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Research Article

New Periodic and Localized Traveling Wave Solutions to a Kawahara-Type Equation: Applications to Plasma Physics

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In this study, some new hypotheses and techniques are presented to obtain some new analytical solutions (localized and periodic solutions) to the generalized Kawahara equation (gKE). As a particular case, some traveling wave solutions to both Kawahara equation (KE) and modified Kawahara equation (mKE) are derived in detail. Periodic and soliton solutions to this family are obtained. The periodic solutions are expressed in terms of Weierstrass elliptic functions (WSEFs) and Jacobian elliptic functions (JEFs). For KE, some direct and indirect approaches are carried out to derive the periodic and localized solutions. For mKE, two different hypotheses in the form of WSEFs are used to derive the periodic and localized solutions. Also, the cnoidal wave solutions in the form of JEFs are obtained. As a realistic physical application, the solutions obtained can be dedicated to studying many nonlinear waves that propagate in plasma.

1. Introduction

Both ordinary and partial differential equations succeed in modelling and describing many complex nonlinear systems that are widely used in various fields of science such as optical fiber, fluid mechanics, nonlinear optics, biology, ecology, astronomy, oceans, economics, and plasma physics [1–10]. Due to the importance of these applications, the great success has achieved by differential equations in clarifying and interpreting the ambiguity of many complex systems, which prompted many authors to look for different analytical and numerical methods in solving such models [5–11]. In recent years, many new analytical and numerical methods have been discovered, and some improvements have been made to many of the existing methods in order to either obtain real solutions related to realistic problems or to obtain more accurate solutions to many integrable and nonintegrable differential equations [12–16]. In particular, there are a large number of partial differential equations (PDEs) that have been used for modelling a lot of nonlinear phenomena such as solitary waves, shock waves, cnoidal waves, peakons, and compactons that arise in different plasma models [5–11]. One of the most important of these equations and the most famous due to its great success not only in the field of fluid mechanics and plasma physics but also in various fields of science is Korteweg–de Vries (KdV) equation [5]:

$$\partial_t u(x,t) + \alpha_1 u(x,t)\partial_x u(x,t) + \beta\partial_x^3 u(x,t) = 0, \qquad (1)$$

where (α_1, β) are the real coefficients which are related to the physical model under study. This equation and its one-dimensional family including a modified KdV (mKdV) equation [5, 10], a Gardner equation, KdV–Burger's equation [5], damped KdV/mKdV equation [11], and so on have been widely used until this day in interpreting the mechanism and properties of many nonlinear phenomena that can propagate in plasma physics. This family is characterized by the third-order dispersion, but there is another family characterized by the fifth-order dispersion which is called the family of Kawahara equation (KE) [17].

$$R_{KE} \equiv \partial_t u(x,t) + \alpha_1 u(x,t)\partial_x u(x,t) + \beta\partial_x^3 u(x,t) - \gamma\partial_x^5 u(x,t) = 0. \tag{2}$$

This is a nonlinear dispersive equation which generalizes the well-known KdV equation. Kawahara equation (2), sometimes referred to as the fifth-order KdV/or super KdV equation [18], is a model that describes solitary waves, cnoidal waves, and periodic waves propagating in nonlinear and high-dispersive media. This equation and many related equations with fifth-order dispersion have been extensively studied in literature [19]. It has important applications in the theory of magnetoacoustic waves in plasma and in the theory of shallow water waves with surface tension [17, 18, 20–30]. However, equation (2) fails to explain the nonlinear waves at some critical values of the plasma compositions due to the disappearance of the nonlinear term, i.e., $\alpha_1 = 0$. Accordingly, modified Kawahara equation (mKE) with higher-order nonlinearity was derived to describe some nonlinear phenomena at the critical plasma compositions:

$$R_{mKE} \equiv \partial_t u(x,t) + \alpha_2 u^2(x,t)\partial_x u(x,t) + \beta\partial_x^3 u(x,t) - \gamma\partial_x^5 u(x,t) = 0. \tag{3}$$

Both KE equation (2) and mKE equation (3) are integrable Hamiltonian systems which are due to the many applications related to this family; many methodologies have been applied for analyzing it [17, 18, 20–24, 27–30]. There remain many secrets about the solutions of this family that appear and become clear day after day as a result of using new analytical and numerical methods for solving this family. This is one of our motives for obtaining a new generation of solutions to this family, which can contribute in understanding the mysterious of many phenomena in plasma physics and other fields related to this family. Thus, our aim is to provide new traveling wave (localized and periodic) solutions to the following generalized KE [29] using several new hypotheses and techniques:

$$\partial_t u(x,t) + \alpha_p u^P(x,t)\partial_x u(x,t) + \beta\partial_x^3 u(x,t) - \gamma\partial_x^5 u(x,t) = 0, \tag{4}$$

where p is a real number. Note that KE equation (2) can be obtained for $p = 1$, while for $p = 2$, mKE equation (3) is recovered.

2. General Analytical Solutions to the Generalized KE

To find a general analytic solution to the evolution equation (4), we suppose

$$\begin{cases} u = v^{(1/p)}, \\ v \equiv v(\xi) \qquad \xi = x + \lambda t, \end{cases} \tag{5}$$

where λ represents the frame velocity.

Inserting ansatz equations (5) into (4) gives the nonlinear ODE:

$$\begin{aligned} &\alpha p^4 v^5 v' + p^4 v^4\left(\beta v^{(3)} + \lambda v' - \gamma v^{(5)}\right) \\ &\quad + (p-1)p^3 v^3\left(5\gamma v^{(4)} v' + 10\gamma v^{(3)} v'' - 3\beta v' v''\right) \\ &\quad + (p-1)p^2(2p-1)v^2 v'\left(-15\gamma (v'')^2 \right.\\ &\quad \left. + \beta (v')^2 - 10\gamma v^{(3)} v'\right) \\ &\quad - \gamma(p-1)(2p-1)(3p-1)(4p-1)(v')^5 \\ &\quad + 10\gamma(p-1)p(2p-1)(3p-1)v(v')^3 v'' = 0, \end{aligned} \tag{6}$$

where $v' \equiv \partial_\xi v$, $v'' \equiv \partial_\xi^2 v$, $v^{(3)} \equiv \partial_\xi^3 v$, $v^{(4)} \equiv \partial_\xi^4 v$, and $v^{(5)} \equiv \partial_\xi^5 v$.

In the following subsections, two important particular cases ($p = 1$ and $p = 2$), i.e., KE equation (2) and mKE equation (3)are analyzed.

2.1. Solutions of the Planar Kawahara Equation.

2.1. Solutions of the Planar Kawahara Equation. In the following sections, we try to find some new solutions including the periodic wave solutions, cnoidal wave solutions, and solitary wave solutions to the planar KE equation (2) ($p = 1$).

2.1.1. Periodic and Solitary Wave Solutions. For planar KE equation (2), ODE equation (6) reduces to

$$-\gamma v^{(5)} + \beta v^{(3)} + \alpha v v' + \lambda v' = 0. \tag{7}$$

Integrating equation (7) once over ξ gives us

$$c_0 - \gamma v^{(4)} + \beta v'' + \frac{1}{2}\alpha v^2 + \lambda v = 0, \tag{8}$$

where c_0 is the integration constant.

Multiplying equation (8) by v' and then integrating it again, we get

$$c_0 v + c_1 + \frac{1}{2}\gamma (v'')^2 + \frac{1}{2}\beta (v')^2 - \gamma v^{(3)} v' + \frac{1}{6}\alpha v^3 + \frac{1}{2}\lambda v^2 = 0, \tag{9}$$

where c_1 the new constant of integration. The solution of equation (2) via several approaches is discussed as follows.

We seek a solution in the ansatz form:

$$v = \sum_{k=0}^{N} d_j \varphi^j, \tag{10}$$

where $\varphi \equiv \varphi(\xi)$ is a solution to the following Helmholtz equation [31].

$$\begin{cases} \varphi'' + A\varphi + B\varphi^2 = 0, \\ (\varphi')^2 = -A\varphi^2 - \frac{2}{3}B\varphi^3 + 2c_2. \end{cases} \quad (11)$$

Balancing the highest linear and nonlinear terms in equation (11) gives $N = 2$, so that

$$v = d_0 + d_1\varphi + d_2\varphi^2. \quad (12)$$

Inserting ansatz equation (12) into equation (9), we obtain

$$\sum_{j=0}^{6} F_j \varphi^j = 0, \quad (13)$$

where the values of $F_j\,(j = 0, 1, \cdots 6)$ are defined in Appendix A, and by solving the following system,

$$F_j = 0, \quad (14)$$

we get

$$\begin{aligned} c_0 &= \frac{-1}{57122\alpha\gamma^2}\left(4598321A^4\gamma^4 - 54418A^2\beta^2\gamma^2 + 1457\beta^4 - 28561\gamma^2\lambda^2\right), \\ c_1 &= \frac{-1}{28960854\alpha^2\gamma^3}\begin{pmatrix} 19394118562A^6\gamma^6 - 183532986A^4\beta^2\gamma^4 + 2331348747A^4\gamma^5\lambda \\ +134862A^2\beta^4\gamma^2 - 27589926A^2\beta^2\gamma^3\lambda + 94922\beta^6 + 738699\beta^4\gamma\lambda - 4826809\gamma^3\lambda^3 \end{pmatrix}, \\ c_2 &= \frac{(13A\gamma + \beta)}{263640B^2\gamma^3}\left(1690A^2\gamma^2 - 403A\beta\gamma + 31\beta^2\right), \\ d_0 &= \frac{-1}{507\alpha\gamma}\left(1183A^2\gamma^2 - 910A\beta\gamma + 31\beta^2 + 507\gamma\lambda\right), \\ d_1 &= \frac{140B(13A\gamma + \beta)}{39\alpha}, \\ d_2 &= \frac{140B^2\gamma}{3\alpha}. \end{aligned} \quad (15)$$

Equation (11) has many formulas for its general solutions such as

$$\varphi = -\frac{A}{2B} - \frac{6}{B}\wp\left(\xi + e_0; A^2/12, g_3\right), \quad (16)$$

where $\wp \equiv \wp(\xi + e_0; A^2/12, g_3)$ indicates the Weierstrass elliptic function (WSEF) and the values of e_0 and g_3 are undetermined parameters which can be obtained from the initial conditions.

Also, the general solution to equation (11) can be expressed by

$$\varphi = e_1 - \frac{6e_1(A + Be_1)}{A + 2Be_1 + 12\wp\left(\xi + e_0; A^2/12, 1/216\left(A^3 - 6B^2e_1^2A - 4B^3e_1^3\right)\right)}, \quad (17)$$

where the values of e_0 and e_1 are determined from the initial conditions.

Thus, a periodic solution to KE equation (2) according to the values of parameters given in system equation (15) and the value of φ given in equation (16) is obtained as

$$\begin{aligned} u = {} & \frac{910\beta A\gamma - 7098A^2\gamma^2 - 910A\beta\gamma - 31\beta^2 - 507\gamma\lambda}{507\alpha\gamma} \\ & - \frac{280\beta}{13\alpha}\wp\left(x + \lambda t + \xi_0; A^2/12, g_3\right) \\ & + \frac{1680\gamma}{\alpha}\wp^2\left(x + \lambda t + \xi_0; A^2/12, g_3\right). \end{aligned} \quad (18)$$

The constants A, B, λ, and ξ_0 are the arbitraries.

Using relation equation (9) with of the parameters given in system equation (15) and the value of φ given in equation (17), the solution to KE equation (2) can be expressed by

$$u=\frac{-31\beta^2-13\gamma\left(39\lambda-70A\beta+91A^2\gamma\right)+1820B\gamma e_0\left(\beta+13A\gamma+13B\gamma e_0\right)}{507\alpha\gamma}-\frac{280Be_0\left(A+Be_0\right)\left(\beta+13A\gamma+26B\gamma e_0\right)}{13\alpha\left[A+2Be_0+12\wp\left(x+\lambda t+\xi_0;A^2/12,1/216\left(A^3-2B^2e_0^2\left(3A+2Be_0\right)\right)\right)\right]}+\frac{1680B^2\gamma e_0^2\left(A+Be_0\right)^2}{\alpha\left[A+2Be_0+12\wp^2\left(x+\lambda t+\xi_0;A^2/12,1/216\left(A^3-2B^2e_0^2\left(3A+2Be_0\right)\right)\right)\right]},\tag{19}$$

where the constants A, B, λ, and e_0 are the arbitraries.

From the periodic solution equation (18), the soliton solutions can be obtained using the following hypotheses:

$$\begin{cases} cg_2=A^2/12=\dfrac{4}{3},\\ g_3=\dfrac{\beta\left(3549A^2\gamma^2-31\beta^2\right)}{4745520\gamma^3}=\dfrac{8}{27},\end{cases}\tag{20}$$

which lead to $(A,\beta)=(4,52\gamma)$, and by rearrange solution (18), the following soliton solution is obtained:

$$=-\frac{1168\gamma+3\lambda}{3\alpha}-\frac{1120\gamma}{\alpha}\wp\left(x+\lambda t;\frac{4}{3},-\frac{8}{27}\right)+\frac{1680\gamma}{\alpha}\wp\left(x+\lambda t;\frac{4}{3},-\frac{8}{27}\right)^2.\tag{21}$$

Moreover, the solitary wave solution can be obtained by

$$\begin{cases} g_2=A^2/12=1/12,\\ g_3=\dfrac{\beta\left(3549A^2\gamma^2-31\beta^2\right)}{4745520\gamma^3}=-1/216,\end{cases}\tag{22}$$

which leads to $(A,\beta)=(1,13\gamma)$; then, the periodic solution (18) reduces to the following soliton solution:

$$=-\frac{73\gamma+3\lambda}{3\alpha}-\frac{280\gamma\wp\left(x+\lambda t+\xi_0;1/12,-1/216\right)}{\alpha}+\frac{1680\gamma\wp\left(x+\lambda t+\xi_0;1/12,-1/216\right)^2}{\alpha}.\tag{23}$$

Also, the periodic solution to KE (2) can be derived directly in terms of WSEF $\wp\equiv\wp\left(x+\lambda t+\xi_0;g_2,g_3\right)$ by inserting the following ansatz in (9).

$$u=a_0+a_1\wp+a_2\wp^2,\tag{24}$$

which leads to

$$\sum_{j=0}^{6}S_j\varphi^j=0,\tag{25}$$

where the values of $S_j\,(j=0,1,\cdots)$ are defined in Appendix B , and by solving the following system,

$$S_j=0,\tag{26}$$

we have

$$\begin{aligned} a_1&=-\frac{280\beta k^2}{13\alpha},\quad a_2=\frac{1680\gamma k^4}{\alpha},\\ c_1&=-\frac{1}{14394744\alpha\gamma^2k}\begin{bmatrix}11661\alpha a_0\gamma k\left(507\alpha a_0\gamma k+1014\gamma\lambda+146\beta^2k\right)\\ -1285245\gamma^2\lambda^2+449159\beta^4k^2+1702506\beta^2\gamma k\lambda\end{bmatrix},\\ c_2&=\frac{1}{21894405624\alpha^2\gamma^3k}\begin{bmatrix}1521\alpha a_0\gamma\begin{pmatrix}507\alpha a_0\gamma k\left(6929\alpha a_0\gamma k+9126\gamma\lambda+2189\beta^2k\right)\\ +5\left(277\beta^2k-507\gamma\lambda\right)\left(507\gamma\lambda+73\beta^2k\right)\end{pmatrix}\\ -7\left(128781549\beta^2\gamma^2\lambda^2+9635389\beta^6k^2+75627162\beta^4\gamma k\lambda\right)\end{bmatrix},\\ g_2&=-\frac{507\alpha a_0\gamma k+507\gamma\lambda+31\beta^2k}{85176\gamma^2k^5},\\ g_3&=-\frac{\beta\left(169\alpha a_0\gamma k+169\gamma\lambda+31\beta^2k\right)}{3163680\gamma^3k^7}.\end{aligned}\tag{27}$$

Collecting both equations (24) and (27), we finally get

$$u = a_0 - \frac{280\beta k^2}{13\alpha}\wp\left(kx + t\lambda + \xi_0; -\frac{31k\beta^2 + 507\gamma\lambda + 507k\alpha\gamma a_0}{85176k^5\gamma^2}, -\frac{\beta\left(31k\beta^2 + 169\gamma\lambda + 169k\alpha\gamma a_0\right)}{3163680k^7\gamma^3}\right) + \frac{1680\gamma k^4}{\alpha}\wp\left(kx + t\lambda + \xi_0; -\frac{31k\beta^2 + 507\gamma\lambda + 507k\alpha\gamma a_0}{85176k^5\gamma^2}, -\frac{\beta\left(31k\beta^2 + 169\gamma\lambda + 169k\alpha\gamma a_0\right)}{3163680k^7\gamma^3}\right)^2. \quad (28)$$

Solution equation (28) satisfies KE equation (2).

2.1.2. Cnoidal and Solitary Wave Solutions. We look for a solution to KE equation (2) in the ansatz form

$$u(x,t) = v(\xi) = \sum_{j=0}^{N} d_j cn^j(\xi), \quad (29)$$

where $\xi = \sqrt{\omega}(x + \lambda t)$ and N is an integer and positive number. From the balance between the highest-order linear $(N + 4)$ and nonlinear $(2N)$ terms of equation (8), we have $N = 4$. Substituting ansatz equation (29) into equation (6) gives a very complicated system. By solving this system using Mathematica package, we found that the coefficients of the odd power in ansatz equation (29) vanish. Thus, the solution of KE equation (2) could be written in the following ansatz:

$$u = A + Bcn^2(\xi, m) + Ccn^4(\xi, m). \quad (30)$$

Inserting ansatz equation (30) into KE equation (2), we get

$$R_{KE} = \sum_{j=0}^{3} W_j cn^{2j} = 0, \quad (31)$$

the values of $W_j\,(j = 0, 1, \cdots 3)$ are given in Appendix C, and by solving the system,

$$W_j = 0, \quad (32)$$

we have

$$A = \frac{1}{507\alpha\gamma}\left[\begin{array}{c} -31\beta^2 + 3640\beta\gamma(1 - 2m)\omega \\ +169\gamma\left(112\gamma(14(m - 1)m - 1)\omega^2 - 3\lambda\right)\end{array}\right],$$

$$B = \frac{-280m\omega}{13\alpha}(52\gamma(2m - 1)\omega - \beta),\ C = \frac{1680\gamma m^2\omega^2}{\alpha}, \quad (33)$$

where ω is a solution to the following cubic equation:

$$31\beta^3 - 56784\beta\gamma^2\left(m^2 - m + 1\right)\omega^2 + 703040\gamma^3(m - 2)(m + 1)(2m - 1)\omega^3 = 0. \quad (34)$$

Finally, the cnoidal wave solutions to KE equation (2) are obtained as

$$u = \frac{-31\beta^2 + 3640\beta\gamma(1 - 2m)\omega + 169\gamma\left(112\gamma(14(m - 1)m - 1)\omega^2 - 3\lambda\right)}{507\alpha\gamma}, - \frac{280m\omega(52\gamma(2m - 1)\omega - \beta)}{13\alpha}cn^2(\xi, m) + \frac{1680\gamma m^2\omega^2}{\alpha}cn^4(\xi, m), \quad (35)$$

where λ is the arbitrary constant, $\xi = \sqrt{\omega}(x + \lambda t)$, and ω is a root to equation (34).

The cnoidal wave solution equation (35) can be directly reduced to the soliton solution for $m \longrightarrow 1$ as

$$u = -\frac{36\beta^2 + 169\gamma\lambda}{169\alpha\gamma} + \frac{105\beta^2}{169\alpha\gamma}\sec h^4\left[\frac{1}{2}\sqrt{\frac{\beta}{13\gamma}}(x + \lambda t)\right]. \quad (36)$$

Moreover, solution equation (36) coincides with the obtained one by means of the tanh method:

$$u = \frac{105\beta^2}{169\alpha\gamma}\sec h^4\left[\frac{1}{2}\sqrt{\frac{\beta}{13\gamma}}\left(x - \frac{36\beta^2}{169\gamma}t\right)\right]. \quad (37)$$

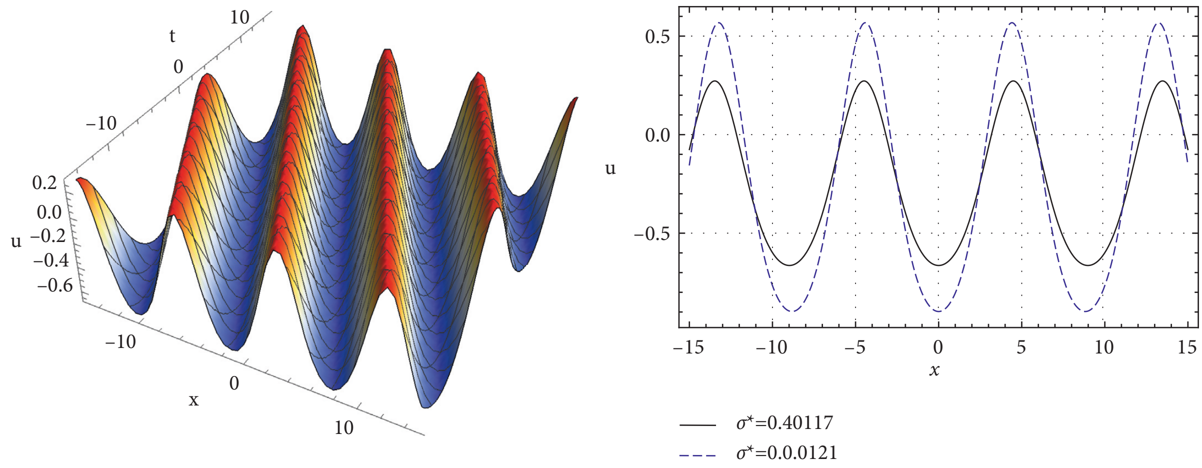
The obtained solutions can be employed for investigating the propagation of nonlinear structures in different plasma models. For instance, we can apply these solutions to study cnoidal and solitary waves in the ultracold neutral plasma (UCNP) which is composed of strongly coupled positive ions and non-Maxwellian electron distributions [32–35]. Based on this model and for Maxwellian electrons, the values of the coefficients (α, β) are given by (to prevent stuffing and repetition, all the details can be found [32])

$$\alpha = \lambda_{ph}\text{and}\beta = \frac{1}{2\lambda_{ph}}, \quad (38)$$

and the phase velocity λ_{ph} of the ion-acoustic waves (IAWs) reads

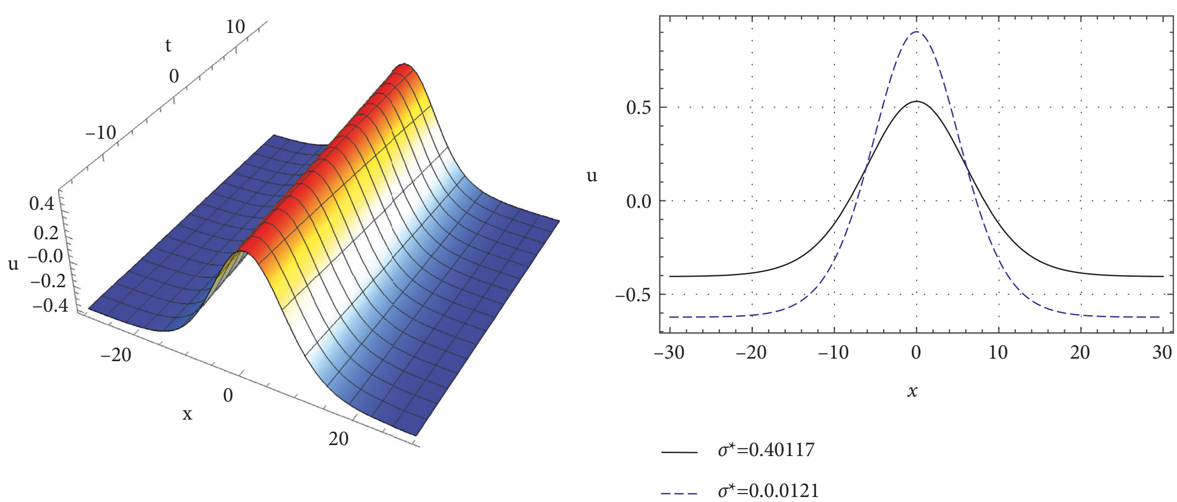
$$\lambda = \sqrt{1 + \sigma_*}, \quad (39a)$$

where $\sigma_* \equiv \sigma_*(T_e, T_i)$ represents the effective temperature ratio which is a function of electron and ion temperatures (T_e, T_i) [32–35]. For $(T_e, T_i) = (25K, 1K)$, we get $\sigma_* = 0.401169$, and for $(T_e, T_i) = (900K, 1K)$, we obtain $\sigma_* = 0.0120837$ [32–35]. With respect to the coefficient of the fifth-order dispersion γ, in general, it has a small value $0 < \gamma \ll 1$. The impact of effective temperature ratio σ^* on the profile of the cnoidal wave solution equation (35) and the solitary wave solution equation (36) for $(\gamma, \lambda) = (0.1, 0.1)$ is shown in Figures 1 and 2, respectively. It is observed that increasing the electron temperature, i.e., decreasing σ^*, leads



(a) (b)

Figure 1: The profile of the periodic wave solution equation (35) to KE equation (2) plotted in (x, t) plane.



(a) (b)

Figure 2: The profile of the solitary wave solution equation (36) to KE equation (2) is plotted in (x, t) plane.

to the enhancement (reduction) of the amplitude (width) of both localized and periodic waves.

2.2. Solutions of Planar Modified Kawahara Equation. The generalized KE equation (4) can reduce the planar mKE equation (3) for $p = 2$. Making the traveling wave transformation $u = v(\xi)$, where $\xi = x + \lambda t + \xi_0$, we get

$$\lambda v' + \alpha v^2 v' + \beta v^{(3)} - \gamma v^{(5)} = 0, \tag{40}$$

and integrating equation (40) twice over ξ, we obtain

$$\begin{aligned} \mathbb{R} \equiv c_0 v + c_1 - \gamma v^{(3)} v' + 1/12 \alpha v^4 + \frac{1}{2} \lambda v^2 \\ + \frac{1}{2} \beta (v')^2 + \frac{1}{2} \gamma (v'')^2 = 0. \end{aligned} \tag{41}$$

Some new localized and periodic solutions to equation (41) are discussed using different approaches in the following sections.

2.2.1. First Ansatz in Terms of WSEFs. The following ansatz is introduced to find a periodic wave solution to equation (41) in terms of WSEFs:

$$v = A + B\wp, \tag{42}$$

where $\wp \equiv \wp(\xi + \xi_0; g_2, g_3)$, g_2 and g_3 denote the elliptic invariants, while the other parameters A, B, g_2, and g_3 are the constants and will be determined later.

Inserting the ansatz equations (42) into (40), we obtain

$$\left(A^2 \alpha + 18 \gamma g_2 + \lambda\right) + (2 A \alpha B + 12 \beta) \wp + \left(\alpha B^2 - 360 \gamma\right) \wp^2 = 0. \tag{43}$$

Equating the coefficients of $\wp^0$, $\wp$, and $\wp^2$ to zero and solving the obtained system, we have

$$A = \pm\frac{\beta}{\sqrt{10\alpha\gamma}}, B = \mp 6\sqrt{\frac{10\gamma}{\alpha}}, \text{and} g_2 = \frac{-\beta^2 - 10\gamma\lambda}{180\gamma^2}. \quad (44)$$

Note that g_3 is an arbitrary constant. Using the initial condition $v(0) = v_0$, we can get

$$A + B\wp(\xi_0; g_2, g_3) = v_0, \quad (45)$$

which leads to

$$\xi_0 = \wp^{-1}\left(\frac{v_0 - A}{B}; g_2, g_3\right). \quad (46)$$

By substituting the values of (A, B, g_2) given in equation (44) into the ansatz equation (42), we finally obtain the solutions of cnoidal wave as

$$u_{1,2}(x,t) = \pm\frac{\beta}{\sqrt{10\alpha\gamma}} \mp 6\sqrt{\frac{10\gamma}{\alpha}}\wp\left(x + \lambda t + \xi_0; \frac{-\beta^2 - 10\gamma\lambda}{180\gamma^2}, g_3\right), \quad (47)$$

and these solutions satisfy the evolution equation (3).

For the following choice,

$$g_2 = \frac{4}{3} \text{and} g_3 = -\frac{8}{27}, \quad (48)$$

the solitary wave solutions are recovered:

$$u_{1,2}(x,t) = \pm\frac{\beta}{\sqrt{10\alpha\gamma}} \mp 6\sqrt{\frac{10\gamma}{\alpha}}\wp\left(x + \lambda t + \xi_0; \frac{4}{3}, -\frac{8}{27}\right). \quad (49)$$

2.2.2. Second Ansatz in Terms of WSEFs. The following rational hypothesis/ansatz is assumed to find some analytical solution to equation (41):

$$v(\xi) = A + \frac{B}{1 + C\wp}, \quad (50)$$

where A, B, and C are the undetermined constants and $\wp \equiv \wp(\xi + \xi_0; g_2, g_3)$.

Inserting the ansatz equations (50) into (41), we have

$$\sum_{j=0}^{6} Z_j \wp^j = 0, \quad (51)$$

where the coefficients $Z_j\,(j = 0, 1, \ldots, 6)$ are defined in Appendix D, and by solving the following system,

$$Z_0 = 0,\ Z_1 = 0, \ldots, Z_6 = 0, \quad (52)$$

we obtain the nontrivial solution:

$$\begin{aligned}
A &= \frac{60\gamma + \beta C}{\sqrt{10\alpha\gamma}C},\\
\lambda &= -\frac{\beta^2}{10\gamma} - \frac{6}{5C^2}\left(\sqrt{10\alpha\gamma}\,BC + 180\gamma\right),\\
c_0 &= \frac{\gamma^{-(3/2)}\sqrt{10/\alpha}\left(86400\gamma^3 + \beta^3C^3 - 2160\beta\gamma^2C\right) - 120BC(\beta C - 60\gamma)}{150C^3},\\
c_1 &= \frac{1}{400\alpha\gamma^2C^4}\Big[-800\gamma^3\left(\alpha B^2C^2 + 36\sqrt{10\alpha\gamma}\,BC + 3240\gamma\right)\\
&\quad + 56\beta^2\gamma^{3/2}C^2\left(\sqrt{10\alpha}\,BC + 180\sqrt{\gamma}\right) + 2880\beta\gamma^{5/2}C\left(\sqrt{10\alpha}\,BC + 120\sqrt{\gamma}\right) + \beta^4\left(-C^4\right)\Big],\\
g_2 &= \frac{\sqrt{(10\alpha/\gamma)}BC + 180}{15C^2},\\
g_3 &= \frac{BC}{15C^3}\left(\sqrt{(10\alpha/\gamma)} + 120\right).
\end{aligned} \quad (53)$$

Thus, the traveling wave solutions to mKE equation (3) are expressed by

$$u = \frac{60\gamma + \beta C}{\sqrt{10\alpha\gamma}C} + \frac{B}{1 + C\wp\left(x - \left(\beta^2/10\gamma + 6\left(\sqrt{10\alpha\gamma}\,BC + 180\gamma\right)/5C^2\right)t + \xi_0; \left(\sqrt{10\alpha/\gamma}\,BC + 180/15C^2\right), \left(\sqrt{10\alpha/\gamma} + 120/15C^3\right)BC\right)}. \quad (54)$$

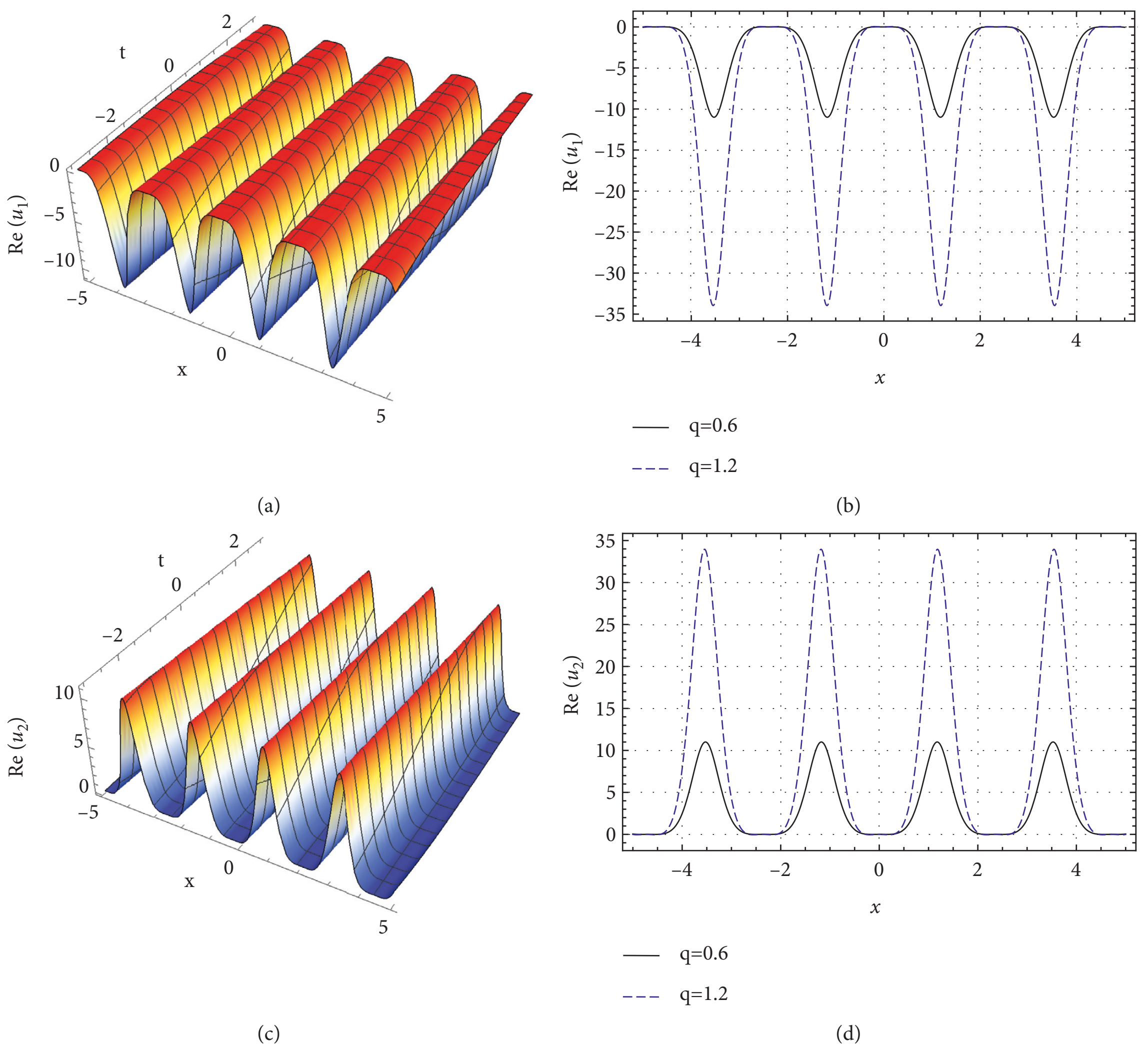


Figure 3: The profile of the cnoidal wave solution equation (47) to mKE equation (3) is plotted in (x, t) plane.

The values of the constants B and C are arbitrary.

The solitary wave solutions can be obtained from the periodic solution equation (54) according to the following choices:

$$\begin{cases} g_2 = 1/12, & g_3 = -1/216, \\ g_2 = \dfrac{4}{3}, & g_3 = -\dfrac{8}{27}. \end{cases} \tag{55}$$

For the choices equation (55), the soliton solutions are obtained:

$$u = \frac{\beta}{\sqrt{10\alpha\gamma}} + \frac{1}{2}\sqrt{\frac{5\gamma}{2\alpha}}\left[4 - \frac{9}{1 + 6\wp\left(x - \left(\beta^2 + 15\gamma^2/10\gamma\right)t + \xi_0;\ (1/12), -(1/216)\right)}\right], \tag{56}$$

$$u = \frac{\beta}{\sqrt{10\alpha\gamma}} + \sqrt{\frac{10\gamma}{\alpha}}\left(1 - \frac{18}{2 + 3\wp\left(x - \left(\beta^2 + 15\gamma^2/10\gamma\right)t + \xi_0;\ (4/3), -(8/27)\right)}\right). \tag{57}$$

2.2.3. Third Ansatz in Terms of JEFs. Using the following ansatz in equation (41),

$$v(\xi) = A + B\,\mathrm{cn}^2, \tag{58}$$

we have

$$\sum_{j=0}^{2n} Y_j cn^j = 0, \quad (59)$$

where the coefficients $Y_j\,(j = 0, 2, 4, 6)$ are defined in Appendix E, and by solving the following system,

$$Y_j = 0, \quad (60)$$

we get

$$A = -\frac{-\beta - 20\gamma\omega + 40\gamma m\omega}{\sqrt{10}\sqrt{\alpha}\sqrt{\gamma}},\ B = \frac{6\sqrt{10}\sqrt{\gamma}m\omega}{\sqrt{\alpha}},$$

$$\lambda = -\frac{\beta^2 + 240\gamma^2\omega^2 + 240\gamma^2 m^2\omega^2 - 240\gamma^2 m\omega^2}{10\gamma},$$

$$c_0 = \frac{1}{15\sqrt{10}\sqrt{\alpha}\gamma^{3/2}}\begin{pmatrix} -240\beta\gamma^2\omega^2 + 3200\gamma^3\omega^3 \\ +3200\gamma^3 m^3\omega^3 - 240\beta\gamma^2 m^2\omega^2 \\ -4800\gamma^3 m^2\omega^3 + 240\beta\gamma^2 m\omega^2 \\ -4800\gamma^3 m\omega^3 + \beta^3 \end{pmatrix}, \quad (61)$$

$$c_1 = -\frac{1}{400\alpha\gamma^2}\begin{pmatrix} \beta^4 - 1120\beta^2\gamma^2\omega^2 - 12800\beta\gamma^3\omega^3 \\ +32000\gamma^4\omega^4 + 32000\gamma^4 m^4\omega^4 \\ -12800\beta\gamma^3 m^3\omega^3 - 64000\gamma^4 m^3\omega^4 \\ -1120\beta^2\gamma^2 m^2\omega^2 + 19200\beta\gamma^3 m^2\omega^3 \\ +96000\gamma^4 m^2\omega^4 + 1120\beta^2\gamma^2 m\omega^2 \\ +19200\beta\gamma^3 m\omega^3 - 64000\gamma^4 m\omega^4 \end{pmatrix}.$$

$\mathrm{cn} \equiv \mathrm{cn}(\xi, m)$ and $\xi = x + \lambda t + \xi_0$.

Using equation (61), the following cnoidal wave solution is obtained:

$$= \frac{\beta + 20\gamma\omega - 40\gamma m\omega}{\sqrt{10}\sqrt{\alpha}\sqrt{\gamma}} + \frac{6\sqrt{10}\sqrt{\gamma}m\omega}{\sqrt{\alpha}} cn^2\left[\sqrt{\omega}\left(x - \frac{t\left(\beta^2 + 240m^2\gamma^2\omega^2 - 240m\gamma^2\omega^2 + 240\gamma^2\omega^2\right)}{10\gamma}\right)|m\right]. \quad (62)$$

For letting $m \longrightarrow 1$, solution equation (62) can recover the soliton solution as

$$u = \frac{\beta - 20\gamma\omega}{\sqrt{10}\sqrt{\alpha}\sqrt{\gamma}} + \frac{6\sqrt{10}\sqrt{\gamma}\omega}{\sqrt{\alpha}} \sec h^2\left[\sqrt{\omega}\left(x - \frac{t\left(\beta^2 + 240\gamma^2\omega^2\right)}{10\gamma}\right)\right]. \quad (63)$$

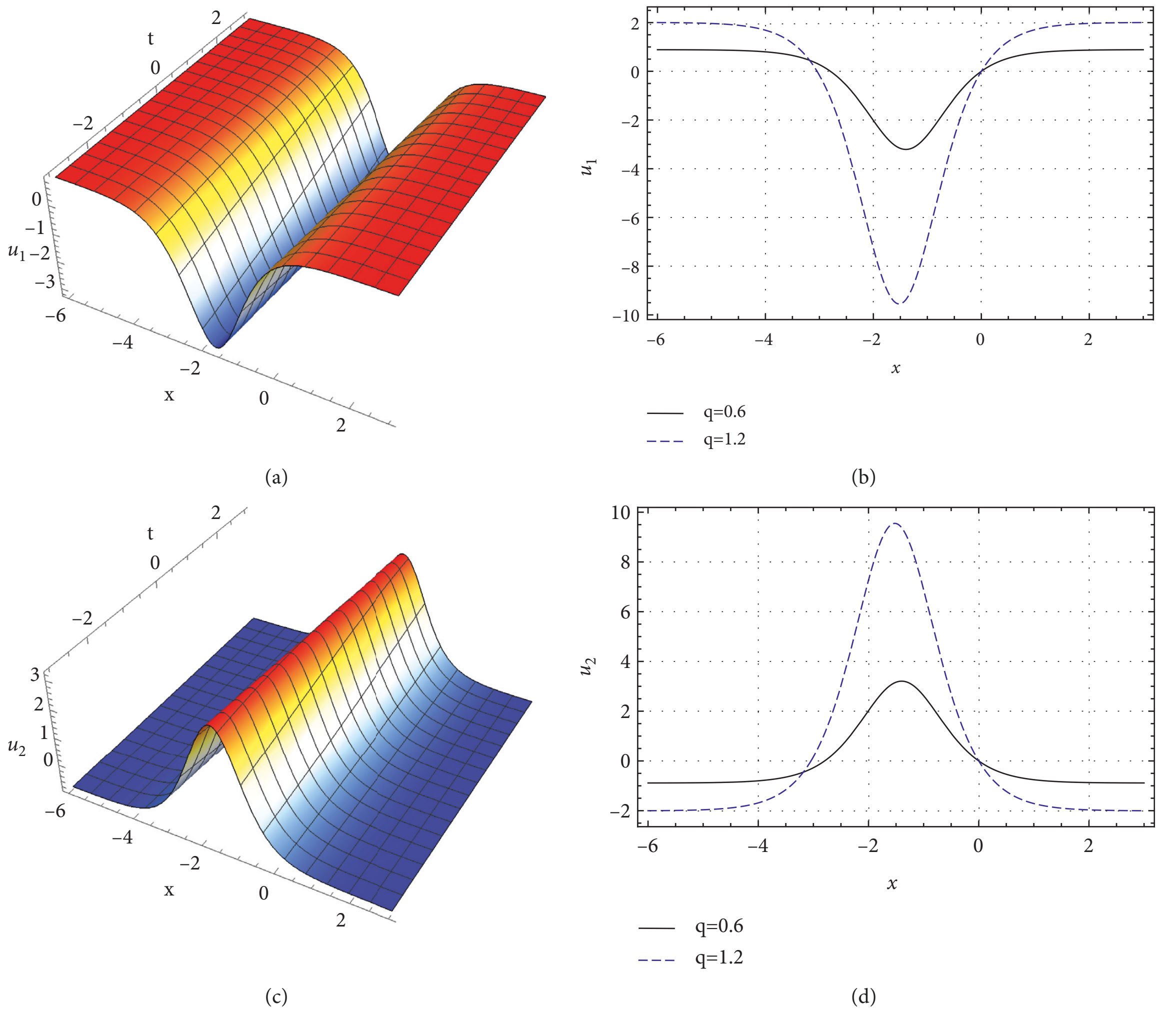


Figure 4: The profile of the solitary wave solution equation (49) to mKE equation (3) is plotted in (x, t) plane.

Furthermore, solution equation (54) can be reduced to the following cnoidal wave solution using the relation between WSEFs and JEFs [36] (more details are inserted in Appendix F):

$$= \frac{60\gamma + \beta C + \beta Cm - 120\gamma m}{\sqrt{10\alpha\gamma}C(m+1)} + \frac{18\sqrt{10\gamma}m}{\sqrt{\alpha}C(m+1)} cn^2\left(\sqrt{\frac{3}{C(m+1)}}\left(x - \left(\frac{\beta^2}{10\gamma} - \frac{216(m^2 - m + 1)\gamma}{C^2(m+1)^2}\right)t\right) + \xi_0, m\right). \quad (64)$$

Also, the soliton solution can be obtained from solution equation (64) for letting $m \longrightarrow 1$:

$$u = \frac{2\beta C - 60\gamma}{2\sqrt{10\alpha\gamma}C} + \frac{9}{C}\sqrt{\frac{10\gamma}{\alpha}}\sec h^2\left(\sqrt{\frac{3}{2C}}\left(x - \frac{\beta^2C^2 - 540\gamma^2}{10\gamma C^2}t\right) + \xi_0\right). \quad (65)$$

The propagation of higher-order ion-acoustic structures in a collisionless and unmagnetized plasma consisting of inertialess nonextensive electrons and positrons and inertial warm ions and nonextensive electrons as well as positrons [10]is investigated. El-Tantawy[10] derived both two-coupled KdV equations and two-coupled modified KdV (mKdV) equations for studying the KdV and mKdV solitons collisions. For some external perturbation or at some certain conditions, the derivatives fifth-order should be taken into consideration which leads to both KE equation (2) and mKE (3). Now, to analyze the obtained solutions, we can use the same values of the coefficients of mKdV equation (26) in [10]. Based on this plasma model $(\alpha_2 = \alpha, \beta) = (2.14663, 0.698771)$ at $q = 0.6$ and $(\alpha_2 = \alpha, \beta) = (0.269511, 0.957799)$ at $q = 1.2$, where q indicates the nonextensive parameter, the profile of the periodic solution equation (47) is illustrated as shown in Figure 3 for the parameter values $(\gamma, \lambda, g_3, \xi_0, \nu_0) = (0.1, -0.1, 5, 0, 0)$. Moreover, the profiles of the solitary wave solutions equation (48) are shown in Figure 4 using the same parameters used in Figure 3 replacing $(g_2, g_3) = (4/3, -8/27)$. It is clear that the two solutions have opposite polarity, i.e., positive and negative potential. Furthermore, it is noticed that both amplitude and width increase with the increase of the nonextensive parameter q.

3. Conclusions

New localized and periodic traveling wave solutions to the generalized KE have been derived in detail using different new approaches and ansatz. As a particular case, several traveling wave solutions to both KE and mKE have been obtained using (in)direct methods. For the indirect method, KE has been solved with the help of Helmholtz equation. After that, we can use any solution to the Helmholtz equation in order to express the solution of the planar KE. We used two different formulas for WSEFs to get some periodic solutions to KE. In the direct method, a new ansatz in the terms of WSEFs has been introduced for getting a cnoidal solution to KE. In all cases and at certain conditions, the periodic solutions have been reduced to the localized solitary wave solutions. In the third (direct) method, the periodic and solitary wave solutions have been derived in the form of JEFs, and it was found that the obtained solutions coincide with that obtained by means of the tanh method. The obtained solutions have been used for interpreting several nonlinear structures that propagate in different plasma models. Furthermore, two new hypotheses in terms of WSEFs have been proposed to find some periodic solutions to mKE. Also, the conditions for reducing the periodic solutions of mKE to the localized solitary waves have been presented. The obtained solutions have been employed for investigating many nonlinear structures in different plasma models.

Appendix

A.

The values of the coefficients $F_j\,(j = 0, 1, \cdots 6)$ are given by

$$
\begin{aligned}
F_0 &= c_1 + c_0 d_0 + \frac{\lambda d_0^2}{2} + \frac{\alpha d_0^3}{6} + \beta c_2 d_1^2 + 2A\gamma c_2 d_1^2 + 8\gamma c_2^2 d_2^2,\\
F_1 &= \frac{1}{2} d_1 \left(2c_0 + 2\lambda d_0 + \alpha d_0^2 + 8B\gamma c_2 d_1 + 8\beta c_2 d_2 + 32A\gamma c_2 d_2\right),\\
F_2 &= -\frac{1}{2}\left(\begin{array}{c} A\beta d_1^2 + A^2\gamma d_1^2 - \lambda d_1^2 - \alpha d_0 d_1^2 - 2c_0 d_2 - 2\lambda d_0 d_2 \\ -\alpha d_0^2 d_2 - 48B\gamma c_2 d_1 d_2 - 8\beta c_2 d_2^2 - 32A\gamma c_2 d_2^2 \end{array}\right),\\
F_3 &= -\frac{1}{6}\left(\begin{array}{c} 2B\beta d_1^2 + 10AB\gamma d_1^2 - \alpha d_1^3 + 12A\beta d_1 d_2 + 36A^2\gamma d_1 d_2 \\ -6\lambda d_1 d_2 - 6\alpha d_0 d_1 d_2 - 160B\gamma c_2 d_2^2 \end{array}\right),\\
F_4 &= -\frac{1}{6}\left(\begin{array}{c} 5B^2\gamma d_1^2 + 8B\beta d_1 d_2 + 80AB\gamma d_1 d_2 - 3\alpha d_1^2 d_2 \\ +12A\beta d_2^2 + 48A^2\gamma d_2^2 - 3\lambda d_2^2 - 3\alpha d_0 d_2^2 \end{array}\right),\\
F_5 &= -\frac{1}{6} d_2 \left(36B^2\gamma d_1 + 8B\beta d_2 + 104AB\gamma d_2 - 3\alpha d_1 d_2\right),\\
F_6 &= -\frac{1}{18} d_2^2 \left(140B^2\gamma - 3\alpha d_2\right).
\end{aligned} \tag{A.1}
$$

B

The values of the coefficients $S_j\,(j = 0, 1, \cdots 6)$ are given by

$$
\begin{aligned}
S_0 &= \frac{1}{24}\left(\begin{array}{c} 24a_0c_1 - 12a_1^2\beta g_3 k^3 + 3a_1^2\gamma g_2^2 k^5 + 48a_2^2\gamma g_3^2 k^5 \\ -48a_1a_2\gamma g_2 g_3 k^5 + 4\alpha a_0^3 k + 12a_0^2\lambda + 24c_2 \end{array}\right),\\
S_1 &= \frac{1}{2} a_1 \left(\begin{array}{c} -a_1\beta g_2 k^3 - 4a_2\beta g_3 k^3 - 3a_2\gamma g_2^2 k^5 + 24a_1\gamma g_3 k^5 \\ +\alpha a_0^2 k + 2a_0\lambda + 2c_1 \end{array}\right),\\
S_2 &= \frac{1}{2}\left(\begin{array}{c} 2a_2c_1 - 4a_1a_2\beta g_2 k^3 - 4a_2^2\beta g_3 k^3 - 3a_2^2\gamma g_2^2 k^5 + 18a_1^2\gamma g_2 k^5 \\ +144a_1a_2\gamma g_3 k^5 + \alpha a_0 a_1^2 k + \alpha a_0^2 a_2 k + a_1^2\lambda + 2a_0a_2\lambda \end{array}\right),\\
S_3 &= \frac{1}{6}\left(\begin{array}{c} -12a_2^2\beta g_2 k^3 + 408a_1a_2\gamma g_2 k^5 + 480a_2^2\gamma g_3 k^5 \\ +12a_1^2\beta k^3 + \alpha a_1^3 k + 6\alpha a_0 a_1 a_2 k + 6a_1a_2\lambda \end{array}\right),\\
S_4 &= \frac{1}{2}\left(168a_2^2\gamma g_2 k^5 + 16a_1a_2\beta k^3 - 60a_1^2\gamma k^5 + \alpha a_0 a_2^2 k + \alpha a_1^2 a_2 k + a_2^2\lambda\right),\\
S_5 &= -\frac{1}{2} a_2 k \left(-\alpha a_1 a_2 - 16a_2\beta k^2 + 432a_1\gamma k^4\right),\\
S_6 &= -\frac{1}{6} a_2^2 k \left(1680\gamma k^4 - \alpha a_2\right).
\end{aligned} \tag{B.1}
$$

C

The values of the coefficients $W_j\,(j = 0, 1, 2, 3)$ are given by

$$
\begin{aligned}
W_3 &= 2sn(\xi,m)cn(\xi,m)dn(\xi,m)\sqrt{\omega}\left(-2C\left(C\alpha - 1680m^2\gamma\omega^2\right)\right.\\
W_2 &= -3\begin{pmatrix} BC\alpha - 20Cm\beta\omega - 1040Cm\gamma\omega^2 \\ -120Bm^2\gamma\omega^2 + 2080Cm^2\gamma\omega^2 \end{pmatrix}\\
W_1 &= \begin{pmatrix} -B^2\alpha - 2AC\alpha - 2C\lambda + 32C\beta\omega + 12Bm\beta\omega \\ -64Cm\beta\omega + 512C\gamma\omega^2 + 240Bm\gamma\omega^2 - 3392Cm\gamma\omega^2 \\ -480Bm^2\gamma\omega^2 + 3392Cm^2\gamma\omega^2 \end{pmatrix}\\
W_0 &= \begin{pmatrix} -AB\alpha - B\lambda + 4B\beta\omega - 12C\beta\omega - 8Bm\beta\omega \\ +12Cm\beta\omega + 16B\gamma\omega^2 - 240C\gamma\omega^2 - 136Bm\gamma\omega^2 \\ +720Cm\gamma\omega^2 + 136Bm^2\gamma\omega^2 - 480Cm^2\gamma\omega^2 \end{pmatrix}.
\end{aligned}
\tag{C.1}
$$

D

The values of the coefficients Z_j $(j = 0, 1, 2 \cdots 6)$ are given by

$$
\begin{aligned}
Z_0 &= 8\alpha AB^3 + 24AB\lambda + 24Ac_0 + 2\alpha B^4 - 96B^2\gamma C^4 g_3^2 + 48B^2\gamma C^3 g_2 g_3 - 12\beta B^2 C^2 g_3\\
&\quad + 3B^2\gamma C^2 g_2^2 + 12B^2\lambda + 24Bc_0 + 24c_1 + 2\alpha A^4 + 8\alpha A^3 B + 12\alpha A^2 B^2 + 12A^2\lambda\\
Z_1 &= 2C\begin{pmatrix} 6\alpha A^4 + 20\alpha A^3 B + 24\alpha A^2 B^2 + 36A^2\lambda + 12\alpha AB^3 + 60AB\lambda + \\ 72Ac_0 + 2\alpha B^4 - 72B^2\gamma C^3 g_2 g_3 - 12\beta B^2 C^2 g_3 + 27B^2\gamma C^2 g_2^2 - \\ 6\beta B^2 C g_2 + 144B^2\gamma C g_3 + 24B^2\lambda + 60Bc_0 + 72c_1 \end{pmatrix},\\
Z_2 &= C^2\begin{pmatrix} 30\alpha A^4 + 80\alpha A^3 B + 72\alpha A^2 B^2 + 180A^2\lambda + 24\alpha AB^3 + 240AB\lambda + \\ 360Ac_0 + 2\alpha B^4 - 12\beta B^2 C^2 g_3 - 45B^2\gamma C^2 g_2^2 - \\ 24\beta B^2 C g_2 + 216B^2\gamma g_2 + 72B^2\lambda + 240Bc_0 + 360c_1 \end{pmatrix},\\
Z_3 &= 4C^2\begin{pmatrix} 10\alpha A^4 C + 20\alpha A^3 BC + 12\alpha A^2 B^2 C + 60A^2 C\lambda + 2\alpha AB^3 C + \\ 60ABC\lambda + 120Ac_0 C + 12\beta B^2 - 3\beta B^2 C^2 g_2 + 120B^2\gamma C^2 g_3 - \\ 84B^2\gamma C g_2 + 12B^2 C\lambda + 60Bc_0 C + 120c_1 C \end{pmatrix},\\
Z_4 &= 2C^2\begin{pmatrix} 15\alpha A^4 C^2 + 20\alpha A^3 BC^2 + 6\alpha A^2 B^2 C^2 + 90A^2 C^2\lambda + \\ 60ABC^2\lambda + 180Ac_0 C^2 - 360B^2\gamma + 108B^2\gamma C^2 g_2 + \\ 6B^2 C^2\lambda + 48\beta B^2 C + 60Bc_0 C^2 + 180c_1 C^2 \end{pmatrix},\\
Z_5 &= 4C^3\begin{pmatrix} 3\alpha A^4 C^2 + 2\alpha A^3 BC^2 + 18A^2 C^2\lambda + 6ABC^2\lambda + 36Ac_0 C^2 + \\ 216B^2\gamma + 12\beta B^2 C + 6Bc_0 C^2 + 36c_1 C^2 \end{pmatrix},\\
Z_6 &= 2C^4\left(\alpha A^4 C^2 + 6A^2 C^2\lambda + 12Ac_0 C^2 + 24B^2\gamma + 12c_1 C^2\right).
\end{aligned}
\tag{D.1}
$$

E

The values of the coefficients Y_j $(j = 0, 2, 4, 6, 8)$ are given by

$$\begin{aligned}
Y_0 &= 1/12\begin{pmatrix} \alpha A^4 + 6A^2\lambda + 12Ac_0 + 24B^2\gamma\omega^2 \\ +24B^2\gamma m^2\omega^2 - 48B^2\gamma m\omega^2 + 12c_1 \end{pmatrix}, \\
Y_2 &= 1/12\begin{pmatrix} 4\alpha A^3B + 12AB\lambda + 24\beta B^2\omega + 96B^2\gamma\omega^2 + \\ 192B^2\gamma m^2\omega^2 - 24\beta B^2 m\omega - 288B^2\gamma m\omega^2 + 12Bc_0 \end{pmatrix}, \\
Y_4 &= 1/12\begin{pmatrix} 6\alpha A^2B^2 - 24\beta B^2\omega - 96B^2\gamma\omega^2 + 6B^2\lambda - \\ 816B^2\gamma m^2\omega^2 + 48\beta B^2 m\omega + 816B^2\gamma m\omega^2 \end{pmatrix}, \\
Y_6 &= 1/12\left(4\alpha AB^3 + 960B^2\gamma m^2\omega^2 - 24\beta B^2 m\omega - 480B^2\gamma m\omega^2\right), \\
Y_8 &= 1/12\left(\alpha B^4 - 360B^2\gamma m^2\omega^2\right).
\end{aligned} \tag{E.1}$$

F

Relation between the Jacobian cn elliptic function and the Weierstrass elliptic function.

It is known that

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \ \wp \equiv \wp(t; g_2, g_3). \tag{F.1}$$

On the other hand, if $v(t) = \operatorname{cn}(\sqrt{\omega}t, m)$, we get

$$\frac{1}{2}\dot{v}^2 - (1-m)\omega + (1-2m)\omega v^2(t) + m\omega v^4(t) = 0. \tag{F.2}$$

Define

$$w(t) = 1 + \frac{B}{1 + C\wp(t; g_2, g_3)}, \tag{F.3}$$

then,

$$\frac{1}{2}\dot{w}^2 - (1-m)\omega + (1-2m)\omega w^2(t) + m\omega w^4(t)$$

$$= \frac{B}{(1+C\wp)^4}\begin{bmatrix} B^3m\omega + 4B^2m\omega - BC^2g_3 + 4Bm\omega + B\omega + 2\omega + \\ C\left(4B^2m\omega - BCg_2 + 8Bm\omega + 2B\omega + 6\omega\right)\wp + \\ +C\omega(4Bm + B + 6)\wp^2 + 2C^2(2B + C\omega)\wp^3 \end{bmatrix}. \tag{F.4}$$

Equating to zero the coefficients of $\wp^j$ $(j = 0, 1, 2, 3)$ gives an algebraic system, and by solving this system, we finally have

$$\begin{aligned}
B &= -\frac{6}{4m+1}, \\
C &= \frac{12}{(4m+1)\omega}, \\
g_2 &= 1/12\left(16m^2 - 16m + 1\right)\omega^2, \\
g_3 &= 1/216(2m-1)\left(32m^2 - 32m - 1\right)\omega^3.
\end{aligned} \tag{F.5}$$

Then,

$$cn(\sqrt{\omega}t, m) = 1 - \frac{6}{(4m+1)\left(1 + 12/(4m+1)\omega\wp\left(t; (1/12)\left(16m^2 - 16m + 1\right)\omega^2, (1/216)(2m-1)\left(32m^2 - 32m - 1\right)\omega^3\right)\right)}. \tag{F.6}$$

This identity shows that the function cn is expressible trough the function $\wp$. Now, if we know the function $\wp$, we want to write it in terms of cn. To this end, we must write ω and m in terms of g_2 and g_3. This is not too easy. Define

$$z = 16m^2 - 16m + 1. \tag{F.7}$$

Now, we eliminate ω and m from the system,

$$\begin{aligned}
g_2 &= 1/12\left(16m^2 - 16m + 1\right)\omega^2, \\
g_3 &= 1/216(2m-1)\left(32m^2 - 32m - 1\right)\omega^3, \\
z &= 16m^2 - 16m + 1,
\end{aligned} \tag{F.8}$$

to obtain

$$27g_2^3 - 27g_2^3z + 4\left(g_2^3 - 27g_3^2\right)z^3 = 0. \tag{F.9}$$

This cubic is solvable by means of Tartaglia formula which leads to

$$m = \frac{1}{4}\left(2 \pm \sqrt{z+3}\right), \omega = 2\sqrt{\frac{3g_2}{z}}. \tag{F.10}$$

Finally, we solve the following equation for $\wp(t; g_2, g_3)$:

$$1 + \frac{B}{1 + C\wp(t; g_2, g_3)} = cn(\sqrt{\omega}t, m). \tag{F.11}$$

The desired expression reads

$$\wp(t; g_2, g_3) = -\frac{\omega}{12}\left(1 + 4m - \frac{6}{1 - \operatorname{cn}(\sqrt{\omega}t|m)}\right). \tag{F.12}$$

In conclusion, if some ODE or some PDE have a solution that is expressible in terms of the Jacobian cn function, then such solution may also be written in terms of the Weierstrass $\wp$ function and vice versa. So, cnoidal waves and $\wp$ solutions have the same meaning. Observe also that the last formula allows us to obtain the main period of the Weierstrass function in the form

$$T = \frac{4K(m)}{\sqrt{\omega}}. \quad \text{(F.13)}$$

Data Availability

The data generated or analyzed during this study are included within the article and available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and approved the final manuscript.

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