Research Article

The Q-Space Deformed Wave Equation

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1. Introduction

The wave equation is a second-order linear hyperbolic partial differential equation that describes the propagation of a variety of waves, such as sound or water waves. It arises in different fields such as acoustics, electromagnetics, or fluid dynamics. In its simplest, the wave equation takes the form:

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},
\]

(1)

where \(x\) is the coordinate, \(t\) is time, \(u = u(x, t)\) is the displacement, and \(c\) is the wave velocity. Due to the ambiguity in the direction of the wave velocity, \(c^2 = (+c)^2 = (-c)^2\), the equation does not contain information about the wave direction and therefore has solutions propagating in both the forward \((+x)\) and backward \((-x)\) directions. In 1746, D’Alembert discovered the one-dimensional wave equation, and he gave the equation of motion of the string as the one-dimensional wave equation in 1747, after ten years Euler discovered the three-dimensional wave equation. This equation is a good description for a wide range of phenomena because it is typically used to model small oscillations about an equilibrium, for which systems can often be well approximated by Hooke law. Solutions to the wave equation are of course important in fluid dynamics, but they also play an important role in electromagnetism, optics, gravitational physics, and heat transfer. The problem of a vibrating string such as that of a musical instrument was studied by Jean le Rond D’Alembert, Leonhard Euler, Daniel Bernoulli, and Joseph-Louis Lagrange. In the area of combinatorics and quantum calculus, the \(q\)-derivative, or Jackson derivative, is a \(q\)-analogue of the ordinary derivative, introduced by Frank Hilton Jackson [1, 2]. In the last decades, the \(q\)-calculus has developed into an interdisciplinary subjects, the \(q\)-calculus has in the last twenty years served as a bridge between mathematics and physics. The majority of scientists in the world who use the \(q\)-calculus today are physicists. The field has expanded explosively, due to the fact that applications of basic hypergeometric series to the diverse subjects of combinatorics, quantum theory, number theory, and statistical mechanics. The formulation of fractional calculus began shortly after the classical calculus was established. Since its definition is based on the concept of a noninteger order either integral or derivative, the fractional calculus had been considered as a subject in pure mathematics with no real applications for a long time. However, the role of fractional calculus has been changed in recent decades. Its applications take place in many fields of mathematical sciences, extended from the fractional...
calculus. The fractional \( q \)-calculus is the \( q \)-extension of the ordinary fractional calculus. Many results of the study on the theory of the \( q \)-calculus operators in recent decades have been applied in various areas such as problems in the ordinary fractional calculus, optimal control, solutions of the \( q \)-difference equations, \( q \)-differential equations, \( q \)-integral equations, and \( q \)-transform analysis, and also in the geometric function theory of complex analysis. \( q \)-calculus has some applications in physics. In theories of quantum gravity, \( q \) can be thought of as a parameter related to the exponential of the cosmological constant. So, when \( q = 1 \), there is no gravity, and we recover ‘classical’ quantum mechanics. However, when \( q \) is not equal to one, we have a theory of quantum mechanics in a spacetime with constant curvature, i.e., a theory where the vacuum has a nonzero energy density, see [3–5]. In this setting, we study a \( q \)-analogue of the wave equation. More precisely, we are interested by the following problem:

\[
\frac{\partial^2 u}{\partial t^2} = c^2 D_{q,x}^2 u, \tag{2}
\]

where \( D_{q,x} \) is defined using the \( q \)-derivative (see the Preliminaries).

2. Preliminaries

We recall some basic notations of the language of \( q \)-calculus (see [6–9]). The natural number \( n \) has the following \( q \)-deformation for \( q \in (0, 1) \):

\[
[n]_q = 1 + q + q^2 + \ldots + q^{n-1}, \text{with } [0]_q = 0. \tag{3}
\]

Occasionally, we shall write \( [\infty]_q \) for the limit of these numbers: \( 1/(1 - q) \). One can get easily that

\[
[n]_q = \frac{1 - q^n}{1 - q}, \quad q \in (0, 1), \tag{4}
\]

for \( n \in \mathbb{N} \). We can give this definition for any real number \( \lambda \). In this case, we call \( [\lambda]_q \) a \( q \)-real. The \( q \) factorials and \( q \) binomial coefficients are defined naturally as

\[
[n]_q! : = [1]_q \cdot [2]_q \cdots [n]_q \text{ with } [0]_q! = 1, \tag{5}
\]

for \( q \in (0, 1) \) and analytic \( f: \mathbb{C} \rightarrow \mathbb{C} \) define operators \( Z \) and \( D_q \) as follows (see [8, 9]):

\[
(Zf)(z) = zf(z), \quad (D_qf)(z) = \begin{cases} \frac{f(z) - f(qz)}{z(1 - q)}, & z \neq 0 \\ f'(0) & \end{cases}, \tag{6}
\]

\( D_{q,x_1} \) and \( D_{q,x_2} \) are given by

\[
D_{q,x_1}g(x_1, x_2) := (D_qg(\cdot, x_2))(x_1), \quad D_{q,x_2}g(x_1, x_2) := (D_qg(x_1, \cdot))(x_2). \tag{7}
\]

The operator \( D_q \) has the following properties:

(i) \( \lim_{q \to 1^{-}} (D_qf)(z) = f'(z) \),
(ii) \( D_q(z^n) = [n]_q z^{n-1} \),
(iii) \( D_q(f(z)g(z)) = (D_qf(z))g(z) + f(qz)(D_qg)(z) \),
(iv) \( D_q(f(z)/g(z)) = (D_qf(z)/g(z) - f(z)(D_qg)(z)/g(z))g(qz) \).

It is well known [8] that the operators \( D_q \) and \( Z \) satisfy

\[
D_q Z - qZD_q = 1. \tag{8}
\]

One of the \( q \)-analogues of classical exponential function \( e^x \) is

\[
e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}, \tag{9}
\]

We see that

\[
e_q(x) = \frac{1}{(1 - (1 - q)x)^\infty}, \tag{10}
\]

where

\[
(1 - y)^\infty = \prod_{k=0}^{\infty} (1 - q^k y). \tag{11}
\]

The \( q \)-exponential functions satisfy the following property:

\[
D_q e_q(x) = e_q(x). \tag{12}
\]

Note that for \( q \in (0, 1) \), the series expansion of \( e_q(x) \) has radius of convergence \( 1/(1 - q) \).

3. \( q \)-Space Wave Equation

In the reminder of this paper, we take \( q \in (0, 1) \). In this section, we introduce the following equation:

\[
\frac{\partial^2 u}{\partial t^2} = c^2 D_{q,x}^2 u. \tag{13}
\]

Equation (13) will be called \( q \)-space wave equation.

Theorem 1. For \( 0 < q < 1 \), we have

(1) \( u \) given by

\[
u(t, x) = \left( A e^{-t \sqrt{c/L}} + B e^{t \sqrt{c/L}} \right) e_q(x \sqrt{L}) \tag{14}
\]

is solution of Eq. (13), for any constants \( A \) and \( B \), where \( L > 0 \).

(2) \( u \) given by

\[
\text{...}
\]
Proof. For $0 < q < 1$, we get

$$D_{q,x}u(t, x) = \frac{u(t, x) - u(t, qx)}{x(1 - q)} = h(t, x),$$

(16)

$$D_{q,x}^2 u(t, x) = \frac{h(t, x) - h(t, qx)}{x(1 - q)}.$$  

(17)

Putting

$$u(t, x) = R(t)S(x).$$  

(18)

We obtain

$$S(x)R''(t) = c^2 R(t)D_{q,x}^2 S(x).$$  

(19)

Then,

$$\frac{D_{q,x}^2 S(x)}{S(x)} = \frac{R''(t)}{c^2 R(t)} = L.$$  

(20)

For a constant $L$, this implies

$$R''(t) = Lc^2 R(t).$$  

(21)

$$D_{q,x}^2 S(x) = LS(x).$$  

(22)

Using Eq. (17) in Eq. (22), we obtain

$$q(x) - (1 + q)S(qx) + S(q^2 x) = qS(qx) - \alpha_x S(qx) = \alpha_x qS(qx) + \sum_{n=0}^{\infty} a_n q^{n+1} x^n.$$

This gives

$$q(1 - x^2 (1 - q)^2) S(x) = (1 + q)S(qx) - S(q^2 x).$$  

(24)

From which, we obtain

$$q(1 - q^2 x^2 (1 - q)^2) S(qx) = (1 + q)S(q^2 x) - S(q^3 x).$$

$$q(1 - q^2 x^2 (1 - q)^2) S(q^2 x) = (1 + q)S(q^3 x) - S(q^4 x).$$

$$q(1 - q^2 x^2 (1 - q)^2) S(q^3 x) = (1 + q)S(q^4 x) - S(q^5 x).$$

$$q(1 - q^2 x^2 (1 - q)^2) S(q^4 x) = (1 + q)S(q^5 x) - S(q^6 x).$$

$$q(1 - q^2 x^2 (1 - q)^2) S(q^5 x) = (1 + q)S(q^6 x) - S(q^7 x).$$

(25)

Then, we deduce that

$$q(1 - \alpha_x) S(x) + qS(qx) - q^3 \alpha_x S(qx) = \alpha_x qS(qx) + \sum_{n=0}^{\infty} a_n q^{n+1} x^n.$$

$$+ qS(q^2 x) - q^2 \alpha_x S(q^2 x) + qS(q^3 x) - \alpha_x S(q^3 x).$$

$$= S(qx) + qS(qx) - S(q^2 x) + qS(q^2 x) - S(q^3 x).$$

$$+ S(q^3 x) + qS(q^3 x) - S(q^4 x) + \cdots + qS(q^n x) - S(q^{n+1} x).$$

$$= S(qx) + qS(qx) - S(q^2 x) + \cdots + qS(q^n x) - S(q^{n+1} x).$$

(26)

where

$$\alpha_x = x^2 (1 - q)^2.$$  

(28)

For $S$ given by

$$S(x) = \sum_{i=0}^{\infty} a_i x^i.$$  

(29)

We get

$$\alpha_x \left( q^2 \sum_{i=0}^{\infty} a_i q^i x^i + \cdots + q^{2n+1} \sum_{i=0}^{\infty} a_i q^{n+1} x^j \right) = q^3 (1 - q)^2 L \left( \sum_{i=0}^{\infty} a_i q^i \left( 1 + q^i + \cdots + (q^2)^i \right) x^{i+2} \right).$$

$$= q^3 (1 - q^2 L) \sum_{i=0}^{\infty} a_i q^i \left( 1 - (q^2)^{i+1} \right) x^{i+2} = \beta.$$  

(30)

As $n \to \infty$, the last term gives

$$\beta \to q^3 (1 - q^2 L) \sum_{i=0}^{\infty} a_i q^i \frac{q^i}{1 - q} x^i.$$  

(31)

Then, Eq. (27) and Eq. (31), we get

$$q(1 - x^2 (1 - q)^3) \sum_{i=0}^{\infty} a_i x^i - \beta = \left( \sum_{i=0}^{\infty} a_i q^i x^i \right) + (q - 1) a_0.$$  

(32)

Therefore, we get

$$\left( \sum_{i=0}^{\infty} a_i q^i x^i \right) - q(1 - q)^3 \sum_{i=0}^{\infty} a_i x^{i+2} - \beta = \left( \sum_{i=0}^{\infty} a_i q^i x^i \right) + (q - 1) a_0.$$  

(33)

which gives
\[ qa_0 + qa_1 x + \left( \sum_{\nu=0}^{\infty} qa_\nu x^\nu \right) - q (1 - q) L \left( \sum_{\nu=0}^{\infty} a_{\nu-2} x^\nu \right) - \beta = a_0 + a_1 q x + \left( \sum_{\nu=0}^{\infty} a_\nu q x^\nu \right) + (q - 1) a_q. \] 

This implies that
\[ qa_i - q (1 - q)^2 L a_{i-2} - q (1 - q)^2 \frac{j^i}{1 - q} a_{i-2} = a_i q^i. \] 

Hence, we obtain
\[ q (1 - q^{-1}) a_i = q (1 - q)^2 L \left( 1 + \frac{q^i}{1 - q} \right) a_{i-2}. \] 

Then,
\[ q [i-1]_q a_i = q (1 - q) L \left( 1 - \frac{q^i}{1 - q} \right) a_{i-2}. \] 

This implies that
\[ [i-1]_q a_i = L \left( \frac{1 - q}{1 - q} \right) a_{i-2}, \] 

which gives
\[ [i-1]_q a_i = L \frac{1}{[i]_q} a_{i-2}. \] 

Therefore, we get
\[ a_i = \frac{L}{[i]_q} a_{i-2}, i \geq 2. \] 

If \( L < 0 \), we take
\[ a_i = \left( \frac{j \sqrt{-L}}{[i]_q} \right)^i, j = \sqrt{-1}. \] 

Hence, we obtain
\[ a_{i+2} = \left( \frac{j \sqrt{-L}}{[i+2]_q} \right)^i = \frac{(j \sqrt{-L})^i}{[i]_q!} \frac{j^i (-L)}{[i + 2]_q [i + 1]_q} \] 

\[ = \frac{L}{[i + 2]_q [i + 1]_q} a_i. \] 

This verifies Eq. (40). Then, we obtain
\[ S(x) = \sum_{i=0}^{\infty} a_i x^i = e_q \left( j x \sqrt{-L} \right). \] 

Hence, we obtain for \( L > 0 \)
\[ u(t, x) = \left( A e^{-t \sqrt{c^2 L}} + B e^{t \sqrt{c^2 L}} \right) e_q (x \sqrt{L}). \] 

If \( L < 0 \), we obtain
\[ u(t, x) = \left( A \cos \left( t \sqrt{-c^2 L} \right) + B \sin \left( t \sqrt{-c^2 L} \right) \right) e_q \left( j x \sqrt{-L} \right), \] 

where \( A \) and \( B \) are constants.

\section*{4. The Limit Case \( q \to 0 \) of the \( q \)-Space Wave Equation}

In this section, we study the following equation:
\[ \frac{\partial^2 u}{\partial t^2} = c^2 D_{q,x}^2 u, \] 

where
\[ D_{0,x} u(x, y) = \frac{u(x, y) - u(0, y)}{x}. \] 

\textbf{Theorem 2}

1. We have
\[ u(t, x) = \frac{a_0}{1 - L k^2} \left( A \cos \left( t \sqrt{-c^2 L} \right) + B \sin \left( t \sqrt{-c^2 L} \right) \right) \] 

is solution of Equation (49), with any constants \( a_0 \), \( A \) and \( B \), where \( k < 0 \).

2. We have
Proof. For \( u(t, x) \) given by

\[
u(t, x) = R(t)S(x).
\]

Then, we get

\[
S(x)R''(t) = c^2R(t)D_{0,x}^2S(x).
\]

Therefore, Equation (55) gives

\[
R''(t) = c^2LR(t), \quad D_{0,x}^2S(x) = LS(x).
\]

The solution of Equation (56) is given by

\[
R(t) = Ae^{-t\sqrt{c^2L}} + Be^{-t\sqrt{c^2L}} \quad \text{if } L > 0,
\]

and if \( L < 0 \), we obtain

\[
R(t) = A \cos(t\sqrt{-c^2L}) + B \sin(t\sqrt{-c^2L}).
\]

for any constants \( A \) and \( B \). Since we have

\[
D_{0,x}S(x) = \frac{S(x) - S(0)}{x} = F_0(x),
\]

we obtain

\[
D_{0,x}^2S(x) = \frac{F_0(x) - F_0(0)}{x} = \frac{S(x) - S(0)/x - 0}{x} = \frac{S(x) - S(0)}{x^2}.
\]

Using Eq. (57), we get

\[
S(x) - S(0) = LS(x),
\]

which implies that

\[
S(x) - S(0) = Lx^2S(x),
\]

which gives

\[
(1 - Lx^2)S(x) = S(0).
\]

Then, we get

\[
S(x) = \frac{S(0)}{1 - Lx^2} \quad \text{for } L < 0.
\]

If \( L > 0 \), we obtain

\[
S(x) = \frac{S(0)}{1 - Lx^2} \quad \text{if } x \neq \frac{1}{\sqrt{L}}
\]

and for \( x = 1/\sqrt{L} \), we should take \( S(0) = 0 \). Then, we get

\[
u(t, x) = \frac{a_0}{1 - Lx^2} \left(A \cos(t\sqrt{-c^2L}) + B \sin(t\sqrt{-c^2L})\right).
\]

For any constants \( a_0, A \) and \( B \), where \( S(0) = 0 \) and \( L > 0 \). On the other hand, we get

\[
u(t, x) = \begin{cases} 
\frac{a_0}{1 - Lx^2} \left(Ae^{-t\sqrt{c^2L}} + Be^{-t\sqrt{c^2L}}\right) \quad \text{if } x \neq \frac{1}{\sqrt{L}} \quad \text{and } L > 0, \\
S(x) \left(Ae^{-t\sqrt{c^2L}} + Be^{-t\sqrt{c^2L}}\right) \quad \text{if } x \neq \frac{1}{\sqrt{L}} \quad \text{and } L > 0,
\end{cases}
\]

For any constants \( a_0, A \) and \( B \), where \( S(0) = 0 \) and \( L > 0 \). This completes the proof.

\[\square\]

5. Conclusion

In this paper, the \( q \)-space wave equation as well as the limit case \( (q \rightarrow 0) \) of the \( q \)-space wave equation are studied. We expect to study the quantum white noise [10–13] case which is now attractive in mathematical physics area. Also, the numerical study, the graphical presentations of the obtained solutions and the representations of these results will be given in future works. The two- and the three-dimensional types of these equations are in our focus [14, 15].

Data Availability

No data availability.

Conflicts of Interest

All authors declare that they have no competing interests.

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