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Research Article

Novel Evaluation of the Fractional Acoustic Wave Model with the Exponential-Decay Kernel

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This study employs a newly developed methodology called the variational homotopy perturbation transformation method to study fractional acoustic wave equations. The motivation for this study is to extend the variational homotopy perturbation technique to the variational homotopy perturbation transformation technique in the sense of the Yang–Caputo–Fabrizio operator. The suggested method demonstrated a straightforward and accurate technique for investigating fractional-order partial differential equations. The technique's validity is demonstrated through the use of several illustrative instances. The obtained answers were found to be extremely near to the precise solutions. Additionally, the proposed strategy achieves the best degree of accuracy. Indeed, the current technique can be seen as one of the analytic strategies for solving nonlinear fractional partial differential equations compared to other analytical techniques.

1. Introduction

Scientists, mathematicians, and engineers have recently been interested in fractional differential equations (FDEs) and fractional calculus. Many significant implementations were assessed in a variety of science and engineering disciplines, including viscoelasticity, material engineering, dynamics physics, electrochemistry, and electromagnetics; fractional partial differential equations (FPDEs) [1] are used to explain all of them. Analytical ways to solve FDEs are gaining traction. FDEs cannot be accurately answered using any method. Approximate methods must be developed utilising series solution or linearization techniques [2], followed by adequate system solvers [3–5] and numerical discretization [6–8].

Nonlinear phenomena may be found in various engineering and scientific domains, including chemical kinetics, solid-state physics, fluid physics, nonlinear spectroscopy, computational biology, thermodynamics, quantum mechanics, etc. Many nonlinear higher-order PDEs define the idea of nonlinearity [9, 10].

The following fractional long-wave equation is investigated with the variational homotopy perturbation transform method.

$$\frac{\partial^\sigma v}{\partial \eta^\sigma} + \frac{1}{2}\frac{\partial v^2}{\partial \wp} - \frac{\partial}{\partial \eta}\left(\frac{\partial^2 v}{\partial \wp^2}\right) = 0, \quad 0 < \wp \le 1, 0 < \sigma \le 1, \eta > 0, \tag{1}$$

with initial condition

$$v(\wp, 0) = \wp,$$

$$\frac{\partial^\sigma v}{\partial \eta^\sigma} + \frac{\partial v}{\partial \wp} + v\frac{\partial v}{\partial \eta} - \frac{\partial}{\partial \eta}\left(\frac{\partial^2 v}{\partial \wp^2}\right) = 0, \quad 0 < \wp \le 1, 0 < \sigma \le 1, \eta > 0, \tag{2}$$

with initial condition

$$v(\wp, 0) = 3\alpha \sec h^2(\sigma\wp), \quad \alpha > 0, \sigma = \frac{1}{2}\sqrt{\frac{\alpha}{1+\alpha}},$$

$$\frac{\partial^\sigma v}{\partial \eta^\sigma} + \frac{\partial v}{\partial \wp} - 2\frac{\partial}{\partial \eta}\left(\frac{\partial^2 v}{\partial \wp^2}\right) = 0, \quad 0 < \wp \leq 1, 0 < \sigma \leq 1, \eta > 0, \tag{3}$$

with initial condition

$$v(\wp, 0) = e^{-\wp}, \tag{4}$$

and

$$\frac{\partial^\sigma v(\wp.\eta)}{\partial \eta^\sigma} + \frac{\partial^4 v(\wp.\eta)}{\partial \wp^4} = 0, \quad 0 < \wp \leq 1, 0 < \sigma \leq 1, \eta > 0, \tag{5}$$

with initial condition

$$v(\wp, 0) = \sin \wp. \tag{6}$$

The nonlinear fractional-order regularised long-wave equation (RLWE) is called equation (1); equation (2) is called the nonlinear fractional general RLWE; and equations (3) and (5) are called linear fractional RLWEs [11].

The regularised long-wave (RLW) equation was also recognized by the Benjamin–Bona–Mahony equation (BBME). This is an upgraded form of the Korteweg–de Vries equation (KdV), representing low amplitude long surface gravity waves propagating unidirectionally in two dimensions. Wave propagation in elastic rods with longitudinal dispersion, stress waves in compressed gas bubble mixtures, ion-acoustic plasma waves, rotational tube flows, and plasma magneto-hydrodynamic waves are examples of RLW equations in action. The RLW equations are suitable models for many major physical structures in applied engineering and physics. It also creates a lot of liquid flow concerns where diffusion is a big deal, regardless of shocks or viscosity. Any dissipation-related nonlinear wave diffusion issue may be modelled using it. Depending on the issue modelling [12, 13], this dissipation might arise in heat conduction, chemical reaction, viscosity, thermal radiation, mass diffusion, or other causes.

The RLW equation is a collection of nonlinear growth models that provide great models for anticipating natural occurrences. The method was first used to describe the undular bore behaviour [14]. It was also discovered via the study of water and ion-acoustic plasma waves. In [15], under boundary circumstances and constrained beginning, an analytic result for the RLW equations was discovered. Many important engineering phenomena and ocean research, for example, long-wave and tiny frequency shallow-water waves, are defined by fractional RLW equations. Several researchers in the field of shallow liquid sea waves are interested in the nonlinear wave described by the fractional equations of RLW. The fractional RLW equations were used to represent nonlinear waves in the sea. Indeed, the tsunami's massive surface waves are characterized by fractional RLW equations. Large internal waves in the core of the ocean generated by temperature differences that might destroy marine ships can be expressed as fractional RLW equations using the present incredibly effective approach [16–18].

In recent years, numerous scientists and researchers have employed analytic approaches to handle issues like homotopy perturbation. The Adomian decomposition method (ADM) [19, 20], Sumudu transform method [11], optimal homotopy perturbation method [21], least-squares method [22], He's homotopy perturbation technique [23], and homotopy perturbation method and variational iteration method [24] have flaws such as the determination of the Lagrange multiplier, the calculation of Adomian polynomials, a large number of calculations, and divergent results. Consequently, VHPTM, a revised analytical technique for solving fractional-order differential equations, was created. VHPTM results from three well-known approaches being combined: variational iteration, homotopy perturbation, and Laplace transform [25–30]. The Lagrange multiplier is used to restrict the number of times an integral operator is implemented in a row and the computing cost. It continues to retain a better level of precision. VHPTM [31–34] has an outstanding scheme and incorporates all of the positive aspects of HPM and VIM.

At last, to construct the homotopy perturbation technique, He's polynomials were employed in the corrective fractional formula. The suggested approach is applied without discretization or transformation, and it is shown to be free of rounding errors. Normally, the variable separable technique requires both starting and boundary points; however, the current approach requires initial circumstances to produce an analytical solution. The proposed technique has the distinct benefit of not requiring the usage of Adomian polynomials, which are needed by the Adomian decomposition method. The proposed approach is shown to yield a solution in a sequence of quick convergence that may lead to a closed solution [35].

2. Basic Definitions

Definition 1. The fractional Caputo–Fabrizio derivative is defined as [36]

$$^{\mathrm{CF}}D_\eta^\sigma[g(\eta)] = \frac{N(\sigma)}{1-\sigma}\int_0^\eta g'(\varrho)K(\eta,\varrho)\mathrm{d}\varrho, \quad n-1 < \sigma \leq n. \tag{7}$$

$N(\sigma)$ is the function of normalisation with $N(0) = N(1) = 1$.

$$^{\mathrm{CF}}D_\eta^\sigma[g(\eta)] = \frac{N(\sigma)}{1-\sigma}\int_0^\eta [g(\eta) - g(\varrho)]K(\eta,\varrho)\mathrm{d}\varrho. \tag{8}$$

Definition 2. For $N(\sigma) = 1$, the following solution represents the Laplace transformation of the Caputo–Fabrizio operator [36]:

$$L\left[^{\mathrm{CF}}D_\eta^\sigma[g(\eta)]\right] = \frac{sL[g(\eta) - g(0)]}{s + \sigma(1-s)}. \tag{9}$$

Definition 3. The Caputo–Fabrizio fractional integral is defined as [36]

$$ {}^{\mathrm{CF}}I_{\eta}^{\sigma}[g(\eta)]t = n\frac{1-\sigma}{N(\sigma)}q\,gh(\eta)_{+}x\frac{\sigma}{N(\sigma)}7\int_{0}^{\eta} g(\varrho)\mathrm{d}\varrho,; \quad \eta \geq 0, \sigma \in (0,1\Big]. \tag{10} $$

Definition 4. The Yang transformation of $g(\eta)$ is given as [37]

$$ \mathscr{Y}[g(\eta)] = \chi(s) = \int_{0}^{\infty} g(\eta)e^{-\eta/s}\mathrm{d}\eta, \quad \eta > 0. \tag{11} $$

2.1. Remarks. The Yang transforms of many relevant terms are listed below.

$$ \begin{aligned} Y[1] &= s, \\ Y[\eta] &= s^{2}, \\ Y\left[\eta^{i}\right] &= \Gamma(i+1)s^{i+1}. \end{aligned} \tag{12} $$

Lemma 1. *(Yang–Laplace duality).*

Let the Laplace transformation of $g(\eta)$ be $F(s)$; then, $\chi(s) = F(1/s)$ [38].

Proof. From equation (11), we can achieve another type of the Yang transformation by putting $\eta/s = \zeta$ as

$$ L[g(\eta)] = \chi(s) = s\int_{0}^{\infty} g(s\zeta)e^{\zeta}\mathrm{d}\zeta, \quad \zeta > 0. \tag{13} $$

Since $L[g(\eta)] = F(s)$,

$$ F(s) = L[g(\eta)] = \int_{0}^{\infty} g(\eta)e^{-s\eta}\mathrm{d}\eta. \tag{14} $$

Put $\eta = \zeta/s$ in (14), and we have

$$ F(s) = \frac{1}{s}\int_{0}^{\infty} g\left(\frac{\zeta}{s}\right)e^{\zeta}\mathrm{d}\zeta. \tag{15} $$

Thus, from (13), we obtain

$$ F(s) = \chi\left(\frac{1}{s}\right). \tag{16} $$

Also, from (11) and (14), we get

$$ F\left(\frac{1}{s}\right) = \chi(s). \tag{17} $$

The connections (16) and (17) show the duality link among the Yang and Laplace transformation. □

Lemma 2. *Let $g(\eta)$ be a continuous function; then, the Caputo–Fabrizio–Yang transform derivative of $g(\eta)$ is given as [38]*

$$ Y[g(\eta)] = \frac{Y[g(\eta) - sg(0)]}{1+\sigma(s-1)}. \tag{18} $$

Proof. The Laplace transform of fractional Caputo–Fabrizio operator is defined as

$$ L[g(\eta)] = \frac{L[sg(\eta) - g(0)]}{s+\sigma(1-s)}. \tag{19} $$

Also, we have the link among Laplace and Yang properties, i.e., $\chi(s) = F(1/s)$. To achieve the necessaries solution, we put s by $1/s$ in (19), and we have

$$ \begin{aligned} Y[g(\eta)] &= \frac{1/sY[g(\eta) - g(0)]}{1/s+\sigma(1-1/s)}, \\ Y[g(\eta)] &= \frac{Y[g(\eta) - sg(0)]}{1+\sigma(s-1)}. \end{aligned} \tag{20} $$

The proof is completed. □

3. The General Implementation of the Technique

To show the basic idea of the current technique [31, 32], we are suggesting

$$ D_{\eta}^{\sigma}\upsilon(\wp,\eta) + \overline{L}\upsilon(\wp,\eta) + \mathrm{N}\upsilon(\wp,\eta) = h(\wp,\eta), \tag{21} $$

with initial condition

$$ \upsilon(\wp,0) = g(\wp). \tag{22} $$

Using Yang transformation of (21), we have

$$ Y\{\upsilon(\wp,\eta)\} - s\upsilon(\wp,0) = -Y\{\overline{L}\upsilon(\wp,\eta) + \mathrm{N}\upsilon(\wp,\eta) - h(\wp,\eta)\}, \tag{23} $$

where $\upsilon(s) = Y(\upsilon(\wp,\eta)) = \int_{0}^{\infty} e^{-s\eta}\upsilon(\eta)\mathrm{d}\eta$.

Using the variation iteration method,

$$ Y\left\{\upsilon_{j+1}(\wp,\eta)\right\} = Y\left\{\upsilon_{j}(\wp,\eta)\right\} + \lambda(s)\left[\frac{1}{(1+\sigma(s-1))}Y\left\{\upsilon_{\eta}(\wp,\eta) + Y\{\overline{L}\upsilon(\wp,\eta) + \mathrm{N}\upsilon(\wp,\eta) - h(\wp,\eta)\}\right\}\right]. \tag{24} $$

The Lagrange multiplier is $\lambda(s)$. Now we put $\lambda(s) = -(1+\sigma(s-1))$ [32].

Applying inverse Laplace of (24),

$$v_{j+1}(\wp,\eta) = v_j(\wp,\eta) - Y^{-1}\left[(1+\sigma(s-1))Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial v}{\partial \eta} + \bar{L}v_j(\wp,\eta) + \mathrm{N}v_j(\wp,\eta) - h(\wp,\eta)\right\}\right]. \quad (25)$$

The result may be expressed as a series in power of p, which is the underlying principle behind the homotopy perturbation approach:

$$v(\wp,\eta) = \sum_{j=0}^{\infty} p^j v_j(\wp,\eta) = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots, \quad (26)$$

where the nonlinear equation is as follows:

$$\mathrm{N}v(\wp,\eta) = \sum_{j=0}^{\infty} p^j \overline{H_j}(v). \quad (27)$$

He's polynomials are $\overline{H_j}$:

$$\overline{H_j}\left(v_0 + v_1 + \cdots + u_j\right) = \frac{1}{j!}\frac{\partial^j}{\partial p^j}\left[\mathrm{N}\left(\sum_{i=0}^{\infty} p^i v_i\right)\right]. \quad (28)$$

The variational homotopy perturbation transform method is applied to (28):

$$\sum_{j=0}^{\infty} p^j v_j(\wp,\eta) = \sum_{j=0}^{\infty} p^j v_j(\wp,\eta) + Y^{-1}\left[\lambda(s)Y\left\{\sum_{j=0}^{\infty} p^j \frac{\partial^\sigma v_j}{\partial \eta^\sigma}(\wp,s) + \sum_{j=0}^{\infty} p^j \bar{L}v_j(\wp,\eta) + \sum_{j=0}^{\infty} p^j \overline{H_j}(v) - h(\wp,\eta)\right\}\right]. \quad (29)$$

We obtain the present technique solution to the given equation by evaluating the coefficient of like power of p (28).

4. Numerical Problems

Problem 1. Consider the fractional nonlinear regularised long-wave equation

$$\frac{\partial^\sigma v}{\partial \eta^\sigma} + \frac{1}{2}\frac{\partial v^2}{\partial \wp} - \frac{\partial \partial^2 v}{\partial \eta\, \partial \wp^2} = 0, \quad 0 < \wp \le 1, 0 < \sigma \le 1, \eta > 0, \quad (30)$$

with the initial condition

$$v(\wp,0) = \wp. \quad (31)$$

The fractional partial differential equation described in (30) can be expressed as

$$\sum_{j=0}^{\infty} p^j v_{j+1}(\wp,\eta) = \sum_{j=0}^{\infty} p^j v_j(\wp,\eta) + Y^{-1}\left[\lambda(s)Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial v_j(\wp,\eta)}{\partial \eta} + \frac{1}{2}\frac{\partial v_j^2(\wp,\eta)}{\partial \wp} - \frac{\partial \partial^2 v_j(\wp,\eta)}{\partial \eta\, \partial \wp^2}\right\}\right]. \quad (32)$$

The Lagrange multiplier $\lambda(s) = -(1+\sigma(s-1))$. Applying variational homotopy perturbation transformation method and applying He's polynomials,

$$\begin{aligned}\sum_{j=0}^{\infty} p^j v_{j+1}(\wp,\eta) = \sum_{j=0}^{\infty} p^j v_j(\wp,\eta) - \sum_{j=0}^{\infty} p^j Y^{-1}\Bigg[(1+\sigma(s-1))Y\Bigg\{\frac{1}{(1+\sigma(s-1))}\left(\frac{\partial v_0}{\partial\eta} + p\frac{\partial v_1}{\partial\eta} + p^2\frac{\partial v_2}{\partial\eta} + \cdots\right)\\ + \frac{1}{2}\frac{\partial}{\partial\wp}\left\{v_0^2 + p(2v_0v_1) + p^2\left(2v_0v_2 + v_1^2\right) + \cdots\right\} - \left\{p^0\frac{\partial\partial^2 v_0}{\partial\eta\,\partial\wp^2} + p^1\frac{\partial\partial^2 v_1}{\partial\eta\,\partial\wp^2} + p^2\frac{\partial\partial^2 v_2}{\partial\eta\,\partial\wp^2} + \cdots\right\}\Bigg\}\Bigg].\end{aligned} \tag{33}$$

Evaluating p coefficient,

$$\begin{aligned}
v_0(\wp,\eta) &= \wp,\\
p^1 v_1(\wp,\eta) &= p^1 v_0(\wp,\eta) - p^1 Y^{-1}\left[(1+\sigma(s-1))Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial v_0(\wp,\eta)}{\partial\eta} + \frac{1}{2}\frac{\partial}{\partial\wp}v_0^2(\wp,\eta) - \frac{\partial\partial^2 v_0(\wp,\eta)}{\partial\eta\,\partial\wp^2}\right\}\right],\\
v_1(\wp,\eta) &= \wp - \wp(1+\sigma\eta-\sigma),\\
p^2 v_2(\wp,\eta) &= p^2 v_1(\wp,\eta) - p^2 Y^{-1}\left[(1+\sigma(s-1))Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial v_1(\wp,\eta)}{\partial\eta} + \frac{1}{2}\frac{\partial}{\partial\wp}(2v_0v_1) - \frac{\partial\partial^2 v_1}{\partial\eta\,\partial\wp^2}\right\}\right],\\
v_2(\wp,\eta) &= \wp - \wp(1+\sigma\eta-\sigma) + 2\wp\left((1-\sigma)2\sigma\eta + (1-\sigma)^2 + \frac{\sigma^2\eta^2}{2}\right),\\
p^3 v_3(\wp,\eta) &= p^3 v_2(\wp,\eta) - p^3 Y^{-1}\left[(1+\sigma(s-1))Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial v_1}{\partial\eta} + \frac{1}{2}\left(2v_0v_2 + v_1^2\right) - \frac{\partial\partial^2 v_2}{\partial\eta\,\partial\wp^2}\right\}\right],\\
v_3(\wp,\eta) &= \wp - \wp(1+\sigma\eta-\sigma) + 2\wp\left((1-\sigma)2\sigma\eta + (1-\sigma)^2 + \frac{\sigma^2\eta^2}{2}\right)\\
&\quad - \wp(2\sigma-1)\left(3\sigma\left(-2\sigma+1+\sigma^2\right)\eta + \frac{\sigma^3\eta^3}{6} - \frac{3\sigma^2(\sigma-1)\eta^2}{2} + 3\sigma^2 - 3\sigma + 1 - \sigma^3\right)\\
&\quad - 4\wp\left(3\sigma\left(-2\sigma+1+\sigma^2\right)\eta + \frac{\sigma^3\eta^3}{6} - \frac{3\sigma^2(\sigma-1)\eta^2}{2} + 3\sigma^2 - 3\sigma + 1 - \sigma^3\right).\\
&\ \ \vdots
\end{aligned} \tag{34}$$

The analytic series solution can be achieved as

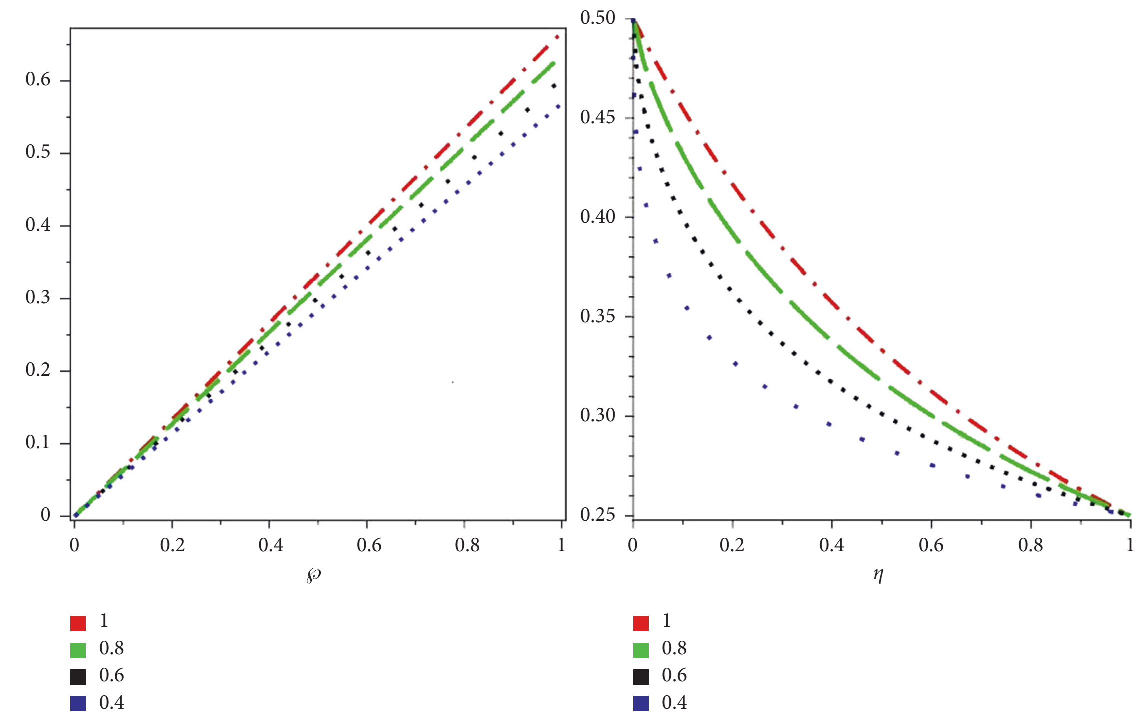

Figure 1: First figure show that the analytic result of various fractional-order of problem 1 and second graph with respect to time.

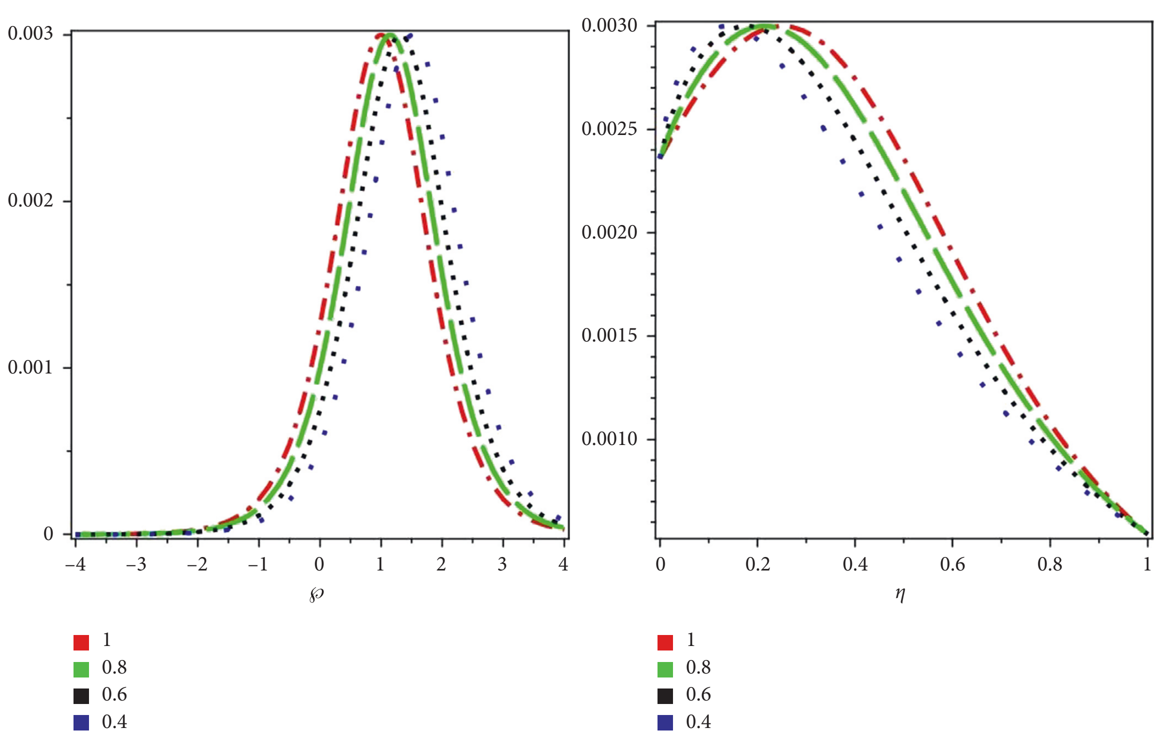

Figure 2: First figure show that the analytic result of various fractional-order of problem 2 and second graph with respect to time.

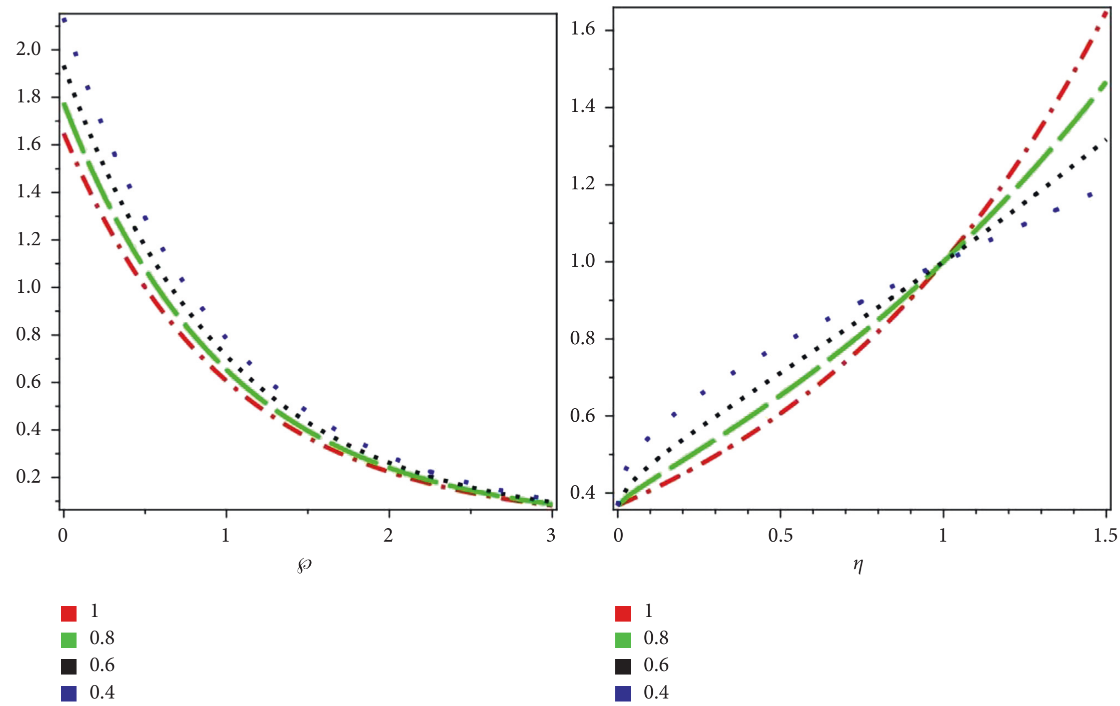

Figure 3: First figure show that the analytic result of various fractional-order of problem 3 and second graph with respect to time.

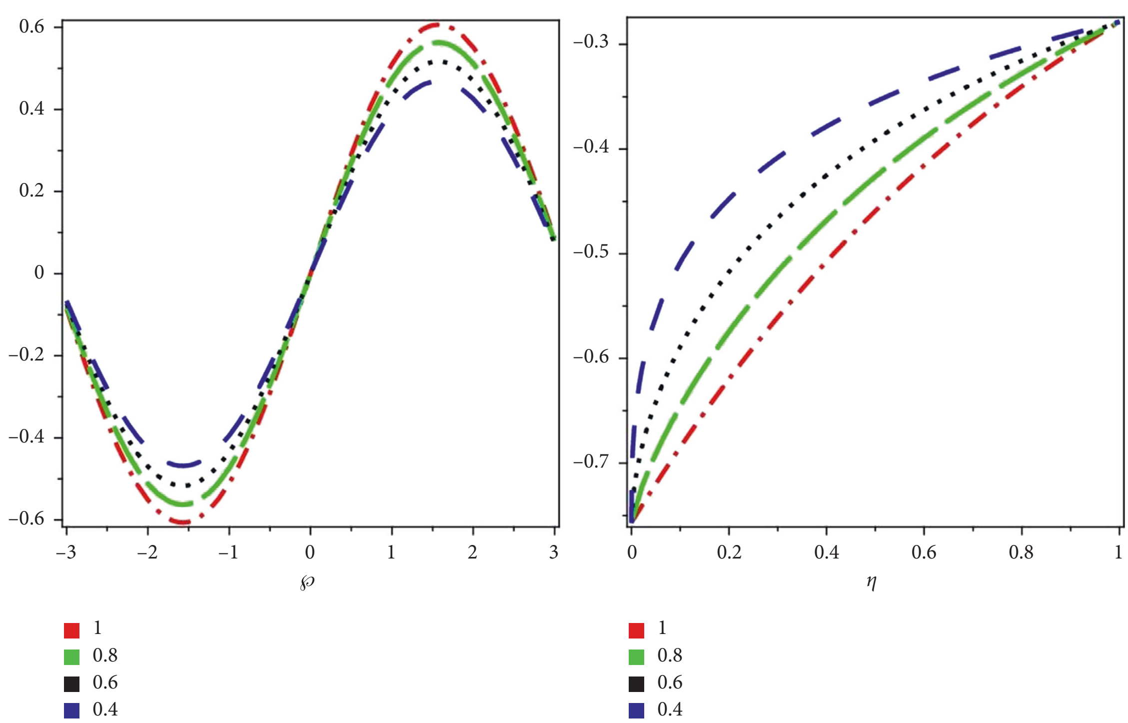

Figure 4: First figure show that the analytic result of various fractional-order of problem 4 and second graph with respect to time.

$$v(\wp,\eta) = \wp - \wp(1+\sigma\eta-\sigma) + 2\wp\left((1-\sigma)2\sigma\eta + (1-\sigma)^2 + \frac{\sigma^2\eta^2}{2}\right)$$
$$-\wp(2\sigma-1)\left(3\sigma\left(-2\sigma+1+\sigma^2\right)\eta + \frac{\sigma^3\eta^3}{6} - \frac{3\sigma^2(\sigma-1)\eta^2}{2} + 3\sigma^2 - 3\sigma + 1 - \sigma^3\right)$$
$$-4\wp\left(3\sigma\left(-2\sigma+1+\sigma^2\right)\eta + \frac{\sigma^3\eta^3}{6} - \frac{3\sigma^2(\sigma-1)\eta^2}{2} + 3\sigma^2 - 3\sigma + 1 - \sigma^3\right) + \cdots. \tag{35}$$

If $\sigma = 1$, the series type solution is given as

$$v(\wp,\eta) = \wp\left(1 - \eta + \eta^2 - \eta^3 + \cdots\right). \tag{36}$$

The exact result at $\sigma = 1$ is

$$v(\wp,\eta) = \frac{\wp}{1+\eta}. \tag{37}$$

Problem 2. Consider the fractional nonlinear regularised long-wave equation

$$\frac{\partial^\sigma v}{\partial\eta^\sigma} + \frac{\partial v}{\partial\wp} + v\frac{\partial v}{\partial\wp} - \frac{\partial\partial^2 v}{\partial\eta\,\partial\wp^2} = 0, \quad 0 < \wp \le 1, 0 < \sigma \le 1, \eta > 0, \tag{38}$$

with the initial condition

$$v(\wp,0) = 3\alpha \sec h^2(\sigma\wp), \quad \alpha > 0, \sigma = \frac{1}{2}\sqrt{\frac{\alpha}{1+\alpha}}. \tag{39}$$

The fractional partial differential equation described in (38) can be expressed as

$$\sum_{j=0}^{\infty} p^j v_{j+1}(\wp,\eta) = \sum_{j=0}^{\infty} p^j v_j(\wp,\eta) + p^j Y^{-1}$$
$$\left[\lambda(s)Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial v_j(\wp,\eta)}{\partial\eta} + \frac{\partial v_j(\wp,\eta)}{\partial\wp} + v_j(\wp,\eta)\frac{\partial v_j(\wp,\eta)}{\partial\wp} - \frac{\partial\partial^2 v_j(\wp,\eta)}{\partial\eta\,\partial\wp^2}\right\}\right]. \tag{40}$$

The Lagrange multiplier $\lambda(s) = -(1+\sigma(s-1))$. Applying variational homotopy perturbation transformation method and applying He's polynomials,

$$\sum_{j=0}^{\infty} p^j v_j(\wp,\eta) = \sum_{j=0}^{\infty} p^j v_j(\wp,\eta) - pY^{-1}\left[(1+\sigma(s-1))Y\left\{\frac{1}{(1+\sigma(s-1))}\left(\frac{\partial v_0}{\partial\eta} + p\frac{\partial v_1}{\partial\eta} + p^2\frac{\partial v_1}{\partial\eta} + \cdots\right)\right.\right.$$
$$+\left\{\frac{\partial v_0}{\partial\wp} + p^1\frac{\partial v_1}{\partial\wp} + p^2\frac{\partial v_2}{\partial\wp} + \cdots\right\} + \left\{v_0\frac{\partial v_0}{\partial\wp} + p\left(v_0\frac{\partial v_1}{\partial\wp} + v_1\frac{\partial v_0}{\partial\wp}\right) + p^2\left(v_0\frac{\partial v_2}{\partial\wp} + v_1\frac{\partial v_1}{\partial\wp} + v_2\frac{\partial v_0}{\partial\wp}\right) + \cdots\right\} \tag{41}$$
$$\left.\left.-\left\{\frac{\partial\partial^2 v_0}{\partial\eta\,\partial\wp^2} + p^1\frac{\partial\partial^2 v_1}{\partial\eta\,\partial\wp^2} + p^2\frac{\partial\partial^2 v_2}{\partial\eta\,\partial\wp^2} + \cdots\right\}\right\}\right].$$

Evaluating p coefficient,

$$
\begin{aligned}
&v_0(\wp,\eta) = 3\alpha \sec h^2(\sigma\wp),\\
&p^1 v_1(\wp,\eta) = p^1 v_0(\wp,\eta) - p^1 Y^{-1}\left[(1+\sigma(s-1))Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial v_0}{\partial\eta} + \frac{\partial v_0}{\partial\wp} + v_0\frac{\partial v_0}{\partial\wp} - \frac{\partial\partial^2 v_0}{\partial\eta\,\partial\wp^2}\right\}\right],\\
&v_1(\wp,\eta) = 3\alpha \sec h^2(\sigma\wp) + 3\alpha\sigma\{1+6\alpha\sigma+\cosh(2\sigma\wp)\}\sec h^4(\sigma\wp)\tanh(\sigma\wp)(1+\sigma\eta-\sigma),\\
&p^2 v_2(\wp,\eta) = p^2 v_1(\wp,\eta) - p^2 Y^{-1}\left[(1+\sigma(s-1))Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial v_1}{\partial\eta} + \frac{\partial v_1}{\partial\wp} + v_0\frac{\partial v_1}{\partial\wp} + v_1\frac{\partial v_0}{\partial\wp} - \frac{\partial\partial^2 v_1}{\partial\eta\,\partial\wp^2}\right\}\right],\\
&v_2(\wp,\eta) = 3\alpha \sec h^2(\sigma\wp) + 3\alpha\sigma\{1+6\alpha\sigma+\cosh(2\sigma\wp)\}\sec h^4(\sigma\wp)\tanh(\sigma\wp)(1+\sigma\eta-\sigma)\\
&\qquad -\frac{3}{32}\alpha\sigma^2\left\{-8-96\alpha-576\alpha^2+3\left(-3-16\alpha+144\alpha^2\right)\cosh(2\sigma\wp)+48\alpha\cosh(4\sigma\wp)+\cosh(6\sigma\wp)\right\}\\
&\qquad \sec h^8(\sigma\wp)\left((1-\sigma)2\sigma\eta + (1-\sigma)^2 + \frac{\sigma^2\eta^2}{2}\right).\\
&\qquad \vdots
\end{aligned}
\tag{42}
$$

The analytic series solution can be achieved as

$$
\begin{aligned}
v(\wp,\eta) &= 3\alpha \sec h^2(\sigma\wp) + 3\alpha\sigma\{1+6\alpha\sigma+\cosh(2\sigma\wp)\}\sec h^4(\sigma\wp)\tanh(\sigma\wp)(1+\sigma\eta-\sigma)\\
&-\frac{3}{32}\alpha\sigma^2\left\{-8-96\alpha-576\alpha^2+3\left(-3-16\alpha+144\alpha^2\right)\cosh(2\sigma\wp)+48\alpha\cosh(4\sigma\wp)+\cosh(6\sigma\wp)\right\}\\
&\sec h^8(\sigma\wp)\left((1-\sigma)2\sigma\eta+(1-\sigma)^2+\frac{\sigma^2\eta^2}{2}\right)+\frac{1}{32}\alpha\sigma^3\\
&\left\{-85-1416\alpha-8496\sigma^2-2937\sigma^3+4\left(-31-432\alpha-1584\alpha^2+3456\alpha^3\right)\right.\\
&\cosh(2\sigma\wp)-4\left(11+54\alpha-540\alpha^2\right)\cosh(4\sigma\wp)-4\cosh(6\sigma\wp)+96\alpha\cosh(6\sigma\wp)+\cosh(8\sigma\wp)\\
&\left.+\sec h^8(\sigma\wp)\tanh(\sigma\wp)\right\}\left(3\sigma\left(-2\sigma+1+\sigma^2\right)\eta+\frac{\sigma^3\eta^3}{6}-\frac{3\sigma^2(\sigma-1)\eta^2}{2}+3\sigma^2-3\sigma+1-\sigma^3\right)+\cdots.
\end{aligned}
\tag{43}
$$

The exact result at $\sigma = 1$ is

$$v(\wp,\eta) = 3\alpha \sec h^2(\sigma(\wp-(1+\alpha)\eta)). \tag{44}$$

Problem 3. Consider the fractional linear regularised long-wave equation

$$\frac{\partial^\sigma v}{\partial\eta^\sigma} + \frac{\partial v}{\partial\wp} - 2\frac{\partial\partial^2 v}{\partial\eta\,\partial\wp^2} = 0, \quad 0<\wp\le 1, 0<\sigma\le 1, \eta>0, \tag{45}$$

with the initial condition

$$v(\wp,0) = e^{-\wp}. \tag{46}$$

The fractional partial differential equation described in (45) can be expressed as

$$\sum_{j=0}^{\infty} p^j v_{j+1}(\wp,\eta) = \sum_{j=0}^{\infty} p^j v_j(\wp,\eta) + p^j Y^{-1}\left[\lambda(s) Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial^\sigma v_j}{\partial \eta^\sigma}(\wp,\eta) + \frac{\partial v_j}{\partial \wp} - 2\frac{\partial \partial^2 v_j}{\partial \eta\, \partial \wp^2}\right\}\right]. \tag{47}$$

The Lagrange multiplier $\lambda(s) = -(1+\sigma(s-1))$. Applying variational homotopy perturbation transformation method and applying He's polynomials,

$$\begin{aligned}\sum_{j=0}^{\infty} p^j v_j(\wp,\eta) = \sum_{j=0}^{\infty} p^j v_j(\wp,\eta) - pY^{-1}\Bigg[(1+\sigma(s-1))Y\Bigg\{&\frac{1}{(1+\sigma(s-1))}\left(\frac{\partial v_0}{\partial \eta} + p\frac{\partial v_1}{\partial \eta} + p^2\frac{\partial v_2}{\partial \eta} + \cdots\right) \\ &+\frac{\partial}{\partial \wp}\left\{v_0 + pv_1 + p^2 v_2 + \cdots\right\} - 2\left\{\frac{\partial\partial^2 v_0}{\partial\eta\,\partial\wp^2} + p^1\frac{\partial\partial^2 v_1}{\partial\eta\,\partial\wp^2} + p^2\frac{\partial\partial^2 v_2}{\partial\eta\,\partial\wp^2} + \cdots\right\}\Bigg\}\Bigg].\end{aligned} \tag{48}$$

Evaluating p coefficient,

$$\begin{aligned}
v_0(\wp,\eta) &= e^{-\wp}, \\
p^1 v_1(\wp,\eta) &= p^1 v_0(\wp,\eta) - p^1 Y^{-1}\left[(1+\sigma(s-1))Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial v_0}{\partial \eta}(\wp,\eta) + \frac{\partial v_0}{\partial \wp} - 2\frac{\partial\partial^2 v_0}{\partial\eta\,\partial\wp^2}\right\}\right], \\
v_1(\wp,\eta) &= e^{-\wp} + e^{-\wp}(1+\sigma\eta-\sigma), \\
p^2 v_2(\wp,\eta) &= p^2 v_1(\wp,\eta) - p^2 Y^{-1}\left[(1+\sigma(s-1))Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial v_1}{\partial \eta}(\wp,\eta) + \frac{\partial v_1}{\partial \wp} - 2\frac{\partial\partial^2 v_1}{\partial\eta\,\partial\wp^2}\right\}\right], \\
v_2(\wp,\eta) &= e^{-\wp} + e^{-\wp}(1+\sigma\eta-\sigma) + e^{-\wp}\left((1-\sigma)2\sigma\eta + (1-\sigma)^2 + \frac{\sigma^2\eta^2}{2}\right), \\
p^3 v_3(\wp,\eta) &= p^3 v_2(\wp,\eta) - p^3 Y^{-1}\left[-(1+\sigma(s-1))Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial v_2}{\partial \eta}(\wp,\eta) + \frac{\partial v_2}{\partial \wp} - 2\frac{\partial\partial^2 v_2}{\partial\eta\,\partial\wp^2}\right\}\right], \\
v_3(\wp,\eta) &= e^{-\wp} + e^{-\wp}(1+\sigma\eta-\sigma) + e^{-\wp}\left((1-\sigma)2\sigma\eta + (1-\sigma)^2 + \frac{\sigma^2\eta^2}{2}\right) \\
&\quad + e^{-\wp}\left(3\sigma\left(-2\sigma+1+\sigma^2\right)\eta + \frac{\sigma^3\eta^3}{6} - \frac{3\sigma^2(\sigma-1)\eta^2}{2} + 3\sigma^2 - 3\sigma + 1 - \sigma^3\right). \\
&\ \ \vdots
\end{aligned} \tag{49}$$

The analytic series solution can be achieved as

$$v(\wp,\eta) = e^{-\wp} + e^{-\wp}(1+\sigma\eta-\sigma) + e^{-\wp}\left((1-\sigma)2\sigma\eta + (1-\sigma)^2 + \frac{\sigma^2\eta^2}{2}\right) + e^{-\wp}\left(3\sigma\left(-2\sigma+1+\sigma^2\right)\eta + \frac{\sigma^3\eta^3}{6} - \frac{3\sigma^2(\sigma-1)\eta^2}{2} + 3\sigma^2 - 3\sigma + 1 - \sigma^3\right) + \cdots. \quad (50)$$

If $\sigma = 1$, the series type solution is given as

$$v(\wp,\eta) = e^{-\wp}\left(1+\eta+\frac{\eta^2}{2!}+\frac{\eta^3}{3!}+\cdots\right). \quad (51)$$

The exact result at $\sigma = 1$ is

$$v(\wp,\eta) = e^{\eta-\wp}. \quad (52)$$

Problem 4. Consider the fractional linear regularised long-wave equation

$$\frac{\partial^\sigma v(\wp.\eta)}{\partial\eta^\sigma} + \frac{\partial^4 v(\wp.\eta)}{\partial\wp^4} = 0, \quad 0<\wp\le 1, 0<\sigma\le 1, \eta>0, \quad (53)$$

with the initial condition

$$v(\wp,0) = \sin\wp. \quad (54)$$

The fractional partial differential equation described in (53) can be expressed as

$$\sum_{j=0}^{\infty} p^j v_{j+1}(\wp,\eta) = \sum_{j=0}^{\infty} p^j v_j(\wp,\eta) + \sum_{j=0}^{\infty} p^j Y^{-1}\left[\lambda(s)Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial^\sigma v_j(\wp\cdot\eta)}{\partial\eta^\sigma}(\wp,\eta) + \frac{\partial^4 v_j(\wp\cdot\eta)}{\partial\wp^4}\right\}\right]. \quad (55)$$

The Lagrange multiplier $\lambda(s) = -(1+\sigma(s-1))$. Applying variational homotopy perturbation transformation method and applying He's polynomials,

$$\sum_{j=0}^{\infty} p^j v_j(\wp,\eta) = \sum_{j=0}^{\infty} p^j v_j(\wp,\eta) - pY^{-1}\left[(1+\sigma(s-1))Y\left\{\frac{1}{(1+\sigma(s-1))}\left(\frac{\partial v_0}{\partial\eta} + p\frac{\partial v_1}{\partial\eta} + p^2\frac{\partial v_2}{\partial\eta} + \cdots\right) + \frac{\partial^4}{\partial\wp^4}\left\{v_0 + pv_1 + p^2 v_2 + \cdots\right\}\right\}\right]. \quad (56)$$

Evaluating p coefficient,

$$
\begin{aligned}
v_0(\wp,\eta) &= \sin\wp,\\
p^1 v_1(\wp,\eta) &= p^1 v_0(\wp,\eta) - p^1 Y^{-1}\left[(1+\sigma(s-1))Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial v_0(\wp.\eta)}{\partial\eta}(\wp,\eta) + \frac{\partial^4 v_0(\wp.\eta)}{\partial\wp^4}\right\}\right],\\
v_1(\wp,\eta) &= \sin\wp - \sin\wp(1+\sigma\eta-\sigma),\\
p^2 v_2(\wp,\eta) &= p^2 v_1(\wp,\eta) - p^2 Y^{-1}\left[(1+\sigma(s-1))Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial v_1(\wp.\eta)}{\partial\eta}(\wp,\eta) + \frac{\partial^4 v_1(\wp.\eta)}{\partial\wp^4}\right\}\right],\\
v_2(\wp,\eta) &= \sin\wp - \sin\wp(1+\sigma\eta-\sigma) + \sin\wp\left((1-\sigma)2\sigma\eta + (1-\sigma)^2 + \frac{\sigma^2\eta^2}{2}\right),\\
p^3 v_3(\wp,\eta) &= p^3 v_2(\wp,\eta) - p^3 Y^{-1}\left[(1+\sigma(s-1))Y\left\{\frac{1}{(1+\sigma(s-1))}\frac{\partial v_2(\wp.\eta)}{\partial\eta}(\wp,\eta) + \frac{\partial^4 v_2(\wp.\eta)}{\partial\wp^4}\right\}\right],\\
v_3(\wp,\eta) &= \sin\wp - \sin\wp(1+\sigma\eta-\sigma) + \sin\wp\left((1-\sigma)2\sigma\eta + (1-\sigma)^2 + \frac{\sigma^2\eta^2}{2}\right)\\
&\quad -\sin\wp\left(3\sigma\left(-2\sigma+1+\sigma^2\right)\eta + \frac{\sigma^3\eta^3}{6} - \frac{3\sigma^2(\sigma-1)\eta^2}{2} + 3\sigma^2 - 3\sigma + 1 - \sigma^3\right).\\
&\vdots
\end{aligned}
\tag{57}
$$

The analytic series solution can be achieved as

$$
\begin{aligned}
v(\wp,\eta) &= \sin\wp - \sin\wp(1+\sigma\eta-\sigma) + \sin\wp\left((1-\sigma)2\sigma\eta + (1-\sigma)^2 + \frac{\sigma^2\eta^2}{2}\right)\\
&\quad - \sin\wp\left(3\sigma\left(-2\sigma+1+\sigma^2\right)\eta + \frac{\sigma^3\eta^3}{6} - \frac{3\sigma^2(\sigma-1)\eta^2}{2} + 3\sigma^2 - 3\sigma + 1 - \sigma^3\right) + \cdots.
\end{aligned}
\tag{58}
$$

If $\sigma = 1$, the series type solution is given as

$$
v(\wp,\eta) = \sin\wp\left(1 - \eta + \frac{\eta^2}{2!} - \frac{\eta^3}{3!} + \cdots\right). \tag{59}
$$

The exact result at $\sigma = 1$ is

$$
v(\wp,\eta) = \sin\wp e^{-\eta}. \tag{60}
$$

5. Results and Discussion

The reliability and applicability of the suggested technique are tested using many numerical examples. The figure representation of the answers in examples 1 to 4 has supplied information regarding the proposed method's correctness and dependability. All of the findings from examples 1–4 revealed a high agreement between VHPTM solutions and the specific answers to the issues. Figures 1–4 show the analytical solution of the different fractional orders of example 1 to space and time. The graphs depicted the answers to each issue at various fractional orders of $\sigma = 1, 0.8, 0.6$, and 0.4. In the case of fractional-order solutions, the convergence of the suggested approach to integer-order solutions attests to the strategy's effectiveness. The solution of fractional-order issues is convergent with the solution of integer-order problems. Furthermore, the recommended method's implementation is basic and uncomplicated throughout the simulation. We hope that the current approach may be adapted to solve various fractional-order differential equations that arise in applied science based on the aforesaid qualities.

6. Conclusions

The fractional analysis of the acoustic wave equation is explored in this article by applying a modified analytic method. The result figure graphs are visualised and analysed to provide a visual representation of the acquired results. Compared to other analytical methods, the graphical representation showed the fastest convergence rate. The fractional-order analysis of the acoustic wave equation is critical for examining the dynamics compared to the classic one. As a result, the proposed method has been important in describing sophisticated solutions to fractional-order partial differential equations that arise in various fields of science and engineering. Additionally, the proposed method uses variational parameters, which simplify the calculations. He's polynomials were employed to produce more precise solutions than Adomian polynomials. The proposed method is found to have a higher rate of convergence than other current methods. As a result, the proposed method can be expanded to solve various fractional nonlinear partial differential equations.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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