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Research Article

Analytical Solution for the Cubic-Quintic Duffing Oscillator Equation with Physics Applications

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The nonlinear differential equation governing the periodic motion of the one-dimensional, undamped, and unforced cubic-quintic Duffing oscillator is solved exactly by obtaining the period and the solution. The period is given in terms of the complete elliptic integral of the first kind and the solution involves Jacobian elliptic functions. We solve the cubic-quintic Duffing equation under arbitrary initial conditions. Physical applications are provided. The solution to the mixed parity Duffing oscillator is also formally derived. We illustrate the obtained results with concrete examples. We give high accurate trigonometric approximations to the Jacobian function cn.

1. Introduction

It is well known that many engineering problems are not linear and their analytical solutions are not easy to obtain. Disturbance methods are among the known methods for solving nonlinear problems, which are based on the existence of small/large parameters, the so-called disturbance parameters. Our approach is different from known solutions to this problem [1–5]. On the contrary, here, we express the solutions without imaginary quantities: both frequency and modulus are real numbers. The quintic term appearing in the cubic-quintic Duffing equation makes this nonlinear oscillator not only more complex but also more interesting to study.

2. The Analytical Solution to the Cubic-Quintic Duffing Equation

Let p, q, r, y_0, and $\dot{y}_0$ be arbitrary real numbers. We will solve the initial value problem:

$$y''(t) + py(t) + qy(t)^3 + ry(t)^5 = 0 \text{ given that } y(0) = y_0 \text{ and } y'(0) = \dot{y}_0. \quad (1)$$

We will assume that $y_0^2 + \dot{y}_0^2 > 0$. Multiplying (1) by $y'(t)$ and integrating it with respect to t gives

$$\frac{1}{2}y'(t)^2 + \frac{p}{2}y(t)^2 + \frac{q}{4}y(t)^4 + \frac{r}{6}y(t)^6 = \frac{1}{2}\dot{y}_0^2 + \frac{p}{2}y_0^2 + \frac{q}{4}y_0^4 + \frac{r}{6}y_0^6. \quad (2)$$

Let

$$y(t) = \frac{c_0 \operatorname{cn}\left(\sqrt{\omega}t + c_1 | m\right)}{\sqrt{1 + \lambda \cdot \operatorname{cn}\left(\sqrt{\omega}t + c_1|m\right)^2 + \mu \cdot \operatorname{cn}\left(\sqrt{\omega}t + c_1|m\right)^4}}, \quad (3)$$

where the parameter values c_0, c_1, m, ω, λ, and μ are to be determined. If the solution in (3) is periodic, it will have the same period as the function cn $(\sqrt{\omega}t + c_1|m)$ and this period may be evaluated by means of the formula

$$T = \frac{4}{\sqrt{\omega}}K(m). \quad (4)$$

In the case, when $-1 \leq m \leq 1/2$, we may approximate the value of $K(m)$ using the formula

$$K(m) \approx \frac{\pi(m(409m - 3984) + 4864)}{50m(41m - 208) + 9728}. \quad (5)$$

The error for this approximation is given by

$$\text{Error} = \max_{-1\le m\le 1/2} |K(m) - K(m)| < 0.000314. \tag{6}$$

On the contrary, we may obtain approximate trigonometric solution making use of the following approximation formula:

$$\begin{aligned}\operatorname{cn}(t,m) &\approx \cos_m(t)\\ &= \frac{\sqrt{1+\kappa}\cos\left(\sqrt{1+\kappa}\,t\right)}{\sqrt{1+\kappa\cos^2\left(\sqrt{1+\kappa}\,t\right)}}, \text{ where } \kappa\\ &= \frac{1}{14}\left(\sqrt{m^2-144m+144}-(m+12)\right).\end{aligned} \tag{7}$$

See Table 1, for the error $= \max_{-2K(m)\le t\le 2K(m)} |\cos_m(t) - \operatorname{cn}(t,m)|$.

A more accurate trigonometric approximation may be obtained using the formula

$$\operatorname{cn}(t,m) \approx \cos_m(t) := \frac{\sqrt{1+\rho+\kappa}\cos\left(\sqrt{w}\,t\right)}{\sqrt{1+\rho\cos^2\left(\sqrt{w}\,t\right)+\kappa\cos^4\left(\sqrt{w}\,t\right)}}, \tag{8}$$

being

$$\begin{aligned}&w = \frac{(\kappa-1)(m-2)}{\kappa+2}, \rho = \frac{(\kappa-7)\kappa-(\kappa-1)^2 m}{\kappa+2},\\ &\textit{and}\\ &\kappa = \frac{8m^2\left(2409m^4-29600m^3+111520m^2-163840m+81920\right)}{35767m^6-831840m^5+6197600m^4-21217280m^3+36823040m^2-31457280m+10485760}.\end{aligned} \tag{9}$$

See Table 2, for the error $= \max_{-2K(m)\le t\le 2K(m)} |\cos_m(t) - \operatorname{cn}(t,m)|$.

Let

$$\begin{aligned}R(t) &= \frac{1}{2}y'(t)^2 + \frac{p}{2}y(t)^2 + \frac{q}{4}y(t)^4 + \frac{r}{6}y(t)^6\\ &-\left(\frac{1}{2}\dot{y}_0^2 + \frac{p}{2}y_0^2 + \frac{q}{4}y_0^4 + \frac{r}{6}y_0^6\right): \text{Residual}.\end{aligned} \tag{10}$$

Introduce the notation:

$$\operatorname{cn} = \operatorname{cn}\left(\sqrt{\omega}\,t|m\right) \text{ and } \operatorname{cn} = \sqrt{\zeta}. \tag{11}$$

Definition 1. The discriminant to the i.v.p. (1) is defined as

$$\begin{aligned}\Delta &= \left(p+qy_0^2+ry_0^4\right)^2\left(3q^2-16pr-4qry_0^2-4r^2y_0^4\right)\\ &+\left(6\left(q+2ry_0^2\right)\left(q^2-6pr-2qry_0^2-2r^2y_0^4\right)\right)\dot{y}_0^2 - 36r^2\dot{y}_0^4.\end{aligned} \tag{12}$$

2.1. First Case: $\Delta > 0$. We define $\mu = 0$. Inserting the ansatz (3) into (10), we obtain

$$R(t) = \frac{1}{12(1+\lambda\zeta)^3}\begin{bmatrix}\left(-6c_0^2m\omega+6c_0^2\omega-6py_0^2-3qy_0^4-2ry_0-6\dot{y}_0^{\,2}\right)+\\ 3\left(4c_0^2m\omega+2c_0^2p-2c_0^2\omega-6\lambda py_0^2-3\lambda qy_0^4-2\lambda ry_0^6-6\lambda\dot{y}_0^{\,2}\right)\zeta,\\ 3\left(-2c_0^2m\omega+4c_0^2\lambda p+c_0^4q-6\lambda^2py_0-3\lambda^2qy_0-2\lambda^2ry_0^6-6\lambda^2\dot{y}_0^{\,2}\right)\zeta^2\\ \left(6c_0^2\lambda^2p+3c_0^4\lambda q+2c_0^6r-6\lambda^3py_0^2-3\lambda^3qy_0^2-2\lambda^3ry_0^2-6\lambda^3\dot{y}_0^{\,2}\right)\zeta^3\end{bmatrix}. \tag{13}$$

Equating to zero the coefficients of ζ^j $(j = 0,1,2,3)$ in the numerator of the last expression gives an algebraic system. Solving it, we obtain

Table 1: Error of approximating the Jacobian cn function by means of the cosine function (7) with $T = 4K(m)$.

m	Error	m	Error
−1	0.0068	0	0
−0.95	0.00624	0.05	0.0000332
−0.9	0.005742	0.1	0.00014
−0.85	0.0053	0.15	0.00033
−0.8	0.0045	0.2	0.00061
−0.75	0.0043	0.25	0.0010
−0.7	0.0039	0.3	0.00153
−0.65	0.00344	0.35	0.00212
−0.6	0.00301	0.4	0.003
−0.55	0.00261	0.45	0.0041
−0.5	0.00222	0.5	0.0054
−0.45	0.00186	0.55	0.007
−0.4	0.001517	0.6	0.009
−0.35	0.0012	0.65	0.0116
−0.3	0.0009	0.7	0.0145
−0.25	0.00066	0.75	0.0188
−0.2	0.00044	0.8	0.0241
−0.15	0.00025	0.85	0.0314
−0.1	0.00012	0.9	0.042
−0.05	0.000031	0.95	0.059

Table 2: Error of approximating the Jacobian cn function by means of the cosine function (8) with $T = 4K(m)$.

m	Error	m	Error
−1	0.000074	0	0
−0.95	0.000063	0.05	$2.002e-9$
−0.9	0.000054	0.1	$3.52e-8$
−0.85	0.0000452	0.15	$1.98e-7$
−0.8	0.0000378	0.2	$6.93e-7$
−0.75	0.00003	0.25	$1.89e-6$
−0.7	0.000025	0.3	$4.40e-6$
−0.65	0.000012	0.35	$9.22e-6$
−0.6	0.000015	0.4	0.000018
−0.55	0.000011	0.45	0.000033
−0.5	$8.37e-6$	0.5	0.00006
−0.45	$5.88e-6$	0.55	0.0000991
−0.4	$3.94e-6$	0.6	0.00017
−0.35	$2.48e-6$	0.65	0.00023
−0.3	$1.44e-6$	0.7	0.00045
−0.25	$7.5e-7$	0.75	0.00074
−0.2	$3.322e-7$	0.8	0.0012
−0.15	$1.145e-7$	0.85	0.0021
−0.1	$2.44e-8$	0.9	0.0036
−0.05	$1.67e-9$	0.95	0.015

$$m = \frac{3\lambda\omega - p + \omega}{(3\lambda + 2)\omega}.\omega = \frac{(3\lambda + 2)\left(6py_0^2 + 3qy_0^4 + 2ry_0^6 + 6\dot{y}_0^2\right)}{6c_0^2} - p. \tag{14}$$

For these choices, we obtain the following system:

$$\begin{aligned}
&\left(-18py_0^2 - 9qy_0^4 - 6ry_0^6 - 18\dot{y}_0^2\right)\lambda^2 \\
&\quad + \left(12c_0^2p - 18py_0^2 - 9qy_0^4 - 6ry_0^6 - 18\dot{y}_0^2\right)\lambda \\
&\quad + \left(6c_0^2p + 3c_0^4q - 6py_0^2 - 3qy_0^4 - 2ry_0^6 - 6\dot{y}_0^2\right) = 0. \\
&\left(-6py_0^2 - 3qy_0^4 - 2ry_0^6 - 6\dot{y}_0^2\right)\lambda^3 \\
&\quad + 6c_0^2p\lambda^2 + 3c_0^4q\lambda + 2c_0^6r = 0.
\end{aligned} \tag{15}$$

Eliminating c_0 from system (15) gives the sextic

$$d_0 + d_1\lambda + d_2\lambda^2 + d_3\lambda^3 + d_4\lambda^4 + d_5\lambda^5 + d_6\lambda^6 = 0, \qquad (16)$$

where

$$\begin{aligned}
d_0 &= 4r^2\left(6py_0^2 + 3qy_0^4 + 2ry_0^6 + 6\dot{y}_0^2\right)^2,\\
d_1 &= 36r\left(6py_0^2 + 3qy_0^4 + 2ry_0^6 + 6\dot{y}_0^2\right)\left(pq + 6pry_0^2 + 3qry_0^4 + 2r^2y_0^6 + 6r\dot{y}_0^2\right),\\
d_2 &= 9\begin{pmatrix} 48p^3r + 240p^2qry_0^2 + 576p^2r^2y_0^4 - 18pq^3y_0^2 + 120pq^2ry_0^4 + 656pqr^2y_0^6 + \\ 240pqr\dot{y}_0^2 + 384pr^3y_0^8 + 1152pr^2\dot{y}_0^2y_0^2 - 9q^4y_0^4 - 6q^3ry_0^6 - 18q^3\dot{y}_0^2 + \\ 144q^2r^2y_0^8 + 192qr^3y_0^{10} + 576qr^2\dot{y}_0^2y_0^4 + 64r^4y_0^{12} + 384r^3\dot{y}_0^2y_0^6 + 576r^2\dot{y}_0^4 \end{pmatrix},\\
d_3 &= 162\begin{pmatrix} 16p^3r - 2p^2q^2 + 48p^2qry_0^2 + 72p^2r^2y_0^4 - 6pq^3y_0^2 + 24pq^2ry_0^4 + 88pqr^2y_0^6 + \\ 48pqr\dot{y}_0^2 + 48pr^3y_0^8 + 144pr^2\dot{y}_0^2y_0^2 - 3q^4y_0^4 - 2q^3ry_0^6 - 6q^3\dot{y}_0^2 + \\ 18q^2r^2y_0^8 + 24qr^3y_0^{10} + 72qr^2\dot{y}_0^2y_0^4 + 8r^4y_0^{12} + 48r^3\dot{y}_0^2y_0^6 + 72r^2\dot{y}_0^4 \end{pmatrix},\\
d_4 &= 27\begin{pmatrix} 208p^3r - 36p^2q^2 + 504p^2qry_0^2 + 576p^2r^2y_0^4 - 78pq^3y_0^2 + 252pq^2ry_0^4 + 744pqr^2y_0^6 + \\ 504pqr\dot{y}_0^2 + 384pr^3y_0^8 + 1152pr^2\dot{y}_0^2y_0^2 - 39q^4y_0^4 - 26q^3ry_0^6 - 78q^3\dot{y}_0^2 + \\ 144q^2r^2y_0^8 + 192qr^3y_0^{10} + 576qr^2\dot{y}_0^2y_0^4 + 64r^4y_0^{12} + 384r^3\dot{y}_0^2y_0^6 + 576r^2\dot{y}_0^4 \end{pmatrix},\\
d_5 &= 324\begin{pmatrix} 16p^3r - 3p^2q^2 + 36p^2qry_0^2 + 36p^2r^2y_0^4 - 6pq^3y_0^2 + \\ 18pq^2ry_0^4 + 48pqr^2y_0^6 + 36pqr\dot{y}_0^2 + 24pr^3y_0^8 + 72pr^2\dot{y}_0^2y_0^2 - 3q^4y_0^4 - \\ 2q^3ry_0^6 - 6q^3\dot{y}_0^2 + 9q^2r^2y_0^8 + 12qr^3y_0^{10} + 36qr^2\dot{y}_0^2y_0^4 + 4r^4y_0^{12} + 24r^3\dot{y}_0^2y_0^6 + 36r^2\dot{y}_0^4 \end{pmatrix},\\
d_6 &= 108\begin{pmatrix} 16p^3r - 3p^2q^2 + 36p^2qry_0^2 + 36p^2r^2y_0^4 - 6pq^3y_0^2 + 18pq^2ry_0^4 + 48pqr^2y_0^6 + \\ 36pqr\dot{y}_0^2 + 24pr^3y_0^8 + 72pr^2\dot{y}_0^2y_0^2 - 3q^4y_0^4 - 2q^3ry_0^6 - 6q^3\dot{y}_0^2 + \\ 9q^2r^2y_0^8 + 12qr^3y_0^{10} + 36qr^2\dot{y}_0^2y_0^4 + 4r^4y_0^{12} + 24r^3\dot{y}_0^2y_0^6 + 36r^2\dot{y}_0^4 \end{pmatrix}.
\end{aligned} \qquad (17)$$

Sextic (16) is solvable by radicals. Indeed, let

$$\lambda = \sqrt{z} - \frac{1}{2}. \qquad (18)$$

Then,

$$s_0 + s_1z + s_2z^2 + s_3z^3 = 0, \qquad (19)$$

where

$$\begin{aligned}
s_0 &= \left(9pq + 6pry_0^2 + 3qry_0^4 + 2r^2y_0^6 + 6r\dot{y}_0^2\right)^2,\\
s_1 &= 9\begin{pmatrix} 48p^3r - 27p^2q^2 - 12p^2qry_0^2 + 36p^2r^2y_0^4 - \\ 18pq^3y_0^2 - 6pq^2ry_0^4 + 32pqr^2y_0^6 - 12pqr\dot{y}_0^2 + \\ 24pr^3y_0^8 + 72pr^2\dot{y}_0^2y_0^2 - 9q^4y_0^4 - 6q^3ry_0^6 - \\ 18q^3\dot{y}_0^2 + 9q^2r^2y_0^8 + 12qr^3y_0^{10} + 36qr^2\dot{y}_0^2y_0^4 + \\ 4r^4y_0^{12} + 24r^3\dot{y}_0^2y_0^6 + 36r^2\dot{y}_0^4 \end{pmatrix},\\
s_2 &= -27\begin{pmatrix} 32p^3r - 9p^2q^2 + 36p^2qry_0^2 - 36p^2r^2y_0^4 - \\ 12pq^3y_0^2 + 18pq^2ry_0^4 - 24pqr^2y_0^6 + 36pqr\dot{y}_0^2 - 24pr^3y_0^8 - \\ 72pr^2\dot{y}_0^2y_0^2 - 6q^4y_0^4 - 4q^3ry_0^6 - 12q^3\dot{y}_0^2 - 9q^2r^2y_0^8 - 12qr^3y_0^{10} - \\ 36qr^2\dot{y}_0^2y_0^4 - 4r^4y_0^{12} - 24r^3\dot{y}_0^2y_0^6 - 36r^2\dot{y}_0^4 \end{pmatrix},\\
s_3 &= 27\begin{pmatrix} 16p^3r - 3p^2q^2 + 36p^2qry_0^2 + 36p^2r^2y_0^4 - 6pq^3y_0^2 + 18pq^2ry_0^4 + 48pqr^2y_0^6 + \\ 36pqr\dot{y}_0^2 + 24pr^3y_0^8 + 72pr^2\dot{y}_0^2y_0^2 - 3q^4y_0^4 - 2q^3ry_0^6 - 6q^3\dot{y}_0^2 + \\ 9q^2r^2y_0^8 + 12qr^3y_0^{10} + 36qr^2\dot{y}_0^2y_0^4 + 4r^4y_0^{12} + 24r^3\dot{y}_0^2y_0^6 + 36r^2\dot{y}_0^4 \end{pmatrix}.
\end{aligned} \qquad (20)$$

The discriminant to cubic (19) is

$$\delta_{\text{cubic}} = 20639121408r^2\left(6py_0^2 + 3qy_0^4 + 2ry_0^6 + 6\dot{y}_0^2\right)^2 \left(16p^3r + 6pq^3y_0^2 + 3q^4y_0^4 + 2q^3ry_0^6 + 6q^3\dot{y}_0^2\right)^2\Delta > 0. \quad (21)$$

Cubic (19) has three real roots and at least one of them must be *positive*. Indeed, let z_1, z_2, and z_3 be the roots to this cubic. Then,

$$z_1z_2z_3 = -\frac{s_0}{s_3} = \frac{\left(9pq + 6pry_0^2 + 3qry_0^4 + 2r^2y_0^6 + 6r\dot{y}_0^2\right)^2}{1728\Delta} > 0. \quad (22)$$

Since $z_1z_2z_3 > 0$, at least one of the numbers z_1, z_2, and z_3 must be positive. We choose the closest to 1/4 positive root to cubic (19) so that, in view of (17), the number λ will be the closest to zero real root to the sextic in (15). Observe also that the condition $\Delta > 0$ implies that $q^2 - 4pr > 0$. Thus, if $q^2 - 4pr \le 0$, then $\Delta \le 0$. Moreover, the discriminant to the cubic (17) and Δ have the same sign.

The numbers c_0 and c_1 are determined from the initial conditions:

$$c_1 = \text{cn}^{-1}\left(\pm\frac{y_0}{\sqrt{c_0^2 - \lambda y_0^2}}\middle| m\right). \quad (23)$$

The number c_0 is a solution to the sextic:

$$\lambda(\lambda+1)(2\lambda+1)y_0^2\left(6py_0^2 + 3qy_0^4 + 2ry_0^6 + 6\dot{y}_0^2\right) + \begin{pmatrix} -24\lambda^2py_0^2 - 24\lambda py_0^2 - 6py_0^2 - 9\lambda^2qy_0^4 - 9\lambda qy_0^4 - \\ 3qy_0^4 - 6\lambda^2ry_0^6 - 6\lambda ry_0^6 - 2ry_0^6 - 18\lambda^2\dot{y}_0^2 - 18\lambda\dot{y}_0^2 - 6\dot{y}_0^2 \end{pmatrix} c_0^2 + 6(2\lambda+1)pc_0^4 + \left(3q + 2ry_0^2\right)c_0^6 = 0. \quad (24)$$

Example 1. Let

$$p = 1, \quad q = 5, \quad r = 1, \quad y_0 = 1 \text{ and } \dot{y}_0 = 1. \quad (25)$$

Sextic (15) reads

$$213084\lambda^6 + 639252\lambda^5 + 598887\lambda^4 + 132354\lambda^3 - 75861\lambda^2 - 35496\lambda - 3364 = 0. \quad (26)$$

The roots to this sextic are

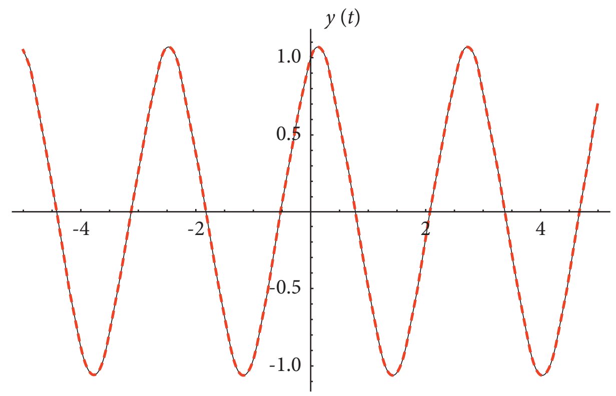

Figure 1: Comparison between the exact and the approximate numerical solution.

$$\begin{aligned}\lambda_1 &= -1.39266,\\ \lambda_2 &= -0.855911,\\ \lambda_3 &= -0.626144.\\ \lambda_4 &= -0.373856,\\ \lambda_5 &= -0.144089,\\ \lambda_6 &= 0.392663.\end{aligned} \quad (27)$$

We choose the value $\lambda = \lambda_5 = -0.144089$. The values of c_0 and c_1 are determined from the initial conditions. They read

$$c_0 = 0.985102 \text{ and } c_1 = -0.327845. \quad (28)$$

The exact solution to the i.v.p.,

$$y''(t) + y(t) + 5y(t)^3 + y(t)^5 = 0 \text{ given that } y(0) = 1 \text{ and } y'(0) = 1, \quad (29)$$

reads

$$y(t) = \frac{0.985102\text{cn}(2.60927t - 0.327845, 0.268447)}{\sqrt{1 - 0.144089\text{cn}(2.60927t - 0.327845, 0.268447)^2}}. \quad (30)$$

The solution is periodic with period $T = 4K(m)/\sqrt{\omega} = 2.5997790766024407$. The approximate trigonometric solution is given by

$$y_{\text{trigo}}(t) = \frac{3.41275\cos(2.41528(t - 0.125818))}{\sqrt{14 - 3.72995\cos^2(2.41528(t - 0.125818))}}. \quad (31)$$

The error of this trigonometric approximant compared with the exact solution on $0 \le t \le T$ equals 0.00352816, see Figure 1.

2.2. Second Case: $\Delta < 0$. Let $\mu \ne 0$. Inserting the ansatz (3) into (10), we obtain

$$\begin{aligned}
&\left(12\zeta^6\mu^3 + 36\zeta^5\lambda\mu^2 + 12\zeta^4\left(3\lambda^2\mu + 3\mu^2\right) + 12\zeta^3\left(\lambda^3 + 6\lambda\mu\right) + 12\zeta^2\left(3\lambda^2 + 3\mu\right) + 36\zeta\lambda + 12\right)R(t) = \\
&\quad - 6c_0^2 m\omega + 6c_0^2\omega - 6py_0^2 - 3qy_0^4 - 2ry_0^6 - 6\dot{y}_0^2 \\
&\quad + 3\begin{pmatrix} 4c_0^2\mu\omega - 4c_0^2\mu m\omega + 2c_0^2 m\omega - 4c_0^2\lambda p - c_0^4 q + \\ 6\lambda^2 py_0^2 + 6\mu py_0^2 + 3\lambda^2 qy_0^4 + 3\mu qy_0^4 + \\ 2\lambda^2 ry_0^6 + 2\mu ry_0^6 + 6\lambda^2\dot{y}_0^2 + 6\mu\dot{y}_0^2 \end{pmatrix}\zeta^2 \\
&\quad + 3\zeta\left(4c_0^2 m\omega + 2c_0^2 p - 2c_0^2\omega - 6\lambda py_0^2 - 3\lambda qy_0^4 - 2\lambda ry_0^6 - 6\lambda\dot{y}_0^2\right) \\
&\quad - \begin{pmatrix} 12c_0^2\mu\omega - 24c_0^2\mu m\omega + 6c_0^2\lambda^2 p + 12c_0^2\mu p + 3c_0^4\lambda q + \\ 2c_0^6 r - 6\lambda^3 py_0^2 - 36\lambda\mu py_0^2 - 3\lambda^3 qy_0^4 - 18\lambda\mu qy_0^4 - \\ 2\lambda^3 ry_0^6 - 12\lambda\mu ry_0^6 - 6\lambda^3\dot{y}_0^2 - 36\lambda\mu\dot{y}_0^2 \end{pmatrix}\zeta^3 \\
&\quad - 3\mu\begin{pmatrix} -2c_0^2\mu\omega + 2c_0^2\mu m\omega - 4c_0^2 m\omega - 4c_0^2\lambda p - \\ c_0^4 q + 6\lambda^2 py_0^2 + 6\mu py_0^2 + 3\lambda^2 qy_0^4 + 3\mu qy_0^4 + \\ 2\lambda^2 ry_0^6 + 2\mu ry_0^6 + 6\lambda^2\dot{y}_0^2 + 6\mu\dot{y}_0^2 \end{pmatrix}\zeta^4 \\
&\quad + 3\mu^2\left(4c_0^2 m\omega + 2c_0^2 p - 2c_0^2\omega - 6\lambda py_0^2 - 3\lambda qy_0^4 - 2\lambda ry_0^6 - 6\lambda\dot{y}_0^2\right)\zeta^5 \\
&\quad - \mu^2\left(6c_0^2 m\omega + 6\mu py_0^2 + 3\mu qy_0^4 + 2\mu ry_0^6 + 6\mu\dot{y}_0^2\right)\zeta^6.
\end{aligned} \tag{32}$$

Equating to zero the coefficients of ζ^j $(j = 0, 1, 2, 3, 4, 5, 6)$ gives an algebraic system. Solving it gives

$$\begin{aligned}
\omega &= \frac{(3\lambda + 2)\left(6py_0^2 + 3qy_0^4 + 2ry_0^6 + 6\dot{y}_0^2\right)}{6c_0^2} - p, \\
m &= \frac{\mu}{\mu - 1}, \\
\mu &= \frac{6c_0^2 p}{6py_0^2 + 3qy_0^4 + 2ry_0^6 + 6\dot{y}_0^2} - 3\lambda - 1.
\end{aligned} \tag{33}$$

The system reduces to

$$\begin{aligned}
&\left(-24pc_0^2 + 3qc_0^4 + 24py_0^2 + 12qy_0^4 + 8ry_0^6 + 24\dot{y}_0^2\right) \\
&\quad + 12\lambda\left(pc_0^2 + 6py_0^2 + 3qy_0^4 + 2ry_0^6 + 6\dot{y}_0^2\right) - 3\lambda^2\left(6py_0^2 + 3qy_0^4 + 2ry_0^6 + 6\dot{y}_0^2\right) = 0, \\
&\quad 2c_0^2\begin{pmatrix} 72p^2c_0^2 - 72p^2y_0^2 + 6prc_0^4y_0^2 - 36pqy_0^4 + 3qrc_0^4y_0^4 - 24pry_0^6 + \\ 2r^2c_0^4y_0^6 - 72p\dot{y}_0^2 + 6rc_0^4\dot{y}_0^2 \end{pmatrix} \\
&\quad - 3\left(48pc_0^2 - qc_0^4 - 24py_0^2 - 12qy_0^4 - 8ry_0^6 - 24\dot{y}_0^2\right)\left(6py_0^2 + 3qy_0^4 + 2ry_0^6 + 6\dot{y}_0^2\right)\lambda \\
&\quad + 6\left(6py_0^2 + 3qy_0^4 + 2ry_0^6 + 6\dot{y}_0^2\right)\left(pc_0^2 + 36py_0^2 + 18qy_0^4 + 12ry_0^6 + 36\dot{y}_0^2\right)\lambda^2 \\
&\quad - \left(6py_0^2 + 3qy_0^4 + 2ry_0^6 + 6\dot{y}_0^2\right)^2\lambda^3 = 0.
\end{aligned} \tag{34}$$

Eliminating c_0 from this system gives the sextic

$$D_0 + D_1\lambda + D_2\lambda^2 + D_3\lambda^3 + D_4\lambda^4 + D_5\lambda^5 + D_6\lambda^6 = 0, \tag{35}$$

where

$$
\begin{aligned}
D_0 &= 64\left(2ry_0^6 + 3qy_0^4 + 6py_0^2 + 6\dot{y}_0^2\right)\left(2r^2y_0^6 + 3qry_0^4 + 6pry_0^2 + 6r\dot{y}_0^2 + 9pq\right)^2,\\
D_1 &= 576\left(2r^2y_0^6 + 3qry_0^4 + 6pry_0^2 + 6r\dot{y}_0^2 + 9pq\right)\\
&\quad quad\begin{pmatrix} 4r^3y_0^{12} + 12qr^2y_0^{10} + 24pr^2y_0^8 + 9q^2ry_0^8 + 24r^2\dot{y}_0^2y_0^6 + \\ 50pqry_0^6 + 21pq^2y_0^4 + 36qr\dot{y}_0^2y_0^4 + 36p^2ry_0^4 + 72pr\dot{y}_0^2y_0^2 + 42p^2qy_0^2 + \\ 36r\dot{y}_0^4 + 24p^3 + 42pq\dot{y}_0^2 \end{pmatrix},\\
D_2 &= 144\begin{pmatrix} 88r^5y_0^{18} + 396qr^4y_0^{16} + 792pr^4y_0^{14} + 594q^2r^3y_0^{14} + \\ 3056pqr^3y_0^{12} + 309q^3r^2y_0^{12} + 792r^4\dot{y}_0^2y_0^{12} + 2376p^2r^3y_0^{10} + \\ 3822pq^2r^2y_0^{10} + 2376qr^3\dot{y}_0^2y_0^{10} + 36q^4ry_0^{10} + 27q^5y_0^8 + \\ 7644p^2qr^2y_0^8 + 4752pr^3\dot{y}_0^2y_0^8 + 1782q^2r^2\dot{y}_0^2y_0^8 + 1602pq^3ry_0^8 + \\ 108pq^4y_0^6 + 2376r^3\dot{y}_0^4y_0^6 + 2664p^3r^2y_0^6 + 11208pqr^2\dot{y}_0^2y_0^6 + \\ 72q^3r\dot{y}_0^2y_0^6 + 7254p^2q^2ry_0^6 + 3564qr^2\dot{y}_0^4y_0^4 + 1809p^2q^3y_0^4 + \\ 108q^4\dot{y}_0^2y_0^4 + 7128p^2r^2\dot{y}_0^2y_0^4 + 6120pq^2r\dot{y}_0^2y_0^4 + \\ 6552p^3qry_0^4 + 7128pr^2\dot{y}_0^4y_0^2 + 3402p^3q^2y_0^2 + \\ 216pq^3\dot{y}_0^2y_0^2 + 12240p^2qr\dot{y}_0^2y_0^2 + 864p^4ry_0^2 + 2376r^2\dot{y}_0^6 + \\ 108q^3\dot{y}_0^4 + 6120pqr\dot{y}_0^4 + 3402p^2q^2\dot{y}_0^2 + 864p^3r\dot{y}_0^2 + 1728p^4q \end{pmatrix},\\
D_3 &= 864\begin{pmatrix} 8r^5y_0^{18} + 36qr^4y_0^{16} + 72pr^4y_0^{14} + 54q^2r^3y_0^{14} + 312pqr^3y_0^{12} + 39q^3r^2y_0^{12} + 72r^4\dot{y}_0^2y_0^{12} + \\ 216p^2r^3y_0^{10} + 450pq^2r^2y_0^{10} + 216qr^3\dot{y}_0^2y_0^{10} + 36q^4ry_0^{10} + 27q^5y_0^8 + 900p^2qr^2y_0^8 + 432pr^3\dot{y}_0^2y_0^8 + \\ 162q^2r^2\dot{y}_0^2y_0^8 + 288pq^3ry_0^8 + 108pq^4y_0^6 + 216r^3\dot{y}_0^4y_0^6 + 232p^3r^2y_0^6 + 1224pqr^2\dot{y}_0^2y_0^6 + \\ 72q^3r\dot{y}_0^2y_0^6 + 1038p^2q^2ry_0^6 + 324qr^2\dot{y}_0^4y_0^4 + 369p^2q^3y_0^4 + 108q^4\dot{y}_0^2y_0^4 + 648p^2r^2\dot{y}_0^2y_0^4 + \\ 864pq^2r\dot{y}_0^2y_0^4 + 888p^3qry_0^4 + 648pr^2\dot{y}_0^4y_0^2 + 522p^3q^2y_0^2 + 216pq^3\dot{y}_0^2y_0^2 + 1728p^2qr\dot{y}_0^2y_0^2 + \\ 48p^4ry_0^2 + 216r^2\dot{y}_0^6 + 108q^3\dot{y}_0^4 + 864pqr\dot{y}_0^4 + 522p^2q^2\dot{y}_0^2 + 48p^3r\dot{y}_0^2 + 216p^4q \end{pmatrix}, \qquad (36)\\
D_4 &= 108\begin{pmatrix} -88r^5y_0^{18} - 396qr^4y_0^{16} - 792pr^4y_0^{14} - 594q^2r^3y_0^{14} - 2496pqr^3y_0^{12} - 145q^3r^2y_0^{12} - 792r^4\dot{y}_0^2y_0^{12} - \\ 2376p^2r^3y_0^{10} - 2142pq^2r^2y_0^{10} - 2376qr^3\dot{y}_0^2y_0^{10} + 456q^4ry_0^{10} + 342q^5y_0^8 - 4284p^2qr^2y_0^8 - \\ 4752pr^3\dot{y}_0^2y_0^8 - 1782q^2r^2\dot{y}_0^2y_0^8 + 642pq^3ry_0^8 + 1368pq^4y_0^6 - 2376r^3\dot{y}_0^4y_0^6 - 2440p^3r^2y_0^6 - \\ 7848pqr^2\dot{y}_0^2y_0^6 + 912q^3r\dot{y}_0^2y_0^6 - 510p^2q^2ry_0^6 - 3564qr^2\dot{y}_0^4y_0^4 + \\ 2223p^2q^3y_0^4 + 1368q^4\dot{y}_0^2y_0^4 - 7128p^2r^2\dot{y}_0^2y_0^4 - 1080pq^2r\dot{y}_0^2y_0^4 - \\ 1176p^3qry_0^4 - 7128pr^2\dot{y}_0^4y_0^2 + 1710p^3q^2y_0^2 + 2736pq^3\dot{y}_0^2y_0^2 - 2160p^2qr\dot{y}_0^2y_0^2 - \\ 192p^4ry_0^2 - 2376r^2\dot{y}_0^6 + 1368q^3\dot{y}_0^4 - 1080pqr\dot{y}_0^4 + 1710p^2q^2\dot{y}_0^2 - 192p^3r\dot{y}_0^2 + 576p^4q \end{pmatrix},\\
D_5 &= 324\begin{pmatrix} 8r^5y_0^{18} + 36qr^4y_0^{16} + 72pr^4y_0^{14} + 54q^2r^3y_0^{14} + 216pqr^3y_0^{12} + 35q^3r^2y_0^{12} + 72r^4\dot{y}_0^2y_0^{12} + 216p^2r^3y_0^{10} + 162pq^2r^2y_0^{10} + \\ 216qr^3\dot{y}_0^2y_0^{10} + 24q^4ry_0^{10} + 18q^5y_0^8 + 324p^2qr^2y_0^8 + 432pr^3\dot{y}_0^2y_0^8 + \\ 162q^2r^2\dot{y}_0^2y_0^8 + 48pq^3ry_0^8 + 72pq^4y_0^6 + 216r^3\dot{y}_0^4y_0^6 + 200p^3r^2y_0^6 + 648pqr^2\dot{y}_0^2y_0^6 + \\ 48q^3r\dot{y}_0^2y_0^6 + 30p^2q^2ry_0^6 + 324qr^2\dot{y}_0^4y_0^4 + 117p^2q^3y_0^4 + 72q^4\dot{y}_0^2y_0^4 + 648p^2r^2\dot{y}_0^2y_0^4 - 24p^3qry_0^4 + \\ 648pr^2\dot{y}_0^4y_0^2 + 90p^3q^2y_0^2 + 144pq^3\dot{y}_0^2y_0^2 - 48p^4ry_0^2 + 216r^2\dot{y}_0^6 + 72q^3\dot{y}_0^4 + 90p^2q^2\dot{y}_0^2 - 48p^3r\dot{y}_0^2 + 24p^4q \end{pmatrix},\\
D_6 &= -27\left(2ry_0^6 + 3qy_0^4 + 6py_0^2 + 6\dot{y}_0^2\right)\begin{pmatrix} 4r^4y_0^{12} + 12qr^3y_0^{10} + 24pr^3y_0^8 + 9q^2r^2y_0^8 + \\ 48pqr^2y_0^6 + 24r^3\dot{y}_0^2y_0^6 - 2q^3ry_0^6 - 3q^4y_0^4 + 36p^2r^2y_0^4 + 36qr^2\dot{y}_0^2y_0^4 + \end{pmatrix},\\
D_6 &= -27\left(2ry_0^6 + 3qy_0^4 + 6py_0^2 + 6\dot{y}_0^2\right).
\end{aligned}
$$

Sextic (34) has at least one real root. Indeed, let z_j $(j = 1, 2, 3, 4, 5, 6)$ be its roots. Then,

$$z_1 z_2 z_3 z_4 z_5 z_6 = \frac{D_0}{D_6} = \frac{64\left(9pq + 6pry_0^2 + 3qry_0^4 + 2r^2y_0^6 + 6r\dot{y}_0^2\right)^2}{27\Delta} < 0. \quad (37)$$

We will choose the closest to zero real root to sextic (34). The values for c_0 and c_1 are determined from the initial conditions:

$$c_1 = \mathrm{cn}^{-1}\left(\sqrt{\frac{c_0^2 - \lambda y_0^2 \pm \sqrt{\left(c_0^2 - \lambda y_0^2\right)^2 - 4\mu y_0^4}}{2\mu y_0^2}} \,\middle|\, m\right). \quad (38)$$

The number c_0 is found from the algebraic equation:

$$\left(6\lambda y_0^6 - 15\lambda y_0^4 + 15\lambda y_0^2 + 3\lambda \dot{y}_0^2 + 2y_0^6 - 5y_0^4 + 5y_0^2 + \dot{y}_0^2 - 5c_0^2\right)$$
$$\left(\left(2y_0^6 - 5y_0^4 + \dot{y}_0^2\right)c_0^2 + 4\lambda y_0^8 - 10\lambda y_0^6 + 10\lambda y_0^4 + 2\lambda \dot{y}_0^2 y_0^2\right)$$
$$\begin{pmatrix} \left(2y_0^6 - 5y_0^4 + 5y_0^2 + \dot{y}_0^2\right)c_0^4 + \\ \left(-4\lambda y_0^8 + 10\lambda y_0^6 - 10\lambda y_0^4 - 2\lambda \dot{y}_0^2 y_0^2 - 20y_0^4\right)c_0^2 + \\ 2\lambda^2 y_0^{10} - 5\lambda^2 y_0^8 + 5\lambda^2 y_0^6 + \lambda^2 \dot{y}_0^2 y_0^4 + 24\lambda y_0^{10} - 60\lambda y_0^8 + \\ 60\lambda y_0^6 + 12\lambda \dot{y}_0^2 y_0^4 + 8y_0^{10} - 20y_0^8 + 20y_0^6 + 4\dot{y}_0^2 y_0^4 \end{pmatrix} = 0. \quad (39)$$

Example 2. Let $p = q = r = y_0 = \dot{y}_0 = 1$. The i.v.p. problem to be solved is

$$y''(t) + y(t) + y(t)^3 + y(t)^5 = 0 \text{ given that } y(0) = 1 \text{ and } y'(0) = 1. \quad (40)$$

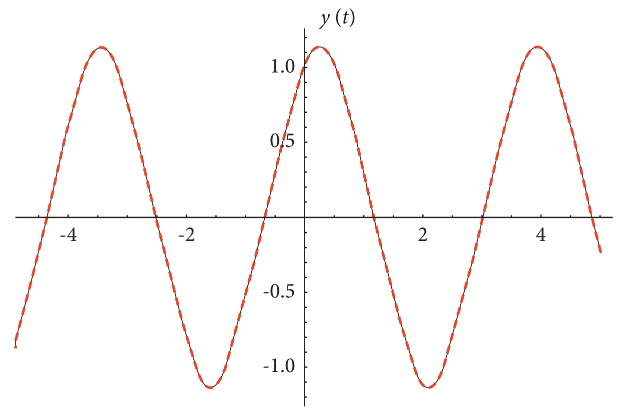

Figure 2: Comparison between the numerical and the exact solution for the i.v.p. (40).

This problem has a negative discriminant $\Delta = -387$. Sextic (35) reads

$$177633\lambda^6 - 1825416\lambda^5 + 5060232\lambda^4 - 12568608\lambda^3 - 16971120\lambda^2 - 6469632\lambda - 735488 = 0. \quad (41)$$

The roots are

$$\begin{aligned} \lambda_1 &= -0.404933 - 0.145779i, \\ \lambda_2 &= -0.404933 + 0.145779i, \\ \lambda_3 &= -0.208601, \\ \lambda_4 &= 1.63821 - 3.26816i, \\ \lambda_5 &= 1.63821 + 3.26816i, \\ \lambda_6 &= 8.01838. \end{aligned} \quad (42)$$

We choose $\lambda = \lambda_3 = -0.208601$. The values for c_0 and c_1 are

$$c_0 - 0.993235, c_1 = 2.72596. \quad (43)$$

The exact solution is given by

$$y(t) = -\frac{0.993235\,\mathrm{cn}(1.71662t + 2.72596|0.0253557)}{\sqrt{-0.0260153\,\mathrm{cn}(1.71662t + 2.72596|0.0253557)^4 - 0.208601\,\mathrm{cn}(1.71662t + 2.72596|0.0253557)^2 + 1}}. \quad (44)$$

See Figure 2, for a comparison with the numerical solution.

2.3. Third Case: $\Delta = 0$. In this case, we have two roots to the cubic:

$$z_1 = \frac{1}{9}\left(\frac{8p^2}{4p^2 + q\left(6py_0^2 + 3qy_0^4 + 2ry_0^6 + 6\dot{y}_0^2\right)} + \frac{4pr + q^2}{q^2 - 4pr}\right),$$
and
$$z_2 = \frac{1}{9}\left(\frac{8p^2}{4p^2 + q\left(6py_0^2 + 3qy_0^4 + 2ry_0^6 + 6\dot{y}_0^2\right)} + \frac{4pr + q^2}{q^2 - 4pr}\right). \quad (45)$$

At least one of these two numbers must be positive. So, we proceed the same way as we did for a positive discriminant.

2.4. Two Particular Cases

2.4.1. First Particular Case: $\dot{y}_0 = 0$. Let

$$y''(t) + py(t) + qy(t)^3 + ry(t)^5 = 0 \text{ given that } y(0) = y_0 \text{ and } y'(0) = 0. \tag{46}$$

The discriminant to the i.v.p. (46) equals

$$\Delta_1 = \left(p + qy_0^2 + ry_0^4\right)^2\left(3q^2 - 4qry_0^2 - 4r^2y_0^4 - 16pr\right). \tag{47}$$

If $\Delta_1 > 0$, then the exact solution is given by

$$y(t) = \frac{\sqrt{\lambda + 1}\,y_0 \operatorname{cn}\left(t\sqrt{\omega} \mid m\right)}{\sqrt{1 + \lambda \operatorname{cn}\left(t\sqrt{\omega} \mid m\right)^2}}, \tag{48}$$

where

$$\begin{aligned}
m &= \frac{2\lambda(3\lambda + 2)p + (\lambda + 1)(3\lambda + 1)qy_0}{(\lambda + 1)\left((6\lambda + 2)p + (3\lambda + 2)qy_0\right)},\\
\omega &= \frac{(\lambda + 1)\left((6\lambda + 2)p + (3\lambda + 2)qy_0\right)}{6\lambda(\lambda + 1) + 2},\\
\lambda &= -\frac{y_0\left(3q + 6ry_0 \pm \sqrt{3}\sqrt{3q^2 - 16pr - 4qry_0 - 4r^2y_0}\right)}{12\left(p + qy_0 + ry_0\right)}.
\end{aligned} \tag{49}$$

Assume now a negative discriminant. The exact solution reads

$$\begin{aligned}
\omega &= \frac{2p(\lambda(3\lambda + 4) - 5\mu + 1) + (3\lambda + 2)qy_0^2(\lambda + \mu + 1)}{6\lambda(\lambda + 1) + 10\mu + 2},\\
m &= \frac{2p(\lambda(3\lambda + 2) - 5\mu) + (3\lambda + 1)qy_0^2(\lambda + \mu + 1)}{2p(\lambda(3\lambda + 4) - 5\mu + 1) + (3\lambda + 2)qy_0^2(\lambda + \mu + 1)},\\
\lambda &= \frac{2\left(3q + 2ry_0^2\right)\left(-2\sqrt{6}\,\Delta_2 + 12p + 9qy_0^2 + 6ry_0^4\right)}{3y_0^2\left(16pr - 3q^2 + 4ry_0^2\left(q + ry_0^2\right)\right)},\\
\mu &= \frac{-96p^2 + 16p\left(\sqrt{6}\,\Delta_2 - 9qy_0^2 - 7ry_0^4\right) + y_0^2\left(3q + 2ry_0^2\right)\left(4\sqrt{6}\,\Delta_2 - 17qy_0^2 - 14ry_0^4\right)}{y_0^4\left(16pr - 3q^2 + 4ry_0^2\left(q + ry_0^2\right)\right)},
\end{aligned} \tag{50}$$

where

$$\Delta_2 = \sqrt{\left(p + qy_0^2 + ry_0^4\right)\left(6p + 3qy_0^2 + 2ry_0^4\right)}.$$

We claim that $\Delta_2 > 0$. Indeed, let

$$\delta = 3q^2 - 4qry_0^2 - 4r^2y_0^4 - 16pr. \tag{51}$$

Then, $\delta < 0$. However,

$$\Delta_2 = \frac{\left(3\left(q + 2ry_0^2\right)^2 - \delta\right)\left(\left(3q + 2ry_0^2\right)^2 - 3\delta\right)}{128r^2}. \tag{52}$$

From the last identity, it is clear that $\Delta_2 > 0$.

Assume now that $\Delta_1 = 0$. In this case, we have the following solutions:

$$y(t)=\frac{y_0\sqrt{3q+2ry_0^2}\cos\left(1/4\sqrt{\left(q+2ry_0^2\right)\left(3q+2ry_0^2\right)/r}\,t\right)}{\sqrt{3q+4ry_0^2-2ry_0^2\cos\left(1/4\sqrt{\left(q+2ry_0^2\right)\left(3q+2ry_0^2\right)/r}\,t\right)}}\ \text{for}\ \frac{\left(q+2ry_0^2\right)\left(3q+2ry_0^2\right)}{r}>0,$$
$$y(t)=\frac{y_0\sqrt{3q+2ry_0^2}\cosh\left(1/4\sqrt{-\left(q+2ry_0^2\right)\left(3q+2ry_0^2\right)/r}\,t\right)}{\sqrt{3q+4ry_0^2-2ry_0^2\cosh\left(1/4\sqrt{-\left(q+2ry_0^2\right)\left(3q+2ry_0^2\right)/r}\,t\right)}}\ \text{for}\ \frac{\left(q+2ry_0^2\right)\left(3q+2ry_0^2\right)}{r}<0. \tag{53}$$

Solutions (53) are called solitons. They arise in soliton theory.

2.4.2. Second Particular Case: $y_0=0$. Let

$$y''(t)+py(t)+qy(t)^3+ry(t)^5=0 \text{ given that } y(0)=0 \text{ and } y'(0)=\dot{v}_0. \tag{54}$$

The solution has the form

$$y(t)=\frac{\dot{v}_0\sqrt{\lambda+\mu+1}\cdot\operatorname{sn}\left(x\sqrt{-p(\lambda+\mu+1)/2\lambda+4\mu-1}\,|\mu/\lambda+\mu+1\right)}{\sqrt{-p(\lambda+\mu+1)/2\lambda+4\mu-1}\sqrt{1+\mu\cdot\operatorname{cn}\left(x\sqrt{-p(\lambda+\mu+1)/2\lambda+4\mu-1}\,|\mu/\lambda+\mu+1\right)^4+\lambda\cdot\operatorname{cn}\left(x\sqrt{-p(\lambda+\mu+1)/2\lambda+4\mu-1}\,|\mu/\lambda+\mu+1\right)^2}}. \tag{55}$$

The values of λ and μ are obtained by solving the two sextics:

$$P(\lambda)=0 \text{ and } Q(\mu)=0, \tag{56}$$

where

$$\begin{aligned}P(\lambda)&=\dot{v}_0^2\left(9q^3-48pqr-64r^2\dot{v}_0^2\right)+12\left(-3p^2q^2+16p^3r-3q^3\dot{v}_0^2+20pqr\dot{v}_0^2+16r^2\dot{v}_0^4\right)\lambda\\&\quad+6\left(-3p^2q^2+16p^3r+3q^3\dot{v}_0^2-12pqr\dot{v}_0^2-16r^2\dot{v}_0^4\right)\lambda^2+4\dot{v}_0^2\left(9q^3-48pqr-32r^2\dot{v}_0^2\right)\lambda^3\\&\quad+3\dot{v}_0^2\left(3q^3-16pqr+16r^2\dot{v}_0^2\right)\lambda^4+48r^2\dot{v}_0^4\lambda^5+8r^2\dot{v}_0^4\lambda^6,\end{aligned} \tag{57}$$

and

$$\begin{aligned}Q(\mu)&=3\dot{v}_0^4\left(3p^2q^2-16p^3r+6q^3\dot{v}_0^2-36pqr\dot{v}_0^2-36r^2\dot{v}_0^4\right)\\&\quad-144\left(4p^6+17p^4q\dot{v}_0^2+21p^2q^2\dot{v}_0^4+14p^3r\dot{v}_0^4+6q^3\dot{v}_0^6+27pqr\dot{v}_0^6+18r^2\dot{v}_0^8\right)\mu\\&\quad+96\left(6p^6+48p^4q\dot{v}_0^2+129p^2q^2\dot{v}_0^4-124p^3r\dot{v}_0^4+114q^3\dot{v}_0^6-252pqr\dot{v}_0^6-198r^2\dot{v}_0^8\right)\mu^2\\&\quad-768\dot{v}_0^2\left(3p^4q+15p^2q^2\dot{v}_0^2-10p^3r\dot{v}_0^2+18q^3\dot{v}_0^4-18pqr\dot{v}_0^4+36r^2\dot{v}_0^6\right)\mu^3\\&\quad+768\dot{v}_0^4\left(3p^2q^2+8p^3r+6q^3\dot{v}_0^2+28pqr\dot{v}_0^2+132r^2\dot{v}_0^4\right)\mu^4-12288r\dot{v}_0^6\left(pq+6r\dot{v}_0^2\right)\mu^5+16384r^2\dot{v}_0^8\mu^6.\end{aligned} \tag{58}$$

3. Applications

The cubic-quintic Duffing oscillator has many interesting applications in soliton theory, optics, nonlinear circuits, plasma physics, and other areas of science and engineering.

3.1. Nonlinear Odd Parity Oscillators. Let us consider the nonlinear oscillator:

$$\begin{aligned} u''(t) + f(u(t)) &= 0 \text{ subjected to } u(0) \\ &= u_0 \text{ and } u'(0) \\ &= 0, \end{aligned} \tag{59}$$

where $f(x)$ is and odd function:

$$f(-x) = -f(x). \tag{60}$$

We may approximate the function $f = f(x)$ by means of a Chebyshev polynomial on some interval $[-A, A]$ $(A > 0)$ as follows:

$$f(x) \approx px + qx^3 + rx^5, \tag{61}$$

where

$$\begin{cases} p = \dfrac{f(-A\sqrt{2}/2) - f(A\sqrt{2}/2) - (5+3\sqrt{3})f(-1/2\sqrt{2-\sqrt{3}}\,A) + (5+3\sqrt{3})f(1/2\sqrt{2-\sqrt{3}}\,A) + (5-3\sqrt{3})(f(-1/2\sqrt{2+\sqrt{3}}\,A) - f(1/2\sqrt{2+\sqrt{3}}\,A))}{3\sqrt{2}A}, \\ q = \dfrac{\sqrt{2}(-8f(-A\sqrt{2}/2) + 8f(A\sqrt{2}/2) + (7+5\sqrt{3})f(-1/2\sqrt{2-\sqrt{3}}\,A) - (7+5\sqrt{3})f(1/2\sqrt{2-\sqrt{3}}\,A) + (5\sqrt{3}-7)(f(-1/2\sqrt{2+\sqrt{3}}\,A) - f(1/2\sqrt{2+\sqrt{3}}\,A)))}{3A^3}, \\ r = \dfrac{4\sqrt{2}(1+\sqrt{3})((\sqrt{3}-1)f(-A\sqrt{2}/2) - (\sqrt{3}-1)f(A\sqrt{2}/2) - f(-1/2\sqrt{2-\sqrt{3}}\,A) + f(1/2\sqrt{2-\sqrt{3}}\,A) + (\sqrt{3}-2)(f(-1/2\sqrt{2+\sqrt{3}}\,A) - f(1/2\sqrt{2+\sqrt{3}}\,A)))}{3A^5}. \end{cases} \tag{62}$$

Then, we may obtain an approximated analytic solution to oscillator (59) by solving the cubic-quintic Duffing oscillator:

$$\begin{aligned} y''(t) + py(t) + qy(t)^3 + ry(t)^5 &= 0 \text{ subjected to } y(0) \\ &= u_0 \text{ and } y'(0) \\ &= 0. \end{aligned} \tag{63}$$

Example 3. Let us obtain an approximate analytical solution to the pendulum equation:

$$\begin{aligned} \theta''(t) + k^2 \sin(\theta(t)) &= 0 \text{ subjected to } \theta(0) \\ &= \theta_0 \text{ and } \theta'(0) \\ &= 0, \end{aligned} \tag{64}$$

having the analytical solution [9]

$$\theta(t) = 2\tan^{-1}\left(\tan\left(\frac{\theta_0}{2}\right)\operatorname{cn}\left(k\cdot t \mid \sin^2\left(\frac{\theta_0}{2}\right)\right)\right). \tag{65}$$

The Chebyshev approximant $P(x)$ of $f(x) = \sin x$ on $[-\pi/2, \pi/2]$ reads

$$\begin{aligned} P(x) = {} & 16\sqrt{2}\left(\sin\left(\frac{\pi}{\sqrt{2}}\right) - \sqrt{3}\sin\left(\frac{1}{2}\sqrt{\frac{3}{2}}\pi\right)\cos\left(\frac{\pi}{2\sqrt{2}}\right) + \sin\left(\frac{\pi}{2\sqrt{2}}\right)\cos\left(\frac{1}{2}\sqrt{\frac{3}{2}}\pi\right)\right) - 3\pi^5 x^5 \\ & + 4\sqrt{2}\left(4\sin\left(\frac{\pi}{\sqrt{2}}\right) - 5\sqrt{3}\sin\left(\frac{1}{2}\sqrt{\frac{3}{2}}\pi\right)\cos\left(\frac{\pi}{2\sqrt{2}}\right) + 7\sin\left(\frac{\pi}{2\sqrt{2}}\right)\cos\left(\frac{1}{2}\sqrt{\frac{3}{2}}\pi\right)\right)3\pi^3 x^3 \\ & - \sqrt{2}x\left(\sin\left(\frac{\pi}{\sqrt{2}}\right) - 6\sqrt{3}\sin\left(\frac{1}{2}\sqrt{\frac{3}{2}}\pi\right)\cos\left(\frac{\pi}{2\sqrt{2}}\right) + 10\sin\left(\frac{\pi}{2\sqrt{2}}\right)\cos\left(\frac{1}{2}\sqrt{\frac{3}{2}}\pi\right)\right)3\pi. \end{aligned} \tag{66}$$

We may rationalize this expression up to 10^{-7} to obtain

$$P(x) = \frac{17x^5}{2926} - \frac{2461x^3}{15601} + \frac{1641x}{1649}. \tag{67}$$

The square mean error is

$$\sqrt{\int_{-\pi}^{\pi} (\sin x - P(x))^2} = 0.0141773. \quad (68)$$

Thus, our aim is to solve the cubic-quintic Duffing oscillator:

$$y''(t) + \frac{1641k^2}{1649} y(t) - \frac{2461k^2}{15601} y^3(t) + \frac{17k^2}{2926} y^5(t) = 0 \text{ subjected to } y(0) = \theta_0 \text{ and } y'(0) = 0. \quad (69)$$

Let $k = 1$ and $\theta_0 = 5\pi/6$. For these data, we have a positive discriminant and the solution to the initial value problem,

$$y''(t) + \frac{1641}{1649} y(t) - \frac{2461}{15601} y^3(t) + \frac{17}{2926} y^5(t) = 0 \text{ subjected to } y(0) = \frac{5\pi}{6} \text{ and } y'(0) = 0, \quad (70)$$

may be obtained making use of formula (48).

3.2. The Duffing–Helmholtz Oscillator. Let us consider the i.v.p.:

$$x''(t) + \alpha x(t) + \beta x^2(t) + \gamma x^3(t) = 0 \text{ given that } x(0) = x_0 \text{ and } x'(0) = \dot{x}_0. \quad (71)$$

Suppose that the function $y = y(t)$ is the solution to the cubic-quintic Duffing equation:

$$y''(t) + \frac{1}{4}\left(\alpha + 2A\beta + 3A^2\gamma\right) y(t) - \frac{1}{3}(A - x_0)(\beta + 3A\gamma) y(t)^3 + \frac{3}{8}(A - x_0)^2 \gamma y(t)^5$$
$$\text{given that } y(0) = 1 \text{ and } y'(0) = \frac{\dot{x}_0}{2(x_0 - A)}. \quad (72)$$

Then, the function,

$$x(t) = A + (x_0 - A) y^2(t), \quad (73)$$

is the solution to the i.v.p. (71) provided that A is a solution to the quartic:

$$3\gamma A^4 + 4\beta A^3 + 6\alpha A^2 - 6\alpha x_0^2 - 4\beta x_0^3 - 3\gamma x_0^4 - 6\dot{x}_0^2 = 0. \quad (74)$$

3.3. Nonlinear Conservative Oscillators. Suppose we are given to sole the i.v.p.:

$$u''(t) + g(u(t)) = 0 \text{ subjected to } u(0) = u_0 \text{ and } u'(0) = \dot{u}_0. \quad (75)$$

Assume that $|u| \leq A$. We approximate the function $g = g(u)$ by means of a cubic polynomial on $-A \leq u \leq A$ using Chebyshev approach so that

$$g(u) \approx a_0 + a_1 u + a_2 u^2 + a_3 u^3, -A \leq u \leq A, \quad (76)$$

where

$$\begin{aligned}
a_0 &= \frac{\cos^2(\pi/8)(g(-A\sin(\pi/8)) + g(A\sin(\pi/8))) - \sin^2(\pi/8)(g(-A\cos(\pi/8)) + g(A\cos(\pi/8)))}{\sqrt{2}},\\
a_1 &= \frac{(\sqrt{2}-1)\left(8\sec(\pi/8)(g(-A\cos(\pi/8)) - g(A\cos(\pi/8))) + \csc^5(\pi/8)(g(A\sin(\pi/8)) - g(-A\sin(\pi/8)))\right)}{32A},\\
a_2 &= \frac{-g(-A\sin(\pi/8)) - g(A\sin(\pi/8)) + g(-A\cos(\pi/8)) + g(A\cos(\pi/8))}{\sqrt{2}A^2},\\
a_3 &= \frac{(\sqrt{2}-1)\csc^2(\pi/8)\sec(\pi/8)(-g(-A\cos(\pi/8)) + g(A\cos(\pi/8)) + \cot(\pi/8)(g(-A\sin(\pi/8)) - g(A\sin(\pi/8))))}{4A^3}.
\end{aligned} \quad (77)$$

We now replace the i.v.p. (75) with the i.v.p.:

$$u''(t) + a_0 + a_1 u(t) + a_2 u^2(t) + a_3 u^3(t) = 0 \text{ subjected to } u(0) = u_0 \text{ and } u'(0) = \dot{u}_0. \quad (78)$$

Let

$$u(t) = \rho + x(t), \text{ where } a_0 + a_1\rho + a_2\rho^2 + a\rho^3. \quad (79)$$

The problem reduces to the i.v.p.:

$$x''(t) + \left(a_1 + 2\rho a_2 + 3\rho^2 a_3\right)x(t) + (a_2 + 3\rho a_3)x^2(t) + a_3 x^3(t) = 0 \text{ given that } x(0) = u_0 - \rho \text{ and } x'(0) = \dot{u}_0. \quad (80)$$

This is a Duffing–Helmholtz oscillator (71) with

$$\alpha = a_1 + 2\rho a_2 + 3\rho^2 a_3, \beta = a_2 + 3\rho a_3 \text{ and } \gamma = a_3. \quad (81)$$

4. Analysis and Discussion

We have solved the cubic-quintic Duffing oscillator equation for any given arbitrary initial conditions. In [1], authors considered the particular case:

$$\frac{d^2x}{dt^2} + a_1 x + a_3 x^3 + a_5 x^5 = 0 \text{ given that } x(0) = A > 0 \text{ and } \frac{dx}{dt}(0) = 0, \quad (82)$$

under the restrictions,

$$a_1 \geq 0, a_3 \geq 0 \text{ and } a_5 \geq 0. \quad (83)$$

These conditions, however, are too restrictive. In [6], author considered the ansatz:

$$y^2(t) = \frac{1}{a + b\,\mathrm{cn}^2\left(\omega t + \phi, k^2\right)}. \quad (84)$$

This approach does not allow to determine the sign of $y(t)$. On the contrary, this ansatz sometimes gives complex values for ω or k, which makes it difficult to interpret the obtained solution physically. Our approach avoids obtaining such complex values.

Other authors solved this equation using perturbative methods [2, 4, 7–9]. In [5], author studied the stability analysis to a cubic-quintic Duffing equation. There are other numerical and analytical methods that allow to solve this oscillator equation.

The cubic-quintic Duffing oscillator may also be solved making use of perturbative methods. One of them is the famous Krylov–Bogoliubov–Mitropolsky method (KBM). For example, let us consider the following oscillator:

$$y''(t) + \omega_0^2 y(t) + \alpha y(t)^3 + \beta y(t)^5 = 0 \text{ given that } y(0) = y_0 \text{ and } y'(0) = \dot{y}_0. \quad (85)$$

Using the Krylov–Bogoliubov–Mitropolsky method gives the following approximate analytical solution:

$$y(t) = A\cos(\omega) - \frac{123\alpha A^4\beta + 63\alpha^2A^2 - 96\alpha\omega_0^2 + 55A^6\beta^2 - 120A^2\beta\omega_0^2}{3072\omega_0^4}A^3\cos(3\omega) + \frac{9\alpha^2 + 5A^4\beta^2 + 9\alpha A^2\beta + 24\beta\omega_0^2}{9216\omega_0^4}A^5\cos(5\omega) + \frac{\beta\left(72\alpha + 95A^2\beta\right)}{294912\omega_0^4}A^7\cos(7\omega) + \frac{\beta^2}{98304\omega_0^4}A^9\cos(9\omega), \quad (86)$$

where

$$\omega = \omega(t) = \left[\omega_0 + \frac{3\alpha}{8\omega_0}A^2 - \frac{5\left(3\alpha^2 - 16\beta\omega_0^2\right)}{256\omega_0^3}A^4 - \frac{5\alpha\beta}{64\omega_0^3}A^6 - \frac{55\beta^2}{3072\omega_0^3}A^8\right]t + B. \quad (87)$$

For the damped oscillator,

$$y''(t) + 2\varepsilon y'(t) + \omega_0^2 y(t) + q y(t)^3 + r y(t)^5 = 0 \text{ given that } y(0) = y_0 \text{ and } y'(0) = \dot{y}_0. \quad (88)$$

The KBM gives

$$y(t) = a(t)\cos(\psi(t)) + \frac{4q + 5ra(t)^2}{128\omega_0^2}a(t)^3\cos(3\psi(t)) + \frac{r}{384\omega_0^2}a(t)^5\cos(5\psi(t)), \tag{89}$$

where

$$a(t) = A\exp(-\varepsilon t) \text{ and } \psi(t) = \omega_0 t + B + \frac{3\left(1 - e^{-2t\varepsilon}\right)}{16\varepsilon\omega_0}qA^2 + \frac{5\left(1 - e^{-4t\varepsilon}\right)}{64\varepsilon\omega_0}rA^4. \tag{90}$$

The constants A and B are determined from the initial conditions. These results are also valid for the cubic Duffing oscillator $(r = 0)$:

$$y''(t) + 2\varepsilon y'(t) + \omega_0^2 y(t) + qy(t)^3 = 0 \text{ given that } y(0) = y_0 \text{ and } y'(0) = \dot{y}_0. \tag{91}$$

5. Conclusions

The cubic-quintic Duffing oscillator has been solved exactly for arbitrary initial conditions. The obtained results may be applied to solve strongly nonlinear conservative oscillators like the pendulum oscillator equation. We may go further by considering a damped oscillator of the form $\ddot{x} + 2\varepsilon\dot{x} + F(x) = 0$. In the case when $F(-x) = -F(x)$, we may approximate the function $F(x)$ by means of some cubic-quintic polynomial using Chebyshev approximation formulas. The solution is then assumed in the form $x(t) = \exp(-\rho t)y(t)$, where ρ is some parameter having a value near $\rho = \varepsilon$ and $y = y(t)$ is the exact solution to some cubic-quintic Duffing oscillator equation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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