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Research Article

Hyers–Ulam Stability, Exponential Stability, and Relative Controllability of Non-Singular Delay Difference Equations

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In this paper, we study the uniqueness and existence of the solutions of four types of non-singular delay difference equations by using the Banach contraction principles, fixed point theory, and Gronwall's inequality. Furthermore, we discussed the Hyers–Ulam stability of all the given systems over bounded and unbounded discrete intervals. The exponential stability and controllability of some of the given systems are also characterized in terms of spectrum of a matrix concerning the system. The spectrum of a matrix can be easily obtained and can help us to characterize different types of stabilities of the given systems. At the end, few examples are provided to illustrate the theoretical results.

1. Introduction

In mathematics, we usually observed that many of the biological systems and models can be resolved by using differential equations. Differential equations have a lot of applications in various fields of natural sciences, economics, statistics, and engineering (see [1–4] and the references therein). Although differential equations are too useful, when we discuss a real-life problem, we need to take the sample in discrete form and show the model in a form of difference equations (for details, see [5, 6]). The applications of difference equation have appeared recently in many fields of sciences and technology, mathematical physics, and biological systems. The theory of difference equations will continue its role in mathematics as a whole because during the period of development of mathematics together with information revolution, there are many difference equations to describe the real problem such as the monographs and wind flow. Similarly, many models were described by fractional-order differential equation (FODE), in which the order of derivative is in fraction form rather than an integer form. These types of differential equations have a lot of applications in real life [7, 8]. In [7], the theoretical study of the Caputo–Fabrizio fractional modelling for hearing loss due to mumps virus with optimal control was discussed which is useful contribution in natural science. Also in [8] some novel mathematical analysis of fractal-fractional model of the AH1N1/09 virus and its generalized Caputo-type version was explained.

Any type of system has some properties (qualitative properties), in which the stability is more important. Every differential system has some qualitative properties, in which the stability plays a vital role. Using this, the system performance can be checked. A differential have various types of stabilities, but here we are interested in Hyers–Ulam stability, because nowadays many researchers wants to know about this stability. The idea of Hyers–Ulam stability started in 1940 [9]. Ulam in a seminar, in his presentation he pointed out some problems associated with the stability of group homomorphism. After a year in [10], Hyers gave a positive answer to the Ulam's question by considering Banach Space in place of that group. The general approach of this stability was given in 1978, by Rassias [11]. He also used this idea in the Cauchy difference system. Obloza [12] used this idea in differential equations, and later Jung [13] and Khan et al. [14] used it in the difference equations. This stability was also discussed in fractional differential equation by Gao et al. [15], and some results on Ulam-type stability of a first-order non-linear delay dynamic system were discussed by Shah et al. in [16]. Recently, the Hyers–Ulam stability of second order differential equations by using Mahgoub transform and generalized Hyers–Ulam stability of a coupled hybrid system of integro-differential equations involving ϕ-caputo fractional operator was studied in [17,18]. The existence and Hyers–Ulam stability of solution for almost periodical fractional stochastic differential equation was discussed in [19]. Also in [20], the existence and Hyers–Ulam stability of random impulsive stochastic functional differential equations with finite delays was discussed, which showed that the Hyers–Ulam stability have a lot of contribution in fractional calculus.

Controllability is one of the fundamental concepts in modern mathematical control theory. Kalman's result [21] on controllability assumes that controls are functions on time having values on some non-empty subset of R^n. This is a qualitative property of control systems and is of particular importance in control theory. Systematic study of controllability was started at the beginning of 1960s and theory of controllability is based on the mathematical description of the dynamical system. Many dynamical systems are such that the control does not affect the complete state of the dynamical system but only a part of it. On the other hand, very often in real industrial processes, it is possible to observe only a certain part of the complete state of the dynamical system. Therefore, it is very important to determine whether or not control of the complete state of the dynamical system is possible. Roughly speaking, controllability generally means that it is possible to steer dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Controllability plays an essential role in the development of the modern mathematical control theory. There are important relationships between controllability, stability, and stabilizability of linear control systems [22, 23]. Controllability is also strongly connected with the theory of minimal realization of linear time-invariant control systems. Moreover, it should be pointed out that there exists a formal duality between the concepts of controllability and observability [24].

The delay difference system can be used in the characterization of automatic engine, control theory, and physiology system. Khusainov et al [25] solved the linear autonomous delay-time system with commutable matrices. Diblik and Khusainov [26] gave the description about the solutions of discrete delayed system using the idea [25]. Then, Wang et al. [27] studied relative controllability and exponential stability of non-singular systems. Recently, the generalized Hyers–Ulam–Rassias stability of impulsive difference equations was demonstrated by Almalki et al. [28]. Kuruklis [29] and Yu [30] studied the asymptotic behavior of the variable type delay difference equation. Kosmala and Teixeira [31] provided a good insight and discussed the behavior of solution of the difference equation of the type $U_{k+1} = (A + U_{k-1})/(BU_k + U_{k-1})$. Liu et al [32] designed the exponential behavior of switch discrete-time delay system. Marwen and Sakly [33] discussed the stability techniques about the switched non-linear time-delay difference equations. Yuanyuan [34] described the stability techniques of high-order difference systems. The stability of higher-order rational difference systems was studied by Khaliq [35].

Our present study is focused on the Hyers–Ulam stability and exponential stability of non-singular delay difference system of the form

$$\begin{cases} \mathrm{E}\mathrm{V}_{n+1} = \mathrm{A}\mathrm{V}_n + \mathrm{B}\mathrm{V}_{n-k}, n \geq 0, k \geq 0, \\ V_n = \Phi_n, -k \leq n \leq 0, \end{cases} \tag{1}$$

$$\begin{cases} E\mathbf{V}_{n+1} = A\mathbf{V}_n + B\mathbf{V}_{n-k} + f(n, \mathbf{V}_n), n \geq 0, k \geq 0, \\ \mathbf{V}_n = \phi_n, -k \leq n \leq 0, \end{cases} \tag{2}$$

and

$$\begin{cases} E\mathbb{V}_{n+1} = A\mathbb{V}_n + B\mathbb{V}_{n-k} + \mathbb{F}(n, \mathbb{V}_{n-k}), n \geq 0, k \geq 0, \\ \mathbb{V}_n = \psi_n, -k \leq n \leq 0, \end{cases} \tag{3}$$

where the commutable constant matrices are $E, A, B \in \mathbb{R}^{n\times n}$ and E is non-singular. $\phi \in \mathbb{B}(\mathbf{Z}_+, \mathbf{X})$, the space of bounded sequences, and $\mathbb{F} \in \mathbf{CS}(\mathbf{Z}_+ \times X, X)$, the space of convergent sequences, where $\mathbf{J} = \{-k, -k+1, \ldots, 0\}$, $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$, and $\mathbf{X} = \mathbb{R}^n$. Also, our focus is on relative controllability of the system

$$\begin{cases} E\mathrm{V}_{n+1} = A\mathrm{V}_n + B\mathrm{V}_{n-k} + y(n, \mathrm{V}_n) + CU_n, n \in \mathbf{I}, k \geq 0, \\ \mathrm{V}_n = \Psi_n, -k \leq n \leq 0, \end{cases} \tag{4}$$

where $\mathbf{I} = \{0, 1, 2, \ldots, n\}, n > 0, C \in \mathbb{R}^{n\times n}, y \in \mathbf{CS}(\mathbf{Z}_{+\times\mathbf{X},\mathbf{X}})$, and the control function $U(\cdot)$ takes values from $L^2(\mathbf{I}, \mathbb{R}^n)$. The continuous form of this work is given in [27]. The Hyers–Ulam stability of (3) was recently presented in [36].

2. Preliminaries

Here, we discuss some notations and definitions, which will be needed for our main work. By $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$, we will denote the n-dimensional Euclidean space with vector norm $\|\cdot\|$ and $n\times n$ matrices with real-valued entries. The vector infinite-norm is defined as $\|v\| = \max_{1\leq i\leq n}|v_i|$ and the matrix infinite-norm is given as $\|A\| = \max_{1\leq i\leq n}\sum_{j=1}^{n}|a_{ij}|$ where $v \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$; also, v_i and a_{ij} are the elements of the vector v and the matrix A. $\mathbb{B}(\mathbf{I}, \mathbf{X})$ will be the space of all bounded sequences from $\mathbf{I}$ to $\mathbf{X}$ with norm $\|v\|_C = \sup_{n\in\mathbf{I}}\|v_n\|$. We will use $\mathbb{R}$, $\mathbf{Z}$ and $\mathbf{Z}_+$ for the set of real, integer, and non-native integer numbers, respectively. Also, we define $\mathbb{B}'(\mathbf{I}, \mathbf{X}) = \{v \in \mathbb{B}(\mathbf{I}, \mathbf{X}); v' \in \mathbb{B}'(\mathbf{I}, \mathbf{X})\}$.

Lemma 1. *The non-singular delay difference systems (1)–(4) have the solutions:*

$$V_n = A^n E^{-n}\Phi_0 + BA^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\Phi_{i-k} + BA^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^iV_{i-k},$$

$$V_n = A^n E^{-n}\phi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\left(B\phi_{i-k} + f\left(i,V_i\right)\right) + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i\left(BV_{i-k} + f\left(i,V_i\right)\right),$$

$$V_n = A^n E^{-n}\psi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\left(B\psi_{i-k} + F\left(i,\psi_{i-k}\right)\right) + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i\left(BV_{i-k} + F\left(i,V_{i-k}\right)\right), \tag{5}$$

and

$$V_n = A^n E^{-n}\Psi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\left(B\Psi_{i-k} + y\left(i,V_i\right) + CU_i\right) + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i\left(BV_{i-k} + y\left(i,V_i\right) + CU_i\right), \tag{6}$$

respectively, where $AE = EA, AB = BA,$ and $EB = BE$. The proofs can easily be obtained by successively putting the values of $n \in \{-k, -k+1, \ldots\}$.

Definition 1. The solution of system (1) is said to be exponentially stable if there exist positive real numbers λ_1 and λ_2, such that

$$\left\|V_n\right\| \le \lambda_1 e^{-\lambda_2 n}, \forall n \ge 0. \tag{7}$$

Definition 2. For a positive number ϵ, the sequence ψ_n is said to be an ϵ-approximate solution of (1)–(3) if the following holds:

$$\begin{cases} \left\|E\psi_{n+1} - A\psi_n - B\psi_{n-k}\right\| \le \epsilon, n \ge 0, k \ge 0 \\ \left\|\psi_n - \Phi_n\right\| \le \epsilon, -k \le n \le 0 \end{cases},$$
$$\begin{cases} \left\|E\psi_{n+1} - A\psi_n - B\psi_{n-k} - f\left(n,\psi_n\right)\right\| \le \epsilon, n \ge 0, k \ge 0 \\ \left\|\psi_n - \phi_n\right\| \le \epsilon, -k \le n \le 0 \end{cases}, \tag{8}$$
$$\begin{cases} \left\|E\psi_{n+1} - A\psi_n - B\psi_{n-k} - \mathbb{F}\left(n,\psi_{n-k}\right)\right\| \le \epsilon, n \ge 0, k \ge 0 \\ \left\|\psi_n - \Psi_n\right\| \le \epsilon, -k \le n \le 0. \end{cases}$$

Definition 3. Systems (1)–(3) are said to be Hyers–Ulam stable if for every ϵ-approximate solutions ψ_n of systems (1)–(3) there are exact solutions $\mathbf{Y}_n$ of (1)–(3) and a non-negative real number $\mathbf{K}$ such that

$$\left\|\mathbf{Y}_n - \psi_n\right\| \le \mathbf{K}\epsilon, n \in I. \tag{9}$$

Definition 4. System (4) is said to be relatively controllable, if for initial vector function $\Psi \in \mathbb{B}'(\mathbf{J},\mathbf{X})$ and final state of the vector function $v_1 \in \mathbf{X}$, there exists a control $u \in \mathscr{L}^2(\mathbf{I},\mathbf{X})$ such that (4) has a solution $v \in \mathbb{B}(\{-v,\ldots,n_1\},\mathbf{X})$ which satisfies the boundary condition $v_{n_1} = v_1$.

Remark 1. It is clear from (5) that $\mathbf{Y} \in \mathbb{B}'(\mathbf{I},\mathbf{X})$ satisfied (5) if and only if there exists $f \in \mathbb{B}(\mathbf{I},\mathbf{X})$ satisfying

$$\begin{cases} \left\|f_n\right\| \le \epsilon, n \in I, \\ Ey_{n+1} = Ay_n + By_{n-k} + f_n, n \in \mathbf{Z}_+, \\ y_n = \Phi_n, -k \le n \le 0. \end{cases}$$
$$\begin{cases} \left\|f_n\right\| \le \epsilon, n \in I, \\ Ey_{n+1} = Ay_n + By_n + f\left(n,y_{n-k}\right) + f_n, n \in \mathbf{Z}_+, \\ y_n = \phi_n, -k \le n \le 0. \end{cases} \tag{10}$$
$$\begin{cases} \left\|f_n\right\| \le \epsilon, n \in I, \\ Ey_{n+1} = Ay_n + By_{n-k} + \mathbb{F}\left(n,y_{n-k}\right) + f_n, n \in \mathbf{Z}_+, \\ y_n = \Psi_n, -k \le n \le 0. \end{cases}$$

3. Existence and Uniqueness of Solutions

Here, we will discuss the existence and uniqueness of the solution of system (1). For this, we need the following assumptions:

Λ_1: the linear system $A\mathbf{G}_{n+1} = M\mathbf{G}_n + N\mathbf{G}_{n-k}$ is well modelled.

Λ_2: $\|A^{n-1}\|\|E^{-n}\|L < 1$.

Theorem 1. *If assumptions Λ_1 and Λ_2 hold, then system (1) has a unique solution $V \in \mathbb{B}(\mathbf{I},\mathbf{X})$.*

Proof. Define $T: \mathbb{B}(\mathbf{I},\mathbf{X}) \longrightarrow \mathbb{B}(\mathbf{I},\mathbf{X})$ by

$$(TV)_n = A^n E^{-n}\Phi_0 + BA^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\Phi_{i-k} + BA^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^iV_{i-k}. \tag{11}$$

Now, for any $V, V' \in \mathbb{B}(\mathbf{I},\mathbf{X})$, we have

$$\left\|(TV)_n-(TV')_n\right\|=\left\|\begin{matrix}A^nE^{-n}\Phi_0+\mathrm{BA}^{n-1}E^{-n}\sum_{i=0}^{k}A^{-i}E^{i}\Phi_{i-k}+\mathrm{BA}^{n-1}E^{-n}\sum_{i=k+1}^{n}A^{-i}E^{i}V_{i-k}\\ -A^{n-1}E^{-n}\Phi_0-\mathrm{BA}^{n-1}E^{-n}\sum_{i=k+1}^{n}A^{-i}E^{i}\Phi_{i-k}+\\ -\mathrm{BA}^{n-1}E^{-n}\sum_{i=k+1}^{n}A^{-i}E^{i}V'_{i-k}.\end{matrix}\right\|. \quad (12)$$

This implies that

$$\begin{aligned}\left\|(TV)_n-(TV')_n\right\|&\le\|B\|\left\|A^{n-1}\right\|\left\|E^{-n}\right\|\sum_{i=k+1}^{n}\left\|A^{-i}\right\|\left\|E^{i}\right\|\left\|V_{i-k}-V'_{i-k}\right\|\\&\le\|B\|\left\|A^{n-1}\right\|\left\|E^{-n}\right\|\|L\|\left\|V-V'\right\|.\end{aligned} \quad (13)$$

Thus, T is contraction if $\|A^n\|\|E^{-n}\|L<1$, so (by BCP) it has a unique fixed point and will be the solution of system (1). Similarly, we can show the existence and uniqueness of solutions of systems (2)–(4). For (3), we also refer to [36]. □

4. Hyers–Ulam Stability over Bounded Discrete Interval

In this part of the paper, we will discuss the Hyers–Ulam stability over bounded discrete interval. Before the result, we will put the following assumptions:

Λ_1: the linear system $\mathrm{EV}_{n+1}=\mathrm{AV}_n+\mathrm{BV}_{n-k}$ is well posed.

Λ_2: there exists a constant η such that

$$\sum_{r=1}^{n-k}\phi_r\le\eta\text{foreachn}\in\mathbf{I}. \quad (14)$$

Theorem 2. *If Λ_1 and Λ_2 and Remark 1 are satisfied, then system (1) is Hyers–Ulam stable over bounded interval.*

Proof. The solution of difference system (1) is

$$\begin{aligned}V_n&=A^nE^{-n}\Phi_0+\mathrm{BA}^{n-1}E^{-n}\sum_{i=0}^{k}A^{-i}E^{i}\Phi_{i-k}\\&\quad+\mathrm{BA}^{n-1}E^{-n}\sum_{i=k+1}^{n}A^{-i}E^{i}V_{i-k}.\end{aligned} \quad (15)$$

From Remark 1, the solution of

$$\begin{cases}\mathrm{EU}_{n+1}=\mathrm{AU}_n+\mathrm{BU}_{n-k}+f_n, n\ge0, k\ge0,\\ U_n=\Phi_n, -k\le n\le0,\end{cases} \quad (16)$$

is

$$\begin{aligned}U_n&=A^nE^{-n}\Phi_0+\mathrm{BA}^{n-1}E^{-n}\sum_{i=0}^{k}A^{-i}E^{i}\Phi_{i-k}\\&\quad+A^{n-1}E^{-n}\sum_{i=k+1}^{n}A^{-i}E^{i}\left(BU_{i-k}+f_{i-k}\right).\end{aligned} \quad (17)$$

Now, we have

$$\begin{aligned}\left\|U_n-V_n\right\|&=\left\|\begin{matrix}A^nE^{-n}\Phi_0+\mathrm{BA}^{n-1}E^{-n}\sum_{i=0}^{k}A^{-i}E^{i}\Phi_{i-k}\\ A^{n-1}E^{-n}\Phi_0-\mathrm{BA}^{n-1}E^{-n}\sum_{i=k+1}^{n}A^{-i}E^{i}\Phi_{i-k}+\\ \mathrm{BA}^{n-1}E^{-n}\sum_{i=k+1}^{n}A^{-i}E^{i}U_{i-k}\end{matrix}\right\|\\&=\left\|A^{n-1}E^{-n}\sum_{i=k+1}^{n}A^{-i}E^{i}\left(BY_{i-k}+f_{i-k}\right)\right.\\&\quad\left.-\mathrm{BA}^{n-1}E^{-n}\sum_{i=k+1}^{n}A^{-i}E^{i}V_{i-k}\right\|\\&\le\left\|A^{n-1}\right\|\left\|E^{-n}\right\|\left\|\sum_{i=k+1}^{n}\left\|A^{-i}\right\|\right\|\left\|E^{-i}\right\|\left\|\mathrm{BU}_{i-k}-\mathrm{BV}_{i-k}\right\|\\&\quad+\left\|A^{n-1}\right\|\left\|E^{-n}\right\|\sum_{i=k+1}^{n}\left\|A^{-i}\right\|\left\|E^{i}\right\|\left\|f_{i-k}\right\|,\\ \left\|U_n-V_n\right\|&=\left\|A^{n-1}\right\|\left\|E^{-n}\right\|\sum_{i=k+1}^{n}\left\|A^{-i}\right\|\left\|E^{i}\right\|\left\|f_{i-k}\right\|\\&\le\left\|A^{n-1}\right\|\left\|E^{-n}\right\|\sum_{i=k+1}^{n}\left\|A^{-i}\right\|\left\|E^{i}\right\|\left\|\epsilon\phi_{i-k}\right\|\\&=\epsilon\left\|A^{n-1}\right\|\left\|E^{-n}\right\|\sum_{r=1}^{n-k}\left\|A^{-k-r}\right\|\left\|E^{k+r}\right\|\phi_r\\&=\epsilon L^4\sum_{r=1}^{n-k}\phi_r\\&\le\epsilon L^4\eta\\&=l\epsilon,\end{aligned} \quad (18)$$

where $l=L^4\eta$. Hence, system (1) is Hyers–Ulam stable over bounded discrete interval.

Next, we will show that system (2) is Hyers–Ulam stable. Again, we need one more assumption:

Λ_3: the map F: $\mathbf{I}\times\mathbf{X}\longrightarrow\mathbf{X}$ satisfies the Carathéodory condition

$$\left\|F(n,\vartheta)-F(n,\vartheta')\right\|\le\mathrm{K}\left\|\left\|\vartheta-\vartheta'\right\|\right\|, \quad (19)$$

for some $\mathrm{K}\ge0$ and for all $\vartheta,\vartheta'\in\mathbb{B}(\mathbf{I},\mathbf{X})$. □

Theorem 3. *If Λ_1–Λ_3 along with (2.6) and Remark 1 are satisfied, then system (2) is Hyers–Ulam stable over bounded interval.*

Proof. The solution of delay difference system (2) is

$$\mathbf{V}_n = A^n E^{-n}\phi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^{i}\left(B\phi_{i-k} + f\left(i,\mathbf{V}_i\right)\right) + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(B\mathbf{V}_{i-k} + f\left(i,\mathbf{V}_i\right)\right). \tag{20}$$

Also, from Remark 1, the solution of

$$\begin{cases} \mathrm{EU}_{n+1} = \mathrm{AU}_n + \mathrm{BU}_{n-k} + f\left(n,U_n\right) + f_n, n \geq 0, k \geq 0, \\ U_n = \phi_n, -k \leq n \leq 0, \end{cases} \tag{21}$$

is

$$U_n = A^n E^{-n}\phi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^{i}\left(B\phi_{i-k} + f\left(i,U_i\right)\right) + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(\mathrm{BU}_{i-k} + f\left(i,U_i\right) + f_{i-k}\right). \tag{22}$$

Now, we have

$$\begin{aligned}
\left\|U_n - \mathbf{V}_n\right\| &= \left\| \begin{matrix} A^n E^{-n}\phi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^{i}\left(B\phi_{i-k} + f\left(i,U_i\right)\right) \\ A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(BU_{i-k} + f\left(i,U_i\right) + f_{i-k}\right) \\ A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(B\mathbf{V}_{i-k} + f\left(i,\mathbf{V}_i\right)\right) \end{matrix} \right\| \\
&= A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(B\mathbf{V}_{i-k} + f\left(i,\mathbf{V}_i\right)\right) \\
&= \left\|A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(BU_{i-k} + f\left(i,U_i\right)\right)\right\| \\
&\quad + \left\|A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}f_{i-k} - A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(B\mathbf{V}_{i-k} + f\left(i,\mathbf{V}_i\right)\right)\right\| \\
&\leq \left\|A^{n-1}\right\|\left\|E^{-n}\right\|\sum_{i=k+1}^{n}\left\|A^{-i}E^{i}\right\|\left\|\left(BU_{i-k} - B\mathbf{V}_{i-k}\right)\right\| \\
&\quad + \left\|f\left(i,U_i\right) - f\left(i,\mathbf{V}_i\right)\right\| + \left\|f_{i-k}\right\|, \\
\left\|U_n - V_n\right\| &\leq \left\|A^{n-1}\right\|\left\|E^{-n}\right\|\sum_{i=k+1}^{n}\left\|A^{-i}E^{i}\right\|\left(\left\|BU_{i-k} - B\mathbf{V}_{i-k}\right\| + L\left\|U_i - \mathbf{V}_i\right\| + \left\|f_{i-k}\right\|\right) \\
&= \left\|A^{n-1}\right\|\left\|E^{-n}\right\|\sum_{i=k+1}^{n}\left\|A^{-i}E^{i}\right\|\left\|f_{i-k}\right\| \\
&\leq \left\|A^{n-1}\right\|\left\|E^{-n}\right\|\sum_{i=k+1}^{n}\left\|A^{-i}E^{i}\right\|\epsilon\phi_{i-k} \\
&= \epsilon\left\|A^{n-1}\right\|\left\|E^{-n}\right\|\sum_{r=1}^{n-k}\left\|A^{-k-r}\right\|\left\|E^{k+r}\right\|\phi_r \\
&= L^4\sum_{r=1}^{n-k}\phi_r \\
&\leq L^4\eta \\
&= K\epsilon.
\end{aligned} \tag{23}$$

Thus, system (2) is Hyers–Ulam stable.

The Hyers–Ulam stability of system (3) over bounded discrete interval is discussed in [36]. □

5. Hyers–Ulam Stability over an Unbounded Discrete Interval

Here, we discuss the Hyers–Ulam stability of systems (1)–(3) over an unbounded discrete interval; we have some assumptions:

A_1: the operator family $\|L^4\| \le Ne^{-\nu n}, n \ge 0, \nu \ge 0, N \ge 1$.

A_2: the linear system $A\mathbf{G}_{n+1} = M\mathbf{G}_n + N\mathbf{G}_{n-k}$ is well posed.

A_3: also, assume that

$$\sum_{r=1}^{n-1} \phi_r \le \eta, \tag{24}$$

for each $n \in \mathbb{Z}_+$, and for $\eta \ge 0$.

Theorem 4. *If A_1–A_3 along with (2.6) and Remark 1 are satisfied, then system (1) is Hyers–Ulam stable over an unbounded interval.*

Proof. The exact solution of non-autonomous difference system (1) is

$$\begin{aligned} V_n &= A^n E^{-n}\Phi_0 + \mathrm{B}A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\Phi_{i-k} \\ &\quad + \mathrm{B}A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^iV_{i-k}. \end{aligned} \tag{25}$$

Let Y be the approximate solution of system (1); then, clearly, for a sequence f_n, with $\|f_n\| \le \epsilon$, we have

$$\begin{cases} \mathrm{E}Y_{n+1} = \mathrm{A}Y_n + \mathrm{B}Y_{n-k} + f_n, n \ge 0, k \ge 0, \\ Y_n = \Phi_n, -k \le n \le 0, \end{cases} \tag{26}$$

and

$$\begin{aligned} Y_n &= A^n E^{-n}\Phi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\Phi_{i-k} \\ &\quad + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i\left(\mathrm{B}Y_{i-k} + f_{i-k}\right). \end{aligned} \tag{27}$$

Now, we have

$$\begin{aligned} \|Y_n - U_n\| &= \Bigg\|A^n E^{-n}\Phi_0 + BA^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\Phi_{i-k} \\ &\quad + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i\left(BY_{i-k} + f_{i-k}\right) \\ &\quad - A^{n-1}E^{-n}\Phi_0 - BA^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i\Phi_{i-k} + \\ &\quad - BA^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^iU_{i-k}\Bigg\| \\ &= \Bigg\|A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i\left(BY_{i-k} + f_{i-k}\right) \\ &\quad - BA^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^iU_{i-k}\Bigg\| \\ &\le \|A^{n-1}\|\|E^{-n}\|\sum_{i=k+1}^{n} \|A^{-i}E^i\|\|BY_{i-k} - BY_{i-k} + f_{i-k}\|, \\ &\le \|A^{n-1}\|\|E^{-n}\|\sum_{i=k+1}^{n} \|A^{-i}E^i\|\|BY_{i-k} - BY_{i-k}\| \\ &\quad + \|A^{n-1}\|\|E^{-n}\|\sum_{i=k+1}^{n} \|A^{-i}E^i\|\|f_{i-k}\| \\ &\le \|A^{n-1}\|\|E^{-n}\|\sum_{i=k+1}^{n} \|A^{-i}E^i\|\|f_{i-k}\| \\ &\le \|A^{n-1}\|\|E^{-n}\|\sum_{i=k+1}^{n} \|A^{-i}E^i\|\epsilon\phi_{i-k} \\ &= \epsilon\|A^{n-1}\|\|E^{-n}\|\sum_{r=1}^{n-k} \|A^{-k-r}\|\|E^{k+r}\|\phi_r \\ &= \epsilon L^4\sum_{r=1}^{n-k} \phi_r \\ &\le \epsilon L^4\eta \\ &\le Ne^{-\nu n}\eta\epsilon \\ &= L, \end{aligned} \tag{28}$$

where $L = \mathrm{M}e^{-\nu k}\eta$. Thus, system (1) is Hyers–Ulam stable over an unbounded interval.

To prove the Hyers–Ulam stability of system (2), we have to add one more assumption:

A_4: the continuous function $\mathbb{H}$: $\mathbb{Z}_+ \times X \longrightarrow X$ satisfies the Carathéodory condition

$$\|\mathbb{H}(n,\omega) - \mathbb{H}(n,\omega')\| \le K\|\omega - \omega'\|, K \ge 0, \tag{29}$$

for every $n \in \mathbb{Z}_+\omega, \omega' \in X$. □

Theorem 5. *If A_1–A_4 along with (2.6) and Remark 1 are satisfied, then system (2) is Hyers–Ulam stable over an unbounded interval.*

Proof. The solution of delay difference system (2) is

$$\begin{aligned}\mathbf{V}_n &= A^n E^{-n}\phi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^{i}\left(B\phi_{i-k} + f\left(i,\mathbf{V}_i\right)\right)\\ &\quad + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(B\mathbf{V}_{i-k} + f\left(i,\mathbf{V}_i\right)\right).\end{aligned} \tag{30}$$

Also, from Remark 1, the solution of

$$\begin{cases} \mathrm{EU}_{n+1} = \mathrm{AU}_n + \mathrm{BU}_{n-k} + f\left(n,U_n\right) + f_n, n\geq 0, k\geq 0,\\ U_n = \phi_n, -k\leq n\leq 0,\end{cases} \tag{31}$$

is

$$\begin{aligned}U_n &= A^n E^{-n}\phi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^{i}\left(B\phi_{i-k} + f\left(i,U_i\right)\right)\\ &\quad + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(\mathrm{BU}_{i-k} + f\left(i,U_i\right) + f_{i-k}\right).\end{aligned} \tag{32}$$

Now, we have

$$\begin{aligned}\left\|U_n - V_n\right\| &= \left\|A^n E^{-n}\phi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^{i}\left(B\phi_{i-k} + f\left(i,U_i\right)\right)\right\| +\\ &\quad A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(BU_{i-k} + f\left(i,U_i\right) + f_{i-k}\right) -\\ &\quad A^n E^{-n}\phi_0 - A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^{i}\left(B\phi_{i-k} + f\left(i,V_i\right)\right) -\\ &\quad A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(BV_{i-k} + f\left(i,V_i\right)\right)\\ &= \left\|A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(BU_{i-k} + f\left(i,U_i\right) + f_{i-k}\right)\right\| -\\ &\quad A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(BV_{i-k} + f\left(i,V_i\right)\right)\\ &= \left\|A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(BU_{i-k} + f\left(i,U_i\right)\right)\right\| +\\ &\quad A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i} f_{i-k} - A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(BV_{i-k} + f\left(i,V_i\right)\right)\\ &\leq \left\|A^{n-1}\right\|\left\|E^{-n}\right\|\sum_{i=k+1}^{n}\left\|A^{-i}E^{i}\right\|\left\|BU_{i-k} - BV_{i-k}\right\|\\ &\quad + \left\|f\left(i,U_i\right) - f\left(i,V_i\right)\right\| + \left\|f_{i-k}\right\|,\\ &\leq \left\|A^{n-1}\right\|\left\|E^{-n}\right\|\sum_{i=k+1}^{n}\left\|A^{-i}E^{i}\right\|\left\|BU_{i-k} - BV_{i-k}\right\| + L\left\|U_i - V_i\right\| + \left\|f_{i-k}\right\|\\ &= \left\|A^{n-1}\right\|\left\|E^{-n}\right\|\sum_{i=k+1}^{n}\left\|A^{-i}E^{i}\right\|\left\|f_{i-k}\right\|\\ &\leq \left\|A^{n-1}\right\|\left\|E^{-n}\right\|\sum_{i=k+1}^{n}\left\|A^{-i}E^{i}\right\|\phi_{i-k}\\ &= \left\|A^{n-1}\right\|\left\|E^{-n}\right\|\sum_{r=1}^{n-k}\left\|A^{-k-r}\right\|\left\|E^{k+r}\right\|\phi_r\\ &= L^4\sum_{r=1}^{n-k}\phi_r\\ &\leq Ne^{-\nu n}\eta\\ &= K,\end{aligned} \tag{33}$$

where $K = Ne^{-\gamma m}\eta$. Thus, system (2) is Hyers–Ulam stable. □

Theorem 6. *System (3) is Hyers–Ulam stable over an unbounded interval.*

For the proof, see [36].

6. Exponential Stability

In this part of the paper, we will present the exponential stability of system (1). First, we recall that a discrete system is said to be exponentially stable if there exist two positive constants M and α such that $\|V_n\| \leq Me^{-\alpha n}$ for all $n \in \mathbf{Z}_+$. Before going to the result, we will consider the following assumptions:

(1) Let $\sigma(AE^{-1}) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the eigenvalues of AE^{-1} with

$$|\lambda_1| \leq |\lambda_2| \leq \cdots |\lambda_n| \leq r, r \in (0, 1). \tag{34}$$

(2) $\|A^n E^{-n}\| \leq Ne^{-\alpha n}$ for some positive number α and for all $n \in \mathbf{Z}_+$.

(3) $N^2\|B\| - \alpha < 0$.

(4) There exists $L > 0$, such that $\|f(i, \mathbf{V}_i)\|\| \leq L\|\|\mathbf{V}_i\|$ fori ≥ 0 and $\mathbf{V}_i \in \mathbb{R}^n$.

(5) There exists $M > 0$, such that $\|\mathbb{F}(i, \mathbb{V}_i)\| \leq M\|\mathbb{V}_i\|$ fori ≥ 0 and $\mathbb{V}_i \in \mathbb{R}^n$.

(6) $N^2\|B\|L^2 - \alpha < 0$.

(7) $N^2 M\|B\| - \alpha < 0$.

Theorem 7. *Assume that (1)–(3) are satisfied. Then, system (1) is exponentially stable.*

Proof. The solution of system (1) is

$$\begin{aligned} V_n &= A^n E^{-n}\Phi_0 + BA^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\Phi_{i-k} \\ &\quad + BA^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i V_{i-k}. \end{aligned} \tag{35}$$

Now,

$$\begin{aligned} \|V_n\| &= \left\| A^n E^{-n}\Phi_0 + BA^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\Phi_{i-k} + BA^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i V \right\| \\ &\leq \|A^n E^{-n}\|\|\Phi_0\| + \|B\|\|A^{n-1}E^{-n}\|\sum_{i=0}^{k}\|A^{-i}E^i\|\|V_{i-k}\| \\ &\quad + \|B\|\|A^{n-1}E^{-n}\|\sum_{i=k+1}^{n}\|A^{-i}E^i\| \\ &\leq Ne^{-\alpha n}\|\Phi_0\| + \|B\|\left\| Ne^{-\alpha n}\sum_{i=0}^{k} Ne^{-\alpha n}\right\|\|V_{i-k}\|, \\ \|V_n\| &\leq Ne^{-\alpha n}\left(\|\Phi_0\| + \|B\|\sum_{i=0}^{k} Ne^{-\alpha n}\|\Phi_{i-k}\| + \|B\|\sum_{i=k+1}^{n} Ne^{-\alpha n}\|V_{i-k}\|\right), \\ e^{\alpha n}\|V_n\| &\leq N\|\Phi_0\| + \|B\|\sum_{i=0}^{k} N^2 e^{-\alpha n}\|\Phi_{i-k}\| + \|B\|\sum_{i=k+1}^{n} N^2 e^{-\alpha n}\|V_{i-k}\| \\ &= M(\phi, \phi_i) + \|B\|\sum_{i=k+1}^{n} \alpha e^{-\alpha n}\|V_{i-k}\|, \end{aligned} \tag{36}$$

where $M(\phi, \phi_i) = N\|\Phi_0\| + \|B\|\sum_{i=0}^{k} N^2 e^{-\alpha n}\|\Phi_{i-k}\|$; now, using the Gronwall inequality, we have

$$e^{\alpha n}\|V_n\| \leq M(\phi, \phi_i)e^{N^2\|B\|n}. \tag{37}$$

From this, we have

$$\|V_n\| \leq M(\phi, \phi_i)e^{(N^2\|B\| - \alpha)n}. \tag{38}$$

Using definition of stability and assumption (3), the result follows. □

Theorem 8. *Assume that (1), (2), (4), and (6) are satisfied. Then, system (2) is exponentially stable.*

Proof. The solution of (2) is in the form of

$$\begin{aligned} \mathbf{V}_n &= A^n E^{-n}\phi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\left(B\phi_{i-k} + f(i, \mathbf{V}_i)\right) \\ &\quad + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i\left(B\mathbf{V}_{i-k} + f(i, \mathbf{V}_i)\right). \end{aligned} \tag{39}$$

Now,

$$\left\|V_n\right\| = \left\| \begin{matrix} A^n E^{-n}\phi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^{i}\left(B\phi_{i-k} + f\left(i,\mathbf{V}_i\right)\right) \\ A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(B\mathbf{V}_{i-k} + f\left(i,\mathbf{V}_i\right)\right) \end{matrix} \right\|,$$

$$\begin{aligned} \left\|\mathbb{V}_n\right\| &\le \left\|A^n E^{-n}\right\|\left\|\phi_0\right\| + \left\|A^{n-1}E^{-n}\right\|\sum_{i=0}^{k}\left\|A^{-i}E^{1}\right\|\left\|B\phi_{i-k} + f\left(i,\mathbf{V}_i\right)\right\| \\ &\quad + \left\|A^{n-1}E^{-n}\right\|\sum_{i=k+1}^{n}\left\|A^{-i}E^{i}\right\|\left\|B\mathbf{V}_{i-k} + f\left(i,\mathbf{V}_i\right)\right\| \\ &\le Ne^{-\alpha n}\left\|\phi_0\right\| + Ne^{-\alpha n}\sum_{i=0}^{k} Ne^{-\alpha n}\left\|B\phi_{i-k}\right\| + Ne^{-\alpha n}\sum_{i=0}^{k} Ne^{-\alpha n}\left\|f\left(i,\mathbf{V_i}\right)\right\| \\ &\quad + Ne^{-\alpha n}\sum_{i=k+1}^{n} Ne^{-\alpha n}\|B\|\left\|\mathbf{V}_{i-k}\right\| + Ne^{-\alpha n}\sum_{i=k+1}^{n} Ne^{-\alpha n}\left\|f\left(i,\mathbf{V_i}\right)\right\|, \\ \left\|\mathbf{V_n}\right\| &\le e^{-\alpha n}\left(N\left\|\phi_0\right\| + \sum_{i=0}^{k} N^2 e^{-\alpha n}\left\|B\phi_{i-k}\right\| + \sum_{i=0}^{k} N^2 e^{-\alpha n}\left\|f\left(i,\mathbf{V_i}\right)\right\|\right. \\ &\quad \left. + \sum_{i=k+1}^{n} N^2 e^{-\alpha n}\|\left\|\mathbf{V}_{i-k}\right\| + \sum_{i=k+1}^{n} N^2 e^{-\alpha n}\left\|f\left(i,\mathbf{V_i}\right)\right\|\right. . \end{aligned} \tag{40}$$

Using (4), we have

$$\begin{aligned} e^{\alpha n}\left\|\mathbf{V_n}\right\| &\le N\left\|\phi_0\right\| + \sum_{i=0}^{k} N^2 e^{-\alpha n}\|B\|\left\|\phi_{i-k}\right\| + \sum_{i=0}^{k} N^2 e^{-\alpha n} L\|\mathbf{V}\|_{\mathbf{i}} \\ &\quad + \sum_{i=k+1}^{n} N^2 e^{-\alpha n}\|B\|\left\|\mathbf{V}_{i-k}\right\| + \sum_{i=k+1}^{n} N^2 e^{-\alpha n} L\left\|\mathbf{V_i}\right\|. \end{aligned} \tag{41}$$

Using the Gronwall inequality, we have

$$e^{\alpha n}\left\|\mathbf{V_n}\right\| \le M\left(\phi,\phi_1\right)e^{\left(N^2\|B\|L^2\right)n}, \tag{42}$$

where $M(\phi,\phi_1) = N\|\phi_0\| + \sum_{i=0}^{k} N^2 e^{-\alpha n}\|B\|\|\phi_{i-k}\| > 0$. From this, we have

$$\left\|\mathbf{V_n}\right\| \le M\left(\phi,\phi_1\right)e^{\left(N^2\|B\|L^2-\alpha\right)n}. \tag{43}$$

From (50), the desired result holds. □

Theorem 9. *Assume that (1), (2), (5), and (7) are satisfied. Then, system (3) is exponentially stable.*

Proof. The solution of (3) is in the form of

$$\begin{aligned} \mathbb{V}_n &= A^n E^{-n}\psi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^{i}\left(B\psi_{i-k} + \mathbb{F}\left(i,\psi_{i-k}\right)\right) \\ &\quad + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(B\mathbb{V}_{i-k} + \mathbb{F}\left(i,\mathbb{V}_{i-k}\right)\right). \end{aligned} \tag{44}$$

Now consider

$$\begin{aligned} \left\|V_n\right\| &= \left\|A^n E^{-n}\psi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^{i}\left(B\psi_{i-k} + F\left(i,\psi_{i-k}\right)\right)\right\| \\ &\quad + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(BV_{i-k} + F\left(i,V_{i-k}\right)\right) \\ &\le \left\|A^n E^{-n}\right\|\left\|\psi_0\right\| \\ &\quad + \left\|A^{n-1}E^{-n}\right\|\sum_{i=0}^{k}\left\|A^{-i}E^{i}\right\|B\psi_{i-k} + F\left(i,\psi_{i-k}\right) \\ &\quad + \left\|A^{n-1}E^{-n}\right\|\sum_{i=k+1}^{n}\left\|A^{-i}E^{i}\right\|BV_{i-k} + F\left(i,V_{i-k}\right) \\ &\le Ne^{-\alpha n}\left\|\psi_0\right\| + Ne^{-\alpha n}\sum_{i=0}^{k} Ne^{-\alpha n}\left\|B\psi_{i-k} + F\left(i,\psi_{i-k}\right)\right\| \\ &\quad + Ne^{-\alpha n}\sum_{i=k+1}^{n} Ne^{-\alpha n}\left\|B\mathbb{V}_{i-k} + F\left(i,V_{i-k}\right)\right\| \\ &= e^{-\alpha n}\left(N\left\|\psi_0\right\| + \sum_{i=0}^{k} N^2 e^{-\alpha n}\left\|B\psi_{i-k} + F\left(i,\psi_{i-k}\right)\right\|\right) \\ &\quad + \sum_{i=k+1}^{n} N^2 e^{-\alpha n}\left\|BV_{i-k}\right\| + \sum_{i=k+1}^{n} N^2 e^{-\alpha n}\left\|F\left(i,V_{i-k}\right)\right\|. \end{aligned} \tag{45}$$

This implies that

$$\begin{aligned} e^{\alpha n}\left\|\mathbb{V}_n\right\| &\leq N\left\|\psi_0\right\| + \sum_{i=0}^{k} N^2 e^{-\alpha n}\left\|B\psi_{i-k} + \mathbb{F}\left(i,\psi_{i-k}\right)\right\| + \sum_{i=k+1}^{n} N^2 e^{-\alpha n}\|B\|\left\|\mathbb{V}_{i-k}\right\| \\ &\quad + \sum_{i=k+1}^{n} N^2 e^{-\alpha n} \\ &= M\left(\psi,\psi_1\right) + \sum_{i=k+1}^{n} N^2 e^{-\alpha n}\|B\|\left\|\mathbb{V}_{i-k}\right\| + \sum_{i=k+1}^{n} N^2 e^{-\alpha n}\left\|\mathbb{F}\left(i,\mathbb{V}_{i-k}\right)\right\|. \end{aligned} \tag{46}$$

Using (5), we have

$$\begin{aligned} e^{\alpha n}\left\|\mathbb{V}_n\right\| &\leq M\left(\psi,\psi_1\right) + \sum_{i=k+1}^{n} N^2 e^{-\alpha n}\|B\|\left\|\mathbb{V}_{i-k}\right\| \\ &\quad + \sum_{i=k+1}^{n} N^2 e^{-\alpha n} M\left\|\mathbb{V}_{i-k}\right\|, \end{aligned} \tag{47}$$

where $M(\psi,\psi_1) = N\|\psi_0\| + \sum_{i=0}^{k} N^2 e^{-\alpha n}\|B\psi_{i-k} + \mathbb{F}(i,\psi_{i-k})\|$.

Using again the Gronwall inequality, we have

$$e^{\alpha n}\left\|\mathbb{V}_n\right\| \leq M\left(\psi,\psi_1\right) e^{\left(N^2 M\|B\|\right)n}. \tag{48}$$

That is,

$$\left\|\mathbb{V}_n\right\| \leq M\left(\psi,\psi_1\right) e^{\left(N^2 M\|B\| - \alpha\right)n}. \tag{49}$$

From (51), the desired result holds. □

7. Controllability

In this portion, we will discuss the controllability of system (4). First, we will discuss the linear problem and then the non-linear problem.

Linear Problem. We assume that $y = 0$; then, (4) reduces to the linear system

$$\begin{cases} \mathrm{EV}_{n+1} = \mathrm{AV}_n + \mathrm{BV}_{n-k} + \mathrm{CU}_n, n \in \mathbf{I}, k \geq 0, \\ \mathrm{V}_n = \psi_n, -k \leq n \leq 0. \end{cases} \tag{50}$$

We define a delay Gramian matrix

$$W_c[0,n_1] = \sum_{i=k+1}^{n}\left(A^{-1}E\right)^i\left(A^{n-1}E^{-n}\right)CC^T\left(A^{n-1}E^{-n}\right)^T\left(\left(A^{-1}E\right)^i\right)^T. \tag{51}$$

Theorem 10. *The linear system (6) is relatively controllable, if and only if $W_c[0,n_1]$ is non-singular.*

Proof. Sufficiency: since $W_c[0,n_1]$ is non-singular, then its inverse is well defined. So, we select a control function as follows:

$$U_n = C^T\left(A^{n-1}E^{-n}\right)^T\left(\left(A^{-1}E\right)^i\right)^T W_c^{-1}[0,n_1]\eta, \tag{52}$$

where

$$\eta = \nu_1 - A^n E^{-n}\Psi_0 - A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\left(B\Psi_{i-k} + CU_i\right). \tag{53}$$

Then,

$$\begin{aligned} \nu_{n_1} &= A^n E^{-n}\Psi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\left(B\Psi_{i-k} + CU_i\right) \\ &\quad + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i CU_i \\ &= A^n E^{-n}\Psi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\left(B\Psi_{i-k} + CU_i\right) \\ &\quad + A^{n-1}E^{-n}\sum_{i=k+1}^{n}\left(A^{-1}E\right)^i CC^T\left(A^{n-1}E^{-n}\right)^T\left(\left(A^{-1}E\right)^i\right)^T W_c^{-1}[0,n_1]\eta \\ &= \nu_1. \end{aligned} \tag{54}$$

Clearly, from Definition 4, we have that (6) is relatively controllable.

Necessity. We will prove by contradiction; assume that $W_c[0,n_1]$ is singular, i.e., there exists at least one non-zero state ṽ ∈∈**X** such that

$$\mathring{v}W_c[0,n_1]\,\mathring{v}=0. \tag{55}$$

So, we obtained

$$\begin{aligned}0&=vW_c[0,n_1]v\\&=\sum_{i=0}^{k}v^T\left(A^{-1}E\right)^i\left(A^{n-1}E^{-n}\right)CC^T\left(A^{n-1}E^{-n}\right)^T\left(\left(A^{-1}E\right)^i\right)^Tv\\&=\sum_{i=0}^{k}\left\|v^T\left(A^{-1}E\right)^i\left(A^{n-1}E^{-n}\right)C\right\|^2,\end{aligned} \tag{56}$$

which implies that

$$\mathring{v}^T\left(A^{-1}E\right)^i\left(A^{n-1}E^{-n}\right)C=\underbrace{(0,\ldots,0)}_{n}=0^T\forall n\in\mathbf{I}. \tag{57}$$

Since (6) is relatively controllable, from Definition 4, there exists $U_1(n)$ that drives the initial state to zero at n_1, that is,

$$\begin{aligned}v_{n_1}&=A^nE^{-n}\Psi_0+A^{n-1}E^{-n}\sum_{i=0}^{k}A^{-i}E^i\left(B\Psi_{i-k}+CU_i\right)\\&\quad+A^{n-1}E^{-n}\sum_{i=k+1}^{n}A^{-i}E^iCU_1(n)=0.\end{aligned} \tag{58}$$

Similarly, there also exists a control $U_2(n)$ that drives the initial state to the state \mathringv at n_1:

$$\begin{aligned}v_{n_1}&=A^nE^{-n}\Psi_0+A^{n-1}E^{-n}\sum_{i=0}^{k}A^{-i}E^i\left(B\Psi_{i-k}+CU_i\right)\\&\quad+A^{n-1}E^{-n}\sum_{i=k+1}^{n}A^{-i}E^iCU_2(n)=\mathring{v}.\end{aligned} \tag{59}$$

From the above, we have

$$v=A^{n-1}E^{-n}\sum_{i=k+1}^{n}A^{-i}E^iC\left[U_2(n)-U_2(n)\right]. \tag{60}$$

Multiplying both sides of (60) by v and via (8), we have

$$v^Tv=A^{n-1}E^{-n}\sum_{i=k+1}^{n}v^{\circ T}A^{-i}E^iC\left[U_2(n)-U_2(n)\right]=0. \tag{61}$$

This implies that $\mathring{v}=0$, which contradicts the fact that $\mathring{v}$ is non-zero. So, the delay Gramian matrix $W_C[0,n_1]$ is non-singular, which completes the proof.

Non-Linear Problem. To discuss the controllability of a non-linear system (4), consider the following conditions:

(1) The operator $W\colon L^2(\mathbf{I},\mathbf{X})\longrightarrow\mathbf{X}$ defined by

$$W_u=\sum_{i=k+1}^{n}\left(A^{-1}E\right)^i\left(A^{n-1}E^n\right)CU_n \tag{62}$$

has inverse operator W^{-1}, which takes values from $L^2(\mathbf{I},\mathbf{X})/\mathrm{ker}W$ and the set $M_1=\|W^{-1}\|_{L_{n_1}(\mathbf{X},L^2(\mathbf{I},\mathbf{X})/\mathrm{ker}W)}$. For the next result, we put another assumption.

(2) The map $u\colon\mathbf{I}\times\mathbf{X}\longrightarrow\mathbf{X}$ is continuous and there exist a constant $p>1$ and $L_y(\cdot)\in L^p(\mathbf{I},\mathbf{X})$ such that

$$\|u(n,b)-u(n,b)\|\Big\|\le L_y(n)\Big\|\|b-a\|,b,a\in\mathbf{X}. \tag{63}$$

□

Theorem 11. *Let us suppose that (1)–(3), (8), and (9) are satisfied. Then, system (4) is relatively controllable if*

$$b\left[1+\frac{NM_1\|C\|\left\|A^{n_1-1}E^{-n_1}\right\|\left(1-e^{-\alpha n_1}\right)}{1-e^{-\alpha}}\right]<1, \tag{64}$$

where $b=\|A^{n_1-1}E^{-n_1}\|N[1-e^{-\alpha qn_1}/1-e^{-\alpha q}]^{1/q}\|L_y\|_{L^p(\mathbf{I},\mathbf{X})}$ *and* $1/q+1/p=1,q,p>1$.

Proof. Using (8), for an arbitrary $v_{(\cdot)}\in\mathbb{B}(\mathbf{I},\mathbf{X})$, we define a control function u_{v_n} by

$$\begin{aligned}u_{v_n}=W^{-1}\Bigg[&v_1-A^nE^{-n}\Psi_0-A^{n-1}E^{-n}\sum_{i=0}^{k}A^{-i}E^i\left(B\Psi_{i-k}+y\left(i,V_i\right)\right)\\&A^{n-1}E^{-n}\sum_{i=k+1}^{n}A^{-i}E^i\left(BV_{i-k}+y\left(i,V_i\right)\right)\Bigg]_n,n\in I.\end{aligned} \tag{65}$$

We show that the operator $P\colon\mathbb{B}(\mathbf{I},\mathbf{X})\longrightarrow\mathbb{B}(\mathbf{I},\mathbf{X})$ defined by

$$(Pv)_n = A^n E^{-n}\Psi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\left(B\Psi_{i-k} + y(i, \mathrm{V}_i)\right) + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i\left(B\mathrm{V}_{i-k} + y(i, \mathrm{V}_i)\right) + A^{n-1}E^{-n}\sum_{i=0}^{n} A^{-i}E^i C U_i \tag{66}$$

has a fixed point, which is the solution of (4), by using the above control function.

We need to check that $(Pv)_{n_1} = v_1$ and $(Pv)_0 = v_0$, which means that u_v steers system (4) from v_0 to v_1 in finite n_1 and this implies that system (4) is relatively controllable on **I**.

Foe every real number r, let $\mathbf{B}_r = \{v \in \mathbb{B}(\mathbf{I}, \mathbf{X}) \colon \|v\|_c \le r\}$. Set $F = \sup_{n\in \mathbf{I}}\|y(n, 0)\|$. We will prove this theorem in following three steps. □

Step 1. We claim that there exists a positive real number r such that $P(\mathbf{B}_r) \subseteq \mathbf{B}_r$.

Note that

$$\sum_{i=0}^{n} e^{-\alpha i}L_y(i) \le \left(\sum_{i=0}^{n} e^{-\alpha i q}\right)^{1/q}\left(\sum_{i=0}^{n} L_y^p(i)\right)^{1/p} \le \left(\frac{1 - e^{-\alpha n q}}{1 - e^{-\alpha q}}\right)^{1/q}\left\|L_y\right\|_{L^p(\mathbf{I},\mathbf{X})}, \tag{67}$$

and $\sum_{i=0}^{n} e^{-\alpha i}\|y(i, 0)\| \le F\sum_{i=0}^{n} e^{-\alpha i} = F(1 - e^{-\alpha n}/1 - e^{-\alpha})$.

Now using (1), (2), (8), and (60), we have

$$\begin{aligned}
\left\|u_{v_n}\right\| &= \left\|W^{-1}\left[v_1 - A^{n_1}E^{-n_1}\Psi_0 - A^{n_1-1}E^{-n_1}\sum_{i=0}^{k} A^{-i}E^i\left(B\Psi_{i-k} + y(i, \mathrm{V}_i)\right) - A^{n_1-1}E^{-n_1}\sum_{i=k+1}^{n} A^{-i}E^i\left(B\mathrm{V}_{i-k} + y(i, \mathrm{V}_i)\right)\right]\right\| \\
&\le \left\|W^{-1}\right\|_{\mathscr{L}_{n_1}(\mathbf{X}, \mathscr{L}^2(\mathbf{I},\mathbf{X})/\ker W)}\Bigg(\left\|v_1\right\| + \left\|A^{n_1}E^{-n_1}\Psi_0\right\| + \left\|A^{n_1-1}E^{-n_1}\right\|\sum_{i=0}^{k}\left\|A^{-i}E^i\right\|\left\|B\Psi_{i-k} + y(i, \mathrm{V}_i)\right\| \\
&\quad + \left\|A^{n_1-1}E^{-n_1}\right\|\sum_{i=k+1}^{n}\left\|A^{-i}E^i\right\|\left\|B\mathrm{V}_{i-k} + y(i, \mathrm{V}_i)\right\|\Bigg) \\
&\le M_1\left(\left\|v_1\right\| + Ne^{-\alpha n_1}\left\|\Psi_0\right\| + \left\|A^{n_1-1}E^{-n_1}\right\|\sum_{i=0}^{k} Ne^{-\alpha i}\left\|B\Psi_{i-k} + y(i, \mathrm{V}_i)\right\| + \left\|A^{n_1-1}E^{-n_1}\right\|\sum_{i=k+1}^{n} Ne^{-\alpha i}\left\|B\mathrm{V}_{i-k} + y(i, \mathrm{V}_i)\right\|\right) \\
&\le M_1\Bigg(\left\|v_1\right\| + Ne^{-\alpha n_1}\left\|\Psi_0\right\| + \left\|A^{n_1-1}E^{-n_1}\right\|\sum_{i=0}^{k} Ne^{-\alpha i}\left\|B\Psi_{i-k}\right\| + \left\|A^{n_1-1}E^{-n_1}\right\|\sum_{i=0}^{n} Ne^{-\alpha i}\left\|y(i, \mathrm{V}_i)\right\| \\
&\quad + \left\|A^{n_1-1}E^{-n_1}\right\|\sum_{i=k+1}^{n} Ne^{-\alpha i}\left\|B\mathrm{V}_{i-k}\right\|\Bigg) \\
&= M_1\Bigg(\left\|v_1\right\| + Ne^{-\alpha n_1}\left\|\Psi_0\right\| + \left\|A^{n_1-1}E^{-n_1}\right\|\sum_{i=0}^{k} Ne^{-\alpha i}\left\|B\Psi_{i-k}\right\| + \left\|A^{n_1-1}E^{-n_1}\right\|\sum_{i=0}^{n} Ne^{-\alpha i}\left\|y(i, \mathrm{V}_i) - y(i, 0) + y(i, 0)\right\| \\
&\quad + \left\|A^{n_1-1}E^{-n_1}\right\|\sum_{i=k+1}^{n} Ne^{-\alpha i}\left\|B\mathrm{V}_{i-k}\right\|\Bigg) \\
&\le M_1\Bigg[\left\|v_1\right\| + Ne^{-\alpha n_1}\left\|\Psi_0\right\| + \left\|A^{n_1-1}E^{-n_1}\right\|\sum_{i=0}^{k} Ne^{-\alpha i}\left\|B\Psi_{i-k}\right\| + \left\|A^{n_1-1}E^{-n_1}\right\|\sum_{i=0}^{n} Ne^{-\alpha i}\left(L_y(i)\left\|\mathrm{V}_i\right\| + \left\|y(i, 0)\right\|\right) \\
&\quad + \left\|A^{n_1-1}E^{-n_1}\right\|\sum_{i=k+1}^{n} Ne^{-\alpha i}\left\|B\mathrm{V}_{i-k}\right\|\Bigg],
\end{aligned}$$

$$
\begin{aligned}
&= M_1\Bigg[\|v_1\| + Ne^{-\alpha n_1}\|\Psi_0\| + \|A^{n_1-1}E^{-n_1}\|\sum_{i=0}^{k} Ne^{-\alpha i}\|B\Psi_{i-k}\| + \|A^{n_1-1}E^{-n_1}\|\sum_{i=0}^{n} Ne^{-\alpha i}L_y(i)\|V_i\| \\
&\quad + \|A^{n_1-1}E^{-n_1}\|\sum_{i=0}^{n} Ne^{-\alpha i}\|y(i,0)\| + \|A^{n_1-1}E^{-n_1}\|\sum_{i=k+1}^{n} Ne^{-\alpha i}\|BV_{i-k}\|\Bigg] \\
&\leq M_1\Bigg[\|v_1\| + Ne^{-\alpha n_1}\|\Psi_0\| + \|A^{n_1-1}E^{-n_1}\|\sum_{i=0}^{k} Ne^{-\alpha i}\|B\Psi_{i-k}\| + \|A^{n_1-1}E^{-n_1}\|N\left(\frac{1-e^{-\alpha n_1 q}}{1-e^{-\alpha q}}\right)^{1/q}\|L_y\|_{L^p(\mathbf{I},\mathbf{X})}\|v\|_c \\
&\quad + \|A^{n_1-1}E^{-n_1}\|NF\left(\frac{1-e^{-\alpha n_1}}{1-e^{-\alpha}}\right) + \|A^{n_1-1}E^{-n_1}\|\sum_{i=k+1}^{n} Ne^{-\alpha i}\|BV_{i-k}\|\Bigg] = M_1\|v_1\| + M_1 a + M_1 b\|v\|_c,
\end{aligned} \tag{68}
$$

where

$$
\begin{aligned}
a &= \|v_1\| + \|{+Ne^{-\alpha n_1}}\|\|\Psi_0\| + \|A^{n_1-1}E^{-n_1}\|\left\|\sum_{i=0}^{k} Ne^{-\alpha i}\right\|\|B\Psi_{i-k}\| + \|A^{n_1-1}E^{-n_1}\|NF\left(\frac{1-e^{-\alpha n_1}}{1-e^{-\alpha}}\right) \\
&\quad + \|A^{n_1-1}E^{-n_1}\|\left\|\sum_{i=k+1}^{n} Ne^{-\alpha i}\right\|\|BV_{i-k}\|,
\end{aligned} \tag{69}
$$

and

$$
b = \|A^{n_1-1}E^{-n_1}\|N\left(\frac{1-e^{-\alpha n_1 q}}{1-e^{-\alpha q}}\right)^{1/q}\|L_y\|_{L^p(\mathbf{I},\mathbf{X})}. \tag{70}
$$

From (57) and (60), we have

$$
\begin{aligned}
\|(Pv)_n\| &= \left\| \begin{array}{c} A^nE^{-n}\Psi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\left(B\Psi_{i-k} + y(i,V_i)\right) \\ +A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i\left(BV_{i-k} + y(i,V_i)\right) + A^{n-1}E^{-n}\sum_{i=0}^{n} A^{-i}E^iCU_i \end{array} \right\| \\
&\quad + \left[1 + \frac{\|A^{n-1}E^{-n}\|\|C\|M_1N\left(1-e^{-\alpha n}\right)}{1-e^{-\alpha}}\right]\|V_1\| \leq r,
\end{aligned} \tag{71}
$$

where

$$
\begin{aligned}
r &= a\left[1 + \frac{\|A^{n-1}E^{-n}\|\|C\|M_1N\left(1-e^{-\alpha n}\right)}{1-e^{-\alpha}}\right] \\
&\quad + b\left[1 + \frac{\|A^{n-1}E^{-n}\|\|C\|M_1N\left(1-e^{-\alpha n}\right)}{1-e^{-\alpha}}\right]\|V\|_c \\
&\quad + +\left[1 + \frac{\|A^{n-1}E^{-n}\|\|C\|M_1N\left(1-e^{-\alpha n}\right)}{1-e^{-\alpha}}\right]\|V_1\|.
\end{aligned} \tag{72}
$$

So, we obtain $P(\mathbf{B})_r \subseteq \mathbf{B}_r$.

Step 2. Now, we define a map P_1 on $\mathbf{B}_r$ and we will show that it is a contraction mapping.

$$
\begin{aligned}
(P_1v)_n &= A^nE^{-n}\Psi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\left(B\Psi_{i-k}\right) \\
&\quad + A^{n-1}E^{-n}\sum_{i=0}^{n} A^{-i}E^iCU_v(i).
\end{aligned} \tag{73}
$$

Let $\rho, \varrho \in \mathbf{B}_r$. Using (57) and (60) for each $n \in \mathbf{I}$, we have

$$\left\|u_\rho(n)-u_\varrho(n)\right\| = \left\| \begin{array}{l} W^{-1}\left[v_1 - A_1^n E^{-n_1}\Psi_0 - A^{n_1-1}E^{-n_1}\sum_{i=0}^{k} A^{-i}E^{i}\left(B\Psi_{i-k}+y(i,\rho_i)\right) - A^{n_1-1}E^{-n_1}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(BV_{i-k}+y(i,\rho_i)\right)\right] \\ -W^{-1}\left[v_1 - A^{n_1}E^{-n_1}\Psi_0 - A^{n_1-1}E^{-n_1}\sum_{i=0}^{k} A^{-i}E^{i}\left(B\Psi_{i-k}+y(i,\varrho_i)\right) - A^{n_1-1}E^{-n_1}\sum_{i=k+1}^{n} A^{-i}E^{i}\left(BV_{i-k}+y(i,\varrho_i)\right)\right] \end{array} \right\|$$
$$\le \left\|W^{-1}\right\|\left\|A^{n_1-1}E^{-n_1}\right\|\left[\sum_{i=0}^{n}\left\|A^{-i}E^{i}\right\|\left\|y(i,\rho_i)-y(i,\varrho_i)\right\|\right] \le M_1\left\|A^{n_1-1}E^{-n_1}\right\|\sum_{i=0}^{n} Ne^{-\alpha i}L_y(i)\|\rho-\varrho\|_c\left\|u_\rho(n)-u_\varrho(n)\right\|$$
$$\le M_1\left\|A^{n_1-1}E^{-n_1}\right\| N\left(\frac{1-e^{-\alpha n_1 q}}{1-e^{-\alpha q}}\right)^{1/q}\left\|L_y\right\|_{L^p(\mathbf{I},\mathbf{X})}\|\rho-\varrho\|_c \le M_1 b\|\rho-\varrho\|_c. \tag{74}$$

Thus,

$$\left\|(P_1\rho)_n-(P_1\varrho)_n\right\| = \left\| \begin{array}{l} A^nE^{-n}\Psi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k}A^{-i}E^{i}\left(B\Psi_{i-k}\right) + A^{n-1}E^{-n}\sum_{i=0}^{n}A^{-i}E^{i}CU_\rho(i) \\ -\left[A^nE^{-n}\Psi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k}A^{-i}E^{i}\left(B\Psi_{i-k}\right) + A^{n-1}E^{-n}\sum_{i=0}^{n}A^{-i}E^{i}CU_\varrho(i)\right] \end{array} \right\|$$
$$= \left\|A^{n-1}E^{-n}\sum_{i=0}^{n}A^{-i}E^{i}C\left(U_\rho(i)-U_\varrho(i)\right)\right\| \le \left\|A^{n-1}E^{-n}\right\|\sum_{i=0}^{n}\left\|A^{-i}E^{i}\right\|\|C\|\left\|U_\rho(i)-U_\varrho(i)\right\|$$
$$\le \left\|A^{n-1}E^{-n}\right\| N\sum_{i=0}^{n}e^{-\alpha i}\|C\|M_1 b\|\rho-\varrho\|_c$$
$$\le \left\|A^{n-1}E^{-n}\right\| N\|C\|M_1 b\|\rho-\varrho_c\|\left(\frac{1-e^{-\alpha n}}{1-e^{-\alpha}}\right) = \mathrm{M}\|\rho-\varrho\|_c, \tag{75}$$

where

$$M = \frac{\left\|A^{n-1}E^{-n}\right\|\|N\|\|C\|M_1 b\left(1-e^{-\alpha n}\right)}{1-e^{-\alpha}}. \tag{76}$$

From (10), $M<1$, so P_1 is contraction.

Step 3. Here we define a map $P_2\colon \mathbf{B}_r \longrightarrow \mathbb{B}(\mathbf{I},\mathbf{X})$ and will show that it is a compact and continuous operator.

$$\left(P_2 v\right)_n = A^{n-1}E^{-n}\sum_{i=0}^{n}A^{-i}E^{i}\left(y\left(i,\mathrm{V}_i\right)\right), \tag{77}$$

for $n\in\mathbf{I}$. Let $v_n\in\mathbf{B}_r$ with $v_n\longrightarrow v$ in $\mathbf{B}_r$ as $n\longrightarrow\infty$. Using (57), we have $y(\cdot,v_n)\longrightarrow y(\cdot,v)$ in $\mathbb{B}(\mathbf{I},\mathbf{X})$ as $n\longrightarrow\infty$, and thus

$$\left\|(P_2 v_n)_n-(P_2 v)_n\right\| = \left\|A^{n-1}E^{-n}\sum_{i=0}^{n}A^{-i}E^{i}\left(y\left(i,\mathrm{V}_{n_i}\right)\right) - A^{n-1}E^{-n}\sum_{i=0}^{n}A^{-i}E^{i}\left(y\left(i,\mathrm{V}_i\right)\right)\right\|$$
$$\le \left\|A^{n-1}E^{-n}\right\|\left\|N\sum_{i=0}^{n}e^{-\alpha i}\right\|\left\|y\left(i,\mathrm{V}_{n_i}\right)-y\left(i,\mathrm{V}_i\right)\right\|$$
$$\longrightarrow 0 \text{asn} \longrightarrow \infty, \tag{78}$$

which implies that P_2 is continuous on $\mathbf{B}_r$.

To show that P_2 is compact on $\mathbf{B}_r$, we have to prove that $P_2(\mathbf{B}_r)$ is equicontinuous and bounded. For any $\nu \in \mathbf{B}_r, n_1 \geq n+h \geq 0$, note that

$$\begin{aligned}
(P_2\nu)_{n+h} - (P_2\nu)_n &= A^{n+h-1}E^{-n+h}\sum_{i=0}^{n+h} A^{-i}E^{i}(y(i,\nu_i)) - A^{n-1}E^{-n}\sum_{i=0}^{n} A^{-i}E^{i}(y(i,\nu_i)),\\
\|(P_2\nu)_{n+h} - (P_2\nu)_n\| &= \left\| A^{n+h-1}E^{-n+h}\sum_{i=0}^{n+h} A^{-i}E^{i}(y(i,\nu_i)) \right\|\\
&\quad A^{n-1}E^{-n}\sum_{i=0}^{n} A^{-i}E^{i}(y(i,\nu_i)),\\
\|(P_2\nu)_{n+h} - (P_2\nu)_n\| &\leq \left\| A^{n+h-1}E^{-n+h}\sum_{i=0}^{n+h} A^{-i}E^{i}(y(i,\nu_i)) \right\|\\
&\quad - \left\| A^{n-1}E^{-n}\sum_{i=0}^{n} A^{-i}E^{i}(y(i,\nu_i)) \right\|\\
&\leq \|S_1\| - \|S_2\|.
\end{aligned} \tag{79}$$

Let

$$S_1 = A^{n+h-1}E^{-n+h}\sum_{i=0}^{n+h} A^{-i}E^{i}(y(i,\nu_i)), \tag{80}$$

and

$$\mathcal{S}_2 = A^{n-1}E^{-n}\sum_{i=0}^{n} A^{-i}E^{i}(y(i,\nu_i)). \tag{81}$$

Now, we have to check $\mathcal{S}_1 \longrightarrow \mathcal{S}_2$ as $h \longrightarrow 0$. Now,

$$\begin{aligned}
\|S_1\| &= \left\| A^{n+h-1}E^{-n+h}\sum_{i=0}^{n+h} A^{-i}E^{i}(y(i,\mathrm{V}_i)) \right\|,\\
&\leq \left\| A^{n+h-1}E^{-n+h} \right\| \left\| N\sum_{i=0}^{n+h} e^{-\alpha i}L_y(i) \right\| \|\nu_i\| + \|y(i,0)\|\\
&\leq \left\| A^{n+h-1}E^{-n+h} \right\| N\left(\frac{1-e^{-\alpha(n+h)q}}{1-e^{-\alpha q}}\right)^{1/q} \|L_y\|_{L^p(\mathbf{I},\mathbf{X})}\\
&\quad + \left\| A^{n+h-1}E^{-n+h} \right\| NF\left(\frac{1-e^{-\alpha(n+h)}}{1-e^{-\alpha}}\right)\\
&\longrightarrow \mathcal{S}_2 \mathrm{ash} \longrightarrow 0,
\end{aligned} \tag{82}$$

which implies that

$$\|(P_2\nu)_{n+h} - (P_2\nu)_n\| \longrightarrow 0 \mathrm{ash} \longrightarrow 0. \tag{83}$$

Therefore, $P_2(\mathbf{B}_r)$ is equicontinuous.

Next, we show that $P_2(\mathbf{B}_r)$ is bounded, and we have

$$\begin{aligned}
\|(P_2\nu)\| &= \left\| A^{n-1}E^{-n}\sum_{i=0}^{n} A^{-i}E^{i}(y(i,\mathrm{V}_i)) \right\|\\
&\leq \left\| A^{n-1}E^{-n} \right\| \left\| N\sum_{i=0}^{n} e^{-\alpha i}L_y(i) \right\| + \|y(i,0)\|\\
&\leq \left\| A^{n-1}E^{-n} \right\| \left\| N\left(\frac{1-e^{-\alpha nq}}{1-e^{-\alpha q}}\right)^{1/q} \right\|\\
&\quad \|L_y\| \|L^p(\mathbf{I},\mathbf{X})\| A^{n-1}E^{-n} \left\| NF\left(\frac{1-e^{-\alpha n}}{1-e^{-\alpha}}\right) \right\|.
\end{aligned} \tag{84}$$

Hence, $P_2(\mathbf{B}_r)$ is bounded. From the Arzelà–Ascoli theorem, $P_2(\mathbf{B}_r)$ is compact in $\mathbb{B}(\mathbf{I},\mathbf{X})$. Thus, $P_2(\mathbf{B}_r)$ is a compact and continuous operator.

Now, Krasnoselskii's fixed point theorem guarantees that P has a fixed point ν in $\mathbf{B}_r$. Clearly, ν is a solution of (4) satisfying $\nu_{n_1} = \nu_1$, and the boundary condition $\nu_n = \Psi_n, -k \leq n \leq 0$ holds from the solution of system (4), which completes the proof.

8. Numerical Examples

In this section, we give some examples on Hyers–Ulam stability and controllability for the theoretical results.

Example 1. Consider the following non-singular delay difference equation:

$$\begin{cases} EV_{n+1} = AV_n + BV_{n-3}, V_0 = 1, n \in \{0,1,2,3\},\\ V_n = \Phi_n, -3 \leq n \leq 0, \end{cases} \tag{85}$$

with inequality

$$\begin{cases} \|EV_{n+1} - AV_n - BV_{n-3}\| \le 0.8, V_0 = 1, n \in \{0,1,2,3\}, \\ \|V_n - \Phi_n\| \le 1, -3 \le n \le 0, \end{cases} \tag{86}$$

where $k = 3$.

If we fixed $A = \begin{pmatrix} -5 & -2 \\ -4 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$, $E = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and $\phi_n = [\cos(n+\pi/2)\cos(n+\pi/2)]^t$, obviously, $\phi_n = [00]^t$, when $n = 0$. Then, we get that $AB = \begin{pmatrix} -24 & -11 \\ -22 & -13 \end{pmatrix} = BA$, $AE = \begin{pmatrix} -10 & -4 \\ -8 & -6 \end{pmatrix} = EA$, $BE = \begin{pmatrix} 8 & 2 \\ 4 & 6 \end{pmatrix} = EB$, $E^{-1} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$, $AE^{-1} = \begin{pmatrix} -2.5 & -1 \\ -2 & -1.5 \end{pmatrix} = E^{-1}A$, and $BE^{-1} = \begin{pmatrix} 2 & 0.5 \\ 1 & 1.5 \end{pmatrix} = E^{-1}B$.

Moreover, if V satisfied (86), then there exists f_n such that $\|f_n\| \le 0.8$, and

$$\begin{cases} EV_{n+1} = AV_n + BV_{n-3} + f_n, V_0 = 1, n \in \{0,1,2,3\} \\ V_n = \Phi_n, -3 \le n \le 0. \end{cases} \tag{87}$$

Also, the solution of (85) is

$$\begin{aligned} V_n = {} & A^n E^{-n}\Phi_0 + BA^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\Phi_{i-k} \\ & + BA^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^iV_{i-k}. \end{aligned} \tag{88}$$

Let $\epsilon = 0.8$, and $f\colon Z_+ \longrightarrow \mathbf{X}$ be as given below.

$f_n = [\,0.6\cos(n+\pi/2)\;\; 0.6\sin(n+\pi/2)\,]^t$.

Then, clearly

$$\begin{aligned} \|f_n\| &= \sqrt{\left(0.6\cos\left(n+\frac{\pi}{2}\right)\right)^2 + \left(0.6\sin\left(n+\frac{\pi}{2}\right)\right)^2} \\ &= \left[(0.6)^2\cos^2\left(n+\frac{\pi}{2}\right) + (0.6)^2\sin^2\left(n+\frac{\pi}{2}\right)\right]^{(1/2)} \\ &= \sqrt{(0.6)^2} \\ &= 0.6 \\ &\le = 0.8. \end{aligned} \tag{89}$$

Now, the perturbed delay difference systems (11)–(13) have the solution

$$\begin{aligned} H_n = {} & A^n E^{-n}\Phi_0 + BA^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\Phi_{i-k} \\ & + BA^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i\left(V_{i-k} + f_{i-k}\right). \end{aligned} \tag{90}$$

The plots of exact and perturbed solutions obtained using Mathematica are shown in Figure 1.

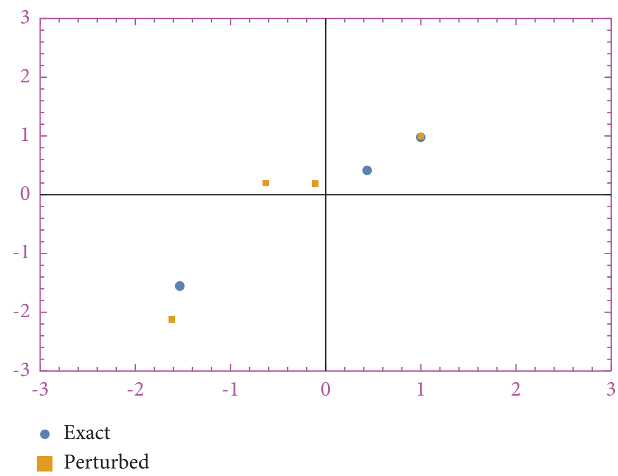

FIGURE 1: The plots of exact and perturbed solutions.

Example 2. Consider the following non-singular delay difference equation:

$$\begin{cases} E\mathbf{V}_{n+1} = A\mathbf{V}_n + B\mathbf{V}_{n-3} + f(n,\mathbf{V}), \\ \mathbf{V}_0 = 1, n \in \{0,1,2,3\}, \\ \mathbf{V}_n = \phi_n, -3 \le n \le 0, \end{cases} \tag{91}$$

with inequality

$$\begin{cases} \|E\mathbf{V}_{n+1} - A\mathbf{V}_n - B\mathbf{V}_{n-3} - f(n,\mathbf{V})\| \\ \le 0.8, \mathbf{V}_0 = 1, n \in \{0,1,2,3\}, \\ \|\mathbf{V}_n - \phi_n\| \le 1, -3 \le n \le 0, \end{cases} \tag{92}$$

where $k = 3$.

If again we fixed $A = \begin{pmatrix} 4 & -3 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix}$, $E = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, and $\phi_n = [\cos(n+\pi/2)\cos(n+\pi/2)]^t$, obviously, $\phi_n = [00]^t$, when $n = 0$. Then, we get $AB = \begin{pmatrix} 2 & -15 \\ 10 & -3 \end{pmatrix} = BA$, $AE = \begin{pmatrix} 12 & -9 \\ 6 & 9 \end{pmatrix} = EA$, $BE = \begin{pmatrix} 6 & -9 \\ 6 & 3 \end{pmatrix} = EB$, $E^{-1} = \begin{pmatrix} 0.333 & 0 \\ 0 & 0.333 \end{pmatrix}$, $AE^{-1} = \begin{pmatrix} 1.332 & -0.999 \\ 0.666 & 0.999 \end{pmatrix} = E^{-1}A$, and $BE^{-1} = \begin{pmatrix} 0.666 & -0.999 \\ 0.666 & 0.333 \end{pmatrix} = E^{-1}B$.

Moreover, if V satisfied (15), then there exists f_n such that $\|f_n\| \le 0.8$, and

$$\begin{cases} E\mathbf{V}_{n+1} = A\mathbf{V}_n + B\mathbf{V}_{n-3} + f(n,\mathbf{V}) + f_n, \\ \mathbf{V}_0 = 1, n \in \{0,1,2,3\}, \\ \mathbf{V}_n = \phi_n, -3 \le n \le 0. \end{cases} \tag{93}$$

Also, the solution of (91) is

$$\mathbf{V}_n = A^n E^{-n}\phi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\left(B\phi_{i-k} + f\left(i, \mathbf{V}_i\right)\right) + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i\left(B\mathbf{V}_{i-k} + f\left(i, \mathbf{V}_i\right)\right). \tag{94}$$

Let $\epsilon = 0.8$, and $f\colon \mathscr{Z}_+ \longrightarrow \mathbf{X}$ be as given below. $f_n = \left[\, 0.7\cos(n + \pi/2) \;\; 0.7\sin(n + \pi/2)\,\right]^t$.

Then, clearly

$$\begin{aligned}\|f_n\| &= \sqrt{\left(0.7\cos\left(n + \frac{\pi}{2}\right)\right)^2 + \left(0.7\sin\left(n + \frac{\pi}{2}\right)\right)^2}\\ &= \left[(0.7)^2\cos^2\left(n + \frac{\pi}{2}\right) + (0.7)^2\sin^2\left(n + \frac{\pi}{2}\right)\right]^{(1/2)}\\ &= \sqrt{(0.7)^2}\\ &= 0.7\\ &\leq\ = 0.8.\end{aligned} \tag{95}$$

Now, the perturbed delay difference systems (91)–(93) have the solution

$$\mathbf{H}_n = A^n E^{-n}\phi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\left(B\phi_{i-k} + f\left(i, \mathbf{V}_i\right)\right) + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i\left(B\mathbf{V}_{i-k} + f\left(i, \mathbf{V}_i\right) + f_{i-k}\right). \tag{96}$$

The plots of exact and perturbed solutions obtained using Mathematica are shown in Figure 2.

Example 3. Set $n_1 = 3$. Consider the following delay difference controlled system:

$$\begin{cases} \mathrm{EV}_{n+1} = \mathrm{AV}_n + \mathrm{BV}_{n-3} + y\left(n, \mathrm{V}_n\right) \\ \qquad + \mathrm{CU}_n, n \in \mathbf{I} = \{0, 1, 2, 3\}, \\ \mathrm{V}_n = \Psi_n, -3 \leq n \leq 0, \end{cases} \tag{97}$$

which has the solution

$$\mathrm{V}_n = A^n E^{-n}\Psi_0 + A^{n-1}E^{-n}\sum_{i=0}^{k} A^{-i}E^i\left(B\Psi_{i-3} + y\left(i, \mathrm{V}_i\right) + CU_i\right) + A^{n-1}E^{-n}\sum_{i=k+1}^{n} A^{-i}E^i\left(B\mathrm{V}_{i-3} + y\left(i, \mathrm{V}_i\right) + CU_i\right), \tag{98}$$

where $k = 3$.

Let $A = \begin{pmatrix} 7 & 3 \\ 6 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$, $E = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $y\left(n, \mathrm{V}_n\right) = \begin{pmatrix} 0.2n\mathrm{V}_2 \\ 0.1n\mathrm{V}_1 \end{pmatrix}$.

Note that $\mathrm{AB} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} = \mathrm{BA}$, $\mathrm{AE} = \begin{pmatrix} 28 & 12 \\ 24 & 16 \end{pmatrix} = \mathrm{EA}$, $\mathrm{BE} = \begin{pmatrix} 4 & -4 \\ -8 & 8 \end{pmatrix} = \mathrm{EB}$, $E^{-1} = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix}$, $\mathrm{AE}^{-1} = \begin{pmatrix} 1.75 & 0.75 \\ 1.5 & 1 \end{pmatrix} = E^{-1}A$, and $\mathrm{BE}^{-1} = \begin{pmatrix} 0.25 & -0.25 \\ -0.5 & 0.5 \end{pmatrix} = E^{-1}B$.

Now $\|C\| = 1$. Also, $\|A^n E^{-n}\| \leq Ne^{-\alpha n}$ with $N = 1, \alpha = 2$, and $n \in \{0, 1, 2, 3\}$.

Now consider

$$\begin{aligned} W_c\left[0, n_1\right] &= \sum_{i=0}^{n_1}\left(A^{-1}E\right)^i\left(A^{n_1-1}E^{-n_1}\right) CC^T\left(A^{n_1-1}E^{-n_1}\right)^T\left(\left(A^{-1}E\right)^i\right)^T\\ &= \sum_{i=0}^{3}\left(A^{-1}E\right)^i\left(A^2E^{-2}\right)CC^T\left(A^2E^{-2}\right)^T\\ &= \left(\left(A^{-1}E\right)^i\right)^T\\ &= \begin{pmatrix} 9.31767 & -14.8336 \\ 4.38789 & -6.31446 \end{pmatrix}. \end{aligned} \tag{99}$$

Then, $W_c\left[0, n_1\right]^{-1} = \begin{pmatrix} -1.00997 & 2.37257 \\ -0.701824 & 1.49032 \end{pmatrix}$ and $M_1 = \sqrt{\|W_c\left[0, n_1\right]^{-1}\|} = 0.399932$. Further for any $\nu, \mu \in \mathbf{X}$, we have

$$\begin{aligned} \|y(n, \nu) - y(n, \mu)\| &= \max\left\{0.2n\left\|\nu_1 - \mu_1\right\|, 0.1n\left\|\nu_2 - \mu_2\right\|\right\}\\ &\leq o.2n\max\left\{\left\|\nu_1 - \mu_1\right\|, \left\|\nu_2 - \mu_2\right\|\right\}\\ &= 0.2n\|\nu - \mu\|. \end{aligned} \tag{100}$$

Now we set $L_y(n) = 0.2n \in L^2(\mathbf{I}, \mathbf{X})$ with $p = q = 2$, so $\|L_y\|_{L^2(\mathbf{I},\mathbf{X})} = \left(\sum_{i=0}^{n_1} L_y(i)\right)^{1/2} = \left(\sum_{i=0}^{3} 0.2i\right)^{1/2} = 1.09545$.

$$\begin{aligned} b &= \left\|A^{n_1-1}E^{-n_1}\right\| N\left(\frac{1 - e^{-\alpha n_1 q}}{1 - e^{-\alpha q}}\right)^{1/q}\left\|L_y\right\|_{L^2(\mathbf{I},\mathbf{X})}\\ &= \left\|A^2E^{-2}\right\|\left(\frac{1 - e^{-12}}{1 - e^{-4}}\right)^{1/2}(1.09545)\\ &= 0.3546, \end{aligned} \tag{101}$$

and

$$\begin{aligned} & b\left[\frac{\left\|A^{n_1-1}E^{-n_1}\right\|\|C\|M_1N\left(1 - e^{-\alpha n_1}\right)}{1 - e^{-\alpha}}\right]\\ &= (0.3546)\left[\frac{\left\|A^2E^{-2}\right\|\left(M_1\right)\left(1 - e^{-6}\right)}{1 - e^{-2}}\right]\\ &= 0.8573\\ &\leq 1. \end{aligned} \tag{102}$$

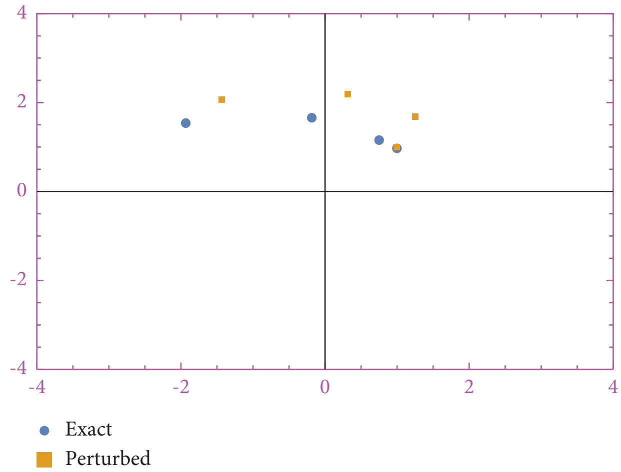

FIGURE 2: The plots of exact and perturbed solutions.

Example 4. Consider the following non-singular delay difference equation:

$$\begin{cases} EV_{n+1} = AV_n + BV_{n-0.3}, n \in \{0, 1, 2, 3, \ldots\}, \\ V_n = (0.3, 0.2)^T, -0.3 \le n \le 0, \end{cases} \quad (103)$$

where $k = 0.3$ and we set $A = \begin{pmatrix} -5 & -2 \\ -4 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$, $E = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and $\phi_n = [\cos(n + \pi/2)\cos(n + \pi/2)]^t$; obviously, $\phi_n = [00]^t$, when $n = 0$. Then, we get that $AB = \begin{pmatrix} -24 & -11 \\ -22 & -13 \end{pmatrix} = BA$, $AE = \begin{pmatrix} -10 & -4 \\ -8 & -6 \end{pmatrix} = EA$, $BE = \begin{pmatrix} 8 & 2 \\ 4 & 6 \end{pmatrix} = EB$, $E^{-1} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$, $AE^{-1} = \begin{pmatrix} -2.5 & -1 \\ -2 & -1.5 \end{pmatrix} = E^{-1}A$, and $BE^{-1} = \begin{pmatrix} 2 & 0.5 \\ 1 & 1.5 \end{pmatrix} = E^{-1}B$.

Now,

$$\|B\| = \left\| \begin{pmatrix} 0.5 & 1 \\ 0.6 & 0.34 \end{pmatrix} \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \right\| = 0.47 < Ne^{-0.3\alpha} =$$
0.656101, choosing$\alpha = 1.4048, N = 1$.

Also, $\|\phi\| = 0.3$; now,

$$\begin{aligned} &M(\phi, \phi_i) = N\|\Phi_0\| + \|B\| \sum_{i=0}^{k} N^2 e^{-\alpha n} \|\Phi_{i-k}\| = 0.4324, \\ &\left(N^2\|B\| - \alpha\right) < -0.3725 < 0, \\ &\|V_n\| \le M(\phi, \phi_i) e^{\left(N^2\|B\| - \alpha\right)n} \longrightarrow 0, as n \longrightarrow \infty. \end{aligned} \quad (104)$$

Hence, system (1) is exponentially stable.

9. Conclusion

In recent years, the qualitative behavior of delay difference equations has a significant contribution in real life. Especially, the discussion regarding the Hyers–Ulam stability, exponential stability, and controllability of delay difference equations has been considered as one of the important topics of the literature, in which different types of conditions have been used in the form of inequalities and mostly results have been obtained through discrete Gronwall inequality. In this paper, we have investigated the existence and uniqueness of the solution through Banach contraction principle, Hyers–Ulam stability over bounded and unbounded discrete interval, exponential stability, and controllability of the delay difference system with the help of Gronwall inequality and Carathéodory condition.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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