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Research Article

Soft Relations Applied to the Substructures of Quantale Module and Their Approximation

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This research article offers a study on a new relation of rough sets and soft sets with an algebraic structure quantale module by using soft reflexive and soft compatible relations. The lower approximation and upper approximation of subsets of quantale module are utilized by aftersets and foresets. As a sequel of this relation, different characterizations of rough soft substructures of quantale modules are obtained. To ensure the results, soft reflective and soft compatible relations are focused and these are interpreted by aftersets and foresets. Furthermore, the algebraic relations between upper (lower) approximation of substructures of quantale module and the upper (lower) approximations of their homomorphic images with the help of quantale module homomorphism are examined. In comparison with the different type of approximations in different type of algebraic structures, it is concluded that this new study is much better.

1. Introduction

The quantale module has piqued the interest of many researchers since it was first proposed by Abramsky and Vickers [1]. The concept of a quantale module was inspired by the concept of module over a ring. Rings are replaced by quantales, while abelian groups are replaced by complete lattices. For the first time, the concept of quantale module appeared out of nowhere as the central concept in Abramsky and Vickers' unified treatment of process semantics. Mulvey [2] proposed the Quantale theory. It is defined on the basis of a complete lattice as an algebraic structure.

Pawlak developed the famous rough set theory [3], which deals with inadequate knowledge. The rough set deals with the categorization and investigation of inadequate information and knowledge. After Pawlak's work, some contributions and a new view on rough set theory were suggested by Zhu [4]. In [5], some properties and characterization of generalized rough sets were presented by Ali et al. Rough sets are now used in a variety of fields, including cognitive sciences, machine learning, pattern recognition, and process control.

Rough sets theory was brought to algebraic structures and soft algebraic structures by a number of authors. Iwinski explored rough set algebraic characteristics [6]. In Q-module [7], Qurashi and Shabir presented the concept of roughness. Xiao and Li [8] proposed the concept of generalized rough quantales (subquantales). Yang and Xu examined rough ideals (prime, semi prime) in quantales [9]. Luo and Wang [10] introduced fuzzy ideals and its type in quantales. Generalized roughness of fuzzy substructures in quantale based on soft relation was studied by Qurashi et al. [11]. Topological structures of lower and upper rough subsets in a hyperring were introduced by Abughazalah et al. [12]. In [13], criteria selection and decision making of hotels using dominance-based rough set theory were presented. Approximations of substructures in partially ordered LA-semihypergroups were presented by Yaqoob and Tang [14].

In [15], roughness of bipolar soft sets and their related applications are discussed. In [16], Feng et al. presented the relationship between soft and rough sets and proposed rough soft sets and soft rough sets. Integrated Best-Worst Method in terms of Green supplier selection based on the information system performance was suggested by Fazlollahtabar and Kazemitash [17].

Many issues emerge in different fields such as engineering, economics, and social sciences where data have some degree of ambiguity. Because well-known mathematical tools are designed for certain situations, they have numerous restrictions. Many theories exist to deal with uncertainty, such as fuzzy set theory, probability theory, rough sets, and ambiguous sets, but they are constrained by their design.

Molodtsov introduced the concept of soft set [18], which is a mathematical tool for overcoming the problems that plague the above theories. Soft set theory is a general mathematical technique for dealing with items that are unclear, imprecise, or not precisely defined. Many authors offer different set operations and attempt to unify the algebraic aspects of soft sets like Maji et al. [19]. A new and different idea of operations was presented by Ali et al. [20]. Soft sets and algebraic structures were combined in various ways by researchers like soft intersection semigroups [21]. Soft linear programming and applications of soft vector spaces were presented in [22]. Khan et al. applied uni-soft structures to ordered Γ-Semihypergroups [23]. Complex intuitionistic fuzzy algebraic structures in groups were introduced by Gulzar et al. [24]. Development of a rough-MABAC-DoE-based metamodel for supplier selection in an iron and steel industry was introduced by Chattopadhyay et al. [25].

The central theme and objective of soft sets is to capture the essence of parametrization, which has been adapted to the creation of soft binary relations (SBRs), which are a parameterized collection of binary relations on a universe of interest. This mentioned the problem of complicated objects that can be interpreted differently from different perspectives.

By using aftersets and foresets and then notions associated to soft binary relation (SBR), a new method of approximation space is widely utilized these days. By using generalized approximation space based on SBR, different soft substructures in semigroups were approximated by Kanawal and Shabir [26]. Motivated by the idea in [26], soft substructures in quantale module are defined and the aftersets and foresets are employed to construct the lower approximation and upper approximation of soft substructures. Since we are dealing with approximation of soft subsets of quantale, further soft substructures are employed for further characterization.

A new generalized approximation space is commonly used these days by utilizing the aftersets and foresets notions related with soft binary relations. Kanawal and Shabir [26] approximated different soft substructures in semigroups using a generalized approximation space based on soft binary relations. Roughness of intuitionistic fuzzy sets by soft relations was discussed by Anwar et al. [27]. Roughness of Pythagorean fuzzy sets based on soft binary relations was proposed in [28] by Bilal and Shabir. Using soft relations, soft substructures were defined by Zhou et al. [29] and these were approximated by soft relations. Soft substructures in quantale modules are defined in this paper, and aftersets and foresets are used to construct the lower and upper approximation of substructures, respectively.

The following scheme is for the remainder of the paper. Section 2 connects some key explanations about quantale modules, their substructures, soft substructures, and their relevant sequels. Section 3 discusses the concept of crisp sub sets approximations over quantale module created by soft binary relations. In Section 4, generalized soft substructures are defined and further fundamental algebraic properties of these phenomena are investigated utilizing these ideas. In Section 5, we also extend this research by defining the relationship between homomorphic images of substructures in quantale module and their approximation by soft binary relations.

2. Preliminaries

In this section, we will review some fundamental concepts related to quantale module and its substructures, soft sets, and rough sets.

2.1. Definition (see [2]). A quantale K_d is a complete lattice equipped with an associative, binary operation $\circledast$ distributing over an arbitrary joins. That is for any $r \in K_d, r_i, s_i \in K_d, (i \in I)$. It holds$r\circledast(\vee_{i\in I}s_i) = \vee_{i\in I}(r\circledast s_i)$ and $(v_{i\varepsilon I}r_i)\otimes s = v_{i\varepsilon i}(r_i \otimes s)$.

Let $X_i, X, Y \subseteq K_d\ (i \in I)$. Then, the followings are defined:

$$\begin{aligned} X\circledast Y &= \{x\circledast y | x \in X, y \in Y\};\\ X\vee Y &= \{x\vee y | x \in X, y \in Y\};\\ \vee_{i\in I}X_i &= \{\vee_{i\in I}x_i | x_i \in X_i\}. \end{aligned} \tag{1}$$

Throughout the paper, quantales are denoted by K_d.

2.2. Definition (see [1]). Let K_d be a quantale and M be a $S_{up}-$ lattice equipped with a left action $\oplus: K_d \times M \longrightarrow M$. Then, M is called left K_d-module over the quantale K_d if for any $a_i, a, b \in K_d, x \in M, \{x_j\} \subseteq M,\ (i \in I),\ (j \in J)$, we have

$$(\vee_{i\in I}a_i)\oplus x = \vee_{i\in I}(a_i \oplus x); \tag{2}$$

Right quantale modules can be defined in the same way. For the rest of the paper, K_d-module M will stand for a left quantale module over the quantale K_d. The symbol T will denote the top element and $\perp$ will stand for the bottom one for quantale module, unless stated otherwise.

2.3. Example. The following are the examples of K_d-modules M:

(1) Let $K_d = \{0, r, s, 1\}$ be a complete lattice where 0 is the bottom element and 1 is the top element of K_d, as shown in Figure 1 and the operation $\circledast$ on K_d is

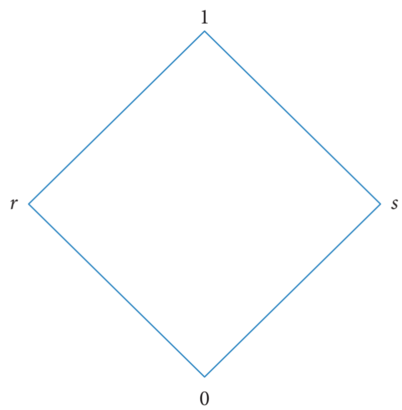

FIGURE 1: Illustration of k_d.

shown in Table 1. Then, it is straightforward to verify that $(K_d, \circledast)$ is a quantale. Let $M = \{\perp, x, T\}$ be a $S_{up}-$lattice. The order relation of M is given in Figure 2 and let $\oplus: K_d \times M \longrightarrow M$ be the left action on M as shown in Table 2. Then, it is straightforward that M is a K_d-module.

(2) Every quantale K_d is certainly a K_d-module over K_d.

TABLE 1: Binary operation subject to $\otimes$.

$\otimes$	0	r	s	I
0	0	0	0	0
r	0	r	0	r
s	0	0	s	s
I	0	r	s	I

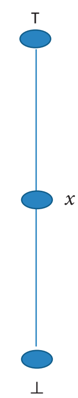

FIGURE 2: Illustration of M.

TABLE 2: Binary opreation subject to $\oplus$.

$\oplus$	$\perp$	x	T
0	$\perp$	$\perp$	$\perp$
r	$\perp$	x	T
s	$\perp$	x	x
I	$\perp$	x	T

2.4. Definition (see [1]). Let M be a K_d-module. A subset $M_1 \subseteq M$ is called a sub-K_d-module of M if for any $r \in M_1$, $\{s_i\} \subseteq M_1$, $k \in K_d$, it holds that $\vee_{i\in I} s_i \in M_1$ and $k \oplus r \in M_1$.

2.5. Definition (see [1]). Let M be a K_d-module and $\varnothing \neq I \subseteq M$. Then, I is a K_d-module ideal of M.

(1) If $r_i \in I\, (i \in I)$, then $\vee_{i\in I} r_i \in I$

(2) $r \in I$ and $c \leq r$ implies $c \in I$

(3) $r \in I$ implies $a \oplus r \in I, \forall a \in K_d$

2.6. Definition (see [1]). Let M be a K_d-module. A binary relation Γ on M is called congruence on M if it is an equivalence relation on M; for any given$\{m_i\}$, $\{n_i \subseteq M\}, m, n\varepsilon M$, and $q \in K_d$, it satisfies the following conditions: for all$i\varepsilon I, m_i \Gamma n_i$implies$\Delta_{i\in I} m_i \Gamma \Delta_{i\in I} n_iandm\Gamma n$ implies$q\Delta m \Gamma q \Delta n$.

2.7. Definition (see [1]). Let M_1 and M_2 be two K_d-modules. A map Ω: $M_1 \longrightarrow M_2$ is a K_d- module homomorphism if it is a sup-lattice homomorphism which also preserves scalar multiplication. That is,

$$\begin{aligned} \Omega\left(\vee_{i\in I} r_i\right) &= \vee_{i\in I} \Omega\left(r_i\right), \\ \Omega(b \oplus r) &= b \oplus \Omega(r), \end{aligned} \tag{3}$$

for any $b \in K_d$, $r \in M$,, $\{r_i\} \subseteq M\,(i \in I)$

A K_d-module homomorphism Ω: $M_1 \longrightarrow M_2$ is called an epimorphism if Ω is onto M_2 and Ω is called a monomorphism if Ω is one-one. It is an isomorphism, if Ω is bijective.

2.8. Theorem (see [1]). Let M_1 and M_2 be two K_d-modules. If Ω: $M_1 \longrightarrow M_2$ is a K_d-module homomorphism, then $\mathrm{Ker}(\Omega) = \{(a, b) \in M_1 \times M_1 : \Omega(a) = \Omega(b)\}$ is a congruence of K_d-modules.

2.9. Definition (see [18]). A pair (F, C) is called soft set over M if F: $C \longrightarrow P(M)$ where C is a subset of E (the set of parameters).

2.10. Definition (see [20]). Let (F, C_1) and (H, C_2) be two soft sets over M. Then, (F, C_1) soft subset (H, C_2) if the following conditions are fulfilled:

(1) $C_1 \subseteq C_2$

(2) $F(c) \subseteq H(c)$, $\forall$ c $\in C_1$

2.11. Definition (see [30]). Let (Γ, C) be a soft set over $M \times M$, i.e., Γ: $C \longrightarrow P(M \times M)$. Then, (Γ, C) is called a soft binary relation (SBYR) over M.

2.12. Definition. Let (Γ, C) be a soft set over quantale module M. Then,

(1) (Γ, C) is called a soft sub-K_d-module over M iff $\Gamma(c)$ is a sub-K_d-module of $M \forall$ c $\in C$

(2) (Γ, C) is a called soft K_d-module ideal over M iff $\Gamma(c)$ is K_d-module ideal of M, $\forall c \in C$

2.13. Definition (see [3]). Let $M \neq \emptyset$ be a finite set and Γ be an equivalence relation on M. Let $[t]_\Gamma$ denote the equivalence class of the relation Γ containing $t \in M$. If a subset of M is expressed as a union of equivalence classes of M, then that is said to be definable set in M. Let a subset R of M cannot be expressed as a union of equivalence classes of M. Then, we say it is undefinable set. However, we can approximate that undefinable set by two definable sets in M. The first one is called Γ-lower approximation $(\Gamma - L_{AP})$ of R, and the second is called Γ-upper approximation $(\Gamma - U_{AP})$ of R. They are defined as follows:

$$\underline{\Gamma}(R) = \{t \in M:\ [t]_\Gamma \subseteq M\} \text{ and } \overline{\Gamma}(R) = \{t \in M:\ [t]_\Gamma \cap M \neq \emptyset\}. \tag{4}$$

A rough set is the pair $(\underline{\Gamma}(R), \overline{\Gamma}(R))$; if $\underline{\Gamma}(R) = \overline{\Gamma}(R)$, then R is definable.

3. Approximation of Subsets of Quantale Module by Soft Binary Relation

In this section, applications of soft relation on quantale module are discussed. A subset of quantale module M can be approximated by soft relations in two ways. Aftersets and foresets are applied to approximate a subset of M. Two sets named as soft set corresponding to each subset are called the lower approximation (L_{AP}) and the upper approximation (U_{AP}) with respect to the aftersets and foresets, respectively.

3.1. Definition (see [11]). Let $\Gamma : C \longrightarrow P(M \times M)$. Then, (Γ, C) is a soft binary relation on a set M, where $\phi \neq C \subseteq E$ (set of parameters). For $\phi \neq S \subseteq M, (\underline{\Gamma}^S, C)$ and $(\overline{\Gamma}^S, C)$ of S with respect to aftersets are basically the two soft sets over M, defined as follows:

$$\underline{\Gamma}^S(u) = \{k \in M | k\Gamma(u) \subseteq S\} \text{ and } \overline{\Gamma}^S(u) = \{k \in M | k\Gamma(u) \cap S \neq \emptyset\} \tag{5}$$

Further, $({}^S\underline{\Gamma}, C)$ and $({}^S\overline{\Gamma}, C)$ of S with respect to foreset are basically the two soft sets over M, defined as follows: ${}^S\underline{\Gamma}(u) = \{k \in M | \Gamma(u)k \subseteq S\}$ and ${}^S\overline{\Gamma}(u) = \{k \in M | \Gamma(u) k \cap S \neq \emptyset\}$.

For all $u \in C$, $k\Gamma(u) = \{r \in M:\ (k, r) \in \Gamma(u)\}$ is called afterset of k and $\Gamma(u)k = \{r \in M:\ (r, k) \in \Gamma(u)\}$ is called foreset of k. Moreover, L_{AP} and U_{AP} are defined as $\underline{\Gamma}^S$: $C \longrightarrow P(M)$ and $\overline{\Gamma}^S$: $C \longrightarrow P(M)$ for aftersets. ${}^S\underline{\Gamma}$: $C \longrightarrow P(M)$ and ${}^S\overline{\Gamma}$: $C \longrightarrow P(M)$ for foresets for each $S \in P(M)$. Generally, $k\Gamma(u) \neq \Gamma(u)k$, $\underline{\Gamma}^S(u) \neq {}^S\underline{\Gamma}(u)$, and $\overline{\Gamma}^S(u) \neq {}^S\overline{\Gamma}(u)$. However, they are equal if $\Gamma(u)$ is a symmetric relation. This is justified in the next example.

3.2. Example. Let $K_d = \{\perp, r, s, T\}$ and $C = \{u_1, u_2\}$. Define Γ: $C \longrightarrow P(M \times M)$ by $\Gamma(u_1) = \{(\perp, \perp), (r, r), (s, s), (T, T), (\perp, r)\}$ and $\Gamma(u_2) = \{(\perp, \perp), (r, r), (s, s), (T, T), (r, T)\}$

Thus, the aftersets of elements M are as follows. $\perp\Gamma(u_1) = \{\perp, r\}$, $r\Gamma(u_1) = \{r\}$, $s\Gamma(u_1) = \{s\}$, $T\Gamma(u_1) = \{T\}$ and $\perp\Gamma(u_2) = \{\perp\}$, $r\Gamma(u_2) = \{r, T\}$, $s\Gamma(u_2) = \{s\}$, and $T\Gamma(u_2) = \{T\}$, and the foresets of elements of M are as follows: $\Gamma(u_1)\perp = \{\perp\}$, $\Gamma(u_1)r = \{\perp, r\}$, $\Gamma(u_1)s = \{s\}$, $\Gamma(u_1)T = \{T\}$ and $\Gamma(u_2)\perp = \{\perp\}$, $\Gamma(u_2)r = \{r\}$, $\Gamma(u_2)s = \{s\}$, $\Gamma(u_2)T = \{r, T\}$. Let $S = \{\perp, T\}$. Then, L_{AP} and U_{AP} of $S \subseteq M$ with respect to aftersets and foresets are as follows: $\underline{\Gamma}^S(u_1) = \{T\}$, $\underline{\Gamma}^S(u_2) = \{\perp, T\}$ and $\overline{\Gamma}^S(u_1) = \{\perp, T\}$, $\overline{\Gamma}^S(u_2) = \{\perp, r, T\}$, ${}^S\underline{\Gamma}(u_1) = \{\perp, T\}$, $\underline{\Gamma}^S(u_2) = \{\perp\}$ and ${}^S\overline{\Gamma}(u_1) = \{\perp, r, T\}$, ${}^S\overline{\Gamma}(u_2) = \{\perp, T\}$. This shows that $(\underline{\Gamma}^S, C) \neq ({}^S\underline{\Gamma}, C)$ and $(\overline{\Gamma}^S, C) \neq ({}^S\overline{\Gamma}, C)$.

3.3. Definition. A SBR Γ on a quantale module M is called soft compatible relation (SCRE), if it satisfies the following conditions: $\forall i \in I$ if $r_i\Gamma s_i \Rightarrow (\vee_{i\in I} r_i)\Gamma(\vee_{i\in I} s_i)$ and $r\Gamma s \Rightarrow (k \oplus r)\Gamma(k \oplus s)$ for any $r, s \in M, \{r_i\}, \{s_i\} \subseteq M, (i \in I)$ and $k \in K_d$.

3.4. Example. Let $K_d = \{\perp, r, s, T\}$ be a complete lattice as shown in Figure 3, and the operation $\circledast$ on K_d is $a \circledast b = \perp$. Then, it is easy to verify that $(K_d, \circledast)$ is a quantale. Let $\oplus$: $K_d \times M \longrightarrow M$ be the left action of K_d on M as shown in Table 3. In this case, $M = K_d$. Then, it is easy to check that K_d is a K_d-module over K_d and represented by M. Let $C = \{u_1, u_2\}$. Define Γ: $C \longrightarrow P(M \times M)$ by

$$\begin{aligned} \Gamma(u_1) &= \{(\perp, \perp), (r, r), (s, s), (\mathrm{T}, \mathrm{T}), (s, \mathrm{T})\}, \\ \Gamma(u_2) &= \{(\perp, \perp), (r, r), (s, s), (\mathrm{T}, \mathrm{T}), (r, \mathrm{T})\}. \end{aligned} \tag{6}$$

Then, (Γ, C) is a soft compatible relation (SCRE) and soft reflexive relation (SRRE) on M.

3.5. Remark. Let (Γ, C) be a SCRE on a K_d-module M. Then, it is easily verified that $k \oplus s\Gamma(u) \subseteq (k \oplus s)\Gamma(u)$ and $r\Gamma(u) \vee s\Gamma(u) \subseteq (r \vee s)\Gamma(u)$ for all $r, s \in M$ and $k \in K_d$.

3.6. Example. Let $(K_d, \oplus)$ be a K_d-module M as given in example 3.4 and let $C = \{u_1, u_2\}$. Define Γ: $C \longrightarrow P(M \times M)$ by $\Gamma(u_1) = \{(\perp, \perp), (r, r), (s, s), (\mathrm{T}, \mathrm{T}), (s, \mathrm{T}), (\mathrm{T}, s), (r, \mathrm{T}), (\mathrm{T}, r)\}$ and $\Gamma(u_2) = \{(\perp, \perp), (r, r), (s, s), (\mathrm{T}, \mathrm{T})\}$. Then, (Γ, C) is a SCRE and SRRE on M. The aftersets calculated by the elements of M are as follows:

$$\perp\Gamma(u_1) = \{\perp\}, r\Gamma(u_1) = \{r, \mathrm{T}\}, s\Gamma(u_1) = \{s, \mathrm{T}\}. \tag{7}$$

$\mathrm{T}\Gamma(u_1) = \{r, s, \mathrm{T}\}$. Thus, we have $r\Gamma(u_1) \vee s\Gamma(u_1) = \{r, \mathrm{T}\} \vee \{s, \mathrm{T}\} = \{\mathrm{T}\} \subseteq \{r, s, \mathrm{T}\} = (r \vee s)$ $\Gamma(u_1)$. Hence, $r\Gamma(u_1) \vee s\Gamma(u_1) \subseteq (r \vee s)\Gamma(u_1)$ Also, similarly we can check that $k \oplus x\Gamma(u) \subseteq (k \oplus x)\Gamma(u) \forall k \in K_d, x \in M \forall u \in C$.

3.7. Remark. If (Γ, C) is a SCRE on a K_d-module M, then $k \oplus \Gamma(u)s \subseteq \Gamma(u)(k \oplus s)$ and $\Gamma(u)r \vee \Gamma(u)s \subseteq \Gamma(u)(r \vee s)$ with respect to foresets.

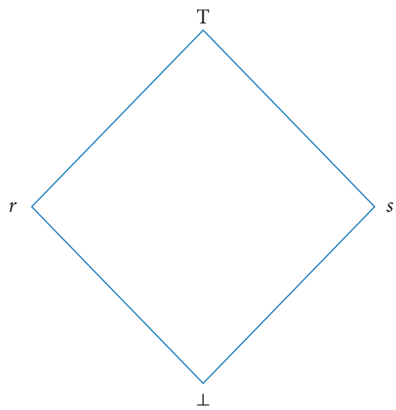

Figure 3: Illustration of M.

Table 3: Binary operation subject to $\oplus$.

$\oplus$	$\perp$	r	s	T
$\perp$	$\perp$	r	s	T
r	$\perp$	r	s	T
s	$\perp$	r	s	T
T	$\perp$	r	s	T

3.8. Definition. A SCRE (Γ, C) on a K_d-module M is called soft join-complete with respect to to aftersets if $p\Gamma(u) \vee q\Gamma(u) = (p \vee q)\Gamma(u)$ and is called soft complete with respect to $\oplus$ if $k \oplus p\Gamma(u) = (k \oplus p)\Gamma(u) \forall k \in K_d p, q \in M \forall u \in C$.

A SCRE (Γ, C) which is both join-complete and $\oplus$-complete with respect to aftersets is called soft complete relation (SCTR) with respect to aftersets.

3.9. Example. Let $(K_d, \oplus)$ be a K_d-module M as given in example 3.4 and let $C = \{u_1, u_2\}$. Define $\Gamma\colon C \longrightarrow P(M \times M)$ by

$$\begin{aligned} \Gamma(u_1) &= \{(\perp, \perp), (r, r), (s, s), (\mathrm{T}, \mathrm{T}), (s, \mathrm{T}), (r, \mathrm{T})\} \\ &\text{and } \Gamma(u_2) = \{(\perp, \perp), (r, r), (s, s), (\mathrm{T}, \mathrm{T})\}. \end{aligned} \tag{8}$$

Then, (Γ, C) is a SCRE and SRRE on M. The aftersets calculated by the elements of M are as follows: $\perp\Gamma(u_1) = \{\perp\}, r\Gamma(u_1) = \{r, \mathrm{T}\}, s\Gamma(u_1) = \{s, \mathrm{T}\}, \mathrm{T}\Gamma(u_1) = \{\mathrm{T}\}$. It is easily checked that $p\Gamma(u_1) \vee q\Gamma(u_1) = (p \vee q)\Gamma(u_1)\ \forall p, q \in M$ That is, $r\Gamma(u_1) \vee s\Gamma(u_1) = \{r, \mathrm{T}\} \vee \{s, \mathrm{T}\} = \{\mathrm{T}\} = (r \vee s)\Gamma(u_1)$ etc. Further, we can check that $k \oplus p\Gamma(u_1) = (k \oplus p)\Gamma(u_1) \forall k \in K_d, p \in M \forall u \in C$.

So, (Γ, C) is a SCTR with respect to aftersets.

3.10. Definition. A SCRE (Γ, C) on a K_d-module M is called soft join-complete with respect to foresets if $\Gamma(u)p \vee \Gamma(u)q = \Gamma(u)(p \vee q)$ and is called soft complete with respect to $\oplus$ if $k \oplus \Gamma(u)p = \Gamma(u)(k \oplus p) \forall k \in K_d p, q \in M \forall u \in C$.

A SCRE (Γ, C) which is both join-complete and $\oplus$-complete is called SCTR with respect to foresets.

3.11. Remark. It has been observed that if we have SCTR for aftersets, not need it is SCTR for foresets. This is demonstrated in the following example.

3.12. Example. Let $(K_d, \oplus)$ be a K_d-module M as given in example 3.4 and let $C = \{u_1, u_2\}$. Define $\Gamma\colon C \longrightarrow P(M \times M)$ by $\Gamma(u_1) = \{(\perp, \perp), (r, r), (s, s), (\mathrm{T}, \mathrm{T}), (s, \mathrm{T}), (r, \mathrm{T})\}$ and $\Gamma(u_2) = \{(\perp, \perp), (r, r), (s, s), (\mathrm{T}, \mathrm{T})\}$. Then, (Γ, C) is a SCRE and SRRE on M. The aftersets and foresets calculated by the elements of M are as follows: $\perp\Gamma(u_1) = \{\perp\}, r\Gamma(u_1) = \{r, \mathrm{T}\}, s\Gamma(u_1) = \{s, \mathrm{T}\}, \mathrm{T}\Gamma(u_1) = \{\mathrm{T}\}$, $\Gamma(u_1)\perp = \{\perp\}$, $\Gamma(u_1)r = \{r\}, \Gamma(u_1)s = \{s\}, \Gamma(u_1)\mathrm{T} = \{r, s, \mathrm{T}\}$. It is observed that $r\Gamma(u_1) \vee s\Gamma(u_1) = \{r, \mathrm{T}\} \vee \{s, \mathrm{T}\} = \{\mathrm{T}\} = (r \vee s)\Gamma(u_1)$. Likewise, we can check that $k \oplus p\Gamma(u_1) = (k \oplus p)\Gamma(u_1) \forall k \in K_d, p \in M, \forall u \in C$ So, (Γ, C) is a SCTR with respect to aftersets. But, $\Gamma(u_1)r \vee \Gamma(u_1)s = \{r\} \vee \{s\} = \{\mathrm{T}\} \neq \{r, s, \mathrm{T}\} = \Gamma(u_1)(r \vee s)$ So, (Γ, C) is not a SCTR with respect to foresets.

In ref [31], the following theorems are helpful for our further study.

3.13. Theorem (see [31]). Let $\varnothing \neq S, \varnothing \neq R$ be the subsets of K_d-module M and (Γ, C) and (γ, C) be SRRE on M. Then, the following hold for all $u \in C$:

(1) $\underline{\Gamma}^S(u) \subseteq S \subseteq \overline{\Gamma}^S(u)$
(2) $S \subseteq R \Rightarrow \underline{\Gamma}^S(u) \subseteq \underline{\Gamma}^R(u)$
(3) $S \subseteq R \Rightarrow \overline{\Gamma}^S(u) \subseteq \overline{\Gamma}^R(u)$
(4) $(\underline{\Gamma}^S, C) \cap (\underline{\Gamma}^R, C) = (\underline{\Gamma}^{S \cap R}, C)$
(5) $(\overline{\Gamma}^S, C) \cap (\overline{\Gamma}^R, C) \supseteq (\overline{\Gamma}^{S \cap R}, C)$
(6) $(\underline{\Gamma}^S, C) \cup (\underline{\Gamma}^R, C) \subseteq (\underline{\Gamma}^{S \cup R}, C)$
(7) $(\overline{\Gamma}^S, C) \cup (\overline{\Gamma}^R, C) = (\overline{\Gamma}^{S \cup R}, C)$
(8) $(\Gamma, C) \subseteq (\gamma, C) \Rightarrow (\underline{\Gamma}^S, C) \supseteq (\underline{\gamma}^S, C)$
(9) $(\Gamma, C) \subseteq (\gamma, C) \Rightarrow (\overline{\Gamma}^S, C) \subseteq (\overline{\gamma}^S, C)$

3.14. Theorem (see [31]). Let $\varnothing \neq S, \varnothing \neq R$ be the subset of K_d-module M and (Γ, C) and (γ, C) be SRRE on M. Then, the following hold for all $u \in C$:

(10) ${}^S\underline{\Gamma}(u) \subseteq S \subseteq {}^S\overline{\Gamma}(u)$
(11) $S \subseteq R \Rightarrow {}^S\underline{\Gamma}(u) \subseteq {}^R\underline{\Gamma}(u)$
(12) $S \subseteq R \Rightarrow {}^S\overline{\Gamma}(u) \subseteq {}^R\overline{\Gamma}(u)$
(13) $({}^S\underline{\Gamma}, C) \cap ({}^R\underline{\Gamma}, C) = ({}^{S \cap R}\underline{\Gamma}, C)$
(14) $({}^S\overline{\Gamma}, C) \cap ({}^R\overline{\Gamma}, C) \supseteq ({}^{S \cap R}\overline{\Gamma}, C)$
(15) $({}^S\overline{\Gamma}, C) \cup ({}^R\overline{\Gamma}, C) \subseteq ({}^{S \cap R}\overline{\Gamma}, C)$
(16) $({}^S\overline{\Gamma}, C) \cup ({}^R\overline{\Gamma}, C) = ({}^{S \cup R}\overline{\Gamma}, C)$
(17) $(\Gamma, C) \subseteq (\gamma, C) \Rightarrow ({}^S\underline{\Gamma}, C) \supseteq ({}^S\underline{\gamma}, C)$
(18) $(\Gamma, C) \subseteq (\gamma, C) \Rightarrow ({}^S\overline{\Gamma}, C) \subseteq ({}^S\overline{\gamma}, C)$

3.15. Theorem (see [31]). Let $\varnothing \neq S, \varnothing \neq R$ be the subsets of K_d-module M and (Γ, C) and (γ, C) be SRRE on M. Then, the following hold for all $u \in C$:

(1) $((\overline{\Gamma \cap \gamma})^S, C) \subseteq (\overline{\Gamma}^S, C) \cap (\overline{\gamma}^S, C)$

(2) $((\underline{\Gamma\cap\gamma})^S, C)\supseteq(\underline{\Gamma}^S, C)\cup(\underline{\gamma}^S, C)$
(3) $({}^S\overline{(\Gamma\cap\gamma)}, C)\subseteq({}^S\overline{\Gamma}, C)\cap({}^S\overline{\gamma}, C)$
(4) $({}^S\underline{(\Gamma\cap\gamma)}, C)\supseteq({}^S\underline{\Gamma}, C)\cup({}^S\underline{\gamma}, C)$

3.16. Theorem. Let (Γ, C) be SRRE and SCRE with respect to the aftersets on a K_d-module M. Then, for non-empty subset S and R of M, we have $k\oplus\overline{\Gamma}^S(u)\subseteq\overline{\Gamma}^{k\oplus S}(u)$ and $\overline{\Gamma}^S(u)\vee\overline{\Gamma}^R(u)\subseteq\overline{\Gamma}^{S\vee R}(u)\forall\ u\in C$.

Proof. Let $x\in k\oplus\overline{\Gamma}^S(u)$ Then, $x=k\oplus p$ where $p\in\overline{\Gamma}^S(u)$ such that $p\Gamma(u)\cap S\neq\varnothing$. Thus, there exist $a\in p\Gamma(u)\cap S$ such that $a\in S$. That is, $(p,a)\in\Gamma(u)$. Since (Γ, C) is SCRE, we have $(k\oplus p, k\oplus a)\in\Gamma(u)$ and $k\oplus a\in k\oplus S$, $\Rightarrow k\oplus a\in(k\oplus p)\Gamma(u)\cap k\oplus S$,
$\Rightarrow\varnothing\neq(k\oplus p)\Gamma(u)\cap k\oplus S\Rightarrow k\oplus p\in\overline{\Gamma}^{k\oplus S}(u)$. So, $x\in\overline{\Gamma}^{k\oplus S}(u)$. Thus, $k\oplus\overline{\Gamma}^S(u)\subseteq\overline{\Gamma}^{k\oplus S}(u)$

Let $y\in\overline{\Gamma}^S(u)\vee\overline{\Gamma}^R(u)$. Then, $y=p\vee q$ where $p\in\overline{\Gamma}^S(u)$ and $q\in\overline{\Gamma}^R(u)$ such that $p\Gamma(u)\cap S\neq\varnothing$ and $q\Gamma(u)\cap R\neq\varnothing$. That is, $a\in p\Gamma(u)\cap S$ and $b\in q\Gamma(u)\cap R$. So, $a\in p\Gamma(u)$, $a\in S$ and $b\in q\Gamma(u)$, $b\in R$, $\Rightarrow(p,a)\in\Gamma(u), a\in S$ and $(q,b)\in\Gamma(u), b\in R$. Since (Γ, C) is SCRE, we have $(p\vee q, a\vee b)\in\Gamma(u)$and $a\vee b\in S\vee R\Rightarrow a\vee b\in(p\vee q)$ $\Gamma(u)\cap S\vee R$, $\Rightarrow\varnothing\neq(p\vee q)\Gamma(u)\cap(S\vee R)\Rightarrow(p\vee q)\in\overline{\Gamma}^{S\vee R}(u)$ So, $y\in\overline{\Gamma}^{S\vee R}(u)$. Thus, $\overline{\Gamma}^S(u)\vee\overline{\Gamma}^R(u)\subseteq\overline{\Gamma}^{S\vee R}(u)$. □

3.17. Example. Let $(K_d, \oplus)$ be a K_d-module M as given in example 3.4 and let $C=\{u_1, u_2\}$. Define $\Gamma: C\longrightarrow P(M\times M)$ by $\Gamma(u_1)=\{(\perp,\perp), (r,r), (s,s), (\mathrm{T},\mathrm{T}), (r,\mathrm{T})\}$ and $\Gamma(u_2)=\{(\perp,\perp), (r,r), (s,s), (\mathrm{T},\mathrm{T})\}$. Then, (Γ, C) is a SCRE and SRRE on M. The aftersets calculated by the elements of M are as follows: $\perp\Gamma(u_1)=\{\perp\}, r\Gamma(u_1)=\{r,\mathrm{T}\}$, $s\Gamma(u_1)=\{s\}, \mathrm{T}\Gamma(u_1)=\{\mathrm{T}\}$, $\perp\Gamma(u_2)=\{\perp\}, r\Gamma(u_2)=\{r\}$, $s\Gamma(u_2)=\{s\}, \mathrm{T}\Gamma(u_2)=\{\mathrm{T}\}$. Let $R=\{s\}$ and $S=\{\mathrm{T}\}$. Then, $\overline{\Gamma}^R(u_1)=\{s\}, \overline{\Gamma}^S(u_1)=\{r,T\}$, and $\overline{\Gamma}^R(u_1)\vee\overline{\Gamma}^S(u_1)=\{T\}$. Now, $S\vee R=\{T\}$and $\overline{\Gamma}^{S\vee R}(u_1)=\{r,T\}$. So, $\overline{\Gamma}^S(u_1)\vee\overline{\Gamma}^R(u_1)\subseteq\overline{\Gamma}^{S\vee R}(u_1)$. Similarly, we can prove that $k\oplus\overline{\Gamma}^S(u)\subseteq\overline{\Gamma}^{k\oplus S}(u)$.

3.18. Theorem. Let (Γ, C) be SRRE and SCRE with respect to the foresets on a K_d-module M. Then, for non-empty subset S and R of M, we have $k\oplus{}^S\overline{\Gamma}(u)\subseteq{}^{k\oplus S}\overline{\Gamma}(u)$ and ${}^S\overline{\Gamma}(u)\vee{}^R\overline{\Gamma}(u)\subseteq{}^{S\vee R}\overline{\Gamma}(u)\forall u\in C$.

Proof. The proof is similar to the proof of Theorem 3.16. □

4. Approximation of Substructures in Quantale Module

In this section, foresets and aftersets are applied to different type of substructures in quantale module through soft relations to discuss their lower and upper approximations. These are then characterized by soft reflexive and soft compatible relations to present different characteristics of them.

4.1. Definition. Let $\varnothing\neq S\subseteq M$ and (Γ, C) be a SBR on a K_d-module M. Then, S is said to be generalized upper soft sub K_d-module of M with respect to aftersets if $(\overline{\Gamma}^S, C)$ is a soft sub-K_d-module of M.

4.2. Theorem. Let (Γ, C) be a SRRE and SCRE on a K_d-module M. Then, S is said to be generalized upper soft sub K_d-module of M with respect to aftersets if S is a soft sub-K_d-module of M.

Proof (1) Suppose that $r_i\in\overline{\Gamma}^S(u)$ for $u\in C$ and for some $i\in I$. Then, $r_i\Gamma(u)\cap S\neq\varnothing$. There are $x_i\in r_i\Gamma(u)\cap S$ such that $x_i\in r_i\Gamma(u)$ and $x_i\in S$. That is, $(r_i, x_i)\in\Gamma(u)$ and $x_i\in S$, $\forall i\in I$. Since S is a sub-K_d-module of M and $\Gamma(u)$ is a SCRE$\forall u\in C$. Thus, we have $(\vee_{i\in I}r_i, \vee_{i\in I}x_i)\in\Gamma(u)$ and $\vee_{i\in I}x_i\in S$. That is, $\vee_{i\in I}x_i\in\vee_{i\in I}r_i\Gamma(u)$ and $\vee_{i\in I}x_i\in S$. So, $\vee_{i\in I}x_i\in\vee_{i\in I}r_i\Gamma(u)\cap S$. Hence, $\vee_{i\in I}r_i\Gamma(u)\cap S\neq\varnothing$. This shows that $\vee_{i\in I}r_i\in\overline{\Gamma}^S(u)$.

(2) Let $r\in M$ such that $r\in\overline{\Gamma}^S(u)$. This shows $r\Gamma(u)\cap S\neq\varnothing$. Then, there is $y\in r\Gamma(u)\cap S$ such that $y\in r\Gamma(u)$ and $y\in S$. That is, $(r,y)\in\Gamma(u)$ and $y\in S$. Since S is a sub-K_d-module of M and $\Gamma(u)$ is a SRRE and SCRE. Thus, we have $k\oplus y\in S$ and $(k\oplus r, k\oplus y)\in\Gamma(u)$, $\Rightarrow k\oplus y\in(k\oplus r)\Gamma(u)$ and $k\oplus y\in S$, $\Rightarrow k\oplus y\in(k\oplus r)\Gamma(u)\cap S$. Hence, $(k\oplus r)\Gamma(u)\cap S\neq\varnothing$. This shows that $k\oplus r\in\overline{\Gamma}^S(u)$. Therefore, $\overline{\Gamma}^S(u)$ is a sub K_d-module of M. That is, S is a generalized upper soft sub-K_d-module of M with respect to aftersets.

It is mentioned in the next example that the converse is not true. □

4.3. Example. Suppose $K_d=\{\perp, r, s, t, T\}$ be a complete lattice as shown in Figure 4 and the operation $\circledast$ on K_d is that $a\circledast b=\perp$. Then, it is easily checked that $(K_d, \circledast)$ is a quantale. Suppose $\circledast$: $K_d\times M\longrightarrow M$ be the left action of K_d on M as shown in Table 4. In this case, $M=K_d$; then, it is easy to check that K_d is a quantale module M. Let $C=\{u_1, u_2\}$. Then, $\Gamma: C\longrightarrow P(M\times M)$ defined by $\Gamma(u_1)=\{(\perp,\perp), (r,r), (s,s),$
$(\mathrm{T},\mathrm{T}), (r,\mathrm{T}), (s,T), (\perp,T), (t,T), (r,t), (r,\perp), (t,s), (t,t)\}$ and $\Gamma(u_2)=\{(\perp,\perp), (r,r), (s,s), (t,t), (\mathrm{T},\mathrm{T}), (T,t), (t,r)\}$.

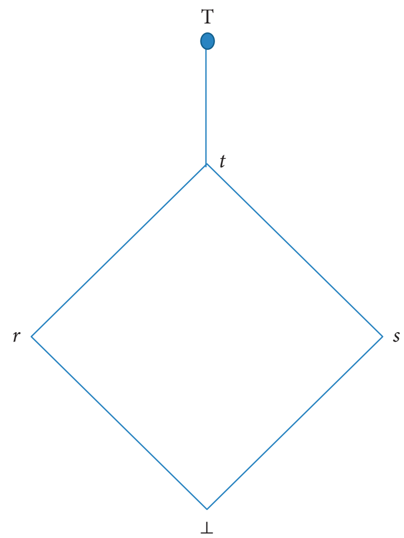

FIGURE 4: Illustration of M.

TABLE 4: Binary operation subject to $\oplus$.

$\oplus$	$\perp$	r	s	t	T
$\perp$	$\perp$	r	s	t	T
r	$\perp$	r	s	t	T
s	$\perp$	r	s	t	T
t	$\perp$	r	s	t	T
T	$\perp$	r	s	t	T

Then, (Γ, C) is a SCRE and SRRE on M. The aftersets calculated by the elements of M are as follows:

$$\begin{aligned}
\perp\Gamma(u_1) &= \{\perp, T\}, r\Gamma(u_1)\\
&= \{\perp, r, t, \mathrm{T}\}, s\Gamma(u_1)\\
&= \{s, T\},\\
t\Gamma(u_1) &= \{s, t, T\}, T\Gamma(u_1)\\
&= \{T\},\\
\perp\Gamma(u_2) &= \{\perp\}, r\Gamma(u_2)\\
&= \{r\}, s\Gamma(u_2)\\
&= \{s\},\\
t\Gamma(u_2) &= \{r, t\}, T\Gamma(u_2)\\
&= \{t, T\}.
\end{aligned} \tag{9}$$

Let $S = \{T, r, s\}$. Then, S is not a sub-K_d-module of M. But, $\overline{\Gamma}^S(u_1) = \{\perp, r, s, t, T\}$ and $\overline{\Gamma}^S(u_2) = \{r, s, t, T\}$ are sub-K_d-module of M. Hence, S is a generalized upper soft sub-K_d-module of M with respect to aftersets.

4.4. Definition. Let $\varnothing \neq S \subseteq M$ and (Γ, C) be a SBR on a K_d-module M. Then, S is said to be generalized upper soft sub K_d-module of M with respect to foresets if $({}^S\overline{\Gamma}, C)$ is a soft sub-K_d-module of M.

4.5. Theorem. Let (Γ, C) be a SRRE and SCRE on a K_d-module M. Then, S is said to be generalized upper soft sub K_d-module of M with respect to foresets if S is a soft sub-K_d-module of M.

Proof. The proof is similar to the proof of Theorem 4.2. □

4.6. Definition. Let $\varnothing \neq S \subseteq M$ and (Γ, C) be a SRRE and SCRE on a K_d-module M. Then, S is said to be generalized lower soft sub-K_d-module of M with respect to aftersets if $(\underline{\Gamma}^S, C)$ is a soft sub-K_d-module of M.

4.7. Theorem. Let (Γ, C) be a SRRE and SCTE on a K_d-module M. Then, S is said to be generalized lower soft sub K_d-module of M with respect to aftersets if S is a soft sub-K_d-module of M.

Proof. (1) Suppose that $r_i \in \underline{\Gamma}^S(u)$ for $u \in C$ and for $i \in I$. This shows that $r_i\Gamma(u) \subseteq S$. Hence, S is a sub-K_d-module of M and $\Gamma(u)$ is SCTR $\forall u \in C$. Thus, $(\vee_{i\in I} r_i)\Gamma(u) = \vee_{i\in I}(r_i\Gamma(u))$ and $(\vee_{i\in I} r_i)\Gamma(u) = \vee_{i\in I}(r_i\Gamma(u)) \subseteq S$. Hence, $(\vee_{i\in I} r_i)\Gamma(u) \subseteq S$. This shows that $\vee_{i\in I} r_i \in \underline{\Gamma}^S(u)$.

(2) Let $r \in M$ be such that $r \in \underline{\Gamma}^S(u)$. This shows $r\Gamma(u) \subseteq S$ for $u \in C$. Hence, S is a sub-K_d-module of M and $\Gamma(u)$ is a SRRE and SCTE. Thus, we have $(k \oplus r)\Gamma(u) = k \oplus r\Gamma(u)$. So, we can write $(k \oplus r)\Gamma(u) = k \oplus r\Gamma(u) \subseteq S$. Hence, $(k \oplus r)\Gamma(u) \subseteq S$. This shows that $k \oplus r \in \underline{\Gamma}^S(u)$. Therefore, $\underline{\Gamma}^S(u)$ is a sub K_d-module of M. That is, S is a generalized lower soft sub-K_d-module of M with respect to aftersets.

Observe in the next example that the converse is not true. □

4.8. Example. Let $(K_d, \oplus)$ be a K_d-module M as given in example 4.3 and let $C = \{u_1, u_2\}$. Define $\Gamma\colon C \longrightarrow P(M \times M)$ by $\Gamma(u_1) = \{(\perp, \perp), (r, r), (s, s), (\mathrm{T}, \mathrm{T}), (r, \mathrm{T}), (s, T)\}$ and $\Gamma(u_2) = \{(\perp, \perp), (r, r), (s, s), (t, t), (\mathrm{T}, \mathrm{T}), (T, t), (r, T), (s, \perp)\}$. The aftersets calculated by the elements of M are as follows: $\perp\Gamma(u_1) = \{\perp\}, r\Gamma(u_1) = \{r, \mathrm{T}\}, s\Gamma(u_1) = \{s, T\}, \mathrm{T}\Gamma(u_1) = \{\mathrm{T}\}$, $\perp\Gamma(u_2) = \{\perp\}, r\Gamma(u_2) = \{r, T\}, s\Gamma(u_2) = \{\perp, s\}, \mathrm{T}\Gamma(u_2) = \{r, \mathrm{T}\}$. Then, (Γ, C) is a SCTR and SRRE with respect to aftersets. Let $S = \{\perp, r, s\}$. Then, S is not a sub-K_d-module of M. However, $\underline{\Gamma}^S(u_1) = \{\perp\}, \underline{\Gamma}^S(u_2) = \{\perp\}$ are sub-K_d-module of M. Hence, S is generalized lower soft sub-K_d-module of M.

4.9. Definition. Let $\varnothing \neq S \subseteq M$ and (Γ, C) be a SRRE and SCRE on a K_d-module M. Then, S is said to be generalized lower soft sub-K_d-module of M with respect to foresets if $({}^S\underline{\Gamma}, C)$ is a soft sub-K_d-module of M.

4.10. Theorem. Let (Γ, C) be a SCTR and SRRE on a K_d-module M. Then, S is said to be generalized lower soft sub K_d-module of M with respect to foresets if S is a soft sub-K_d-module of M.

Proof. The proof is obvious. □

4.11. Remark. The results in this section related to foresets are similar to the results with respect to aftersets.

4.12. Definition. Let $\emptyset \neq S \subseteq M$ and (Γ, C) be a SRRE and SCRE on a K_d-module M if $(\overline{\Gamma}^S, C)$ is a soft K_d-module ideal M. Then, S is said to be generalized upper soft K_d-module ideal of M with respect to aftersets.

4.13. Theorem. Let $\emptyset \neq S \subseteq M$ and (Γ, C) be a SRRE, SCRE and soft join-complete relation on a K_d-module M. Then, S is said to be generalized upper soft K_d-module ideal of M with respect to aftersets if S is an K_d-module ideal of M.

Proof (1) Suppose that $r, s \in M$ such that $r, s \in \overline{\Gamma}^S(u)$ for $u \in C$. Therefore, $r\Gamma(u) \cap S \neq \emptyset$ and $s\Gamma(u) \cap S \neq \emptyset$. Then ,there are $x \in r\Gamma(u) \cap S$ and $y \in s\Gamma(u) \cap S$ such that $x \in r\Gamma(u)$, $y \in s\Gamma(u)$, and $x \in S, y \in S$. This implies $(r, x) \in \Gamma(u)$, $(s, y) \in \Gamma(u)$, and $x, y \in S$. S is a K_d-module ideal of M and $\Gamma(u)$ is a SCRE $\forall u \in C$. Hence, we have $(r \vee s, x \vee y) \in \Gamma(u)$ and $x \vee y \in S$. That is, $x \vee y \in (r \vee s)\Gamma(u)$ and $x \vee y \in S$. So, $x \vee y \in (r \vee s)\Gamma(u) \cap S$. Hence, $(r \vee s)\Gamma(u) \cap S \neq \emptyset$. This shows that $(r \vee s) \in \overline{\Gamma}^S(u)$.

(2) Let $s \in \overline{\Gamma}^S((u))$ and $r \leq s$. Therefore, $s\Gamma(u) \cap S \neq \emptyset$. Then, there is $y \in s\Gamma(u) \cap S$ such that $y \in s\Gamma(u)$ and $y \in S$. Since $\Gamma(u)$ is a SCRE and soft join-complete relation, we have $s\Gamma(u) = (r \vee s)\Gamma(u) = r\Gamma(u) \vee s\Gamma(u)$. Then, there is $u \in r\Gamma(u)$ and $v \in s\Gamma(u)$ such that $y = u \vee v.S$ is a K_d-module ideal of M and $u \leq y \in S$. We have $u \in S$ so $r \in \overline{\Gamma}^S(u)$

(3) Let $k \in M$ be such that $r \in \overline{\Gamma}^S(u)$. This shows $r\Gamma(u) \cap S \neq \emptyset$. Then, there are $y \in r\Gamma(u) \cap S$ such that $y \in r\Gamma(u)$ and $y \in S$. That is, $(r, y) \in \Gamma(u)$ and $y \in S$. S is a K_d module ideal of M and $\Gamma(u)$ is a SRRE and SCRE. We have $k \oplus y \in S$ and $(k \oplus r, k \oplus y) \in \Gamma(u) \Rightarrow k \oplus y \in (k \oplus r)\Gamma(u)$ and $k \oplus y \in S$, $\Rightarrow k \oplus y \in (k \oplus r)\Gamma(u) \cap S$. Hence, $(k \oplus r)\Gamma(u) \cap S \neq \emptyset$. This shows that $k \oplus r \in \overline{\Gamma}^S(u)$. Therefore, $\overline{\Gamma}^S(u)$ is a K_d-module ideal of M. That is, S is a generalized upper soft K_d-module ideal of M with respect to aftersets.

It is observed in the next example that the converse is not true. □

4.14. Example. Let M be the quantale module as given in example 4.3 and $C = \{u_1, u_2\}$. Then, $\Gamma(u_1) = \{(\perp, \perp), (r, r), (s, s), (\mathrm{T}, \mathrm{T}), (r, \mathrm{T}), (s, T), (\perp, T), (t, T), (r, t), (r, \perp), (t, s), (t, t)\}$ and $\Gamma(u_2) = \{(\perp, \perp), (r, r), (s, s), (t, t), (\mathrm{T}, \mathrm{T}), (T, t), (t, r)\}$.

The aftersets calculated by the elements of M are as follows:

$$\begin{aligned}\perp\Gamma(u_1) &= \{\perp, T\}, r\Gamma(u_1)\\ &= \{\perp, r, t, \mathrm{T}\}, s\Gamma(u_1)\\ &= \{s, T\}, t\Gamma(u_1)\\ &= \{s, t, T\}, \mathrm{T}\Gamma(u_1)\\ &= \{\mathrm{T}\}\perp\Gamma(u_2)\\ &= \{\perp\}, r\Gamma(u_2)\\ &= \{r, T\}, s\Gamma(u_2) = \{s\}, \mathrm{T}\Gamma(u_2)\\ &= \{\mathrm{T}\}, t\Gamma(u_2)\\ &= \{t\},\end{aligned} \tag{10}$$

then (Γ, C) is a SCRE and SRRE and soft join-complete relation with respect to aftersets on M. Let $S = \{T, t, s, \perp\}$. Then, S is not a K_d-module ideal of M. However, $\overline{\Gamma}^S(u_1) = \{\perp, r, s, t, T\}$ and $\overline{\Gamma}^S(u_1) = \{\perp, r, s, t, \mathrm{T}\}$ are K_d-module ideal of M. Hence, S is a generalized upper soft K_d-module ideal of M with respect to aftersets.

4.15. Definition. Let $\emptyset \neq S \subseteq M$ and (Γ, C) be a SRRE and SCRE on a K_d-module M if $({}^S\overline{\Gamma}, C)$ is a soft K_d-module ideal M. Then, S is said to be generalized upper soft K_d-module of M with respect to foresets if S is a K_d-module ideal of M.

4.16. Theorem. Let $\emptyset \neq S \subseteq M$ and (Γ, C) be a SRRE, SCRE, and soft join-complete relation on a K_d-module M. Then, S is said to be generalized upper soft K_d-module ideal of M with respect to foresets if S is an K_d-module ideal of M.

Proof. The proof is obvious. □

4.17. Definition. Let $\emptyset \neq S \subseteq M$ and (Γ, C) be a SRRE and SCRE on a K_d-module M if $(\underline{\Gamma}^S, C)$ is a soft K_d-module ideal M. Then, S is said to be generalized lower soft K_d-module of M with respect to aftersets.

4.18. Theorem. Let $\emptyset \neq S \subseteq M$ and (Γ, C) be a SRRE and SCTR on a K_d-module M. Then, S is said to be generalized lower soft K_d-module ideal of M with respect to aftersets if S is an K_d-module ideal of M.

Proof (1) Suppose that $r, s \in M$ such that $r, s \in \underline{\Gamma}^S(u)$ for $u \in C$. This shows $r\Gamma(u) \subseteq S$ and $s\Gamma(u) \subseteq S$. Hence, S is a K_d-module ideal of M and $\Gamma(u)$ is SCTR $\forall u \in C$. So, we have $(r \vee s)\Gamma(u) = r\Gamma(u) \vee s\Gamma(u)$ and $r\Gamma(u) \vee s\Gamma(u) = (r \vee s)\Gamma(u) \subseteq S$. Hence, $(r \vee s)\Gamma(u) \subseteq S$. This shows that $(r \vee s) \in \underline{\Gamma}^S(u)$.

(2) Let $s \in \underline{\Gamma}^S(u)$ and $r \leq s$. Therefore, $s\Gamma(u) \subseteq S$. For $w \in r\Gamma(u)$, we have $(r, w) \in \Gamma(u)$. Since $(r, w) \in \Gamma(u)$ is a SRRE and SCRE, we have $(r \vee s, w \vee s) \in \Gamma(u)$. That is, $(s, w \vee s) \in \Gamma(u)$. Thus,

$(w \vee s) \in S$. Since S is a K_d-module ideal of M, we have $w \in S$. Thus, $r\Gamma(u) \subseteq S$ and $r \in \overline{\Gamma}^S(u)$.

(3) Let $k \in M$ be such that $r \in \underline{\Gamma}^S(u)$. This shows $r\Gamma(u) \subseteq S$ for $u \in C$. Hence, S is a K_d-module ideal of M. Then, for all $k \in M$, we have $k \oplus r\Gamma(u) \subseteq k \oplus S \subseteq S$. Also given that $\Gamma(u)$ is a SRRE and SCTE. Thus, we can write $k \oplus r\Gamma(u) = (k \oplus r)\Gamma(u) \subseteq S$. Hence, $k \oplus r\Gamma(u) = (k \oplus r)\Gamma(u) \subseteq S$. This shows that $k \oplus r \in \underline{\Gamma}^S(u)$. Therefore, $\underline{\Gamma}^S(u)$ is a K_d-module ideal of M. That is, S is a generalized lower soft K_d-module ideal of M with respect to aftersets.

Observe in the next example that the converse is not true. □

4.19. Example. Let $(K_d, \oplus)$ be a K_d-module M as given in example 3.4 and let $C = \{u_1, u_2\}$. Define $\Gamma: C \longrightarrow P(M \times M)$ by $\Gamma(u_1) = \{(\perp, \perp), (r, r), (s, s), (\mathrm{T}, \mathrm{T}), (r, \mathrm{T}), (s, T)\}$ and $\Gamma(u_2) = \{(\perp, \perp), (r, r), (s, s), (\mathrm{T}, \mathrm{T}), (T, r), (r, T), (s, \perp)\}$. The aftersets calculated by the elements of M are as follows: $\perp\Gamma(u_1) = \{\perp\}, r\Gamma(u_1) = \{r, \mathrm{T}\}, s\Gamma(u_1) = \{s, T\}, \mathrm{T}\Gamma(u_1) = \{\mathrm{T}\}$, $\perp\Gamma(u_2) = \{\perp\}, r\Gamma(u_2) = \{r, T\}, s\Gamma(u_2) = \{\perp, s\}$, $\mathrm{T}\Gamma(u_2) = \{r, \mathrm{T}\}$. Then, (Γ, C) is a SCTR and SRRE with respect to aftersets. Let $S = \{\perp, r, s\}$. Then, S is not an ideal of K_d-module of M. However, $\underline{\Gamma}^S(u_1) = \{\perp\}, \underline{\Gamma}^S(u_2) = \{\perp\}$ are K_d-module ideal of M. Hence, S is generalized lower soft K_d-module ideal of M with respect to aftersets.

4.20. Definition. Let $\varnothing \neq S \subseteq M$ and (Γ, C) be a SBR on a K_d-module M if $({}^S\underline{\Gamma}, C)$ is a soft K_d-module ideal M. Then, S is said to be generalized lower soft K_d-module of M with respect to foresets.

4.21. Theorem. Let $\varnothing \neq S \subseteq M$ and (Γ, C) be a SRRE and SCTR on a K_d-module M. Then, S is said to be generalized lower soft K_d-module ideal of M with respect to foresets if S is an K_d-module ideal of M.

Proof. The proof is simple. □

5. Homomorphic Images of Generalized Rough Soft Substructures

The relationship between the upper and lower generalized soft substructures of the K_d- module, as well as the images of upper (lower) approximations under K_d-module homomorphism, is being discussed in this section. Further, we study some properties of these approximations.

5.1. Lemma. Let M_1 and M_2 be K_d-modules and (β_2, C) be a SBR on M_2. Let $\Omega: M_1 \longrightarrow M_2$ be a surjective K_d-module homomorphism. Set $\beta_1(u) = \{(s, t) \in M_1 \times M_1: (\Omega(s), \Omega(t)) \in \beta_2(u) \forall u \in C\}$. Then, the following holds:

(1) (β_1, C) is a SBR on M_1

(2) (β_1, C) is SRRE if (β_2, C) is SRRE

(3) (β_1, C) is SCRE if (β_2, C) is SCRE

Proof. The proof is obvious. □

5.2. Lemma. Let M_1 and M_2 be a K_d-modules and (β_2, C) be a SBR on M_2. Let $\Omega: M_1 \longrightarrow M_2$ be a surjective K_d-module homomorphism. Set $\beta_1(u) = \{(s, t) \in M_1 \times M_1: (\Omega(s), \Omega(t)) \in \beta_2(u) \forall u \in C\}$ Then, (β_1, C) is SCTR with respect to aftersets if (β_2, C) is SCTR with respect to aftersets and Ω is one-one.

Proof. Clearly $r\beta_1(u) \vee s\beta_1(u) \subseteq (r \vee s)\beta_1(u) \forall u \in C$.

Conversely, suppose that $x \in (r \vee s)\beta_1(u)$. Then, by definition of aftersets $(r \vee s, x) \in \beta_1(u)$. Thus, $\Omega(r \vee s), \Omega(x) \in \beta_2(u)$, that is, $\Omega(r) \vee \Omega(s), \Omega(x) \in \beta_2(u)$. This implies that $\Omega(x) \in \Omega(r) \vee \Omega(s)\beta_2(u)$. Since (β_2, C) is SCTR with respect to aftersets, $\Omega(x) \in \Omega(r)\beta_2(u) \vee \Omega(s)\beta_2(u)$. Then, there is $t_1 \in \Omega(r)\beta_2(u)$ and $t_2 \in \Omega(r)\beta_2(u)$ such that $\Omega(x) = t_1 \vee t_2$. As Ω is onto, we have $s_1, s_2 \in M_1, \Omega(s_1) = t_1$ and $\Omega(s_2) = t_2$. Thus, $\Omega(x) = \Omega(s_1) \vee \Omega(s_2)$. This implies, $\Omega(x) = \Omega(s_1 \vee s_2)$. As Ω is one-one, we have $x = s_1 \vee s_2$. Now, $t_2 \in \Omega(r)\beta_2(u)$, $\Rightarrow$, $\Omega(s_1) \in \Omega(r)\beta_2(u)ww$, $\Rightarrow (\Omega(r), \Omega(s_1)) \in \beta_2(u)$, $\Rightarrow (r, s_1) \in \beta_1(u)$, $\Rightarrow s_1 \in r\beta_1(u)$. Also now, $t_2 \in \Omega(s)\beta_2(u)$, $\Rightarrow$, $\Omega(s_2) \in \Omega(s)\beta_2(u)$, $\Rightarrow (\Omega(s), \Omega(s_2)) \in \beta_2(u)$, $\Rightarrow (s, s_2) \in \beta_1(u)$, $\Rightarrow s_2 \in s\beta_1(u)$. Now, $x = s_1 \vee s_2 \in r\beta_1(u) \vee s\beta_1(u)$. That is, $x \in r\beta_1(u) \vee s\beta_1(u)$. This implies that $(r \vee s)\beta_1(u) \subseteq r\beta_1(u) \vee s\beta_1(u)$. Hence, $(r \vee s)\beta_1(u) = r\beta_1(u) \vee s\beta_1(u) \forall u \in C$. Similarly, we can prove that $k \oplus r\beta_1(u) = (k \oplus r)\beta_1(u) \forall u \in C$. This show that (β_1, C) is SCTR with respect to aftersets. □

5.3. Lemma. Let M_1 and M_2 be a K_d-modules and (β_2, C) be a SBR on M_2. Let $\Omega: M_1 \longrightarrow M_2$ be a surjective K_d-module homomorphism. Set $\beta_1(u) = \{(s, t) \in M_1 \times M_1: (\Omega(s), \Omega(t)) \in \beta_2(u) \forall u \in C\}$. Then, (β_1, C) is SCTR with respect to foresets if (β_2, C) is SCTR with respect to foresets and Ω is one-one.

Proof. The proof is obvious. □

5.4. Lemma. Let M_1 and M_2 be K_d-modules and (β_2, C) be a SBR on M_2. Let $\Omega: M_1 \longrightarrow M_2$ be a surjective K_d-module homomorphism. Set $\beta_1(u) = \{(s, t) \in M_1 \times M_1: (\Omega(s), \Omega(t)) \in \beta_2(u) \forall u \in C\}$ Then, $\Omega(\overline{\beta}_1^S(u)) = \overline{\beta}_2^{\Omega(S)}(u)$ for $S \subseteq M_1$ and $\forall u \in C$.

Proof. Let $s \in \overline{\beta}_2^{\Omega(S)}(u)$. Then, $s\beta_2(u) \cap \Omega(S) \neq \varnothing$. Then, there is $r \in s\beta_2(u) \cap \Omega(S)$ such that $r \in s\beta_2(u)$ and $r \in \Omega(S)$. So, $(s, r) \in \beta_2(u)$ and $r \in \Omega(S)$. Since Ω is onto, there exist $x \in S$ and $y \in M_1$ such that $r = \Omega(x)$ and $s = \Omega(y)$. Thus, $(s, r) = (\Omega(y), \Omega(x)) \in \beta_2(u)$, that is, $(\Omega(y), \Omega(x) \in \beta_2(u)$. Hence, $(yx) \in \beta_1(u)$, that is, $x \in y\beta_1(u)$ and $x \in S$. This implies $x \in y\beta_1(u) \cap S$. So, we have $y \in \overline{\beta}_1^S(u)$, $\Rightarrow \Omega(y) \in \Omega(\overline{\beta}_1^S(u))$, $\Rightarrow s \in \Omega(\overline{\beta}_1^S(u))$. This show that $\Rightarrow s \in \Omega(\overline{\beta}_1^S(u))$.

Conversely, let $z \in \Omega(\overline{\beta}_1^S(u))$. Then, there is $r \in \overline{\beta}_1^S(u)$ such that $r\beta_1(u) \cap S \neq \varnothing$ and $z \in \Omega(r)$. Thus, $x \in r\beta_1(u) \cap S$ such that $x \in r\beta_1(u)$ and $x \in S$. So, $(r, x) \in \beta_1(u)$ and $x \in S$,

$\Rightarrow \Omega(x) \in \Omega(S)$ and $(\Omega(r), \Omega(x) \in \beta_2(u)$, $\Rightarrow \Omega(x) \in \Omega(r)\beta_2(u)$ and $\Omega(x) \in \Omega(S)$. Hence, $\Omega(x) \in \Omega(r)\beta_2(u) \cap \Omega(S)$, that is, $\varnothing \neq \Omega(r)\beta_2(u) \cap \Omega(S)$, $\Rightarrow \Omega(r) \in \overline{\beta}_2^{\Omega(S)}$, $\Rightarrow z \in \overline{\beta}_2^{\Omega(S)}(u)$. This shows that $\Omega(\overline{\beta}_1^S(u)) \subseteq \overline{\beta}_2^{\Omega(S)}(u)$. Hence, $\Omega(\overline{\beta}_1^S(u)) = \overline{\beta}_2^{\Omega(S)}(u) \forall u \in C$. □

5.5. Lemma. Let M_1 and M_2 be K_d-modules and (β_2, C) be a SBR on M_2. Let Ω: $M_1 \longrightarrow M_2$ be a surjective K_d-module homomorphism. Set $\beta_1(u) = \{(s,t) \in M_1 \times M_1: \Omega(s), \Omega(t) \in \beta_2(u) \forall u \in C\}$. Then, the following holds:

(1) $\Omega(\underline{\beta_1}^S(u)) \subseteq \underline{\beta_2}^{\Omega(s)}(u) \forall u \in C$; if Ω is one-one, then $\Omega(\underline{\beta_1}^S(u)) = \underline{\beta_2}^{\Omega(s)}(u) \forall u \in C$

(2) If Ω: $M_1 \longrightarrow M_2$ is one-one, then $\Omega(x) \in \Omega(\overline{\beta}_1^S(u))$ if and only if $x \in (\overline{\beta}_1^S(u)) \forall u \in C$

Proof (1) Let $r \in \Omega(\underline{\beta_1}^S(u))$ for $u \in C$. Then, there is $s \in \underline{\beta_1}^S(u)$ such that $s\beta_1(u) \subseteq S$ and $r = \Omega(x)$. Suppose that $y \in r\beta_2(u)$. Then, there is $x \in M_1$ such that $\Omega(x) = y$ and $\Omega(x) \in \Omega(s)\beta_2(u)$, that is, $(\Omega(s), \Omega(x)) \in \beta_2(u)$. Hence, $(s,x) \in \beta_1(u)$ by definition of aftersets $x \in s\beta_1(u)$. Hence, $s\beta_1(u) \subseteq S$ and $x \in S$. So, $\Omega(x) \in \Omega(S)$ using above $\Omega(x) = y$, $\Rightarrow y \in \Omega(S)$. Thus, $r\beta_2(u) \subseteq \Omega(S)$. Hence, $r \in \underline{\beta_2}^{\Omega(s)}(u)$which shows that $\Omega(\underline{\beta_1}^S(u)) \subseteq \underline{\beta_2}^{\Omega(s)}(u) \forall u \in C$. Conversely, let $s \in \underline{\beta_2}^{\Omega(s)}(u) \forall u \in C$. Then, $s\beta_2(u) \subseteq \Omega(S)$. Suppose that Ω is one-one. Then, there is $r \in M_1$ such that $s = \Omega(r)$ and $\Omega(r)\beta_2(u) \subseteq \Omega(S)$. Let $x \in r\beta_1(u)$ by definition of aftersets, we have $(r,x) \in \beta_1(u)$. Then, $\Omega(r), \Omega(x) \in \beta_2(u)$, that is, $\Omega(x) \in \Omega(r)\beta_2(u)$. Since $\Omega(r)\beta_2(u) \subseteq \Omega(S)$, $\Omega(x) \in \Omega(S)$, $\Rightarrow x \in S$. Thus, $r\beta_1(u) \subseteq S$, that is, $r \in \underline{\beta_1}^S(u)$, $\Rightarrow \Omega(r) \in \Omega(\underline{\beta_1}^S(u))$ since $s = \Omega(r)$. Hence, $s \in \Omega(\underline{\beta_1}^S(u))$. So,$\underline{\beta_2}^{\Omega(s)}(u) \subseteq \Omega(\underline{\beta_1}^S(u)) \forall u \in C$. Thus, $\Omega(\underline{\beta_1}^S(u)) = \underline{\beta_2}^{\Omega(s)}(u) \forall u \in C$.

(2) Let $\Omega(r) \in \Omega(\overline{\beta}_1^S(u))$. Then, there is $s \in \overline{\beta}_1^S(u)$ such that $\Omega(r) = \Omega(s)$. Since Ω is one-one, we get $r = s$. Thus, $r \in \overline{\beta}_1^S(u)$. Conversely, let $r \in \overline{\beta}_1^S(u) \forall u \in C$. Then, $\Omega(r) \in \Omega(\overline{\beta}_1^S(u)) \forall u \in C$. □

5.6. Theorem. Let $S \subseteq M_1$ and let Ω be a surjective K_d-module homomorphism on a K_d-modules M_1, M_2 and (β_2, C) be a SBR and SCTR with respect to aftersets on M_2. Set $\beta_1(u) = \{(s,t) \in M_1 \times M_1: (\Omega(s), \Omega(t)) \in \beta_2(u) \forall u \in C\}$. Then, the following holds:

(1) $\overline{\beta}_1^S(u)$ is a K_d-module ideal of M_1 if and only if $\overline{\beta}_2^{\Omega(S)}(u)$ is a K_d-module ideal of $M_2 \forall u \in C$

(2) $\overline{\beta}_1^S(u)$ is a sub-K_d-module of M_1 if and only if $\overline{\beta}_2^{\Omega(S)}(u)$ is a sub-K_d-module of $M_2 \forall u \in C$

Proof. (1) Let $\overline{\beta}_1^S(u)$ be a K_d-module ideal of M_1. Then, we have to show that $\overline{\beta}_2^{\Omega(S)}(u)$ is a K_d-module ideal of $M_2 \forall u \in C$. By Lemma 5.4, we have $\Omega(\overline{\beta}_1^S(u)) = \overline{\beta}_2^{\Omega(S)}(u)$ for all $S \subseteq M_1$ and $\forall u \in C$.

Let $r_i \in \Omega(\overline{\beta}_1^S(u)) \forall u \in C$. Then, there is $t_i \in \overline{\beta}_1^S(u)$ such that $\Omega(t_i) = r_i$. Hence, Ω is a K_d-module homomorphism and $\overline{\beta}_1^S(u)$ is a K_d-module ideal of $M_1 \forall u \in C$. So, we have $\vee_{i \in I} t_i \in \overline{\beta}_1^S(u)$, $\Rightarrow \Omega(\vee_{i \in I} t_i) \in \Omega(\overline{\beta}_1^S(u))$. Thus, $\vee_{i \in I} \Omega(t_i) \in \Omega(\overline{\beta}_1^S(u))$. Hence, $\vee_{i \in I} r_i \in \Omega(\overline{\beta}_1^S(u))$.

Let $r \leq s$ and $s \in \Omega(\overline{\beta}_1^S(u))$. Then, there is $t \in M_1$ and $u \in \overline{\beta}_1^S(u)$ such that $\Omega(t) = r$ and $\Omega(u) = s$. Hence, $r \leq s$, that is, $\Omega(t) \leq \Omega(u)$ and $\Omega(t) \vee \Omega(u) = \Omega(u)$. Further, Ω is a K_d-module homomorphism. Thus, we have $\Omega(t \vee u) = \Omega(t) \vee \Omega(u)$, $= \Omega(u) \in \Omega(\overline{\beta}_1^S(u))$, $\Rightarrow \Omega(t \vee u) \in \Omega(\overline{\beta}_1^S(u))$. By Lemma 5.5(2), we have $(t \vee u) \in \overline{\beta}_1^S(u)$. Hence, $\overline{\beta}_1^S(u)$ is a lower set and $t \leq t \vee u$. Thus, $t \in \overline{\beta}_1^S(u)$ and $\Omega(t) \in \Omega(\overline{\beta}_1^S(u))$ so $r \in \Omega(\overline{\beta}_1^S(u))$. Hence, $\Omega(\overline{\beta}_1^S(u))$ is a lower set.

Now, we show that $k \oplus r \in \Omega(\overline{\beta}_1^S(u))$ for all $r \in \Omega(\overline{\beta}_1^S(u))$ and $\forall k \in K_d$. Let $k = \Omega(s) \in K_d$ and $r = \Omega(t) \in \Omega(\overline{\beta}_1^S(u))$. So, $\Omega(t) \in \Omega(\overline{\beta}_1^S(u))$. Then, by Lemma 5.5(2), we have $t \in \overline{\beta}_1^S(u)$. Hence, $\overline{\beta}_1^S(u)$ is an K_d-module ideal of M_1. Thus, $s \oplus t \in \overline{\beta}_1^S(u)$. Thus, $\Omega(s \oplus t) \in \Omega(\overline{\beta}_1^S(u))$. Hence, Ω is an K_d-module homomorphism. Thus, $\Omega(s \oplus t) = \Omega(s) + \Omega(t)$, that is, $\Omega(s) \oplus \Omega(t) \in \Omega(\overline{\beta}_1^S(u))$. Hence, $k \oplus r \in \Omega(\overline{\beta}_1^S(u))$. Thus, $\Omega(\overline{\beta}_1^S(u)) = \overline{\beta}_2^{\Omega(S)}(u)$ is a K_d-module ideal of $M_2 \forall u \in C$.

Conversely, suppose $\Omega(\overline{\beta}_1^S(u)) = \overline{\beta}_2^{\Omega(S)}(u)$ is a K_d-module ideal of $M_2 \forall u \in C$.

Suppose $s_i \in \overline{\beta}_1^S(u) \forall u \in C$ for some $i \in I$. Then, $\Omega(s_i) \in \Omega(\overline{\beta}_1^S(u)) \forall u \in C$. Hence, $\Omega(\overline{\beta}_1^S(u))$ is a K_d-module ideal and Ω is a K_d-module homomorphism. So, $\vee_{i \in I} \Omega(s_i) = \Omega(\vee_{i \in I} s_i) \in \Omega(\overline{\beta}_1^S(u))$, $\forall u \in C$. Then, by Lemma 5.5(2), we have $\vee_{i \in I} s_i \in \overline{\beta}_1^S(u)$ and $\overline{\beta}_1^S(u)$ is directed.

Let $r \leq s$and $s \in \overline{\beta}_1^S(u)$. Then, $\Omega(r) \leq \Omega(s)$ and $\Omega(s) \in \Omega(\overline{\beta}_1^S(u))$. Since $\Omega(\overline{\beta}_1^S(u))$ is a lower set, we have $\Omega(r) \in \Omega(\overline{\beta}_1^S(u))$. Thus, by Lemma 5.5(2), we have $r \in \overline{\beta}_1^S(u)$. So, $\overline{\beta}_1^S(u)$ is a lower set. Suppose $k \in K_d$ and $v \in \overline{\beta}_1^S(u)$ Then, $\Omega(k) \in K_d$ and $\Omega(v) \in \Omega(\overline{\beta}_1^S(u))$. Since $\Omega(\overline{\beta}_1^S(u))$ is an K_d-module ideal of M_2, we have $\Omega(k \oplus v) = k \oplus \Omega(v) \in \Omega(\overline{\beta}_1^S(u))$, $\Rightarrow \Omega(k \oplus v) \in \Omega(\overline{\beta}_1^S(u)) \forall u \in C$ and then by Lemma 5.5(2), we have $k \oplus v \in \overline{\beta}_1^S(u)$. By the above discussion, we have $\overline{\beta}_1^S(u)$ is K_d-module ideal of M_1.

(2) Let $\overline{\beta}_1^{\Omega(S)}(u)$ is a sub-K_d-module of $M_1 \forall u \in C$. Then, we have to show that $\overline{\beta}_2^{\Omega(S)}(u)$ is sub-K_d-module of $M_2 \forall u \in C$. By Lemma 5.4, we have $\Omega(\overline{\beta}_1^S(u)) = \overline{\beta}_2^{\Omega(S)}(u)$ for all $S \subseteq M_1$ and $\forall u \in C$.

Let $r_i \in \Omega(\overline{\beta}_1^S(u)) \forall u \in C$ Then, there is $t_i \in \overline{\beta}_1^S(u)$ such that $\Omega(t_i) = r_i$. Hence, Ω is a K_d-module homomorphism and $\overline{\beta}_1^S(u)$ is a sub-K_d-module of M_1, $\forall u \in C$. So, $\vee_{i\in I} t_i \in \overline{\beta}_1^S(u)$, $\Rightarrow \Omega(\vee_{i\in I} t_i) \in \Omega(\overline{\beta}_1^S(u))$. So, we have $\vee_{i\in I}\Omega(t_i) \in \Omega(\overline{\beta}_1^S(u))$. Hence, $\vee_{i\in I} r_i \in \Omega(\overline{\beta}_1^S(u))$.

Now, we show that$k \oplus r \in \Omega(\overline{\beta}_1^S(u))$ for all $r \in \Omega(\overline{\beta}_1^S(u))$ and $\forall k \in K_d$. Let$k = \Omega(s) \in K_d$ and $r = \Omega(t) \in \Omega(\overline{\beta}_1^S(u))$.So, $\Omega(t) \in \Omega(\overline{\beta}_1^S(u))$. Then, by Lemma 5.5(2), we have $t \in \overline{\beta}_1^S(u)$. Hence, $\overline{\beta}_1^S(u)$ is a sub-K_d-module of M_1. This shows that $k \oplus t \in \overline{\beta}_1^S(u)$. Thus, $\Omega(k \oplus t) \in \Omega(\overline{\beta}_1^S(u))$. Hence, Ω is an K_d-module homomorphism. Thus, $\Omega(k \oplus t) = \Omega(k) + \Omega(t)$, that is, $\Omega(k) \oplus \Omega(t) \in \Omega(\overline{\beta}_1^S(u))$. Hence, $k \oplus t \in \Omega(\overline{\beta}_1^S(u))$. Thus, we have that $\Omega(\overline{\beta}_1^S(u)) = \overline{\beta}_2^{\Omega(S)}(u)$ is a sub-K_d-module of $M_2 \forall u \in C$.

Conversely, suppose $\Omega(\overline{\beta}_1^S(u)) = \overline{\beta}_2^{\Omega(S)}(u)$ is a sub-K_d-module of $M_2 \forall u \in C$. Suppose, $s_i \in \overline{\beta}_1^S(u) \forall u \in C$ for some $i \in I$.Then, $\Omega(s_i) \in \Omega(\overline{\beta}_1^S(u)) \forall u \in C$. Hence, $\Omega(\overline{\beta}_1^S(u))$ is a sub-K_d-module and Ω is a K_d-module homomorphism. So, $\vee_{i\in I}\Omega(s_i) = \Omega(\vee_{i\in I} s_i) \in \Omega(\overline{\beta}_1^S(u))$, $\forall u \in C$. Then, by Lemma 5.5(2), we have $\vee_{i\in I} s_i \in \overline{\beta}_1^S(u)$.

Suppose $k \in K_d$and $v \in \overline{\beta}_1^S(u)$. Then, $\Omega(k) \in K_d$ and $\Omega(v) \in \Omega(\overline{\beta}_1^S(u))$. Since $\Omega(\overline{\beta}_1^S(u))$ is an sub-K_d-module ideal of M, we have $\Omega(k \oplus v) = k \oplus \Omega(v) \in \Omega(\overline{\beta}_1^S(u)) \Rightarrow \Omega(k \oplus v) \in \Omega(\overline{\beta}_1^S(u)) \forall u \in C$ and then by Lemma 5.5(2) we have $k \oplus v \in \overline{\beta}_1^S(u)$. From the above discussion, we get $\overline{\beta}_1^S(u)$ is sub-K_d-module of M_1. □

5.7. Remark. With similar arguments, Theorem 5.6 can be similarly proved but for the foresets.

5.8. Theorem. Let $S \subseteq M_1$ and let Ω be a surjective K_d-module homomorphism on a K_d-module M and (β_2, C) be a SBR and SCTR and with respect to aftersets on M_2. Set $\beta_1(u) = \{(s,t) \in M_1 \times M_1 \colon (\Omega(s), \Omega(t)) \in \beta_2(u) \forall u \in C\}$. Then, the following holds:

(1) $\underline{\beta}_1^S(u)$ is a K_d-module ideal of M_1 if and only if $\underline{\beta}_2^{\Omega(S)}(u)$ is a K_d-module ideal of $M_2 \forall u \in C$

(2) $\underline{\beta}_1^S(u)$ is a sub-K_d-module of M_1 if and only if $\underline{\beta}_2^{\Omega(S)}(u)$ is a sub-K_d-module of $M_2 \forall u \in C$

Proof. The proof is similar in view of Theorem 5.6. □

6. Comparison

Qurashi and Shabir presented the roughness in quantale modules with the help of congruence relation [7]. Furthermore, generalized roughness of fuzzy substructures in quantale with respect to soft relations in quantale was defined in [11]. It is clear that equivalence relation is a hurdle while evaluating roughness. In order to avoid this this hurdle, soft binary relations are presented in this paper. Since suitable soft binary relations are easy to find out, it is an easy approach to observe soft rough properties to discuss different characterizations of soft rough substructures in quantale modules with the help of aftersets and foresets. Different characterization of soft substructures in semigroups and their approximation based on soft relation was discussed by Kanwal and Shabir [26]. We are actually motivated from the paper roughness in quantale module and taken help from [11] to develop the idea of this paper.

7. Conclusion

In this paper, we have suggested a new relation of substructures of quantale module with rough set and soft sets. The properties of rough soft substructures in quantale module are discussed for the first time. On the one hand, we have presented different characterizations for soft relations to approach quantale module subsets, as well as the use of aftersets and foresets in this regard. Structural features of soft relations under aftersets and foresets are discussed. Furthermore, in the quantale module, foresets and aftersets are applied to various types of substructures using soft relations to explore their lower and upper approximations. The following work can be done in future:

(1) Soft relations applied to the fuzzy substructures of quantale module and their approximations

(2) Some studies of soft substructures of quantale module and their approximations

Data Availability

The paper includes the information used to verify the study's findings.

Conflicts of Interest

The authors state that they have no conflicts of interest.

Authors' Contributions

All authors have contributed equally to this paper in all aspects. All authors have read and agreed to the published version of the manuscript.

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