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*Research Article*

# Existence of $C^1$-Positive Solutions for a Class of Second-Order Impulsive Differential Equations

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In this study, under some inequality conditions, necessary and sufficient conditions, using fixed-point theorem in cones, are established for the existence of $C^1$-positive solutions for a class of second-order impulsive differential equations. Two examples are given in the last section to illustrate the abstract results.

## 1. Introduction

The theory of differential equations with impulsive effects has an extensive application in realistic mathematical models. It has been used to describe many evolution processes, containing abrupt change, such as biological systems, population dynamics, and optimal control. Hence, in recent years, more and more attention has been paid on this topic. For the general theory of impulsive differential equations, one can see the monographs of Lakshmikantham et al. [1], Bainov and Simeonov [2], and Benchohra et al. [3]. There are also some studies focusing on impulsive differential equations. In [4], Ye investigated the existence of mild solutions for first-order impulsive semilinear neutral functional differential equations with infinite delay in Banach spaces by using the Hausdorff measure of noncompactness conditions. In [5], Hernández et al. concerned with the existence of solutions for partial neutral functional differential equations of first and second order with impulses by using fixed-point theorems. The existence of solutions for fractional differential equations have also been studied widely. In [6], Gu et al. studied the existence of positive solutions for impulsive fractional differential equations attached with integral boundary conditions via global bifurcation techniques. In [7], Benchohra and Seba demonstrated the existence and uniqueness of solutions for the initial value problem of fractional differential equations by utilizing fixed-point theorems.

The existence of solutions for second-order differential equations, involving different boundary conditions, has been studied by many authors. Chu and Nieto in [8], utilizing the nonlinear alternative principle of Leray-Schauder type and Schauder's fixed-point theorem, presented existence results of positive $T$-periodic solutions for second-order differential equations. In 2019, Ma and Zhang in [9] proved sharp conditions for the existence of positive solutions of second-order singular differential equation with integral boundary conditions. Recently, Zhang and Tian [10] established sharp conditions for the existence of positive solutions of second-order impulsive differential equations. But in their work, a key assumption is that the nonlinearity $f(t,x)$ is nondecreasing with respect to $x \geq 0$. Clearly, if $f(t,x) = t^2 x^{1/3} + t^3 x^{-1/3}$, the results obtained in [9, 10] are not valid. The aim of this study is to extend the results in [10] to more general cases. We establish necessary and sufficient conditions for the existence of $C^1$-positive solutions for a class of second-order impulsive differential equations. The results obtained in this study extend and improve some existing works.

In the present work, we consider the boundary value problem (BVP for short) of second-order impulsive differential equation:

$$\begin{cases} -x''(t) = \tau(t) f(t, x(t), y(t)), & t \in I, t \neq t_k, \\ \Delta x|_{t=t_k} = \delta_k x(t_k), & k = 1, 2, \ldots, n, \\ \alpha x(0) - \beta x'(0) = 0, \gamma x(1) + \omega x'(1) = 0, \end{cases} \tag{1}$$

where $\alpha, \beta, \gamma, \omega$ are the nonnegative constants satisfying $(\alpha + \beta)\gamma + \alpha\omega > 0$, $I = (0,1)$, $\tau \in L^p[0,1]\,(1 \leq p < +\infty)$, $y(t) = \varsigma(t)x^{-1}(t)$, and $\varsigma$ is a positive continuous function on $I$, $f \in C(I \times (0,+\infty) \times (0,+\infty), [0,+\infty))$, $f(t,x,y)$ may be singular at $t = 0, 1$, $t_0 = 0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = 1$, $\{\delta_k\}_{k=1}^n$ is a real sequence, and $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k)\,(k = 1, 2, \ldots, n)$ represents the impulsive term, where $x(t_k) = x(t_k^-)$ and $x(t_k^+)$ denotes the right-hand limit of $x(t)$ at $t = t_k$.

In addition, $\tau$ and $\{\delta_k\}_{k=1}^n$ satisfy the following conditions:

$(A1)\tau \in L^p[0,1]$ for some $1 \leq p < +\infty$, and there exists $\xi > 0$, such that $\tau(t) \geq \xi$, i.e., $t \in I$.

$(A2)\,(\delta_k)_{k=1}^n$ is a real sequence with $\delta_k > -1, k = 1, 2, \ldots, n$, and denoted by

$$\delta(t) = \prod_{0<t_k<t} (1 + \delta_k), \quad t \in I^* \coloneqq [0,1]. \tag{2}$$

Clearly, $\delta(t)$ is continuous on $I^*$. Denote by $\delta_M = \max_{t\in I^*}\delta(t), \delta_m = \min_{t\in I^*}\delta(t)$. From $(A2)$, we have

$$\begin{aligned} &\delta(t) \geq \delta_m > 0, \forall t \in I^*, \\ &\delta(t) = 1, \forall t \in [0, t_1], \end{aligned} \tag{3}$$

$$\delta'(t) \equiv 0, \forall t \in I^*. \tag{4}$$

Furthermore, $\delta^{-1}(t) = \prod_{0<t_k<t}(1 + \delta_k)^{-1}$ for any $t \in I^*$. The main results of this work are summarized as follows:

(i) The necessary and sufficient conditions on $\tau$ and $f$ are established for the existence of $C^1$-positive solutions of the BVP (1) (Theorems 1 and 2)

(ii) We assume that the nonlinear term $f(t,x,y)$ satisfies $(A3)$, which implies that $f(t,x,y)$ is nondecreasing with respect to $x$ and nonincreasing with respect to $y$ (Remark 2). But $f(t,x,y)$ does not have monotonicity in whole;

(iii) Correct examples are given in the last section to illustrate our abstract results, which show that the results obtained in this study contain some existing works (Theorem 3 and 4).

The rest of this study is organized as follows. Some preliminaries and notations are presented in Section 2. Particularly, we transform the BVP (1) to a problem without impulse in this section. In Section 3, we prove the main results by using the fixed-point theorem of cone mapping and give some remarks. Examples are given in Section 4 to illustrate the abstract results.

## 2. Preliminaries

In this section, some preliminaries and notations, which are useful in the proof of the main results, are presented. In order to discuss the BVP (1) more clearly, we first transform the BVP (1) to a problem without impulse. Let

$$x(t) = \delta(t)u(t), \quad \forall t \in I^*. \tag{5}$$

Then, $y(t) = \delta(t)u^{-1}(t)$ with $\varsigma(t) = \delta^2(t)$. The BVP (1) is rewritten into the BVP:

$$\begin{cases} -u''(t) = \delta^{-1}(t)\tau(t)f\big(t, \delta(t)u(t), \delta(t)u^{-1}(t)\big), & t \in I, \\ \alpha u(0) - \beta u'(0) = 0, \gamma\delta(1)u(1) + \omega\delta(1)u'(1) = 0. \end{cases} \tag{6}$$

**Lemma 1.** *Let the assumptions $(A1)$ and $(A2)$ hold. Then, $u(t)$ is a solution of BVP (2) on $I$ if and only if $x(t) = \delta(t)u(t)$ is a solution of BVP (1) on $I$.*

*Proof.* (Necessity). Let $u(t)$ be a solution of the BVP (2) on $I$. Then, on each interval $(t_k, t_{k+1}], k = 0, 1, 2, \ldots, n$, $x(t) = \delta(t)u(t)$ is absolutely continuous. For $t \neq t_k$, we have

$$\begin{aligned} x'(t) &= \delta(t)u'(t), \\ x''(t) &= \delta(t)u''(t). \end{aligned} \tag{7}$$

So,

$$-x''(t) = -\delta(t)u''(t) = \tau(t)f\big(t, \delta(t)u(t), \delta(t)u^{-1}(t)\big) = \tau(t)f(t, x(t), y(t)). \tag{8}$$

When $t = t_k$, we have

$$\begin{aligned} x(t_k^+) &= \lim_{t\longrightarrow t_k^+} \delta(t)u(t) = \prod_{0<t_i\leq t_k} (1 + \delta_i)u(t_k), \\ x(t_k) = x(t_k^-) &= \lim_{t\longrightarrow t_k^-} \delta(t)u(t) = \prod_{0<t_i\leq t_{k-1}} (1 + \delta_i)u(t_k). \end{aligned} \tag{9}$$

It follows that

$$\begin{aligned} \Delta x|_{t=t_k} &= x(t_k^+) - x(t_k) \\ &= \prod_{0<t_i\leq t_{k-1}} (1 + \delta_i)\delta_k u(t_k) \\ &= \delta_k \delta(t_k)u(t_k) \\ &= \delta_k x(t_k). \end{aligned} \tag{10}$$

Obviously, $x(t)$ satisfies the boundary conditions. Then, $x(t)$ is a solution of the BVP (1) on $I$.

(Sufficiency). Let $x(t)$ be a solution of the BVP (1) on $I$. Then, $u(t) = \delta^{-1}(t)x(t)$ and $u''(t) = \delta^{-1}(t)x''(t)$, which follows

$$-\delta(t)u''(t) = -x''(t) = \tau(t)f(t, x(t), y(t)) = \tau(t)f\left(t, \delta(t)u(t), \delta(t)u^{-1}(t)\right), \quad \forall t \in I. \tag{11}$$

When $t = t_k$, since $x(t_k) = x(t_k^-)$ and $\delta^{-1}(t_k^+) = (1/1 + \delta_k)\delta^{-1}(t_k^-)$, we have

$$\begin{aligned} u(t_k^+) - u(t_k^-) &= \delta^{-1}(t_k^+)x(t_k^+) - \delta^{-1}(t_k^-)x(t_k^-) \\ &= \delta^{-1}(t_k^+)\left[x(t_k^-) + \delta_k x(t_k^-)\right] - \delta^{-1}(t_k^-)x(t_k^-) \\ &= (1 + \delta_k)\delta^{-1}(t_k^+)x(t_k^-) - \delta^{-1}(t_k^-)x(t_k^-) \\ &= 0. \end{aligned} \tag{12}$$

Direct calculation shows that $u(t)$ satisfies all boundary conditions. Then, $u(t)$ is a solution of the BVP (2) on $I$. □

*Definition 1.* A function $u \in C[0,1] \cap C^2(0,1)$ is called a positive solution of the BVP (2) if $u(t)$ satisfies all the equations in (2) and $u(t) > 0$ for all $t \in I$. If $u(t)$ is the positive solution of the BVP (2) and $u \in C^1[0,1]$, namely, $u'(0^+)$ and $u'(1^-)$ exist, then $u$ is called a $C^1$-positive solution of the BVP (2).

Throughout this study, the following assumptions on $f$ are needed.

$(A3)$ $f(t, \delta_m, \delta_M) > 0$ for any $t \in I$, and there exist constants $\sigma_1 \geq \sigma_2 > 1$, such that, for every $\ell \in (0,1]$,

$$\ell^{\sigma_1} f(t, x, y) \leq f\left(t, \ell x, \ell^{-1} y\right) \leq \ell^{\sigma_2} f(t, x, y), \tag{13}$$

for any $t \in I$ and $x, y \in (0, +\infty)$.

$(A4)$ $\int_0^1 \delta^{-1}(t) f(t, \delta_M, \delta_m) \mathrm{d}t < +\infty$.

*Remark 1.* The condition $(A3)$ implies, for every $\ell \geq 1$,

$$\ell^{\sigma_2} f(t, x, y) \leq f\left(t, \ell x, \ell^{-1} y\right) \leq \ell^{\sigma_1} f(t, x, y), \tag{14}$$

for any $t \in I, x, y \in (0, +\infty)$.

*Remark 2.* If $f(t, x, y)$ satisfies $(A3)$, then for any $t \in I$, $f(t, x, y)$ is nondecreasing with respect to $x \in (0, +\infty)$ and nonincreasing with respect to $y \in (0, +\infty)$.

Denote by $E: = C[0,1]$ the Banach space of all continuous functions on $[0,1]$ equipped with the norm $\|u\| = \max_{t \in I^*} |u(t)|$. Define an operator $Q: E \longrightarrow E$ by

$$(\mathrm{Q}u)(t) = \int_0^1 G(t,s)\delta^{-1}(s)\tau(s)f\left(s, \delta(s)u(s), \delta(s)u^{-1}(s)\right)\mathrm{d}s, \quad t \in I^*, \tag{15}$$

where $G(t,s)$ is Green's function of the BVP (2) with $u'' = 0$, and

$$G(t,s) = \frac{1}{\rho}\begin{cases} (\gamma + \omega - \gamma t)(\beta + \alpha s), & 0 \leq s \leq t \leq 1, \\ (\gamma + \omega - \gamma s)(\beta + \alpha t), & 0 \leq t \leq s \leq 1, \end{cases} \tag{16}$$

where $\rho = (\alpha + \beta)\gamma + \alpha\omega > 0$.

*Remark 3.* By the definition of $Q$, $u \in C[0,1] \cap C^2(0,1)$ is a positive solution of the BVP (2) if and only if $u \in C[0,1]$ is a positive fixed point of $Q$.

**Lemma 2** (See [10]). *Green's function $G(t,s)$ of the BVP (2) satisfies*

$$0 < \alpha^* \leq q(t)G(s,s) \leq G(t,s) \leq G(s,s) \leq \beta^*, \quad \forall t, s \in I^*, \tag{17}$$

*where*

$$\begin{aligned} q(t) &= \min\left\{\frac{\alpha t + \beta}{\alpha + \beta}, \frac{\gamma + \omega - \gamma t}{\gamma + \omega}\right\}, \\ \alpha^* &= \frac{\omega\beta}{\rho}, \\ \beta^* &= \frac{(\gamma + \omega)\beta}{\rho}. \end{aligned} \tag{18}$$

For any $\nu \in (0, 1/2)$, let $I_0 = [\nu, 1 - \nu]$ and $\eta = \min_{t \in I_0} q(t)$. Then, $\eta > 0$. Define a cone $P$ by

$$P = \left\{u \in E\colon \exists \zeta \geq 1 \ s.t. 0 < u(t) \leq \zeta G(t,t), \forall t \in I^*, \min_{t \in I_0} u(t) \geq \eta \|u\|\right\}. \tag{19}$$

Then, $P$ is a closed convex cone in $E$.

**Lemma 3.** *Let the assumptions* $(A1)$–$(A4)$ *hold. Then, the operator* $Q$: $P \longrightarrow E$, *defined by (3), is well defined.*

*Proof.* Assume that $(A1)$–$(A4)$ hold. For fixed $u \in P$ with $u(t) > 0$ for any $t \in I^*$, choosing a constant $a \in (0,1)$ satisfying $0 < \mathrm{au}(t) < 1$ for $t \in I^*$, we have

$$\begin{aligned} f\left(t, \delta(t)u(t), \delta(t)u^{-1}(t)\right) &\le \left(\frac{1}{a}\right)^{\sigma_1} f\left(t, a\delta(t)u(t), \frac{1}{a}\delta(t)u^{-1}(t)\right) \\ &\le \left(\frac{1}{a}\right)^{\sigma_1} (\mathrm{au}(t))^{\sigma_2} f(t, \delta(t), \delta(t)) \\ &\le a^{\sigma_2-\sigma_1}\|u\|^{\sigma_2} f(t, \delta_M, \delta_m). \end{aligned} \tag{20}$$

So, for $t \in I^*$, by (17), we have

$$\begin{aligned} 0 < (\mathrm{Qu})(t) &= \int_0^1 G(t,s)\delta^{-1}(s)\tau(s) f\left(s, \delta(s)u(s), \delta(s)u^{-1}(s)\right)\mathrm{d}s \\ &\le a^{\sigma_2-\sigma_1}\|u\|^{\sigma_2}\beta^*\|\tau\|_p \int_0^1 \delta^{-1}(s) f(s, \delta_M, \delta_m)\mathrm{d}s \\ &< +\infty. \end{aligned} \tag{21}$$

This implies that the operator $Q$: $P \longrightarrow E$ is well defined.

To end this section, we state a fixed-point theorem of cone mapping, which is useful in the proof of our main results. □

**Lemma 4** (See [11]). *Let $E$ be a Banach space, and $P \subset E$ be a cone in $E$. Assume that $\Omega_1$ and $\Omega_2$ are two bounded and open subsets of $E$ with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. If*

$$Q: P \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow P, \tag{22}$$

*is a completely continuous operator such that either*

(i) $\|Qu\| \le \|u\|, \forall u \in P \cap \partial\Omega_1$ *and* $\|Qu\| \ge \|u\|, \forall u \in P \cap \partial\Omega_2$, *or*

(ii) $\|Qu\| \ge \|u\|, \forall u \in P \cap \partial\Omega_1$ *and* $\|Qu\| \le \|u\|, \forall u \in P \cap \partial\Omega_2$,

*then $Q$ has at least one fixed point in* $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

## 3. Main Results

In this section, we establish necessary and sufficient conditions for the existence of $C^1$-positive solutions of the BVP (1) The proof is based on Lemma 4.

**Theorem 1.** *Let the assumptions* $(A1)$–$(A4)$ *hold. Then, the BVP (1) has at least one positive solution $x \in C^1[0,1]$ if and only if*

$$\begin{aligned} 0 < \int_0^1 &\delta^{-1}(s)\tau(s) f(s, \delta(s)G(s,s), \\ &\delta\left(s\right)G^{-1}(s,s)\big)\mathrm{d}s < +\infty. \end{aligned} \tag{23}$$

*Proof.* (Necessity). Let $x \in C^1[0,1]$ be a positive solution of the BVP (1) By Lemma 1, $u \in C^1[0,1]$ is a positive solution of the BVP (2). So, $u'(0)$ and $u'(1)$ exist and are finite.

Since $u''(t) \le 0$ for $t \in I^*$, it follows that $u'(t)$ is nonincreasing on $I^*$ and $u'(0) \ge u'(1)$. Hence, by (2.1) of [6], there exists $b > 0$, such that

$$u(s) \ge bG(s,s), \quad s \in I^*. \tag{24}$$

Then,

$$u^{-1}(s) \le \frac{1}{b}G^{-1}(s,s), \quad s \in I^*. \tag{25}$$

Let $\ell = \min\{b, 1\}$. Then,

$$\begin{aligned} &\delta(s)u(s) \ge \ell\delta(s)G(s,s), \delta(s)u^{-1}(s) \\ &\le \frac{1}{\ell}\delta(s)G^{-1}(s,s), \quad s \in I^*. \end{aligned} \tag{26}$$

Hence, on the one hand, by Remark 1, we have

$$\begin{aligned}\int_0^1 \delta^{-1}(s)\tau(s)f\left(s,\delta(s)G(s,s),\delta(s)G^{-1}(s,s)\right)\mathrm{d}s &\leq \int_0^1 \delta^{-1}(s)\tau(s)f\left(s,\frac{1}{\ell}\delta(s)u(s),\ell\delta(s)u^{-1}(s)\right)\mathrm{d}s\\ &\leq \left(\frac{1}{\ell}\right)^{\sigma_1}\int_0^1 \delta^{-1}(s)\tau(s)f\left(s,\delta(s)u(s),\delta(s)u^{-1}(s)\right)\mathrm{d}s\\ &= \left(\frac{1}{\ell}\right)^{\sigma_1}\int_0^1[-u''(s)]\mathrm{d}s = \left(\frac{1}{\ell}\right)^{\sigma_1}[u'(0)-u'(1)] < +\infty.\end{aligned} \tag{27}$$

On the other hand, if $f(s,\delta(s)u(s),\delta(s)u^{-1}(s)) \equiv 0$ for $s \in I^*$, then

$$\delta^{-1}(s)\tau(s)f\left(s,\delta(s)u(s),\delta(s)u^{-1}(s)\right) \equiv 0. \tag{28}$$

By (15) and Lemma 1, $u(t) \equiv 0$ for $t \in I^*$. So, $x(t) \equiv 0$ for $t \in I^*$, which contradicts the positivity of $x$. Hence, there exists $t^* \in (0,1)$, such that

$$f\left(t^*,\delta(t^*)u(t^*),\delta(t^*)u^{-1}(t^*)\right) > 0. \tag{29}$$

By (2.1) of [6], there exists $c > 0$, such that $u(s) \leq cG(s,s)$ for $s \in I^*$. Then,

$$u^{-1}(s) \geq \frac{1}{c}G^{-1}(s,s), \quad s \in I^*. \tag{30}$$

Let $\ell_1 = \max\{c,1\}$. Then,

$$\begin{aligned}&\delta(t^*)u(t^*) \leq \ell_1\delta(t^*)G(t^*,t^*), \delta(t^*)u^{-1}(t^*)\\ &\quad \geq \frac{1}{\ell_1}\delta(t^*)G^{-1}(t^*,t^*).\end{aligned} \tag{31}$$

Hence, by $(A1)$ and Remark 1, we have the following:

Case (1): if $\tau(t^*) > 0$, then

$$\begin{aligned}0 &< \delta^{-1}(t^*)\tau(t^*)f\left(t^*,\delta(t^*)u(t^*),\delta(t^*)u^{-1}(t^*)\right)\\ &\leq \delta^{-1}(t^*)\tau(t^*)f\left(t^*,\ell_1\delta(t^*)G(t^*,t^*),\frac{1}{\ell_1}\delta(t^*)G^{-1}(t^*,t^*)\right)\\ &\leq (\ell_1)^{\sigma_1}\delta^{-1}(t^*)\tau(t^*)f\left(t^*,\delta(t^*)G(t^*,t^*),\delta(t^*)G^{-1}(t^*,t^*)\right).\end{aligned} \tag{32}$$

Case (2): if $\tau(t^*) = 0$, choose a small neighborhood $[a_1,b_1]$ of $t^*$, such that $\tau(s) \equiv 0$ for $s \in [a_1,b_1] \subset I$. So,

$$\int_{a_1}^{b_1} \delta^{-1}(s)\tau(s)f\left(s,\delta(s)G(s,s),\delta(s)G^{-1}(s,s)\right)\mathrm{d}s > 0. \tag{33}$$

Consequently, under both cases, we have

$$\int_0^1 \delta^{-1}(s)\tau(s)f\left(s,\delta(s)G(s,s),\delta(s)G^{-1}(s,s)\right)\mathrm{d}s > 0. \tag{34}$$

(Sufficiency). At first, we prove that $Q$: $P \longrightarrow P$ is a completely continuous operator. In fact, on the one hand, for any $u \in P$, by (15) and (17), we have

$$(\mathrm{Qu})(t) \geq \int_0^1 q(t)G(s,s)\delta^{-1}(s)\tau(s)f\left(s,\delta(s)u(s),\delta(s)u^{-1}(s)\right)\mathrm{d}s. \tag{35}$$

Then,

$$\begin{aligned}\min_{t\in I_0}(\mathrm{Qu})(t) &\geq \min_{t\in I_0} q(t)\int_0^1 G(s,s)\delta^{-1}(s)\tau(s)f\left(s,\delta(s)u(s),\delta(s)u^{-1}(s)\right)\mathrm{d}s\\ &\geq \eta\|\mathrm{Qu}\|, \quad \forall t \in I^*.\end{aligned} \tag{36}$$

On the other hand, since $u \in P$, there exists a constant $\zeta \geq 1$, such that $u(s) \leq \zeta G(s,s)$ for any $s \in I^*$. Hence,

$$\delta(s)u(s) \le \zeta\delta(s)G(s,s), \delta(s)u^{-1}(s) \ge \frac{1}{\zeta}\delta(s)G^{-1}(s,s), \quad s \in I^*. \tag{37}$$

Consequently, by $(A3)$ and Remark 1, we have

$$\begin{aligned}(\mathrm{Qu})(t) &\le \int_0^1 G(t,s)\delta^{-1}(s)\tau(s)f\left(s, \zeta\delta(s)G(s,s), \frac{1}{\zeta}\delta(s)G^{-1}(s,s)\right)\mathrm{d}s \\ &\le \zeta^{\sigma_1}\int_0^1 G(t,s)\delta^{-1}(s)\tau(s)f\left(s,\delta(s)G(s,s),\delta(s)G^{-1}(s,s)\right)\mathrm{d}s \\ &\le \zeta^{\sigma_1}G(t,t)\int_0^1 \delta^{-1}(s)\tau(s)f\left(s,\delta(s)G(s,s),\delta(s)G^{-1}(s,s)\right)\mathrm{d}s, \quad \forall t \in I^*.\end{aligned} \tag{38}$$

Let $\zeta^* = \max\left\{\zeta^{\sigma_1}\int_0^1 \delta^{-1}(s)\tau(s)f(s,\delta(s)G(s,s),\delta(s)G^{-1}(s,s))\mathrm{d}s, 1\right\}$. Then,

$$(\mathrm{Qu})(t) \le \zeta^* G(t,t), \quad \forall t \in I^*. \tag{39}$$

Hence, $Q(P) \subset P$. By Ascoli-Arzela's theorem, one can prove that $Q$: $P \longrightarrow P$ is completely continuous.

Second, we prove that there exists a constant $r > 0$, such that

$$\|\mathrm{Qu}\| \le \|u\|, \quad \forall u \in P \cap \partial\Omega_r, \tag{40}$$

where $\Omega_r = \{u \in E: \|u\| < r\}$ and $\partial\Omega_r = \{u \in E: \|u\| = r\}$.

For $u \in P$ with $\|u\| \le 1$, we have $u(t) \le \|u\| \le 1$ for any $t \in I^*$. Hence, by $(A3)$, we have

$$\begin{aligned}f\left(t,\delta(t)u(t),\delta(t)u^{-1}(t)\right) &\le u^{\sigma_2}(t)f(t,\delta(t),\delta(t)) \\ &\le \|u\|^{\sigma_2}f(t,\delta_M,\delta_m), \quad \forall t \in I^*.\end{aligned} \tag{41}$$

So,

$$\begin{aligned}\|Qu\| &\le \max_{t\in I^*}\int_0^1 G(t,s)\delta^{-1}(s)\tau(s)\|u\|^{\sigma_2}f(t,\delta_M,\delta_m)\mathrm{d}s \\ &\le \beta^*\|\tau\|_p\|u\|^{\sigma_2}\int_0^1 \delta^{-1}(s)f(t,\delta_M,\delta_m)\mathrm{d}s = A\|u\|^{\sigma_2},\end{aligned} \tag{42}$$

where $A = \beta^*\|\tau\|_p\int_0^1 \delta^{-1}(s)f(t,\delta_M,\delta_m)\mathrm{d}s$.

If $A > 1$, set $r = (1\ A)^{1/(\sigma_2 - 1)} < 1$; when $u \in P \cap \partial\Omega_r$, we have

$$\|\mathrm{Qu}\| \le A\|u\|^{\sigma_2} = A^{1-(\sigma_2/(\sigma_2-1))} = r = \|u\|. \tag{43}$$

If $A \le 1$, set $r = 1$; when $u \in P \cap \partial\Omega_r$, we have

$$\|\mathrm{Qu}\| \le A\|u\|^{\sigma_2} = A \le 1 = r = \|u\|. \tag{44}$$

Third, we prove that there exists a constant $R > r$, such that

$$\|\mathrm{Qu}\| \ge \|u\|, \forall u \in P \cap \partial\Omega_R, \tag{45}$$

where $\Omega_R = \{u \in E: \|u\| < R\}$ and $\partial\Omega_R = \{u \in E: \|u\| = R\}$.

For $u \in P$ with $u(t) \ge 1$ for $t \in I^*$, we have

$$\delta(t)u(t) \ge \delta_m, \delta(t)u^{-1}(t) \le \delta(t) \le \delta_M. \tag{46}$$

It follows that

$$\begin{aligned}f\left(t,\delta(t)u(t),\delta(t)u^{-1}(t)\right) &\ge u^{\sigma_2}(t)f(t,\delta(t),\delta(t)) \\ &\ge q^{\sigma_2}(t)\|u\|^{\sigma_2}f(t,\delta_m,\delta_M), \quad \forall t \in I^*.\end{aligned} \tag{47}$$

Hence, by the definition of operator $Q$ and cone $P$, for any $t \in I^*$, we have

$$\begin{aligned}(\mathrm{Qu})(t) &\ge \alpha^*\xi\int_0^1 \delta^{-1}(s)q^{\sigma_2}(s)\|u\|^{\sigma_2}f(s,\delta(s),\delta(s))\mathrm{d}s \\ &\ge \alpha^*\xi\|u\|^{\sigma_2}\int_\nu^{1-\nu}\delta^{-1}(s)q^{\sigma_2}(s)f(s,\delta(s),\delta(s))\mathrm{d}s \\ &\ge \|u\|^{\sigma_2}\alpha^*\xi\eta^{\sigma_2}\int_\nu^{1-\nu}\delta^{-1}(s)f(s,\delta_m,\delta_M)\mathrm{d}s = B\|u\|^{\sigma_2},\end{aligned} \tag{48}$$

where $B = \alpha^*\xi\eta^{\sigma_2}\int_\nu^{1-\nu}\delta^{-1}(s)f(s,\delta_m,\delta_M)\mathrm{d}s$ and $\eta = \min_{t\in I_0} q(t)$.

If $B < 1$, setting $R = (1/B)^{1/(\sigma_2 - 1)}$, $R > 1 \ge r$. For $u \in P \cap \partial\Omega_R$, we have

$$\|\mathrm{Qu}\| \ge (\mathrm{Qu})(t) \ge B\|u\|^{\sigma_2} = B^{1-(\sigma_2/\sigma_2-1)} = R = \|u\|. \tag{49}$$

If $B \ge 1$, setting $R = B + 1$, $R > 1 \ge r$. For $u \in P \cap \partial\Omega_R$, we have

$$\|\mathrm{Qu}\| \ge (\mathrm{Qu})(t) \ge B\|u\|^{\sigma_2} \ge B\|u\| \ge \|u\|. \tag{50}$$

By Lemma 4, $Q$ has at least one fixed point $u^* \in P \cap (\overline{\Omega}_R \setminus \Omega_r)$ satisfying $0 < r \le \|u^*\| \le R$. Hence, $\min_{t\in I_0} u^*(t) \ge \eta\|u^*\| > 0$, and it is a positive solution of the BVP (2).

Finally, we prove $u^* \in C^1[0,1]$. Since $u^* \in P$, there exists a constant $\zeta_1 \ge 1$, such that $u^*(s) \le \zeta_1 G(s,s)$ for $s \in I^*$. Then, by Remark 1, one has

$$\begin{aligned}
\int_0^1 \left|(u^*)''(s)\right| \mathrm{d}s &= \int_0^1 \delta^{-1}(s)\tau(s) f\left(s, \delta(s)u^*(s), \delta(s)(u^*)^{-1}(s)\right)\mathrm{d}s \\
&\leq \int_0^1 \delta^{-1}(s)\tau(s) f\left(s, \zeta_1\delta(s)G(s,s), \frac{1}{\zeta_1}\delta(s)G^{-1}(s,s)\right)\mathrm{d}s \\
&\leq (\zeta_1)^{\sigma_1}\int_0^1 \delta^{-1}(s)\tau(s) f\left(s, \delta(s)G(s,s), \delta(s)G^{-1}(s,s)\right)\mathrm{d}s < +\infty.
\end{aligned} \tag{51}$$

Hence, $u^*$ is absolutely integrable on $I^*$. So, $(u^*)'(0^+)$ and $(u^*)'(1^-)$ exist and are finite. Then, $u^* \in C^1[0,1]$, and it is a positive solution of the BVP (2). By Lemma 1, $x^*(t) = \delta(t)u^*(t), \forall t \in I^*$ belongs to $C^1[0,1]$, and it is a positive solution of the BVP (1). □

**Theorem 2.** *Assume that $\tau$ and $f$ satisfy the conditions $(A1)$, $(A2)$, $(A4)$, and $(A5) f(t,\delta_m,\delta_M) > 0, t \in I$, and there exist constants $0 < \sigma_3 \leq \sigma_4 < 1$, such that for every $\ell \in (0,1]$,*

$$\ell^{\sigma_4} f(t,x,y) \leq f\left(t, \ell x, \ell^{-1}y\right) \leq \ell^{\sigma_3} f(t,x,y), \tag{52}$$

*for any $t \in I, x, y \in (0,+\infty)$.*

Then, the BVP (1) has at least one positive solution $x \in C^1[0,1]$ if and only if (23) holds.

*Proof.* If the condition $(A3)$ is replaced by $(A5)$, we can also obtain the conclusion of Lemma 3. The proof of this theorem is the same as the one of Theorem 1. So, we omit the details here. □

*Remark 4.* Condition (52) is equivalent to

$$\ell^{\sigma_3} f(t,x,y) \leq f\left(t, \ell x, \ell^{-1}y\right) \leq \ell^{\sigma_4} f(t,x,y), \quad \ell \geq 1. \tag{53}$$

If $f(t,x,y)$ satisfies $(A5)$, then $f(t,x,y)$ is nondecreasing with respect to $x$ and nonincreasing with respect to $y$ for every $t \in I$.

*Remark 5.* In [10], the authors assumed that $f(t,x)$ must be nondecreasing with respect to $x$. In the present work, we assume that $f(t,x,y)$ is nondecreasing with respect to $x$ and nonincreasing with respect to $y$. But $f(t,x,y)$ is not monotonous with respect to both $x$ and $y$. Therefore, our results extend the ones of [10].

*Remark 6.* If $\tau \in L^\infty[0,1]$, the results in Theorems 1 and 2 are still true.

## 4. Examples

*Example 1.* Consider the second-order impulsive boundary value problem (IBVP):

$$\begin{cases}
-x''(t) = (3 + \sin t)\left(e^{t^2}x^3 + e^t x^{-3}\right), & t \in (0,1), t \neq t_k, \\
\Delta x|_{t=t_k} = x(t_k), \quad k = 1,2,\ldots,n, \\
\alpha x(0) - \beta x'(0) = 0, \gamma x(1) + \omega x'(1) = 0,
\end{cases} \tag{54}$$

where $\alpha, \beta, \gamma, \omega \geq 0$ satisfy $(\alpha+\beta)\gamma + \alpha\omega > 0$.

**Theorem 3.** *The IBVP (7) has at least one positive solution $x \in C^1[0,1]$ if and only if*

$$0 < \int_0^1 (3 + \sin s) f\left(s, 2^n G(s,s), G^{-1}(s,s)\right)\mathrm{d}s < +\infty, \tag{55}$$

*where*

$$G(s,s) = \frac{(\gamma + \omega - \gamma s)(\beta + \alpha s)}{(\alpha+\beta)\gamma + \alpha\omega}, \quad 0 \leq s \leq 1. \tag{56}$$

*Proof.* Let $f(t,x,y) = e^{t^2}x^3 + e^t y^{-3}$. Obviously, $f(t,x,x) = e^{t^2}x^3 + e^t x^{-3}$ and $\tau(t) = 3 + \sin t$. Then, the condition $(A1)$ holds with $\xi = 2$. Since $\delta_k = 1$, it follows that the condition $(A2)$ holds with $\delta(t) = \Pi_{0<t_k<t} 2$ and $\delta_m = 1, \delta_M = 2^n$. Then, $f(t,\delta_m,\delta_M) = e^{t^2} + 2^{-3n}e^t > 0, \forall t \in (0,1)$, and for $\ell \in (0,1]$, we have

$$\begin{aligned}
f\left(t, \ell x, \ell^{-1}y\right) &= e^{t^2}(\ell x)^3 + e^t\left(\ell^{-1}y\right)^{-3} \\
&= \ell^3 e^{t^2}x^3 + \ell^4 e^t y^{-3},
\end{aligned} \tag{57}$$

which implies that

$$\ell^5 f(t,x,y) \leq f\left(t, \ell x, \ell^{-1}y\right) \leq \ell^2 f(t,x,y), \quad \forall t \in (0,1). \tag{58}$$

Clearly, $\int_0^1 2^{3n}e^{t^2} + e^t \mathrm{d}t < +\infty$. Hence, the conditions $(A3)$ and $(A4)$ are satisfied. Therefore, by Theorem 1, the

IBVP (7) has at least one positive solution $x \in C^1[0,1]$ if and only if (55) holds. □

*Example 2.* Consider the second-order impulsive boundary value problem (IBVP):

$$\begin{cases} -x''(t) = \left(2-\cos t^2\right)\left(t^2 x^{1/3} + t^3 x^{-(1/3)}\right), & t \in (0,1), t \neq t_k, \\ \Delta x|_{t=t_k} = x(t_k), & k = 1,2,\ldots,n, \\ \alpha x(0) - \beta x'(0) = 0, \gamma x(1) + \omega x'(1) = 0, \end{cases} \tag{59}$$

where $\alpha, \beta, \gamma, \omega \geq 0$ satisfy $(\alpha+\beta)\gamma + \alpha\omega > 0$.

**Theorem 4.** *The IBVP (10) has at least one positive solution $x \in C^1[0,1]$ if and only if*

$$0 < \int_0^1 \left(2-\cos s^2\right) f\left(s, 2^n G(s,s), G^{-1}(s,s)\right) ds < +\infty, \tag{60}$$

*where $G(s,s)$ is given in (56).*

*Proof.* Let $f(t,x,y) = t^2 x^{1/3} + t^3 y^{-1/3}$. Obviously, $f(t,x,x) = t^2 x^{1/3} + t^3 x^{-(1/3)}$ and $\tau(t) = 2-\cos t^2$. Then, for $\ell \in (0,1]$, we have

$$\begin{aligned} f\left(t, \ell x, \ell^{-1} y\right) &= t^2 (\ell x)^{1/3} + t^3 \left(\ell^{-1} y\right)^{-1/3} \\ &= \ell^{1/3} t^2 x^{1/3} + \ell^{1/4} t^3 y^{-1/3}, \end{aligned} \tag{61}$$

which implies that

$$\ell^{1/2} f(t,x,y) \leq f\left(t, \ell x, \ell^{-1} y\right) \leq \ell^{1/5} f(t,x,y), \quad \forall t \in (0,1). \tag{62}$$

The remain proof is similar to the one of Theorem 3, and we omit the details here. □

*Remark 7.* If $f(t,x,x) = t^2 x^{1/3} + t^3 x^{-1/3}$ or $f(t,x,x) = t^2 x^{1/3} + t^3 x^{-1/3}$, it is clear that $F(t,x) = f(t,x,x)$ does not have monotonicity with respect to both $x$. Hence, the results in [10] are not valid to Examples 1 and 2.

## 5. Conclusions

This study deals with the existence of $C^1$-positive solutions for a class of second-order impulsive differential equations. By using the fixed-point theorem of cone mapping, we establish the necessary and sufficient conditions for the existence of $C^1$-positive solutions of the BVP (1) Two examples are given at last to illustrate the application of the obtained abstract results.

## Data Availability

The data used to support the findings of the study are available from the corresponding author upon request.

## Conflicts of Interest

The author declares that there are no conflicts of interest.

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