J, \end{aligned} \tag{9}$$

where if $x_i(t) \in \mathscr{M}(s_k(t), \sigma_k)$, then $s_{\bar{i}}(t) = s_k(t)$, $i = 1, 2, \ldots, N, k = 1, 2, \ldots, m$.

Now let's note $\overline{\varepsilon}_i(t) = x_i(t) - s_{\bar{i}}(t)$ is the error signal, where $i = 1, 2, \ldots, N$, and devise a new pinned pulse controller as shown below:

$$u_i(t) = \begin{cases} \sum_{h=1}^{+\infty} \theta_k \overline{\varepsilon}_i(t) \delta(t - t_h), & i \in \mathfrak{I}_k(t_h), \#\mathfrak{I}_k(t_h) = \omega_k, \\ 0, & i \notin \mathfrak{I}_k(t_h), \end{cases} \tag{10}$$

where $\delta(\cdot)$ and θ_k represents the Dirac impulsive function and impulsive gain, respectively, and $t_h (h = 0, 1, 2, \ldots)$ indicates the instant of the pulse that satisfies $0 = t_1 < t_2 < \cdots < t_h < \cdots, \lim_{t_h\longrightarrow+\infty} t_h = +\infty$. The node set on $t = t_h$ is represented by $\sum_{k=1}^{m} \mathfrak{I}_k(t_h) = \{\mathfrak{I}_1(t_h), \mathfrak{I}_2(t_h), \ldots, \mathfrak{I}_m(t_h)\} \subset \{D_1, D_2, \ldots, D_m\} \subset \{1, 2, \ldots, N\}$, and make $0 < \omega_k \leq \kappa_k, k = 1, 2, \ldots, m$, namely, $\mathfrak{I}_k(t_h)$ is a subset of D_k, and $\mathfrak{I}_k(t_h)$ represents the set of pinned nodes at $t = t_h$. Assume the error vector $\overline{\varepsilon}_{i1} \geq \overline{\varepsilon}_{i2} \geq \cdots \geq \overline{\varepsilon}_{in}$. Under the pulse controller (10), the system of errors can be described:

$$\begin{cases} {}^{c}D_{t_0,t}^{\alpha} \overline{\varepsilon}_i(t) = -(\widehat{P} + \Delta\mathscr{P}(t)) \overline{\varepsilon}_i(t) + (\widehat{Q} + \Delta\mathscr{Q}(t)) \widetilde{f}_i(\overline{\varepsilon}_i(t)) \\ +(\widehat{R} + \Delta\mathscr{R}(t)) \widetilde{f}_i(\overline{\varepsilon}_i(t - \tau)) + \sum_{j=1}^{N} \widehat{G}_{ij} \Gamma \overline{\varepsilon}_i(t), t \neq t_h \\ \overline{\varepsilon}_i(t_h^+) = (1 + \theta_k) \overline{\varepsilon}_i(t_h^-), i \in \sum_{k=1}^{m} \mathfrak{I}_k(t_h), \\ \overline{\varepsilon}_i(t_h^+) = \overline{\varepsilon}_i(t_h^-), i \notin \sum_{k=1}^{m} \mathfrak{I}_k(t_h), \end{cases} \tag{11}$$

where $h=0,1,2,\ldots,\overline{\varepsilon}_i(t_h^+)=\lim_{t\longrightarrow t_h^+}\overline{\varepsilon}_i(t),\overline{\varepsilon}_i(t_h^-)=\lim_{t\longrightarrow t_h^-}\overline{\varepsilon}_i(t_h),\widetilde{f}_i(\overline{\varepsilon}_i(t))=f_i(x_i(t))-f_{\overline{i}}(s_{\overline{i}}(t)),\widetilde{f}_i(\overline{\varepsilon}_i(t-\tau))=f_i(x_i(t-\tau))-f_{\overline{i}}(s_{\overline{i}}(t-\tau))$, and $\widetilde{f}_i(0)=0$.

The initial value condition of the above error system (11) is as follows:

$$\overline{\varepsilon}_i(s)=\phi_i(s), s\in[-\tau,0], \tag{12}$$

where $\phi_i(s)\in C([-\tau,0],R^n)$ and $i=1,2,\ldots,N$.

Lemma 1 (see [16]). *If $x(t)\in R^n$ is a vector-valued function that is differentiable and continuous for t, next for any $\alpha\in(0,1)$ and $t\geq t_0$, we have the following relationship:*

$$^cD_{t_0,t}^{\alpha}\left(x^T(t)Px(t)\right)\leq 2x^T(t)P\,^cD_{t_0,t}^{\alpha}x(t), \tag{13}$$

where $P\in R^{n\times n}$ is a constant matrix that is symmetric and positive definite.

Lemma 2 (see [17]). *Let $\widehat{\mathscr{R}},\widehat{\mathscr{W}}$, and $\widehat{\mathscr{S}}(t)$ be the real matrices corresponding matching dimensions, then if there is $\widehat{\mathscr{S}}^T(t)\widehat{\mathscr{S}}(t)\leq I$, then there is the following equation:*

$$\widehat{\mathscr{R}}^T\widehat{\mathscr{S}}^T(t)\widehat{\mathscr{W}}^T+\widehat{\mathscr{W}}\widehat{\mathscr{S}}(t)\widehat{\mathscr{R}}\leq\frac{1}{\xi}\widehat{\mathscr{R}}^T\widehat{\mathscr{R}}+\xi\widehat{\mathscr{W}}\widehat{\mathscr{W}}^T, \tag{14}$$

where $\xi>0$ is the constant.

Lemma 3 (see [17]). *Let $\widehat{\mathscr{R}}$ and $\widehat{\mathscr{W}}$ be the real matrices corresponding matching dimensions, then*

$$\widehat{\mathscr{R}}^T\widehat{\mathscr{W}}+\widehat{\mathscr{W}}^T\widehat{R}\leq\xi\widehat{\mathscr{R}}^T\widehat{\mathscr{R}}+\frac{1}{\xi}\widehat{\mathscr{W}}^T\widehat{\mathscr{W}}, \tag{15}$$

where $\xi>0$ is the constant.

Lemma 4 (see [18]). *For arbitrary vector $x_1,x_2\in R^n$ and $Q\in R^{n\times n}$ which is a positive definite matrix, we have the below inequalities hold:*

$$2x_1^Tx_2\leq x_1^TQ^{-1}x_1+x_2^TQx_2. \tag{16}$$

Lemma 5 (see [19]). *For positive definite matrix R, vector x_i with proper dimensionality and symmetric matrix W, then we have the following:*

$$\lambda_{\min}\left(R^{-1}W\right)x_i^TRx_i\leq x_i^TWx_i\leq\lambda_{\max}\left(R^{-1}W\right)x_i^TRx_i, \tag{17}$$

where $\lambda_{\max}(\cdot),\lambda_{\min}(\cdot)$ denote the maximum and minimum eigenvalues, respectively, and R^{-1} stands for the inverse of a matrix R.

Definition 2. The definition of the pinning rate η_k at $t=t_h$ is as follows:

$$\frac{\sum_{i\in\mathfrak{J}_k(t_h)}\overline{\varepsilon}_i^T(t_h^-)\overline{\varepsilon}_i(t_h^-)}{\sum_{i\in D_k}\overline{\varepsilon}_i^T(t_h^-)\overline{\varepsilon}_i(t_h^-)}=\eta_k, \tag{18}$$

here the pinning rate η_k is related to time and impulse instants, and we can also determine the lower bound of the pinning rate η_k.

Lemma 6 (see [20]). *Consider the following system, where the system has a time delay:*

$$\begin{cases}^cD_{t_0,t}^{\alpha}V_k(t)\leq -K_1V_k(t)+K_2V_k(t-\tau), & t>0,\ i\in D_k,\\ V_k(s)=\Phi_k(s), & s\in[-\tau,0],\end{cases} \tag{19}$$

and the linear fractional-order delay differential system is as follows:

$$\begin{cases}^cD_{t_0,t}^{\alpha}W_k(t)=-K_1W_k(t)+K_2W_k(t-\tau), & t>0,\ i\in D_k,\\ W_k(t)=\Phi_k(s), & s\in[-\tau,0],\end{cases} \tag{20}$$

where except for the point $t_k,k=1,2,\ldots,V_k(t),W_k(t)\in R^n$ is continuous everywhere, and $\Phi_k(s)\geq 0$ is continuous in $[-\tau,0]$. If $K_1>0,K_2>0$, then $V_k(t)\leq W_k(t),t\in[0,+\infty]$.

3. Main Result

We will derive several synchronization standards in this section. Under the action of the pinning impulsive controller, D_k, $i\in D_k,D_k\in\{D_1,D_2,\ldots,D_m\}$, system (7) and reference trajectory $s_k(t)\in\{s_1(t),s_2(t),\ldots,s_m(t)\}$ to achieve multiple quasisynchronization.

Theorem 1. *Let $\xi_i>0\,(i=1,2,3)$. For any $i\in D_k,k=1,2,\ldots,m$, under the pinning impulsive control (10), system (7) can achieve multiple quasisynchronization if Assumptions (A1) and (A2) hold, there exist symmetric matrices $M_i\in R^{n\times n}>0\,(i=1,2)$ and positive definite matrix $P\in R^{n\times n}$ and such that*

$$\begin{pmatrix}\widehat{G} & 0\\ 0 & P\Gamma\end{pmatrix}<0, \tag{21}$$

$$\omega_1=\begin{pmatrix}P\hat{P}+\hat{P}^TP+\xi_1P\widehat{\mathscr{A}}_1\widehat{\mathscr{A}}_1^TP+\frac{1}{\xi_1}\widehat{\mathscr{B}}_1^T\widehat{\mathscr{B}}_1-P\hat{Q}M_1^{-1}\hat{Q}^TP-\mathscr{L}_i^TM_1\mathscr{L}_i\\ -\xi_2P\widehat{\mathscr{A}_2}\widehat{\mathscr{A}_2}^TP-\frac{1}{\xi_2}\mathscr{L}_i^T\widehat{\mathscr{B}}_2^T\widehat{\mathscr{B}}_2\mathscr{L}_i-P\hat{R}M_2^{-1}\hat{R}^TP-\xi_3P\widehat{\mathscr{A}}_3\widehat{\mathscr{A}}_3^TP\end{pmatrix}\geq K_1P>0, \tag{22}$$

$$\omega_2 = \mathscr{L}_i^T M_2 \mathscr{L}_i + \frac{1}{\xi_3}\mathscr{L}_i^T \widehat{\mathscr{B}}_3^T \widehat{\mathscr{B}}_3 \mathscr{L}_i \leq K_2 P, \tag{23}$$

$$\omega_3 = (1+\theta_k)^2 \eta_k \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} + (1-\eta_k)\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \leq \rho_k \in (0,1), \tag{24}$$

where $K_1>0, K_2>0, K_1>\sqrt{2}K_2$ *and* $\sqrt{(\lambda_{\max}(P)\kappa/\lambda_{\min}(P))} < \sigma_k$. *Define the Lyapunov function as follows:*

$$V_k(t) = \sum_{i\in D_k} \overline{\varepsilon}_i^T(t) P \overline{\varepsilon}_i(t). \tag{25}$$

For $t \in [t_{h-1}, t_h), h = 0, 1, 2, \ldots$, from Lemma 1 we obtained the following:

$$\begin{aligned}
{}^c D_{t_0,t}^{\alpha} V_k(t) &\leq \sum_{i\in D_k} \overline{2\varepsilon}_i^T(t) P\, {}^c D_{t_0,t}^{\alpha} \overline{\varepsilon}_i(t) \\
&= \sum_{i\in D_k} \overline{2\varepsilon}_i^T(t) P\big(-(\widehat{P} + \Delta\mathscr{P}(t))\overline{\varepsilon}_i(t) + (\widehat{Q} + \Delta\mathscr{Q}(t))\widetilde{f}_i(\overline{\varepsilon}_i(t)) \\
&\quad + (\widehat{R} + \Delta\mathscr{R}(t))\widetilde{f}_i(\overline{\varepsilon}_i(t-\tau)) + \sum_{j\in D_k} \widehat{G}_{ij}\Gamma\overline{\varepsilon}_j(t)\big) \\
&= 2\sum_{i\in D_k} \overline{\varepsilon}_i^T(t) P\big(-(\widehat{P} + \Delta\mathscr{P}(t))\big)\overline{\varepsilon}_i(t) + 2\sum_{i\in D_k} \overline{\varepsilon}_i^T(t) P(\widehat{Q} + \Delta\mathscr{Q}(t))\widetilde{f}_i(\overline{\varepsilon}_i(t)) \\
&\quad + 2\sum_{i\in D_k} \overline{\varepsilon}_i^T(t) P(\widehat{R} + \Delta\mathscr{R}(t))\widetilde{f}_i(\overline{\varepsilon}_i(t-\tau)) + 2\sum_{i\in D_k} \overline{\varepsilon}_i^T(t) P \sum_{j\in D_k} \widehat{G}_{ij}\Gamma\overline{\varepsilon}_j(t).
\end{aligned} \tag{26}$$

By Assumptions (A1) and (A2) and Lemmas 1-4, we obtain the following:

$$\begin{aligned}
&\sum_{i\in D_k} 2\overline{\varepsilon}_i^T(t) P\big(-(\hat{P} + \Delta\mathscr{P}(t))\big)\overline{\varepsilon}_i(t) \\
&\quad \leq \sum_{i\in D_k} \overline{\varepsilon}_i^T(t)\Big(-\Big(P\hat{P} + \hat{P}^T P + P\widehat{\mathscr{A}}_1\widehat{\mathscr{C}}_1(t)\widehat{\mathscr{B}}_1 + \widehat{\mathscr{B}}_1^T\widehat{\mathscr{C}}_1^T(t)\widehat{\mathscr{A}}_1^T P\Big)\Big)\overline{\varepsilon}_i(t) \\
&\quad \leq \sum_{i\in D_k} \overline{\varepsilon}_i^T(t)\Big(-\Big(P\hat{P} + \hat{P}^T P + \xi_1 P\widehat{\mathscr{A}}_1\widehat{\mathscr{A}}_1^T P + \frac{1}{\xi_1}\widehat{\mathscr{B}}_1^T\widehat{\mathscr{B}}_1\Big)\Big)\overline{\varepsilon}_i(t),
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\sum_{i\in D_k} 2\overline{\varepsilon}_i^T(t) P(\hat{Q} + \Delta\mathscr{Q}(t))\widetilde{f}_i(\overline{\varepsilon}_i(t)) \\
&\quad \leq \sum_{i\in D_k} 2\overline{\varepsilon}_i^T(t) P\hat{Q}\widetilde{f}_i(\overline{\varepsilon}_i(t)) + \sum_{i\in D_k} 2\overline{\varepsilon}_i^T(t) P\widehat{\mathscr{A}}_2\widehat{\mathscr{C}}_2(t)\widehat{\mathscr{B}}_2\widetilde{f}_i(\overline{\varepsilon}_i(t)) \\
&\quad \leq \sum_{i\in D_k} \overline{\varepsilon}_i^T(t) P\hat{Q}M_1^{-1}\hat{Q}^T P\overline{\varepsilon}_i(t) + \sum_{i\in D_k} \widetilde{f}_i^T(\overline{\varepsilon}_i(t)) M_1 \widetilde{f}_i(\overline{\varepsilon}_i(t)) \\
&\qquad + \sum_{i\in D_k} \overline{\varepsilon}_i^T(t) P\widehat{\mathscr{A}}_2\widehat{\mathscr{C}}_2(t)\widehat{\mathscr{B}}_2\widetilde{f}_i(\overline{\varepsilon}_i(t)) + \sum_{i\in D_k} \widetilde{f}_i^T(\overline{\varepsilon}_i(t))\widehat{\mathscr{B}}_2^T\widehat{\mathscr{C}}_2^T(t)\widehat{\mathscr{A}}_2^T P\overline{\varepsilon}_i(t) \\
&\quad \leq \sum_{i\in D_k} \overline{\varepsilon}_i^T(t) P\hat{Q}M_1^{-1}\hat{Q}^T P\overline{\varepsilon}_i(t) + \sum_{i\in D_k} \overline{\varepsilon}_i^T(t)\mathscr{L}_i^T M_1 \mathscr{L}_i\overline{\varepsilon}_i(t) \\
&\qquad + \sum_{i\in D_k} \frac{1}{\xi_2}\overline{\varepsilon}_i^T(t)\mathscr{L}_i^T\widehat{\mathscr{B}}_2^T\widehat{\mathscr{B}}_2\mathscr{L}_i\overline{\varepsilon}_i(t) + \sum_{i\in D_k} \xi_2\overline{\varepsilon}_i^T(t) P\widehat{\mathscr{A}}_2\widehat{\mathscr{A}}_2^T P\overline{\varepsilon}_i(t) \\
&\quad = \sum_{i\in D_k} e_i^T(t)\Big(P\hat{Q}M_1^{-1}\hat{Q}^T P + L_i^T M_1 L_i + \frac{1}{\xi_2}L_i^T\hat{B}_2^T\hat{B}_2 L_i + \xi_2 P\hat{A}_2\hat{A}_2^T P\Big)e_i(t),
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\sum_{i\in D_k} 2\bar{\varepsilon}_i^T(t)P(\hat{R}+\Delta\mathscr{R}(t))\widetilde{f}_i(\bar{\varepsilon}_i(t-\tau))\\
&\quad\le \sum_{i\in D_k} 2\bar{\varepsilon}_i^T(t)P\hat{R}\widetilde{f}_i(\bar{\varepsilon}_i(t)) + \sum_{i\in D_k} 2\bar{\varepsilon}_i^T(t)P\widehat{\mathscr{A}}_3\widehat{\mathscr{C}}_3(t)\widehat{\mathscr{B}}_3\widetilde{f}_i(\bar{\varepsilon}_i(t-\tau))\\
&\quad\le \sum_{i\in D_k} \bar{\varepsilon}_i^T(t)P\hat{R}M_2^{-1}\hat{R}^T P\bar{\varepsilon}_i(t) + \sum_{i\in D_k} \widetilde{f}_i^T(\bar{\varepsilon}_i(t-\tau))M_2\widetilde{f}_i(\bar{\varepsilon}_i(t-\tau))\\
&\qquad + \sum_{i\in D_k} \bar{\varepsilon}_i^T(t)P\widehat{\mathscr{A}}_3\widehat{\mathscr{C}}_3(t)\widehat{\mathscr{B}}_3\widetilde{f}_i(\bar{\varepsilon}_i(t-\tau)) + \sum_{i\in D_k} \widetilde{f}_i^T(\bar{\varepsilon}_i(t-\tau))\widehat{\mathscr{B}}_3^T\widehat{\mathscr{C}}_3^T(t)\widehat{\mathscr{A}}_3^T P\bar{\varepsilon}_i(t)\\
&\quad\le \sum_{i\in D_k} \bar{\varepsilon}_i^T(t)P\hat{R}M_2^{-1}\hat{R}^T P\bar{\varepsilon}_i(t) + \sum_{i\in D_k} \bar{\varepsilon}_i^T(t-\tau)\mathscr{L}_i^T M_2\mathscr{L}_i\bar{\varepsilon}_i(t-\tau)\\
&\qquad + \sum_{i\in D_k} \frac{1}{\xi_3}\bar{\varepsilon}_i^T(t-\tau)\mathscr{L}_i^T\widehat{\mathscr{B}}_3^T\widehat{\mathscr{B}}_3\mathscr{L}_i\bar{\varepsilon}_i(t-\tau) + \sum_{i\in D_k} \xi_3\bar{\varepsilon}_i^T(t)P\widehat{\mathscr{A}}_3\widehat{\mathscr{A}}_3^T P\bar{\varepsilon}_i(t)\\
&\quad= \sum_{i\in D_k} \bar{\varepsilon}_i^T(t)\left(P\hat{R}M_2^{-1}\hat{R}^T P + \xi_3 P\widehat{\mathscr{A}}_3\widehat{\mathscr{A}}_3^T P\right)\bar{\varepsilon}_i(t) + \sum_{i\in D_k} \bar{\varepsilon}_i^T(t-\tau)\left(\mathscr{L}_i^T M_2\mathscr{L}_i + \frac{1}{\xi_3}\mathscr{L}_i^T\widehat{\mathscr{B}}_3^T\widehat{\mathscr{B}}_3\mathscr{L}_i\right)\bar{\varepsilon}_i(t-\tau).
\end{aligned} \tag{29}$$

Form (21), we have the following:

$$\sum_{i\in D_k} 2\bar{\varepsilon}_i^T(t)P\sum_{j\in D_k}\widehat{G}_{ij}\Gamma\bar{\varepsilon}_j(t) = 2\bar{\varepsilon}^T(t)(\widehat{G}\otimes P\Gamma)\bar{\varepsilon}(t)\le 0. \tag{30}$$

Substituting (27)-(30) into (27282930) yields the following equation:

$$\begin{aligned}
{}^cD_{t_0,t}^{\alpha}V_k(t) \le &\sum_{i\in D_k} -\bar{\varepsilon}_i^T(t)\omega_1\bar{\varepsilon}_i(t) + \sum_{i\in D_k}\bar{\varepsilon}_i^T(t-\tau)\omega_2\bar{\varepsilon}_i(t-\tau)\\
&+ 2\bar{\varepsilon}^T(t)(\widehat{G}\otimes P\Gamma)\bar{\varepsilon}(t).
\end{aligned} \tag{31}$$

From (21)-(24), we have the following equation:

$$\begin{aligned}
{}^cD_{t_0,t}^{\alpha}V_k(t) &\le \sum_{i\in D_k} -\bar{\varepsilon}_i^T(t)K_1P\bar{\varepsilon}_i(t) + \bar{\varepsilon}_i^T(t-\tau)K_2P\bar{\varepsilon}_i(t-\tau)\\
&\le -K_1V_k(t) + K_2V_k(t-\tau).
\end{aligned} \tag{32}$$

When $t = t_h$, from (11) and (24), Lemma 5, and Definition 2, we obtain the following:

$$\begin{aligned}
V_k(t_h^+) &= \sum_{i\in D_k}\bar{\varepsilon}_i^T(t_h^+)P\bar{\varepsilon}_i(t_h^+)\\
&= \sum_{i\in\mathfrak{J}_k(t_h)}\bar{\varepsilon}_i^T(t_h^+)P\bar{\varepsilon}_i(t_h^+) + \sum_{i\notin\mathfrak{J}_k(t_h)}\bar{\varepsilon}_i^T(t_h^+)P\bar{\varepsilon}_i(t_h^+)\\
&= \sum_{i\in\mathfrak{J}_k(t_h)}(1+\theta_k)^2\bar{\varepsilon}_i^T(t_h^-)P\bar{\varepsilon}_i(t_h^-) + \sum_{i\notin\mathfrak{J}_k(t_h)}\bar{\varepsilon}_i^T(t_h^-)P\bar{\varepsilon}_i(t_h^-)\\
&\le (1+\theta_k)^2\lambda_{\max}(P)\sum_{i\in\mathfrak{J}_k(t_h)}\bar{\varepsilon}_i^T(t_h^-)\bar{\varepsilon}_i(t_h^-) + \lambda_{\max}(P)\sum_{i\notin\mathfrak{J}_k(t_h)}\bar{\varepsilon}_i^T(t_h^-)\bar{\varepsilon}_i(t_h^-)\\
&\le (1+\theta_k)^2\lambda_{\max}(P)\sum_{i\in D_k}\bar{\varepsilon}_i^T(t_h)\bar{\varepsilon}_i(t_h) + (1-\eta_k)\lambda_{\max}(P)\sum_{i\in D_k}2\,\bar{\varepsilon}_i^T(t_h)\bar{\varepsilon}_i(t_h)\\
&\le \omega_3V_k(t_h)\\
&\le \rho_kV_k(t_h).
\end{aligned} \tag{33}$$

Now, consider the following system with time lag:

$$\begin{cases} {}^cD_{t_0,t}^{\alpha}W_k(t) = -K_1W_k(t) + K_2W_k(t-\tau), & t>0,\\ W_k(t) = \Phi_k(s), & s\in[-\tau,0]. \end{cases} \tag{34}$$

If $\lim_{t\longrightarrow\infty}W_k(t) = 0, \Phi_k(s)\ge 0$, afterward through Lemma 6, we can have $\lim_{t\longrightarrow\infty}V_k(t) = 0, \Phi_k(s)\ge 0$.

Next, we will prove that when $K_1 > \sqrt{2}K_2\,(K_1>0, K_2>0)$, there is $\lim_{t\longrightarrow\infty}W_k(t) = 0, \Phi_k(s)\ge 0$.

In order to distinguish the subsystem subscript in this article from the original imaginary unit, here, we change the original inherent imaginary unit $\widetilde{i}$. The characteristic (34) can be changed to the following form according to Corollary 3 in Reference [21].

$$\overline{v}_k^{\alpha} + K_1 - K_2 e^{-\overline{v}_k \tau} = 0, \tag{35}$$

If $K_1 > \sqrt{2}K_2$ and (35) has no pure imaginary roots, so the zero solution of equation (34) is globally Lyapunov asymptotically stable, namely, $\lim_{t\longrightarrow\infty} W_k(t) = 0, \Phi_k(s) \geq 0$.

Next, we will use contradiction analysis to show that (35) does not have pure imaginary roots. Then, suppose (35) has pure imaginary roots $\overline{v}_k$, and $\overline{v}_k = v_k \tilde{i} = |v_k|(\cos(\pi/2) + \tilde{i}\sin(\pi/2))$, where v_k is a real number. If $v_k \leq 0$, then $\overline{v}_k = v_k\tilde{i} = |v_k|(\cos(\pi/2) - \tilde{i}\sin(\pi/2))$, and if $v_k > 0$, then $\overline{v}_k = v_k\tilde{i} = |v_k|(\cos(\pi/2) + \tilde{i}\sin(\pi/2))$.

By substituting $\overline{v}_k = v_k\tilde{i}$ into (35), we can obtain the following:

$$\left|(v_k\tilde{i})^{\alpha} + K_1\right|^2 = \left|K_2 e^{-\tau v_k \tilde{i}}\right|^2, \tag{36}$$

that is,

$$\begin{aligned} |v_k|^{2\alpha} + 2K_1 \cos\left(\frac{\alpha\pi}{2}\right)|v_k|^{\alpha} + K_1^2 &= |K_2\cos(\tau v_k)|^2 \\ &+ |K_2\sin(\tau v_k)|^2 \leq 2(K_2)^2. \end{aligned} \tag{37}$$

Let

$$\begin{aligned} h_k(x_k) = x_k^2 + 2K_1\cos\left(\frac{\alpha\pi}{2}\right)x_k + K_1^2 \\ -\left((K_2\cos(\tau v_k))^2 + (K_2\sin(\tau v_k))^2\right). \end{aligned} \tag{38}$$

So,

$$h_k(0) = K_1^2 - \left((K_2\cos(\tau v_k))^2 + (K_2\sin(\tau v_k))^2\right) \geq K_1^2 - 2(K_2)^2. \tag{39}$$

Because $K_1 > \sqrt{2}K_2\,(K_1 > 0, K_2 > 0)$, so $h_k(0) > 0$. We know that h_k is a second-order polynomial, so we have $h_k(|v_k|^{\alpha}) > 0$, which contradicts (37). That is, (37) has no solution, which means that (35) has no pure imaginary roots, namely, $\lim_{t\longrightarrow\infty} V_k(t) = 0$.

Therefore, there exists T_k, and for arbitrary $\kappa > 0$ and for all $t > T_k$, we have the following equation:

$$V_k(t) < \lambda_{\max}(P)\kappa, \quad t > T_k, \tag{40}$$

where $V_k(t) = \sum_{i\in D_k} \overline{\varepsilon}_i^T(t) P \overline{\varepsilon}_i(t)$, so we have the following equation:

$$\lambda_{\min}(P)\|\overline{\varepsilon}_i(t)\|^2 < \lambda_{\max}(P)\kappa, \quad t > T_k, \tag{41}$$

that is,

$$\|\overline{\varepsilon}_i(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)\kappa}{\lambda_{\min}(P)}} < \sigma_k, \tag{42}$$

where $i \in D_k$ and $\sum_{k=1}^m T_k = \{T_1, T_2, \ldots, T_N\}$. So there exits $T = \max\{T_1, T_2, \ldots, T_N\}$, for $\forall t > T$ and any small positive number $\sigma_k > 0$, such that $0 < \|x_i(t) - s_k(t)\| < \sigma_k, k = 1, 2, \ldots, m$.

If $\Delta\mathscr{P}(t) = 0, \Delta\mathscr{Q}(t) = 0, \Delta\mathscr{R}(t) = 0$, (2.1) will degrade into

$$\begin{aligned} {}^{c}D_{t_0,t}^{\alpha} x_i(t) = -\widehat{P}x_i(t) + \widehat{Q}f_i(x_i(t)) + \widehat{R}f_i(x_i(t-\tau)) \\ + \sum_{j=1}^{N}\widehat{G}_{ij}\Gamma x_j(t) + J. \end{aligned} \tag{43}$$

Corollary 1. *For any $i \in D_k, k = 1, 2, \ldots, m$, under the pinning impulsive control (10), system (43) can achieve multiple quasisynchronization if Assumptions (A1) and (A2) hold, there exist symmetric matrices $M_i \in R^{n\times n} > 0\,(i = 1, 2)$, and positive definite matrix $P \in R^{n\times n}$ such that*

$$\begin{aligned} &\begin{pmatrix} \widehat{G} & 0 \\ 0 & P\Gamma \end{pmatrix} < 0, \\ &\omega_1 = \left(P\widehat{P} + \widehat{P}^T P - P\widehat{Q}M_1^{-1}\widehat{Q}^T P - \mathscr{L}_i^T M_1 \mathscr{L}_i - P\widehat{R}M_2^{-1}\widehat{R}^T P\right) \geq K_1 P > 0, \\ &\omega_2 = \mathscr{L}_i^T M_2 \mathscr{L}_i \leq K_2 P, \\ &\omega_3 = (1+\theta_k)^2 \eta_k \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} + (1-\eta_k)\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \leq \rho_k \in (0, 1), \end{aligned} \tag{44}$$

where $K_1 > 0$, $K_2 > 0$, $K_1 > \sqrt{2}K_2$, and $\sqrt{(\lambda_{\max}(P)\kappa/\lambda_{\min}(P))} < \sigma_k$.

Remark 3. We give general theoretical results for multiple quasisynchronization of FCNNs with uncertain terms and delays in Theorem 1 Among existing references, the quasisynchronization problem of FCNNs with uncertainty is rarely discussed. Moreover, unlike the analytical method of Reference [22], the model (7) in this article is a fractional-order system instead of the integer-order model in Reference [22]. The analysis and processing method of fractional-order system is unlike that of integer-order system, so it cannot be applied directly.

Remark 4. Multiple quasisynchronization is the extension of quasisynchronization. When $m = 1$ in reference trajectory indicates that there is only one reference track, next, the multiple quasisynchronization is reduced to quasisynchronization.

Remark 5. Corollary 1 gives the sufficient conditions for multiple quasisynchronization of DFCNN when the uncertainty term is zero.

Remark 6. In Reference [23], the author solved the synchronization problem through adaptive control method. In this article to reduce the control cost and realize the pinned pulse control, only some nodes need to be controlled to be in the bounded field where they share the reference trajectory.

Remark 7. Compared with Reference [20], the difference of this article is that we consider parameter uncertainty satisfying bounded conditions in the model, and the advantage is that the model considered in this article is more practical in practical systems and applications. In particular, our model is fractional-order, and parameter uncertainties and coupling terms are taken into account in the model.

4. Examples

In this section, we give two numerical simulation examples to illustrate the abovementioned theoretical values.

Example 1. We design the FCNNs with uncertain terms and delays (7), where $f_i(x_i(t)) = \tanh(x_i(t))$, $i = 1, 2$, $m = 1, 2$, time-delay $\tau = 1$, and $\alpha = 0.96$, and the parameter matrix of the network is as follows:

$$\begin{aligned}
&\widehat{P} = \begin{pmatrix} 5.2 & 0 \\ 0 & 5.2 \end{pmatrix}, \widehat{Q} = \begin{pmatrix} 4.8 & -2 \\ -3 & 2.5 \end{pmatrix}, \widehat{R} = \begin{pmatrix} 0.25 & 0 \\ 0 & 1.5 \end{pmatrix},\\
&\Delta\mathscr{P} = \cos(t)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \Delta\mathscr{Q} = \sin(t)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \Delta\mathscr{R} = \cos(t)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\\
&\widehat{G} = \begin{pmatrix} -5 & 0.2 \\ 0.4 & -1 \end{pmatrix}, J = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned} \tag{45}$$

Two corresponding reference trajectories (9), where $k = 1, 2, \alpha = 0.96, \tau = 1, f_{\bar{i}}(s_{\bar{i}}(t)) = \tanh(s_{\bar{i}}(t))$.

We select $\Gamma = \mathrm{diag}(1, 1)$, pulse gain $\theta_1 = -0.2$, $\theta_2 = -0.6$, $\eta_1 = 0.7$, $\eta_2 = 0.4$, $\rho_1 = 0.84$, and $\rho_2 = 0.74$. It can be proved that the conditions (21)-(24) in Theorem 1 are established and can be obtained by the following calculation:

$$P = \begin{pmatrix} 1.0017 & 0.0022 \\ 0.0022 & 1.0945 \end{pmatrix}, \tag{46}$$

at the same time, $(\lambda_{\max}(P)/\lambda_{\min}(P)) = 10/9$, $\sigma_1 = 0.08$, and $\sigma_2 = 0.12$, after that we can get $\omega_1 = \begin{pmatrix} 173.0088 & -0.0003 \\ -0.0003 & 21.0120 \end{pmatrix}, \omega_2 = \begin{pmatrix} 2.4908 & 0 \\ 0 & 4.4362 \end{pmatrix}, \omega_3 = \begin{pmatrix} 0.8311 \\ 0.3271 \end{pmatrix}$ through the abovementioned formulas (22)-(24). Then, the uncertain fractional-order neural network can realize multiple quasisynchronization, and the convergence of its error signals $e_{1m}(t)$, $e_{2m}(t)$, and $m = 1, 2$, under the fixed pulse controller is shown in Figure 1.

Example 2. We design the FCNNs with uncertain terms and delays (6), where $f_i(x_i(t)) = \tanh(x_i(t))$, $i = 1, 2$, $m = 1, 2$, time delay $\tau = 1$, and $\alpha = 0.98$, and the parameter matrix of the network is as follows:

$$\begin{aligned}
&\widehat{P} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \widehat{Q} = \begin{pmatrix} 5 & -5 \\ -5 & 3.8 \end{pmatrix}, \widehat{R} = \begin{pmatrix} 0.25 & 0 \\ 0 & 1.5 \end{pmatrix},\\
&\Delta\mathscr{P} = \sin(t)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \Delta\mathscr{Q} = \cos(t)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \Delta\mathscr{R} = \sin(t)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\\
&\widehat{G} = \begin{pmatrix} -5 & 0.2 \\ 0.4 & -1 \end{pmatrix}, J = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned} \tag{47}$$

Two corresponding reference trajectories (9), where $k = 1, 2, \alpha = 0.98, \tau = 1, f_{\bar{i}}(s_{\bar{i}}(t)) = \tanh(s_{\bar{i}}(t))$.

We select $\Gamma = \mathrm{diag}(1, 1)$, pulse gain $\theta_1 = -0.3$, $\theta_2 = -0.7$, $\eta_1 = 0.5$, $\eta_2 = 0.6$, $\rho_1 = 0.83$, and $\rho_2 = 0.51$. It can be proved that conditions (21)-(24) in Theorem 1 are established and can be obtained by the following calculation:

$$P = \begin{pmatrix} 1.0017 & 0.0022 \\ 0.0022 & 1.0945 \end{pmatrix}, \tag{48}$$

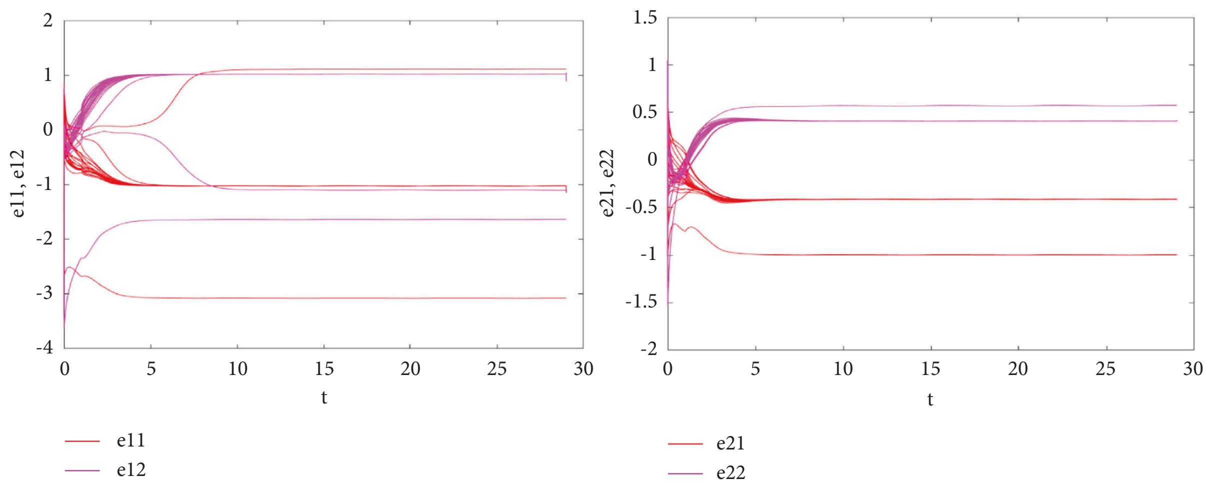

FIGURE 1: Trajectory of e_{11}, e_{12}, e_{21}, and e_{22} of the system in Example 1 under the impulsive controller.

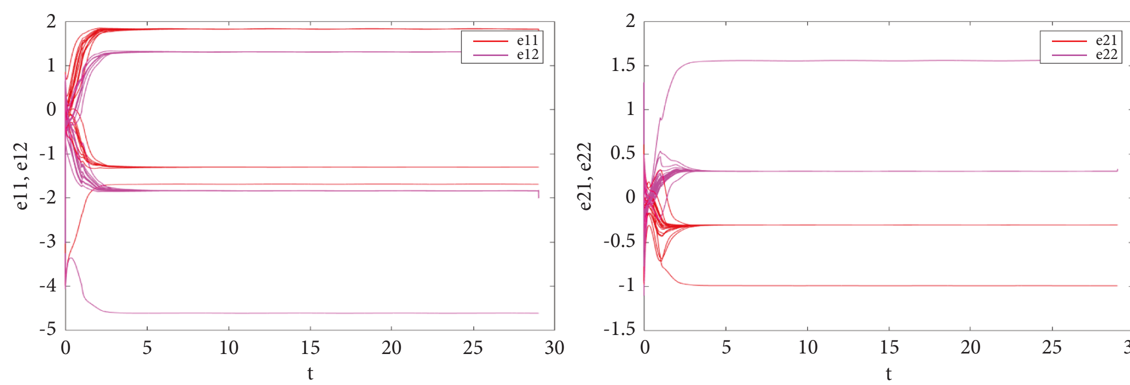

FIGURE 2: Trajectory of e_{11}, e_{12}, e_{21}, and e_{22} of the system in Example 2 under the impulsive controller.

at the same time, $(\lambda_{\max}(P)/\lambda_{\min}(P)) = (10/9)$, $\sigma_1 = 0.06$, and $\sigma_2 = 0.18$. After that we can get $\omega_1 = \begin{pmatrix} 135.2047 & -0.0012 \\ -0.0012 & 74.7350 \end{pmatrix}$, $\omega_2 = \begin{pmatrix} 3.0918 & 0 \\ 0 & 1.1362 \end{pmatrix}$, $\omega_3 = \begin{pmatrix} 0.8278 \\ 0.5044 \end{pmatrix}$ through the abovementioned formulas (22)-(24). Then, the uncertain fractional-order neural network can realize multiple quasisynchronization, and the convergence of its error signals $e_{1m}(t)$, $e_{2m}(t)$, $m = 1, 2$, and under the fixed pulse controller is shown in Figure 2.

5. Conclusion

In this article, the multiple quasisynchronization problem of FCNNs with uncertainty and time-delay is studied. Firstly, our main theoretical method is to construct an impulse controller to control some nodes and then divide the nodes into several disjoint subsets, so as to make the system achieve multiple quasisynchronization. Secondly, using the relevant knowledge of the comparison principle and the method of constructing the Lyapunov function, we obtain the sufficient conditions for the system to realize multiple quasisynchronization. Finally, two examples are given to carry out simulation operations to demonstrate the validity of the theoretical results in this article.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] P. Schirmer and I. Mporas, “Energy disaggregation using fractional calculus,” in *Proceedings of the 2020 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pp. 3257–3261, Barcelona, Spain, May 2020.

[2] A. Bendaoud and F. Hachouf, “Fractional calculus for improving edge-based active contour models,” in *Proceedings of the 2021 International Conference on Recent Advances in*

Mathematics and Informatics (ICRAMI), pp. 1–5, Tebessa, Algeria, September 2021.
[3] H. B. Bao and J. D. Cao, "Projective synchronization of fractional-order memristor-based neural networks," *Neural Networks*, vol. 63, pp. 1–9, 2015.
[4] 李. Li Xin-bo, 付. Fu Yun-bo, 姜. Jiang Liang-xu, 石 DuShi Yao-wu, and 杨 Yang Zhi-gang, "Application of neural network fractional order PIDto piezoelectric stack control," *Optics and Precision Engineering*, vol. 23, no. 12, pp. 3439–3445, 2015.
[5] S. Xu, X. Wang, and X. Ye, "A new fractional-order chaos system of Hopfield neural network and its application in image encryption," *Chaos, Solitons & Fractals*, vol. 157, p. 111889, 2022.
[6] W. Du, Q. Y. Miao, L. Tong, and Y. Tang, "Identification of fractional-order systems with unknown initial values and structure," *Physics Letters A*, vol. 381, no. 23, pp. 1943–1949, 2017.
[7] H. Liu and J. Zhang, "Stabilization of Uncertain Stochastic Nonlinear Delay Systems Subject to Event-Triggered Mechanism," in *Proceedings of the 2019 IEEE Symposium Series on Computational Intelligence (SSCI)*, pp. 1263–1270, Xiamen, China, December 2019.
[8] S. Wang, X. Wang, and Y. Zhou, "A memristor-based complex lorenz system and its modified projective synchronization," *Entropy*, vol. 17, no. 12, pp. 7628–7644, 2015.
[9] M. Xu, Y. Li, P. Liu, P. Yu, S. Li, and J. Sun, "On synchronization of coupled fractional-order neural networks in finite-time sense," in *Proceedings of the 2021 International Conference on Neuromorphic Computing (ICNC)*, pp. 41–46, Wuhan, China, October 2021.
[10] C. Chen, Z. Ding, S. Li, and L. Wang, "Synchronization of Fractional-Order Memristive Neural Networks with Time Delays," in *Proceedings of the 2019 Chinese Automation Congress (CAC)*, pp. 2754–2759, Hangzhou, China, November 2019.
[11] S. Wang, Y. Huang, and S. Ren, "Synchronization and robust synchronization for fractional-order coupled neural networks," *IEEE Access*, vol. 5, pp. 12439–12448, 2017.
[12] J. Feng, S. Chen, J. Wang, and Y. Zhao, "Quasi-Synchronization of coupled nonlinear memristive neural networks with time delays by pinning control," *IEEE Access*, vol. 6, pp. 26271–26282, 2018.
[13] X. Lv, X. Li, J. Cao, and M. Perc, "Dynamical and static multisynchronization of coupled multistable neural networks via impulsive control," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 29, no. 12, pp. 6062–6072, 2018.
[14] C. P. Li and F. R. Zhang, "A survey on the stability of fractional differential equations," *The European Physical Journal - Special Topics*, vol. 193, no. 1, pp. 27–47, 2011.
[15] D. X. He, G. Ling, Z. H. Guan, B. Hu, and R. Q. Liao, "Multisynchronization of coupled heterogeneous genetic oscillator networks via partial impulsive control," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 29, no. 2, pp. 335–342, 2018.
[16] M. A. Duarte-Mermoud, N. Aguila-Camacho, J. A. Gallegos, and R. Castro-Linares, "Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 22, no. 1-3, pp. 650–659, 2015.
[17] Y. L. Wang and C. E. de Souza, "Robust control of a class of uncertain nonlinear systems," *Systems & Control Letters*, vol. 19, no. 2, pp. 139–149, 1992.
[18] L. Pan and J. Cao, "Stochastic quasi-synchronization for delayed dynamical networks via intermittent control," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 3, pp. 1332–1343, 2012.
[19] L. Huang, *Linear Algebria in Systems and Control Theory*, Science Press, Beijing, China, 1984.
[20] X. Ruan and A. Wu, "Multi-quasi-synchronization of coupled fractional-order neural networks with delays via pinning impulsive control," *Advances in Difference Equations*, no. 1, p. 359, 2017.
[21] W. Deng, C. Li, and J. Lü, "Stability analysis of linear fractional differential system with multiple time delays," *Nonlinear Dynamics*, vol. 48, no. 4, pp. 409–416, 2006.
[22] W. He, F. Qian, and J. Cao, "Pinning-controlled synchronization of delayed neural networks with distributed-delay coupling via impulsive control," *Neural Networks*, vol. 85, pp. 1–9, 2017.
[23] A. Jajarmi, M. Hajipour, and D. Baleanu, "New aspects of the adaptive synchronization and hyperchaos suppression of a financial model," *Chaos, Solitons & Fractals*, vol. 99, pp. 285–296, 2017.