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Research Article

The New Generalized Exponentiated Fréchet–Weibull Distribution: Properties, Applications, and Regression Model

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Statistical probability distributions are commonly used by data analysts and statisticians to describe and analyze their data. It is possible in many situations that data would not fit the existing classical distributions. A new distribution is therefore required in order to accommodate the complexities of different data shapes and enhance the goodness of fit. A novel model called the new generalized exponentiated Fréchet–Weibull distribution is proposed in this paper by combing two methods, the transformed transformer method and the new generalized exponentiated method. This novel modeling approach is capable of modeling complex data structures in a wide range of applications. Some statistical properties of the new distribution are derived. The parameters have been estimated using the method of maximum likelihood. Then, different simulation studies have been conducted to assess the behavior of the estimators. The performance of the proposed distribution in modeling has been investigated by means of applications to three real datasets. Further, a new regression model is proposed through reparametrization of the new generalized exponentiated Fréchet–Weibull distribution using the log-location-scale technique. The effectiveness of the proposed regression model is also investigated with two simulation studies and three real censored datasets. The results demonstrated the superiority of the proposed models over other competing models.

1. Introduction

Statistical distributions are extremely useful in describing many world phenomena. Specifically, finding an appropriate distribution is a fundamental requirement to analyze and interpret data properly. The suitable selection of distributions leads to valid inference and right conclusion. Many statistical distributions have been proposed and applied to fit real data in many applications, such as education, physics, chemistry, demography, management, and engineering. However, in many of these areas, data may display a complex pattern which cannot be adequately fit using the classical and traditional distributions. This complexity in data patterns has led to the need to develop statistical distributions that are more flexible, practical, and accurate in modeling them in the literature. Recently, several studies have attempted to extend the classical models and generate some new families of distributions (for a good review of these methods, see [1]).

Alzaatreh et al. [2] introduced a new general method for generating families of continuous distributions, called the transformed transformer (T-X) method. This method generalizes the beta-G [3] and Kumaraswamy-G [4] families by replacing the beta distribution and Kumaraswamy distribution with any continuous distribution for a random variable T defined on [a, b]. Particularly, the cumulative distribution function (cdf) of the T-X family can be defined as

$$F(x) = \int_{a}^{W[G(x)]} r(t)\,dt, \tag{1}$$

where a is a real number and $W[G(x)]$ is a function of any cdf of a random variable X.

Many new statistical distributions have been proposed using this method, such as the gamma-normal distribution in [5], the odds generalized exponential-exponential distribution in [6], the new Weibull–Pareto distribution in [7], the Weibull–Burr type X distribution in [8], the Lindley–Pareto distribution in [9], the odd log-logistic logarithmic normal distribution in [10], the odd Lindley–Burr XII distribution in [11], the Topp–Leone-exponential Poisson distribution in [12], the Lomax–Gumbel (Fréchet) distribution in [13], the Weibull-gamma distribution in [14], the Topp–Leone generalized odd log-logistic Weibull distribution in [15], the Burr III-Marshall–Olkin–Weibull distribution in [16], the Hjorth uniform distribution in [17], the Xgamma-Lindley distribution in [18], the Fréchet–Topp–Leone–Kumaraswamy distribution in [19], and the Marshall–Olkin–Weibull exponential distribution in [20].

Abd-Elmonem et al. [21] applied the T-X method to introduce a new extended distribution, called Fréchet–Weibull distribution which is based on the Fréchet distribution as a generator. The cdf and the probability density function (pdf) of Fréchet–Weibull distribution with four parameters, namely, $\alpha, \lambda > 0$ as scale parameters and $\beta, k > 0$ as shape parameters, respectively, are given as

$$H(x; \alpha, \beta, k, \lambda) = e^{-\left(\alpha (\lambda/x)^{k}\right)^{\beta}}, \ x > 0, \tag{2}$$

$$h(x; \alpha, \beta, k, \lambda) = k\beta\alpha^{\beta}\lambda^{k\beta}x^{-(k\beta+1)}e^{-\left(\alpha (\lambda/x)^{k}\right)^{\beta}}, x > 0. \tag{3}$$

On the other hand, Gupta et al. [22] introduced the exponentiated method for which the existing distribution is generalized by adding an extra shape parameter to its pdf. Consequently, Cordeiro et al. [23] proposed a new class that generalizes the exponentiated method by adding two extra shape parameters to an existing distribution. Recently, Rezaei et al. [24] introduced a more general method by adding three extra shape parameters to an existing distribution. The cdf and pdf of this new exponentiated family are defined, respectively, as

$$F(x) = 1 - \left(1 - \{1 - [1 - G(x)]^{a}\}^{b}\right)^{\theta}, a, b, \theta > 0, \tag{4}$$

$$f(x) = ab\theta g(x)[1 - G(x)]^{a-1}\{1 - [1 - G(x)]^{a}\}^{b-1}\left(1 - \{1 - [1 - G(x)]^{a}\}^{b}\right)^{\theta-1}, a, b, \theta > 0, \tag{5}$$

where $G(x)$ and $g(x)$ are the cdf and pdf of any statistical distribution and a, b, and θ are positive real numbers. The exponentiated generalized half logistic Fréchet distribution introduced in [25] and the exponentiated generalized exponential Dagum distribution proposed in [26] can be regarded as members of this family. Another distribution is the exponentiated generalized extended Gompertz distribution in [27] that generalizes the Gompertz distribution.

It is the purpose of this paper to increase the flexibility of some existing distributions in order to accommodate the complexity of certain data. To that end, we combine the classical Fréchet–Weibull distribution with the new generalized exponentiated distribution class, providing the new generalized exponentiated Fréchet–Weibull distribution (NGEFWD). The proposed distribution can be used as an alternative to several existing distributions in modeling different applications. Another goal is related to the significance of regression modeling. Specifically, real data are frequently explained by other variables, which are referred to as explanatory variables or covariates. Hence, researchers have shown an increasing interest in investigating these relationships by considering regression analysis. Many regression models have been constructed in the literature recently based on some distributions of the response variable. In particular, log-location-scale regression models have been considered by many authors based on different distributions. Among these, Silva et al. [28] studied the log-Burr XII regression model, Carrasco et al. [29] introduced the log-modified Weibull regression model, Ortega et al. [30] proposed the log generalized modified Weibull regression model, Pescim et al. [31] developed a log-linear regression model based on the odd log-logistic generalized half-normal distribution, Altun et al. [32] suggested the log Zografos–Balakrishnan BXII distribution, Korkmaz et al. [33] proposed the log odd power Lindley–Weibull regression model, Baharith et al. [34] introduced the log odds exponential Pareto IV regression model, Cordeiro et al. [18] discussed the log-Xgamma Weibull regression model, Eliwa et al. [35] proposed the log odd Lindley half logistic regression model, Altun et al. [36] proposed the log additive odd log-logistic odd Weibull-Weibull regression model, Shama et al. [37] suggested the log gamma Gompertz regression model, and Anwaar Dhiaa and Sunbul Rasheed [38] provided two regression models derived from the Burr XII family of distributions. Then, a further objective of this paper includes introducing a new regression model based on the NGEFWD distribution.

This paper is organized as follows. In Section 2, the NGEFWD is introduced and some plots for the pdf and hazard rate function (hrf) of NGEFWD are provided. In Section 3, we derive the expansion of the pdf for the NGEFWD. In Section 4, we discuss some of the statistical properties of the new distribution. The maximum likelihood estimates (MLEs) of the model parameters are determined in Section 5. Section 6 discusses the simulation results. In Section 7, the NGEFWD is applied to three real datasets. In Section 8, we propose the log-NGEFW regression model and estimate the model parameters using the maximum likelihood estimation. Section 9 presents some simulation studies to estimate log-NGEFW regression model parameters. In Section 10, three real datasets are investigated to show the flexibility of the new regression model. Finally, Section 11 offers some concluding remarks.

2. The New Generalized Exponentiated Fréchet–Weibull Distribution

The NGEFWD can be obtained by replacing $G(x)$ in equation (4) by the cdf in equation (2) and $g(x)$ in equation (5) by the pdf in equation (3). That is, a random variable X is said to have NGEFWD with seven parameters $a, b, \theta, \beta, k > 0$ as shape parameters and $\alpha, \lambda > 0$ as scale parameters if its cdf and pdf are defined, respectively, as

$$F(x) = 1 - \left(1 - \left\{1 - \left[1 - e^{-\left(\alpha(\lambda/x)^k\right)^\beta}\right]^a\right\}^b\right)^\theta, x > 0, \quad (6)$$

and

$$f(x) = ab\theta k\beta\alpha^\beta\lambda^{k\beta}x^{-(k\beta+1)}e^{-\left(\alpha(\lambda/x)^k\right)^\beta}\left[1 - e^{-\left(\alpha(\lambda/x)^k\right)^\beta}\right]^{a-1} \times\left\{1 - \left[1 - e^{-\left(\alpha(\lambda/x)^k\right)^\beta}\right]^a\right\}^{b-1}\left(1 - \left\{1 - \left[1 - e^{-\left(\alpha(\lambda/x)^k\right)^\beta}\right]^a\right\}^b\right)^{\theta-1}, x > 0. \quad (7)$$

The reliability function and hrf of NGEFWD can be obtained, respectively, as

$$R(x) = \left(1 - \left\{1 - \left[1 - e^{-\left(\alpha(\lambda/x)^k\right)^\beta}\right]^a\right\}^b\right)^\theta, x > 0,$$
$$\mathrm{hrf}(x) = \frac{ab\theta k\beta\alpha^\beta\lambda^{k\beta}x^{-(k\beta+1)}e^{-\left(\alpha(\lambda/x)^k\right)^\beta}\left[1 - e^{-\left(\alpha(\lambda/x)^k\right)^\beta}\right]^{a-1}}{1 - \left\{1 - \left[1 - e^{-\left(\alpha(\lambda/x)^k\right)^\beta}\right]^a\right\}^b} \times\left\{1 - \left[1 - e^{-\left(\alpha(\lambda/x)^k\right)^\beta}\right]^a\right\}^{b-1}. \quad (8)$$

For various values of the distribution's parameters, Figures 1 and 2 illustrate the shapes of the NGEFWD's pdf and hrf, receptively. It can be seen that the NGEFWD can demonstrate left skewed, symmetrical, right skewed, and reversed-J shaped densities. Also, it can take a form of decreasing, upside down bathtub, reversed bathtub, and reversed-J shaped hazard rates. Accordingly, NGEFWD can be considered as an appropriate model for fitting a variety of lifetime data in applied areas.

3. Expansion of pdf for NGEFWD

In the following, we can express the pdf of NGEFWD in equation (7) with an expanded form using the binomial expansion defined for a positive real power as

$$(1 - z)^{\gamma-1} = \sum_{i=0}^{\infty}(-1)^i\binom{\gamma - 1}{i}z^i, \quad (9)$$

for $|z| < 1$ and $\gamma > 0$ (γ is a nonnegative integer).

Specifically, applying the binomial expansion in equation (9) three times, the pdf of the NGEFWD can be rewritten as

$$f(x) = ab\theta k\beta\alpha^\beta\lambda^{k\beta}\sum_{i,j,h=0}^{\infty}w_{\mathrm{ijh}}x^{-(k\beta+1)}e^{-(h+1)\left(\alpha(\lambda/x)^k\right)^\beta}, x > 0, \quad (10)$$

where

$$w_{\mathrm{ijh}} = (-1)^{i+j+h}\binom{\theta - 1}{i}\binom{b(i + 1) - 1}{j}\binom{a(j + 1) - 1}{h}. \quad (11)$$

4. Statistical Properties

In this section, we derive some useful statistical properties of the NGEFWD.

4.1. The Quantile Function and Median. The quantile function of the NGEFWD is defined as

$$Q(u) = \frac{\lambda}{\left[[-\log(z(u))]^{1/\beta}/\alpha\right]^{1/k}}, \quad (12)$$

where $z(u) = 1 - (1 - \{1 - [1 - u]^{1/\theta}\}^{1/b})^{1/a}$ and u is a uniformly distributed random variable. If we use $u = 0.25$ or 0.75, we get the first quantile or the third quantile of the NGEFWD, respectively.

The median of the NGEFWD is given as

$$\mathrm{Med} = \frac{\lambda}{\left[[-\log(z(0.5))]^{1/\beta}/\alpha\right]^{1/k}}, \quad (13)$$

where $z(0.5) = 1 - (1 - \{1 - [0.5]^{1/\theta}\}^{1/b})^{1/a}$.

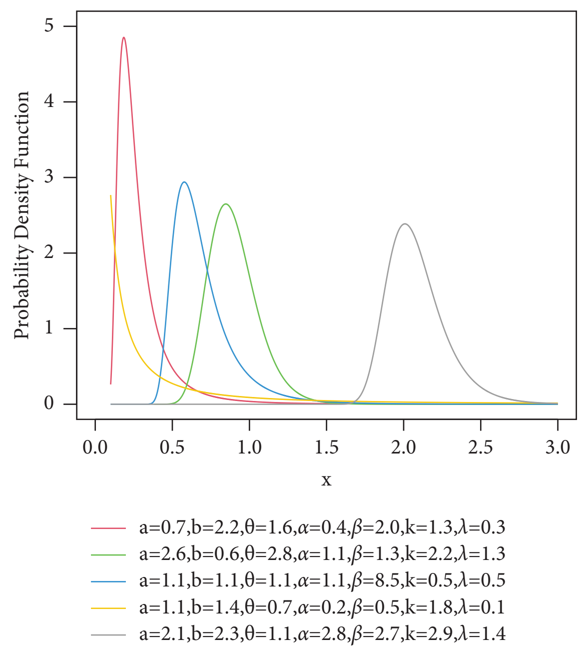


Figure 1: Probability density function plots of the NGEFWD.

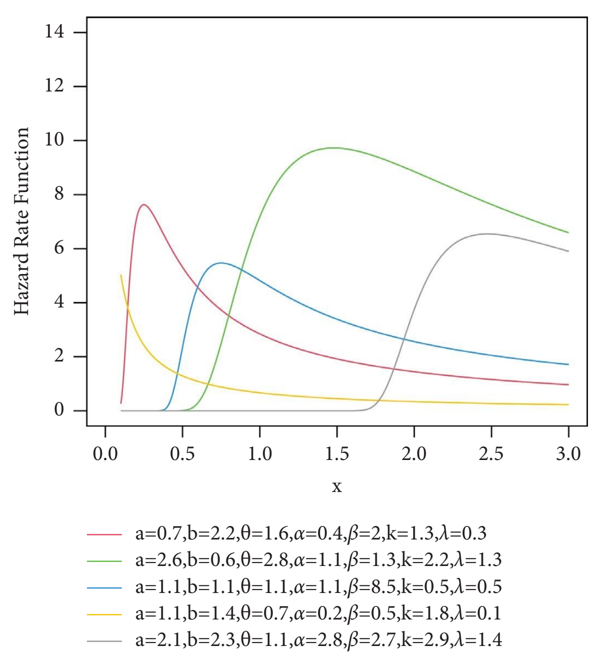


Figure 2: Hazard rate function plots of the NGEFWD.

4.2. The Galton Skewness and Moors Kurtosis. The Galton skewness (GS) measures the degree of the long tail (towards left if GS < 0 or right side if GS > 0). It is defined in [39] as

$$GS = \frac{Q(6/8) - 2Q(4/8) + Q(2/8)}{Q(6/8) - Q(2/8)}, \tag{14}$$

and the Moors kurtosis (MK) measures the degree of tail heaviness (if MK increases, the tail of the distribution becomes heavier). It is defined in [40] as

$$MK = \frac{(Q(7/8) - Q(5/8)) + (Q(2/8) - Q(1/8))}{(Q(6/8) - Q(2/8))}, \tag{15}$$

where $Q(.)$ is the quantile function in equation (12).

From Figure 3, the NGEFWD can be right skewed, and for fixed α, the MK is a decreasing function of θ.

4.3. The r^{th} Moment. The r^{th} moment of the NGEFWD can be obtained as

$$\begin{aligned} \mu_r &= E(x^r) \\ &= \int_0^\infty x^r f(x)\mathrm{d}x \\ &= ab\theta\lambda^r\alpha^{r/k}\sum_{i,j,h=0}^{\infty} w_{ijh}(h+1)^{r/k\beta-1}\Gamma\left(1-\frac{r}{k\beta}\right), r<k\beta, \end{aligned} \tag{16}$$

where w_{ijh} is defined in equation (11).

Then, the mean and variance of the NGEFWD are, respectively, given as

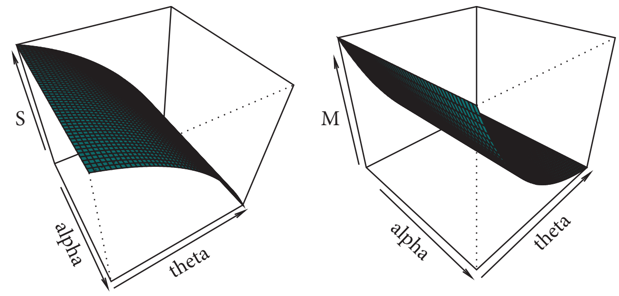
$$\begin{aligned} \mu_1 &= E(x) \\ &= ab\theta\lambda\alpha^{1/k}\sum_{i,j,h=0}^{\infty} w_{ijh}(h+1)^{1/k\beta-1}\Gamma\left(1-\frac{1}{k\beta}\right), \\ \sigma^2 &= \mu_2 - [\mu_1]^2 \\ &= ab\theta\lambda^2\alpha^{2/k}\sum_{i,j,h=0}^{\infty} w_{ijh}(h+1)^{2/k\beta-1}\Gamma\left(1-\frac{2}{k\beta}\right) - [\mu_1]^2, \end{aligned} \tag{17}$$

where w_{ijh} is defined in equation (11).

4.4. The Moment Generating Function and Characteristic Function. Based on the expansion of $e^{tx} = \sum_{r=0}^{\infty} t^r x^r / r!$, the moment generating function can be calculated based on the r^{th} moment of the NGEFWD as

$$\begin{aligned} \mu_x(t) &= E\left(e^{tx}\right) \\ &= \int_0^\infty e^{tx} f(x)\mathrm{d}x, \\ \mu_x(t) &= ab\theta\lambda^r\alpha^{r/k}\sum_{i,j,h,r=0}^{\infty} w_{ijh}\frac{t^r(h+1)^{r/k\beta-1}}{r!}\Gamma\left(1-\frac{r}{k\beta}\right), r<k\beta, \end{aligned} \tag{18}$$

where w_{ijh} is defined in equation (11).



(a) (b)

Figure 3: GS (a) and MK (b) for the NGEFWD.

Similarly, we can obtain the characteristic function based on r^{th} moment of the NGEFWD as

$$\phi_x(t) = E\left(e^{\mathrm{it}x}\right) = \int_0^\infty e^{\mathrm{it}x} f(x)\mathrm{d}x, \tag{19}$$
$$\phi_x(t) = ab\theta\lambda^r\alpha^{r/k}\sum_{i,j,h,r=0}^{\infty} w_{ijh}\frac{(\mathrm{it})^r (h+1)^{r/k\beta-1}}{r!}\Gamma\left(1-\frac{r}{k\beta}\right), r<k\beta,$$

where w_{ijh} is defined in equation (11).

4.5. Order Statistics. Suppose $x_1, x_2, \ldots, x_n$ is a random sample from NGEFWD, where X_L is the L^{th} order statistic; then, the pdf of this L^{th} order statistic is defined as

$$f_{L:n}(x) = \frac{n!}{(L-1)!(n-L)!}[F(x)]^{L-1}[1-F(x)]^{n-L} f(x), \tag{20}$$

where $f(x)$ and $F(x)$ are the pdf and cdf of NGEFWD defined, respectively, in equations (7) and (6). By using the binomial expansion, we can write

$$[1-F(x)]^{n-L} = \sum_{m=0}^{n-L}(-1)^m\binom{n-L}{m}[F(x)]^m. \tag{21}$$

Thus,

$$f_{L:x}(x) = \frac{n!}{(L-1)!(n-L)!}\sum_{m=0}^{n-L}(-1)^m\binom{n-L}{m} \cdot f(x)[F(x)]^{L+m-1}. \tag{22}$$

Substituting by equations (6) and (7) and applying the binomial expansion in equation (9) four times, we obtain

$$f_{L:x}(x) = ab\theta k\beta\alpha^\beta\lambda^{k\beta}\sum_{m=0}^{n-L}\sum_{i,j,h,g=0}^{\infty} \cdot w_{\mathrm{mijhg}} x^{-(k\beta+1)} e^{-(g+1)\left(\alpha(\lambda/x)^k\right)^\beta}, x>0, \tag{23}$$

where

$$w_{\mathrm{mijhg}} = \frac{n!(-1)^{m+i+j+h+g}}{(L-1)!(n-L)!}\binom{n-L}{m}\binom{L+m-1}{i}\binom{\theta(i+1)-1}{j}\binom{b(j+1)-1}{h}\binom{a(h+1)-1}{g}. \tag{24}$$

4.6. Rényi Entropy. The Rényi entropy of a random variable represents a measure of variation of the uncertainty, and it is defined as

$$I_R(x) = \frac{1}{1-R}\log\left[\int_0^\infty\left[\frac{f}{(x)^R}\right]\mathrm{d}x\right], R>0, R\neq 1. \tag{25}$$

The Rényi entropy of the NGEFWD can be given as

$$[f(x)]^R = \left[ab\theta k\beta\alpha^{\beta}\lambda^{k\beta}\right]^R x^{-R(k\beta+1)} e^{-R\left(\alpha(\lambda/x)^k\right)^{\beta}} \left[1 - e^{-\left(\alpha(\lambda/x)^k\right)^{\beta}}\right]^{R(a-1)} \times\left\{1 - \left[1 - e^{-\left(\alpha(\lambda/x)^k\right)^{\beta}}\right]^a\right\}^{R(b-1)} \left(1 - \left\{1 - \left[1 - e^{-\left(\alpha(\lambda/x)^k\right)^{\beta}}\right]^a\right\}^b\right)^{R(\theta-1)}. \quad (26)$$

By using the binomial expansion in equation (9) three times, we get

$$[f(x)]^R = \left[ab\theta k\beta\alpha^{\beta}\lambda^{k\beta}\right]^R \cdot \sum_{i,j,h=0}^{\infty} w_{ijhR} x^{-R(k\beta+1)} e^{-(h+R)\left(\alpha(\lambda/x)^k\right)^{\beta}}, \quad (27)$$

where

$$w_{\mathrm{ijhR}} = (-1)^{i+j+h}\binom{R(\theta-1)}{i}\binom{b(R+i)-R}{j}\binom{a(R+j)-R}{h}. \quad (28)$$

Thus,

$$\int_0^{\infty}[f(x)]^R \mathrm{d}x = [ab\theta]^R[k\beta]^{R-1}\alpha^{(1-R/k)}\lambda^{(1-R)}\sum_{i,j,h=0}^{\infty} w_{ijhR}\left(\frac{1}{h+R}\right)^{((R-1/k\beta)+R)}\Gamma\left(\frac{R-1}{k\beta}+R\right). \quad (29)$$

Then, the Rényi entropy of the NGEFWD can be obtained as

$$I_R(x) = \frac{R}{1-R}\log(ab\theta) + \log\left(\frac{\lambda}{k\beta}\right) + \frac{1}{k}\log(\alpha) + \frac{1}{1-R}\log\left[\sum_{i,j,h=0}^{\infty} w_{ijhR}\left(\frac{1}{h+R}\right)^{(R-1/k\beta+R)}\Gamma\left(\frac{R-1}{k\beta}+R\right)\right]. \quad (30)$$

5. Estimation of the NGEFWD Parameters

This section provides the estimation of the unknown parameters using the maximum likelihood technique, which is the most widely used estimation method. Let $x_1, x_2, \ldots, x_n$ be a random sample from the NGEFWD with unknown parameters $\Theta = (a, b, \theta, \alpha, \beta, k, \lambda)$; then, the log likelihood function is given as

$$\log L(\Theta) = n\log\left[ab\theta k\beta\alpha^{\beta}\lambda^{k\beta}\right] - (k\beta+1)\sum_{i=1}^{n}\log(x_i) - \alpha^{\beta}\lambda^{k\beta}\sum_{i=1}^{n}x_i^{-k\beta} + (a-1)\sum_{i=1}^{n}\log[1-\xi_i(\alpha,\beta,k,\lambda)] + (b-1)\sum_{i=1}^{n}\log\{1-[1-\xi_i(\alpha,\beta,k,\lambda)]^a\} + (\theta-1)\sum_{i=1}^{n}\log\left(1-\{1-[1-\xi_i(\alpha,\beta,k,\lambda)]^a\}^b\right). \quad (31)$$

By taking the first partial derivatives of the log likelihood function with respect to $a, b, \theta, \alpha, \beta, k,$ and λ, we obtain

$$\frac{\partial \log L(\Theta)}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \log\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]\left[1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}\right. \times\left(\frac{b-1}{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}} - \frac{b(\theta-1)\left\{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}\right\}^{b-1}}{1-\left\{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}\right\}^{b}}\right)\Bigg], \tag{32}$$

$$\frac{\partial \log L(\Theta)}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \log\left\{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}\right\} \times \left[1 - \frac{(\theta-1)\left\{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}\right\}^{b}}{1-\left\{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}\right\}^{b}}\right], \tag{33}$$

$$\frac{\partial \log L(\Theta)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log\left(1-\left\{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}\right\}^{b}\right), \tag{34}$$

$$\frac{\partial \log L(\Theta)}{\partial \alpha} = \beta\left\{\frac{n}{\alpha} - \alpha^{\beta-1}\lambda^{k\beta}\sum_{i=1}^{n} x_i^{-k\beta}\left[1-\xi_i(\alpha,\beta,k,\lambda)\left(\frac{a-1}{\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]} - a\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a-1}\right.\right.\right. \cdot\left(\frac{b-1}{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}} - \frac{b(\theta-1)\left\{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}\right\}^{b-1}}{1-\left\{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}\right\}^{b}}\right)\Bigg)\Bigg]\Bigg\}, \tag{35}$$

$$\frac{\partial \log L(\Theta)}{\partial \beta} = n\left(\frac{1}{\beta} + \log(\alpha) + k\log(\lambda)\right) - k\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\log\left(\alpha\left(\frac{\lambda}{x_i}\right)^{k}\right)\left(\alpha\left(\frac{\lambda}{x_i}\right)^{k}\right)^{\beta} \times\left\{1-\xi_i(\alpha,\beta,k,\lambda)\left[\frac{a-1}{\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]} - a\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a-1}\right.\right. \times\left(\frac{b-1}{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}} - \frac{b(\theta-1)\left\{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}\right\}^{b-1}}{1-\left\{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}\right\}^{b}}\right)\Bigg]\Bigg\}, \tag{36}$$

$$\frac{\partial \log L(\Theta)}{\partial k} = n\left(\frac{1}{k} + \beta\log(\lambda)\right) - \beta\left\{\sum_{i=1}^{n}\log(x_i) + \alpha^{\beta}\sum_{i=1}^{n}\left(\frac{\lambda}{x_i}\right)^{k\beta}\log\left(\frac{\lambda}{x_i}\right) \times\left[1-\xi_i(\alpha,\beta,k,\lambda)\left(\frac{(a-1)}{\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]}\right.\right.\right. - a\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a-1} \times\left(\frac{b-1}{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}} - \frac{b(\theta-1)\left\{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}\right\}^{b-1}}{1-\left\{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}\right\}^{b}}\right)\Bigg)\Bigg]\Bigg\}, \tag{37}$$

$$\frac{\partial \log L(\Theta)}{\partial \lambda} = k\beta\left\{\frac{n}{\lambda} - \alpha^{\beta}\lambda^{k\beta-1}\sum_{i=1}^{n} x_i^{-k\beta}\left[1-\xi_i(\alpha,\beta,k,\lambda)\times\left(\frac{a-1}{\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]} - a\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a-1}\right.\right.\right. \left(\frac{b-1}{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}} - \frac{b(\theta-1)\left\{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}\right\}^{b-1}}{1-\left\{1-\left[1-\xi_i(\alpha,\beta,k,\lambda)\right]^{a}\right\}^{b}}\right)\Bigg)\Bigg]\Bigg\}, \tag{38}$$

where $\xi_i(\alpha,\beta,k,\lambda) = e^{-(\alpha(\lambda/x_i)^{k})^{\beta}}$.

The MLEs of the unknown parameters can then be obtained by solving the system of nonlinear equations (32)–(38) numerically. Alternatively, equation (31) might be directly maximizing using an optimization technique in any software, such as the statistical R program.

6. Simulation Studies for the NGEFWD

In this section, some simulation studies are performed to examine the accuracy of the MLEs of the NGEFWD. The results were obtained by generating $N = 1000$ samples from the NGEFWD with different sample sizes, $n = 25, 50, 100, 200$, and 500, and with various cases for the true parameter values as

Case I: $a = 1.5, b = 5.0, \theta = 0.6, \alpha = 1.3, \beta = 0.6, k = 0.1, \lambda = 10.0$.

Case II: $a = 0.5, b = 0.9, \theta = 1.6, \alpha = 0.7, \beta = 1.6, k = 5.0, \lambda = 0.8$.

Case III: $a = 1.3, b = 5.5, \theta = 0.8, \alpha = 1.6, \beta = 0.9, k = 0.06, \lambda = 7.0$.

The quantile function in equation (12) is applied to generate random samples from the NGEFWD where u is uniformly distributed. The mean square error (MSE) and the root mean square error (RMSE) were computed for each parameter in order to investigate its accuracy using the following relations:

$$\mathrm{MSE} = \mathrm{var}(\hat{\theta}) + [\mathrm{Bias}(\hat{\theta})]^2 = \frac{1}{N}\sum_{i=1}^{n}\left(\hat{\theta} - \theta_{tr}\right)^2, \quad (39)$$

where Bias $= 1/N\sum_{i=1}^{n}(\hat{\theta} - \theta_{tr})$.

$$\mathrm{RMSE} = \sqrt{\mathrm{MSE}} = \sqrt{\frac{1}{N}\sum_{i=1}^{n}\left(\hat{\theta} - \theta_{tr}\right)^2}, \quad (40)$$

where N is the number of generated samples, n is the size for each sample, $\hat{\theta}$ is the MLE, and θ_{tr} is the true value of each parameter.

From Table 1, it can be seen that when the sample size n increases, the MLEs become closer to the true value of parameters, and hence the MSE and RMSE decrease and tend to zero. The results demonstrate that the maximum likelihood method provides an accurate estimation of the parameters for the NGEFWD.

7. Applications for the NGEFWD

In this section, some applications of the NGEFWD are provided to illustrate its usefulness, using three real datasets. The goodness of fit of the NGEFWD is compared with some of its submodels and a related distribution. Specifically, the fit of NGEFWD is compared to the following distributions.

(i) The Weibull distribution (WD) with pdf as

$$g(x; k, \lambda) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}, x \geq 0. \quad (41)$$

(ii) The Fréchet–Weibull distribution (FWD) with pdf in equation (3).

(iii) The exponentiated Fréchet–Weibull distribution (EFWD) with pdf as

$$f(x) = ak\beta\alpha^{\beta}\lambda^{k\beta}x^{-(k\beta+1)}e^{-(\alpha(x/\lambda)^k)^{\beta}}\left[1 - e^{(\alpha(x/\lambda)^k)^{\beta}}\right]^{a-1}, x > 0. \quad (42)$$

(iv) The exponentiated generalized Fréchet–Weibull distribution (EGFWD) with pdf as

$$f(x) = abk\beta\alpha^{\beta}\lambda^{k\beta}x^{-(k\beta+1)}e^{-(\alpha(x/\lambda)^k)^{\beta}}\left[1 - e^{(\alpha(x/\lambda)^k)^{\beta}}\right]^{a-1} \times\left\{1 - \left[1 - e^{-(\alpha(\lambda/x)^k)^{\beta}}\right]^{a}\right\}^{b-1}, x > 0. \quad (43)$$

(v) The Kumaraswamy–Weibull–Burr XII distribution (KWBXIID) in [41] with pdf as

$$f(x) = \mathrm{abc}\alpha\beta\sigma\mu^{-c}x^{c-1}e^{-\alpha\left[\left(1+(x/\mu)^c\right)^{\sigma}-1\right]^{\beta}}\left(1 + \left(\frac{x}{\mu}\right)^{c}\right)^{\sigma-1}\left[\left(1 + \left(\frac{x}{\mu}\right)^{c}\right)^{\sigma} - 1\right]^{\beta-1} \times\left[1 - e^{-\alpha\left[\left(1+(x/\mu)^c\right)^{\sigma}-1\right]^{\beta}}\right]^{a-1}\left\{1 - \left[1 - e^{-\alpha\left[\left(1+(x/\mu)^c\right)^{\sigma}-1\right]^{\beta}}\right]^{a}\right\}^{b-1}, x > 0. \quad (44)$$

The comparison is based on some different criteria, namely, the negative log likelihood function ($-\mathrm{Log}L$), the Akaike information criterion (AIC), the consistent Akaike information criteria (CAIC), the Bayesian information criteria (BIC), Hannan–Quinn information criterion (HQIC), and the Kolmogorov–Smirnov (KS) statistic as $D_n = \sup_x \{|F_n(x) - F(x \mid \Theta)|\}$ with its p value.

The best model to fit data is the model with lowest values of AIC, CAIC, BIC, HQIC and KS and highest p value. The MLEs of the model parameters were computed by using "optim" function in R program. Furthermore, the observed frequencies for the data are plotted and compared with the expected frequencies for each model. Tables 2–4 summarize each dataset while the results of the analyzed datasets are reported in Tables 5–7 and Figures 4–6.

7.1. First Dataset. We will consider the dataset discussed in [42], which represents the ages of 155 patients suffering from breast tumors from June to October in 2014.

Table 1: Simulation study: MLEs, MSE, and RMSE for three different cases with different sample sizes.

Sample size	Parameter	Case I Estimate	Case I MSE	Case I RMSE	Case II Estimate	Case II MSE	Case II RMSE	Case III Estimate	Case III MSE	Case III RMSE
	a	1.3302627	0.2310003	0.4806249	0.7098322	0.22594640	0.4753382	1.0341103	0.3119100	0.5584891
	b	5.0254171	0.3790335	0.6156570	0.9549841	0.21674231	0.4655559	5.5650413	0.4904028	0.7002877
	θ	0.6497745	0.1248266	0.3533080	1.5756128	0.28784549	0.5365123	0.9003465	0.1797108	0.42392311
$n = 25$	α	1.3687668	0.2749363	0.5243437	0.7904611	0.21949550	0.4685035	1.7381738	0.3727898	0.6105651
	β	0.4638483	0.1751859	0.4185522	1.5508786	0.17313109	0.4160902	0.7285084	0.2455925	0.4955729
	k	0.1850151	0.1279683	0.3577266	4.9833872	0.30380098	0.5511814	0.1605225	0.1081410	0.3288479
	λ	10.0304642	0.4350544	0.6595865	0.8604375	0.05077811	0.2253400	7.0393220	0.4453141	0.6673186
	a	1.3841632	0.14444984	0.3800656	0.5824516	0.07897752	0.2810294	1.0895784	0.20193324	0.4493698
	b	5.0318294	0.24107254	0.4909914	0.9173159	0.10051765	0.3170452	5.5601210	0.27597257	0.5253309
	θ	0.6210948	0.06582067	0.2565554	1.5858719	0.15269341	0.3907600	0.8800108	0.10854783	0.3294660
$n = 50$	α	1.3560579	0.15792424	0.3973968	0.7435987	0.09881569	0.3143496	1.6856973	0.23470139	0.4844599
	β	0.5001187	0.08632563	0.2938122	1.5736894	0.09046936	0.3007812	0.7431364	0.17562871	0.4190808
	k	0.1873957	0.06652333	0.2579212	4.9759597	0.15078465	0.3883100	0.1509327	0.07580858	0.2753336
	λ	10.0076624	0.21310196	0.4616297	0.8308467	0.02330085	0.1526462	6.9812627	0.24297030	0.4929202
	a	1.4275938	0.07556618	0.2748930	0.5530802	0.04951151	0.22251118	1.18166020	0.12340647	0.3512926
	b	5.0105125	0.12315316	0.3509318	0.9077659	0.06344974	0.2518923	5.52253129	0.15642598	0.3955072
	θ	0.6241116	0.03440801	0.1854940	1.5655599	0.13531506	0.3678520	0.84968161	0.07704126	0.2775631
$n = 100$	α	1.3310322	0.07727870	0.2779905	0.7221272	0.07152993	0.2674508	1.64853786	0.13237073	0.3638279
	β	0.5478059	0.04677035	0.2162645	1.5811369	0.06879237	0.2622830	0.79451929	0.10964770	0.3311309
	k	0.1408675	0.03109033	0.1763245	4.9988484	0.09047408	0.3007891	0.09807134	0.02780139	0.1667375
	λ	9.9897468	0.11598187	0.3405611	0.8205469	0.01392076	0.1179863	7.00767353	0.14686000	0.3832232
	a	1.4717346	0.03085088	0.1756442	0.5157006	0.01769658	0.1330285	1.22406740	0.06414683	0.2532722
	b	5.0083261	0.04544610	0.2131809	0.8921629	0.01937106	0.1391799	5.50970835	0.06811091	0.2609807
	θ	0.6084602	0.01529972	0.1236920	1.5939706	0.04259567	0.2063872	0.82393319	0.02626152	0.1620541
$n = 200$	α	1.3152695	0.03916740	0.1979076	0.7007405	0.02085480	0.1444119	1.62521423	0.05286219	0.2299178
	β	0.5552519	0.02942743	0.1715442	1.5973126	0.02007826	0.1416978	0.84287738	0.05629045	0.2372561
	k	0.1333687	0.01846685	0.1358928	0.9871575	0.02512169	0.1584982	0.08296335	0.01068387	0.1033628
	λ	10.0031101	0.05145884	0.2268454	0.8135793	0.01158567	0.1076367	6.99074715	0.07438004	0.2727270
	a	**1.48991752**	**0.008830536**	**0.09397093**	**0.5069023**	**0.009870441**	**0.09935009**	**1.27064558**	**0.022093272**	**0.14863806**
	b	**5.0010744**	**0.018112638**	**0.13458320**	**0.9005842**	**0.006237313**	**0.07897666**	**5.50426758**	**0.0240531109**	**0.15509065**
	θ	**0.6039007**	**0.004857594**	**0.06969644**	**1.5938484**	**0.036725101**	**0.19163794**	**0.81195641**	**0.010164072**	**0.10081702**
$n = 500$	α	**1.3086590**	**0.011894367**	**0.10906130**	**0.6984757**	**0.006347208**	**0.07966937**	**1.60616706**	**0.018561516**	**0.13624066**
	β	**0.5797716**	**0.010918925**	**0.10449366**	**1.5961551**	**0.011755609**	**0.10842329**	**0.87472187**	**0.019218204**	**0.13862974**
	k	**0.1108573**	**0.003060211**	**0.05531918**	**4.9989203**	**0.006693424**	**0.08181335**	**0.06979647**	**0.003304211**	**0.05748226**
	λ	**9.9995578**	**0.016498606**	**0.12844690**	**0.8049456**	**0.004936801**	**0.07026238**	**6.98928041**	**0.019606893**	**0.14002462**

If the number of clusters found out is not correct, it is in italics. The best benchmark is written in bold in the condition of right cluster number.

Table 2: Descriptive statistics of the first dataset.

Sample size	Mean	Median	IQR	SD	Skewness	Kurtosis
155	43.6500	42.0000	14.0000	12.0855	0.6847	5.1627

Table 3: Descriptive statistics of the second dataset.

Sample size	Mean	Median	IQR	SD	Skewness	Kurtosis
202	69.0200	58.6000	46.5000	32.5653	1.1747	4.3651

Table 4: Descriptive statistics of the third dataset.

Sample size	Mean	Median	IQR	SD	Skewness	Kurtosis
101	133.8000	132.5000	26.5000	22.6134	0.3715	4.1199

Table 5: The goodness-of-fit measures for the first dataset.

Distribution	−LogL	AIC	CAIC	BIC	HQIC	KS	*P* value
WD	−635.8449	1275.690	1283.777	1281.777	1278.162	0.3186704	$4.252154e-14$
FWD	−646.5227	1301.045	1317.219	1313.219	1305.990	0.1729225	$1.884781e-04$
EFWD	−616.4790	1242.958	1263.175	1258.175	1249.139	0.1346185	$7.264767e-03$
EGFWD	−622.9793	1257.959	1282.219	1276.219	1265.376	0.1303286	$1.033342e-02$
KWBXIID	−662.5651	1339.130	1367.434	1360.434	1347.783	0.2317489	$1.175748e-07$
NGEFWD	−608.8457	1231.691	1259.995	1252.995	1240.345	0.1070294	$5.738245e-02$

Table 6: The goodness-of-fit measures for the second dataset.

Distribution	−LogL	AIC	CAIC	BIC	HQIC	KS	*P* value
WD	−975.2433	1954.487	1963.103	1961.103	1957.164	0.1007178	0.033204688
FWD	−965.6735	1939.347	1956.580	1952.580	1944.701	0.1320700	0.001740441
EFWD	−960.1731	1930.346	1951.888	1946.888	1937.039	0.1151473	0.009433857
EGFWD	−987.2047	1986.409	2012.259	2006.259	1994.441	0.1265340	0.003103469
KWBXIID	−960.2395	1934.479	1964.637	1957.637	1943.849	0.07766784	0.174720566
NGEFWD	−953.3925	1920.785	1950.943	1943.943	1930.155	0.07446313	0.212642709

Table 7: The goodness-of-fit measures for the third dataset.

Distribution	−LogL	AIC	CAIC	BIC	HQIC	KS	*P* value
WD	−527.4639	1058.9278	1066.1382	1064.1382	1061.0366	0.5350169	$<2.2e-16$
FWD	−470.9345	949.8690	964.2897	960.2897	954.0864	0.1323509	0.0601910
EFWD	−458.5464	927.0928	945.1186	940.1186	932.3646	0.1192493	0.1163526
EGFWD	−458.5728	929.1456	950.7766	944.7766	935.4718	0.1049159	0.2209811
KWBXIID	−455.1496	924.2993	949.5354	942.5354	931.6798	0.0835741	0.4872319
NGEFWD	−452.8640	919.7280	944.9642	937.9642	927.1085	0.0654688	0.7847161

7.2. Second Dataset. This dataset was discussed in [43] in which it contains sums of skin folds in 202 athletes collected at the Australian Institute of Sport.

7.3. Third Dataset. Data from [44], representing the fatigue times of 6061-T6 aluminum coupons comprising 101 observations with maximum stress per cycle of 31,000 psi, is considered.

From Tables 5–7, it can be seen that the NGEFWD is the best model to fit all of the considered datasets in which it has the smallest AIC, CAIC, BIC, HQIC, and KS and the largest *P* value. Also, from Figures 4–6, it is clear that the NGEFWD is the closest to the actual distribution of all data. Thus, the NGEFWD can be considered as the best model for all real datasets considered.

8. The Log New Generalized Exponentiated Fréchet–Weibull Regression Model

Assume that X is a random variable from the NGEFWD given in equation (7) and let $Y = \log(x)$. Then, the cdf and pdf of the log new generalized exponentiated Fréchet–Weibull (LNGEFW) regression model with the transformation parameters $\mu = \log(\lambda)$ and $\sigma = 1/k$ can be expressed, respectively, as

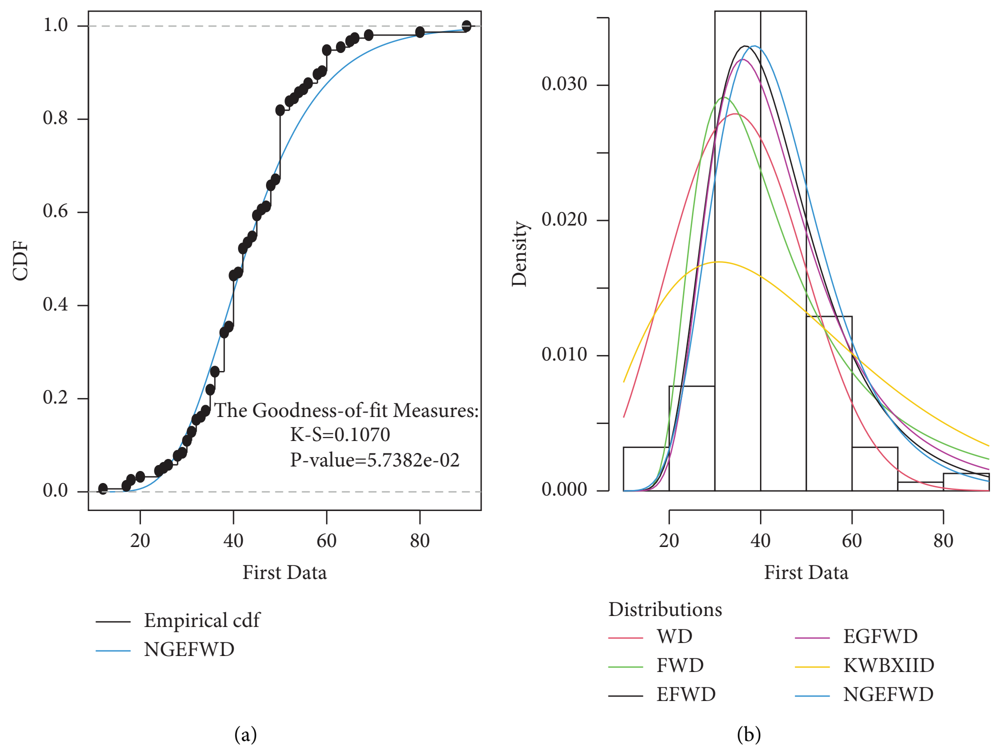


Figure 4: Comparison of NGEFWD with the other distributions for the first dataset. (a) cdf for NGEFWD. (b) Observed and expected frequencies for each model.

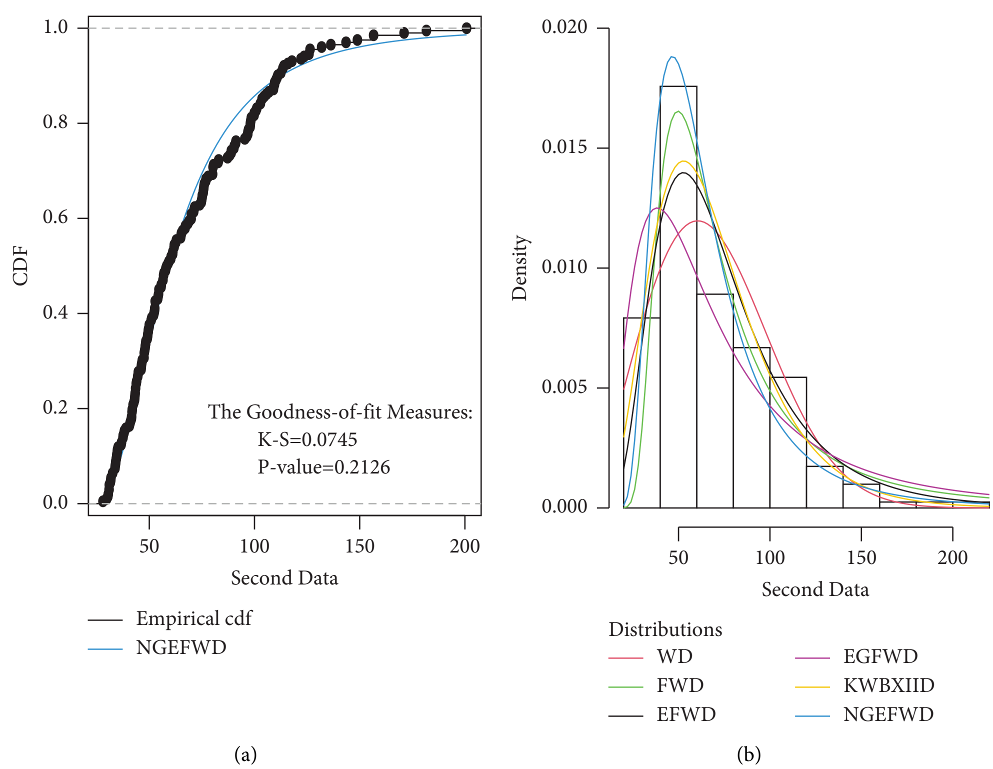


Figure 5: Comparison of NGEFWD with the other distributions for the second dataset. (a) cdf for NGEFWD. (b) Observed and expected frequencies for each model.

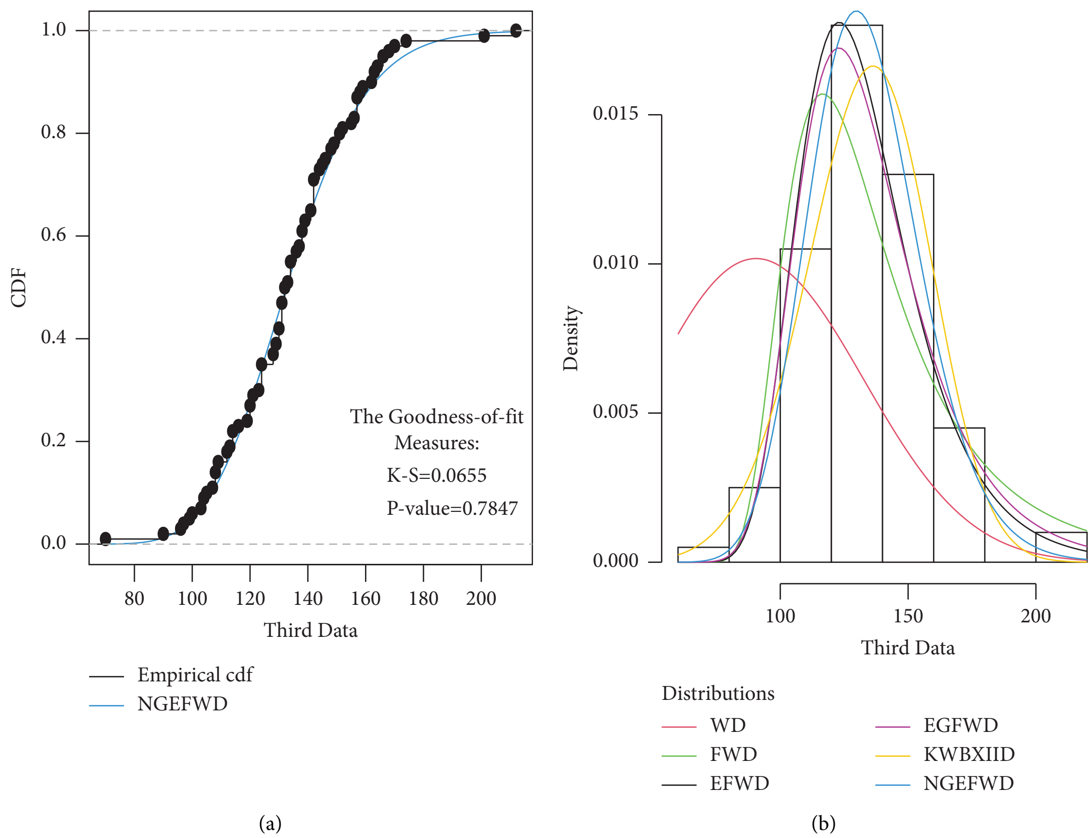


FIGURE 6: Comparison of NGEFWD with the other distributions for the third dataset. (a) cdf for NGEFWD. (b) Observed and expected frequencies for each model.

$$F(y) = 1 - \left(1 - \left\{1 - \left[1 - e^{-\alpha^{\beta} e^{-\beta(y-\mu/\sigma)}}\right]^{a}\right\}^{b}\right)^{\theta}, -\infty < y < \infty,$$

$$f(y) = \frac{ab\theta\alpha^{\beta}\beta}{\sigma} e^{-\beta(y-\mu/\sigma)} e^{-\alpha^{\beta} e^{-\beta(y-\mu/\sigma)}} \left[1 - e^{-\alpha^{\beta} e^{-\beta(y-\mu/\sigma)}}\right]^{a-1} \left\{1 - \left[1 - e^{-\alpha^{\beta} e^{-\beta(y-\mu/\sigma)}}\right]^{a}\right\}^{b-1} \times \left(1 - \left\{1 - \left[1 - e^{-\alpha^{\beta} e^{-\beta(y-\mu/\sigma)}}\right]^{a}\right\}^{b}\right)^{\theta-1}, -\infty < y < \infty, \tag{45}$$

where $\alpha, \sigma > 0$ are the scale parameters, $a, b, \theta, \beta > 0$ are the shape parameters, and $-\infty < \mu < \infty$ is the location parameter.

The survival function of the LNGEFW regression model is given as

$$SF(y) = \left(1 - \left\{1 - \left[1 - e^{-\alpha^{\beta} e^{-\beta(y-\mu/\sigma)}}\right]^{a}\right\}^{b}\right)^{\theta}. \tag{46}$$

The standardized random variable $z = y - \mu/\sigma$ has the following pdf:

$$f(z) = ab\theta\alpha^{\beta}\beta e^{-\beta z} e^{-\alpha^{\beta} e^{-\beta z}} \left[1 - e^{-\alpha^{\beta} e^{-\beta z}}\right]^{a-1} \left\{1 - \left[1 - e^{-\alpha^{\beta} e^{-\beta z}}\right]^{a}\right\}^{b-1} \times \left(1 - \left\{1 - \left[1 - e^{-\alpha^{\beta} e^{-\beta z}}\right]^{a}\right\}^{b}\right)^{\theta-1}, \tag{47}$$

with survival function as

Table 8: Simulation study: parameter estimates, MSE, and RMSE for two different cases with different sample sizes.

Sample size	Censoring levels	Parameter	Case I			Case II		
			Estimate	MSE	RMSE	Estimate	MSE	RMSE
	0.1		2.9043933	2.6614730	1.6314022	5.6845956	1.03592587	1.0178044
	0.3	$v_{\circ}$	2.9278856	2.7395549	1.6551601	5.5772473	1.03927840	1.0194500
	0.5		2.901076	3.4508950	1.8576585	5.542598	1.29479051	1.1378886
	0.1		2.7206096	0.9024097	0.9499524	5.8380756	0.90159038	0.9495211
	0.3	v_1	2.6354913	0.9860313	0.9929911	5.8191731	0.86893008	0.9321642
	0.5		2.562997	1.1834458	1.0878630	5.745187	1.17554787	1.0842269
	0.1		3.8970807	1.9223031	1.3864715	4.3927280	0.87424540	0.9350109
	0.3	σ	3.9002495	2.2882914	1.5127100	4.3459740	0.95543534	0.9774637
	0.5		3.907139	2.0797213	1.4421239	4.378631	1.04485774	1.0221828
	0.1		3.0254633	1.7874736	1.3369643	1.8655000	0.56437980	0.7512522
	0.3	a	3.1408139	2.0114723	1.4182638	1.9165543	0.60654664	0.7788110
$n = 25$	0.5		3.065488	1.9679849	1.4028489	1.974780	0.81995551	0.9055513
	0.1		3.9195261	2.6839551	1.6382781	1.2829982	0.36081994	0.6006829
	0.3	b	3.8973590	2.6720920	1.6346535	1.2683916	0.39114900	0.6254191
	0.5		3.908507	3.0387139	1.7431907	1.205630	0.45243282	0.6726313
	0.1		4.7249332	2.2070034	1.4855987	5.8717707	0.96394350	0.9818062
	0.3	θ	4.7779629	2.4544725	1.5666756	5.8890157	1.04411668	1.0218203
	0.5		4.704226	2.9311686	1.7120656	5.972209	1.25457336	1.1200774
	0.1		0.9885457	0.5672057	0.7531306	2.7996376	0.79562811	0.8949799
	0.3	α	1.0072347	0.6970529	0.8348969	2.7615884	0.86115263	0.9278721
	0.5		0.951108	0.6907800	0.8311318	2.797765	1.01185428	1.0048097
	0.1		0.9387424	0.3240579	0.5692608	0.4608604	0.02841255	0.1685602
	0.3	β	0.9558267	0.3068450	0.5539359	0.4712861	0.03310275	0.1819416
	0.5		1.047202	0.4502492	0.6710061	0.504577	0.05122002	0.2263184
	0.1		3.0227126	1.6715526	1.2928854	5.6415868	0.61427315	0.7837558
	0.3	$v_{\circ}$	2.9777252	1.9567024	1.3988218	5.6139958	0.62181675	0.7885536
	0.5		3.0016343	2.3167199	1.5220775	5.578085	0.75130644	0.8667793
	0.1		2.7972976	0.4732802	0.6879536	5.7993681	0.48862527	0.6990174
	0.3	v_1	2.6202428	0.6206988	0.7878444	5.7793403	0.58021786	0.7617203
	0.5		2.6255316	0.6337823	0.7961045	5.798081	0.78124596	0.8838812
	0.1		3.8318455	1.3271562	1.1520227	4.3111049	0.54353314	0.7372470
	0.3	σ	3.8049443	1.5086522	1.2282720	4.2782971	0.64956723	0.8059573
	0.5		3.8517816	1.6567050	1.2871305	4.336392	0.75183506	0.8670842
	0.1		2.9165939	1.1331452	1.0644929	1.8412220	0.35719848	0.5976608
	0.3	a	3.0396639	1.5057311	1.2270823	1.8927729	0.42411431	0.6512406
$n = 50$	0.5		3.0317066	1.6512523	1.2850106	1.918655	0.53085885	0.7286006
	0.1		3.9110509	1.6909376	1.3003606	1.2432150	0.21843222	0.4673673
	0.3	b	3.8941866	2.0086997	1.4172860	1.2144004	0.24390234	0.4938647
	0.5		3.9938509	1.9639256	1.4014013	1.180116	0.26662241	0.5163549
	0.1		4.6562724	1.4085955	1.1868426	5.8760465	0.57177868	0.7561605
	0.3	θ	4.7046446	1.6316498	1.2773605	5.8869475	0.69104106	0.8312888
	0.5		4.7249758	2.0666813	1.4375957	5.876770	0.76012718	0.8718527
	0.1		0.8840577	0.3791098	0.6157139	2.8920951	0.56118532	0.7491230
	0.3	α	0.9480049	0.5208052	0.7216683	2.8461792	0.60497547	0.7778017
	0.5		0.8964323	0.5162093	0.7184771	2.817710	0.75980550	0.8716682
	0.1		0.8436658	0.1422018	0.3770966	0.4381209	0.01394487	0.1180884
	0.3	β	0.8734317	0.1755586	0.4189971	0.4501908	0.01819325	0.1348824
	0.5		0.9203076	0.2561941	0.5061562	0.472226	0.02684176	0.1638346

Table 8: Continued.

Sample size	Censoring levels	Parameter	Case I			Case II		
			Estimate	MSE	RMSE	Estimate	MSE	RMSE
$n = 100$	0.1	$v_{\circ}$	3.1642139	1.04054128	1.0200693	5.677084	0.30353040	0.55093593
	0.3		3.2158323	1.2740339	1.1287311	5.6785632	0.356225538	0.59684633
	0.5		3.1032675	1.3248641	1.1510274	5.6374634	0.47862084	0.6918243
	0.1	v_1	2.7737849	0.27107984	0.5206533	5.8510653	0.29469539	0.54285854
	0.3		2.7359416	0.3179754	0.5638931	5.8165178	0.358101215	0.59841559
	0.5		2.7021140	0.3600679	0.6000566	5.7912564	0.36851783	0.6070567
	0.1	σ	3.7305464	0.76113013	0.8724277	4.2741117	0.31611624	0.56224215
	0.3		3.8017303	0.9344663	0.9666780	4.2505760	0.345111102	0.58746158
	0.5		3.7613606	1.1286111	1.0623611	4.2881742	0.40169096	0.6337909
	0.1	a	2.8308030	0.81054129	0.9003007	1.8115091	0.20252678	0.45002976
	0.3		2.8861780	0.9511048	0.9752460	1.8116953	0.224020403	0.47330794
	0.5		2.8958280	0.9760603	0.9879577	1.8592521	0.35632134	0.5969266
	0.1	b	4.0071315	0.96300670	0.9813290	1.2258156	0.10773646	0.32823233
	0.3		4.0137198	1.2121655	1.1009839	1.1940681	0.128708063	0.35875906
	0.5		4.0183085	1.3535788	1.1634341	1.1718397	0.15466014	0.3932685
	0.1	θ	4.6898522	0.83929346	0.9161296	5.8965239	0.29650077	0.54451885
	0.3		4.7262508	1.1605253	1.0772768	5.9120494	0.341166165	0.58409431
	0.5		4.7543391	1.2699561	1.1269233	5.9100774	0.45371531	0.6735839
	0.1	α	0.7996097	0.21342963	0.4619845	2.8846715	0.29980339	0.54754305
	0.3		0.8015444	0.2309840	0.4806079	2.8549099	0.323736521	0.56897849
	0.5		0.8068490	0.258196	0.5038448	2.8584534	0.39246868	0.6264732
	0.1	β	0.7822646	0.07032744	0.2651932	0.4210997	0.00602759	0.07763756
	0.3		0.8091867	0.1080392	0.328691	0.4335915	0.008939374	0.094544826
	0.5		0.8263019	0.1029935	0.3209260	0.4457516	0.01392451	0.1180022
$n = 200$	0.1	$v_{\circ}$	3.2831027	0.46584873	0.6825311	5.707396	0.13915172	0.37303045
	0.3		3.2625632	0.57806441	0.7603055	5.6912210	0.170022789	0.41233820
	0.5		3.2084384	0.76207639	0.8729699	5.6863497	0.263120855	0.51295307
	0.1	v_1	2.8172459	0.12775621	0.3574300	5.889795	0.11782954	0.34326307
	0.3		2.7883470	0.16657634	0.4081376	5.8774773	0.165930661	0.40734587
	0.5		2.7519225	0.23381239	0.4835415	5.8546933	0.214052948	0.46265857
	0.1	σ	3.6693791	0.39187763	0.6260013	4.238972	0.15433716	0.39285768
	0.3		3.6684719	0.52083528	0.7216892	4.2337038	0.178340055	0.42230327
	0.5		3.6953952	0.65139175	0.8070884	4.2364246	0.255737369	0.50570482
	0.1	a	2.7640314	0.39978511	0.6322856	1.783730	0.14312302	0.37831603
	0.3		2.8216567	0.51737420	0.7192873	1.8020098	0.134272860	0.36643261
	0.5		2.8106163	0.63390234	0.7961798	1.8033532	0.176685916	0.42034024
	0.1	b	4.1060451	0.52692668	0.7258972	1.219803	0.07007836	0.26472317
	0.3		4.1282699	0.63700072	0.7981232	1.2061720	0.062227251	0.24945391
	0.5		4.0730701	0.86067050	0.9277233	1.1949768	0.092675894	0.30442716
	0.1	θ	4.6977551	0.39134128	0.6255728	5.893700	0.14749300	0.38404817
	0.3		4.6452835	0.61272306	0.7827663	5.9075710	0.210778914	0.45910665
	0.5		4.6814896	0.67242297	0.8200140	5.8929179	0.254179561	0.504162224
	0.1	α	0.7183217	0.09819633	0.3133629	2.886101	0.13731345	0.37055829
	0.3		0.7252527	0.11583277	0.3403421	2.8868772	0.168593355	0.41060121
	0.5		0.7352590	0.12356240	0.3515144	2.8517374	0.223913097	0.47319457
	0.1	β	0.7347036	0.03328719	0.1824478	0.410907	0.00301374	0.05489754
	0.3		0.7450192	0.03871637	0.1967648	0.4150026	0.003782556	0.06150249
	0.5		0.7711087	0.05354971	0.2314081	0.4222837	0.005523873	0.07432276

Table 8: Continued.

Sample size	Censoring levels	Parameter	Case I			Case II		
			Estimate	MSE	RMSE	Estimate	MSE	RMSE
$n = 500$	0.1	$\upsilon_\circ$	3.3597835	0.141368723	0.37599032	5.7018866	0.0462670611	0.21509779
	0.3		3.3471661	0.22589030	0.4752792	5.7044293	0.071697492	0.26776387
	0.5		3.3204382	0.25960322	0.5095127	5.6891000	0.097119333	0.31163975
	0.1	υ_1	2.8641073	0.049095482	0.22157500	5.8845428	0.0352950108	0.18786966
	0.3		2.8473067	0.06493988	0.2548330	5.8733283	0.060251605	0.24546202
	0.5		2.8243258	0.08860445	0.2976650	5.8757648	0.083550678	0.28905134
	0.1	σ	3.6122456	0.152271097	0.39021929	4.218779	0.0468807311	0.21651959
	0.3		3.6335732	0.20466201	0.4523939	4.2107234	0.065322095	0.25558188
	0.5		3.6241897	0.29808154	0.5459684	4.2190077	0.105580797	0.32493199
	0.1	a	2.6790965	0.141086199	0.37561443	1.7378668	0.0335137365	0.18306757
	0.3		2.6730413	0.17483708	0.4131352	1.7653428	0.061942796	0.24888310
	0.5		2.7004400	0.25357107	0.5035584	1.7422480	0.062046756	0.24909186
	0.1	b	4.1654236	0.145740752	0.38176007	1.2078451	0.0183719520	0.13554317
	0.3		4.1327885	0.24877179	0.4987703	1.2093048	0.031615916	0.17780865
	0.5		4.0795612	0.35826402	0.5985516	1.1864855	0.037784216	0.19438162
	0.1	θ	4.6341592	0.145598521	0.38157374	5.9088753	0.0404305785	0.20107356
	0.3		4.6510384	0.20518025	0.4529683	5.8985200	0.073209057	0.27057172
	0.5		4.6403844	0.32043231	0.5660674	5.9014690	0.088708047	0.297838896
	0.1	α	0.6425492	0.022786799	0.15095297	2.9058145	0.0556298080	0.23585972
	0.3		0.6495571	0.03418508	0.1848921	2.8936659	0.058059832	0.24095608
	0.5		0.6643961	0.04945985	0.2223957	2.8826119	0.098654934	0.31409383
	0.1	β	0.7075132	0.009596957	0.09796406	0.4038434	0.0008223338	0.02867636
	0.3		0.7239492	0.01304295	0.1142058	0.4060673	0.001329486	0.03646212
	0.5		0.7351679	0.02183635	0.1477219	0.4119129	0.002277972	0.04772811

$$SF(z) = \left(1 - \left\{1 - \left[1 - e^{-\alpha^{\beta} e^{-\beta z}}\right]^{a}\right\}^{b}\right)^{\theta}. \tag{48}$$

Based on the LNGEFW regression model, a linear regression model can be defined as

$$y_i = \upsilon \eta_i^T + \sigma z_i, i = 1, 2, 3, \ldots, n, \tag{49}$$

where z_i is the random error with pdf in equation (47), $\upsilon = (\upsilon_1, \ldots, \upsilon_p)^T$, $\sigma > 0$, $a > 0$, $b > 0$, $\theta > 0$, $\alpha > 0$, and $\beta > 0$ are unknown parameters, and $\eta_i^T = (\eta_{i1}, \ldots, \eta_{ip})^T$ is the explanatory variable vector. The parameter $\mu_i = \upsilon \eta_i^T$ is the location of y_i. Then, the location parameter vector $\mu = (\mu_1, \ldots, \mu_n)^T$ is defined as a linear model $\mu = \upsilon\eta$, where $\eta = (\eta_1, \ldots, \eta_n)^T$ is a known model matrix.

8.1. Maximum Likelihood Estimation of the LNGEFW Regression Model. For the right-censored lifetime data, let $(y_1, \eta_1), \ldots, (y_n, \eta_n)$ be random sample of n observations where each random response variable is obtained as $y_i = \min\{\log(x_i), \log(C_i)\}$. Let f_i be the log-lifetime and C_i be the log-censoring time which are independent and random; then, the likelihood function $L(\Theta)$ for the parameter vector $\Theta = (a, b, \theta, \alpha, \beta, \sigma, \upsilon^T)$ is given as

$$L(\Theta) = \prod_{i=1}^{n} \left\{ f(y_i)^{\tau_i} SF(y_i)^{1-\tau_i} \right\}, \tag{50}$$

where $\tau_i = \begin{cases} 1 & \text{if } y_i = \log(f_i) \\ 0 & \text{if } y_i = \log(C_i) \end{cases}$ is the indicator random variable. Then, the log likelihood function can be obtained as

$$\log L(\Theta) = \sum_{i=1}^{n} \left\{ \tau_i \log \frac{1}{\sigma} + \tau_i \log f_0(z_i) + (1 - \tau_i) \log SF_0(z_i) \right\}, \tag{51}$$

where $f(z_i)$ and $SF(z_i)$ are given in equations (47) and (48), respectively. Thus, we have

$$\begin{aligned} \log L(\Theta) = {} & r \log\left(ab\theta\alpha^{\beta}\beta\right) - r\log(\sigma) - \sum_{i=1}^{n} \tau_i \left(\beta z_i + \alpha^{\beta} e^{-\beta z_i}\right) + (a-1) \sum_{i=1}^{n} \tau_i \log\left[1 - e^{-\alpha^{\beta} e^{-\beta z_i}}\right] \\ & + (b-1) \sum_{i=1}^{n} \tau_i \log\left\{1 - \left[1 - e^{-\alpha^{\beta} e^{-\beta z_i}}\right]^{a}\right\} + (\theta - 1) \sum_{i=1}^{n} \tau_i \log\left(1 - \left\{1 - \left[1 - e^{-\alpha^{\beta} e^{-\beta z_i}}\right]^{a}\right\}^{b}\right) \\ & + \theta \sum_{i=1}^{n} (1 - \tau_i) \log\left(1 - \left\{1 - \left[1 - e^{-\alpha^{\beta} e^{-\beta z_i}}\right]^{a}\right\}^{b}\right), \end{aligned} \tag{52}$$

where r denotes the number of uncensored observations and $z_i = (y_i - \upsilon\eta_i^T)/\sigma$.

The MLE of the parameter vector $\Theta = (a, b, \theta, \alpha, \beta, \sigma, \upsilon^T)^T$ can be obtained by maximizing the log

Table 9: MLEs and SEs in () for the voltage data.

Distribution	MLE and SEs in ()							
LBXII	$\widehat{\upsilon}_{\circ} = 10.112274115$ (6.2252052)	$\widehat{\upsilon}_1 = -0.008535644$ (0.1259612)	$\widehat{\sigma} = 0.969051924$ (0.1863925)	$\widehat{k} = 14.494247364$ (10.2626716)	—	—	—	—
LTLF	$\widehat{\upsilon}_{\circ} = 16.2169778$ (5.3776494)	$\widehat{\upsilon}_1 = -0.2546604$ (0.1100871)	$\widehat{\sigma} = 2.8660289$ (1.0869977)	$\widehat{\alpha} = 24.7915560$ (5.2852845)	—	—	—	—
LTLGW	$\widehat{\upsilon}_{\circ} = 8.8649865$ (7.732865)	$\widehat{\upsilon}_1 = -0.4161175$ (0.215204)	$\widehat{\sigma} = 12.5239339$ (6.996055)	$\widehat{\alpha} = 26.1415810$ (20.959309)	$\widehat{\theta} = 32.2899846$ (27.627779)	—	—	—
LNGEFW	$\widehat{\upsilon}_{\circ} = \mathbf{2.52736032}$ **(5.93905411)**	$\widehat{\upsilon}_1 = \mathbf{-0.01301879}$ **(0.07516281)**	$\widehat{\sigma} = \mathbf{1.96287994}$ **(1.91600978)**	$\widehat{a} = \mathbf{0.33685012}$ **(0.89320700)**	$\widehat{b} = \mathbf{2.01869220}$ **(0.19374290)**	$\widehat{\theta} = \mathbf{3.76597072}$ **(8.75756093)**	$\widehat{\alpha} = \mathbf{5.61822064}$ **(2.40028946)**	$\widehat{\beta} = \mathbf{2.39417980}$ **(2.65229628)**

The bold values represent the best results in the tables.

Table 10: The information criteria for the voltage data.

Distribution	AIC	CAIC	BIC	HQIC
LBXII	190.8785	203.2559	199.2559	194.1553
LTLF	193.8972	206.2746	202.2746	197.1741
LTLGW	189.3901	204.8619	199.8619	193.4862
LNGEFW	178.4749	203.2296	195.2296	185.0286

likelihood function in equation (52). The "optim" function in the statistical program R might be applied to obtain the MLEs.

8.2. Residual Analysis. After the regression model has been formulated, it is important to perform a residual analysis. It is derived to evaluate the adequacy of the fitted model and check outlier observations. In this study, we conducted residual analysis based on the martingale residual and deviance residual.

8.2.1. Martingale Residual. Barlow and Prentice [45] introduced the martingale residual as

$$r_{M_i} = \delta_i + \log\left(SF\left(y_i, \widehat{\Theta}\right)\right), \tag{53}$$

where δ_i is the censor indicator; $\delta_i = 1$ if the i^{th} observation is lifetime and $\delta_i = 0$ if the i^{th} observation is censored.

The martingale residual of the LNGEFW regression model is

$$r_{M_i} = \begin{cases} 1 + \left\{\left(1 - \left\{1 - \left[1 - e^{-\alpha^\beta e^{-\beta \widehat{z}_i}}\right]^a\right\}^b\right)^\theta\right\}, & \text{if } i \in \text{lifetime},\\ \log\left\{\left(1 - \left\{1 - \left[1 - e^{-\alpha^\beta e^{-\beta \widehat{z}_i}}\right]^a\right\}^b\right)^\theta\right\}, & \text{if } i \in \text{censored}, \end{cases} \tag{54}$$

where $\widehat{z}_i = (y_i - \widehat{\upsilon}\eta_i^T)/\widehat{\sigma}$. r_{M_i} takes value between $-\infty$ and $+1$ and has skewness.

8.2.2. Deviance Residual. Therneau et al. [46] defined the deviance residual to reduce the skewness symmetrically distributed around zero as

$$r_{D_i} = \operatorname{sign}\left(r_{M_i}\right)\left\{-2\left[r_{M_i} + \delta_i \log\left(\delta_i - r_{M_i}\right)\right]\right\}^{1/2}, \tag{55}$$

where r_{M_i} is given in equation (54). The deviance residual of the LNGEFW regression model is

$$r_{D_i} = \begin{cases} \operatorname{sign}\left(1 + \log\left\{\left(1 - \left\{1 - \left[1 - e^{-\alpha^\beta e^{-\beta \widehat{z}_i}}\right]^a\right\}^b\right)^\theta\right\}\right) \times \left\{\begin{array}{l} -21 + \log\left\{\left(1 - \left\{1 - \left[1 - e^{-\alpha^\beta e^{-\beta \widehat{z}_i}}\right]^a\right\}^b\right)^\theta\right\} \\ +\log\left(-\log\left\{\left(1 - \left\{1 - \left[1 - e^{-\alpha^\beta e^{-\beta \widehat{z}_i}}\right]^a\right\}^b\right)^\theta\right\}\right) \end{array}\right\}, & \text{if i} \in \text{lifetime},\\ \operatorname{sign}\left(\log\left\{\left(1 - \left\{1 - \left[1 - e^{-\alpha^\beta e^{-\beta \widehat{z}_i}}\right]^a\right\}^b\right)^\theta\right\}\right) \times \left\{-2\left[\log\left\{\left(1 - \left\{1 - \left[1 - e^{-\alpha^\beta e^{-\beta \widehat{z}_i}}\right]^a\right\}^b\right)^\theta\right\}\right]\right\}^{1/2}, & \text{if i} \in \text{censored}. \end{cases} \tag{56}$$

9. Simulation Studies for the Log New Generalized Exponentiated Fréchet–Weibull Regression Model

We conduct Monte Carlo simulation studies for various values of sample size (n), parameter values, and different censoring percentages to investigate the accuracy of the MLE in the LNGEFW regression model. The lifetimes $x_1, \ldots, x_n$ are sampled from the NGEFWD in equation (7) considering the following reparametrization: $\mu = \log(\lambda)$ and $\sigma = 1/k$, and by taking $\mu_i = \upsilon_\circ + \upsilon_1\eta_i$, where η_i is the explanatory variable generated from a standard uniform distribution. The considered values for the parameters are

Case I: $\upsilon_\circ = 3.4, \upsilon_1 = 2.9, \sigma = 3.6, a = 2.6, b = 4.2, \theta = 4.6, \alpha = 0.6, \beta = 0.7$

Table 11: MLEs and SEs in () for the leukemia data.

Distribution	MLE and SEs in ()								
LBXII	$\widehat{\upsilon}_{\circ} = 21.58190220$ (24.5405116)	$\widehat{\upsilon}_1 = 11.78233771$ (6.9842642)	$\widehat{\upsilon}_2 = -0.01565953$ (0.1088288)	$\widehat{\sigma} = 19.24728186$ (1.2172049)	$\widehat{k} = 3.85887867$ (3.9641882)	—	—	—	—
LTLF	$\widehat{\upsilon}_{\circ} = 17.3498523$ (25.9687875)	$\widehat{\upsilon}_1 = 19.2735648$ (4.6322606)	$\widehat{\upsilon}_2 = -0.1348475$ (0.1032461)	$\widehat{\sigma} = 25.8796636$ (2.7233180)	$\widehat{\alpha} = 0.6205973$ (0.9577270)	—	—	—	—
LTLGW	$\widehat{\upsilon}_{\circ} = 4.0261718$ (5.61580152)	$\widehat{\upsilon}_1 = 9.4664027$ (3.07310125)	$\widehat{\upsilon}_2 = -0.1295746$ (0.02832091)	$\widehat{\sigma} = 12.5686718$ (2.79058555)	$\widehat{\alpha} = 0.1151792$ (0.07128416)	$\widehat{\theta} = 15.3964928$ (9.41081790)	—	—	—
LNGEFW	$\widehat{\upsilon}_{\circ} = \mathbf{3.44364060}$ **(0.50481503)**	$\widehat{\upsilon}_1 = \mathbf{1.27260552}$ **(0.50025649)**	$\widehat{\upsilon}_2 = \mathbf{-0.02295856}$ **(0.00774572)**	$\widehat{\sigma} = \mathbf{6.73142242}$ **(0.29526719)**	$\widehat{a} = \mathbf{1.65623373}$ **(1.79335101)**	$\widehat{b} = \mathbf{0.15901645}$ **(0.07967806)**	$\widehat{\theta} = \mathbf{2.84130998}$ **(1.48737764)**	$\widehat{\alpha} = \mathbf{1.90262911}$ **(0.12286804)**	$\widehat{\beta} = \mathbf{3.04453943}$ **(0.29512317)**

The bold values represent the best results in the tables.

Table 12: The information criteria for the leukemia data.

Distribution	AIC	CAIC	BIC	HQIC
LBXII	261.7738	274.2563	269.2563	264.2915
LTLF	256.8065	269.2891	264.2891	259.3242
LTLGW	221.0307	236.0097	230.0097	224.0518
LNGEFW	127.4377	149.9063	140.9063	131.9695

Case II: $\upsilon_\circ = 5.7, \upsilon_1 = 5.9, \sigma = 4.2, a = 1.7, b = 1.2, \theta = 5.9, \alpha = 2.9, \beta = 0.4$

Noninformative censoring is commonly used in different studies. The censoring times $C_1, \ldots, C_n$ are generated from a uniform distribution $(0, \tau)$, where the indicator random variable is given as

$$\tau_i = \begin{cases} 1, & \text{if } f_i \leq C_i, \\ 0, & \text{if } f_i > C_i. \end{cases} \quad (57)$$

This is adjusted until the censoring percentages of 0.1, 0.3, and 0.5 are reached. The lifetimes considered in each fit are calculated as $y_i = \min\{\log(x_i), \log(C_i)\}$. This simulation was repeated $N = 1000$ times, and for each parameter, the mean estimate, MSE, and RMSE are calculated. The results are listed in Table 8.

From Table 8, it is shown that when sample sizes increase, the MSE and RMSE of estimates decrease and the estimates tend to the true values of the parameters. Also, when censoring levels increase, the MSE and RMSE of parameter estimates increase for the same sample size. The results indicate that the maximum likelihood method provides consistent estimation for the parameters of the LNGEFW regression model.

10. Applications for the Log New Generalized Exponentiated Fréchet–Weibull Regression Model

In this section, three real datasets are applied to illustrate the usefulness of the LNGEFW regression model. For three applications, the maximum likelihood method is applied to obtain the estimates of the parameters for the LNGEFW regression model. The estimates and their standard errors (SEs) are calculated along with the AIC, CAIC, BIC, and HQIC to compare the LNGEFW regression model with some competitive models, namely, log-Burr XII (LBXII) regression model in [28], log Topp–Leone–Fréchet (LTLF) regression model in [47], and log Topp–Leone generated Weibull (LTLGW) regression model in [48]. The estimates and their SEs are reported in Tables 9–11, while Tables 12–14 summarize the information criteria for each analyzed dataset.

10.1. Voltage Data. Lawless [49] introduced an experiment in which specimens of solid epoxy electrical insulation were considered in an accelerated voltage life test. The sample size of data is $n = 60$ with a percentage of 10% censored observations, and it has three levels of voltage: 52.5, 55.0, and 57.5 kV. The variables considered in the study are as follows: y_i: failure times for epoxy insulation specimens, cens_i: censoring indicator (0 = censoring, 1 = lifetime observed), and η_{i1}: voltage (kV). The results are presented by the fitting model

$$y_i = \upsilon_\circ + \upsilon_1\eta_{i1} + \sigma z_i, i = 1, 2, \ldots, 60, \quad (58)$$

where y_i follows the NGEFWD.

10.2. Leukemia Data. Leukemia data are presented in [49]. These data contain information of 33 patients who were diagnosed with leukemia. The variables involved in the study are as follows: t_i: survival time, y_i: log survival time, cens_i: censoring indicator (0 = censoring, 1 = lifetime), η_{i1}: white blood cell characteristics test (0 = negative, 1 = positive), and η_{i2}: white blood cell count. The fit of the regression model is described as

$$y_i = \upsilon_\circ + \upsilon_1\eta_{i1} + \upsilon_2\eta_{i2} + \sigma z_i, i = 1, 2, \ldots, 33, \quad (59)$$

where y_i follows the NGEFWD.

10.3. Stanford Heart Transplant Data. Kalbfleisch and Prentice [50] considered the Stanford heart transplant dataset. The dataset contains the survival time of 103 patients since acceptance into transplant program to death. The following variables are displayed in the study: y_i: log survival time, cens_i: censoring indicator (0 = censoring, 1 = dead), η_{i1}: the age of patients, η_{i2}: the previous surgery (0 = No, 1 = Yes) and η_{i3}: the transplant (0 = No, 1 = Yes). The model fitted can be written as

$$y_i = \upsilon_\circ + \upsilon_1\eta_{i1} + \upsilon_2\eta_{i2} + \upsilon_3\eta_{i3} + \sigma z_i, i = 1, 2, \ldots, 103, \quad (60)$$

where y_i follows the NGEFWD.

The results in Tables 12–14 show that the LNGEFW regression model has smallest values of AIC, CAIC, BIC, and HQIC for the voltage, leukemia, and Stanford heart transplant data compared to the other competitive models. Therefore, the LNGEFW regression model might provide the best fit to the three data among other models. Figures 7–9 represent the deviance residuals against the index of the observations for all datasets. It can be noted that all observations fall within the interval (−3, 3), except observation 26 in Figure 9. Thus, observation 26 in Figure 9 is a possible outlier. In addition, from these figures, it can be seen that all points lie inside the envelope, which indicates that the LNGEFW regression model provides good fit to all datasets.

Table 13: MLEs and SEs in () for the Stanford heart transplant data.

Distribution	MLE and SEs in ()									
LBXII	$\hat{\upsilon}_\circ = 10.16735896$ (2.2639559)	$\hat{\upsilon}_1 = 0.02818055$ (0.1404552)	$\hat{\upsilon}_2 = 0.25876298$ (0.6836249)	$\hat{\upsilon}_3 = 2.24754211$ (0.5280168)	$\hat{\sigma} = 2.00441906$ (0.2769117)	$\hat{k} = 20.91184142$ (17.0586543)	—	—	—	—
LTLF	$\hat{\upsilon}_\circ = 7.3238441$ (3.6369104)	$\hat{\upsilon}_1 = 0.1061086$ (0.1974065)	$\hat{\upsilon}_2 = 3.0613653$ (4.9455403)	$\hat{\upsilon}_3 = 2.9030265$ (0.8958325)	$\hat{\sigma} = 3.4498448$ (3.8377002)	$\hat{\alpha} = 0.2427554$ (0.1679167)	—	—	—	—
LTLGW	$\hat{\upsilon}_\circ = 4.99076353$ (4.93219199)	$\hat{\upsilon}_1 = 2.16712352$ (1.31629897)	$\hat{\upsilon}_2 = 5.89378864$ (8.42843133)	$\hat{\upsilon}_3 = 12.12149664$ (4.51282522)	$\hat{\sigma} = 14.60801547$ (1.99785096)	$\hat{\alpha} = 0.05103121$ (0.01579519)	$\hat{\theta} = 17.38546822$ (5.37791836)	—	—	—
LNGEFW	$\hat{\upsilon}_\circ = \mathbf{6.9175974}$ **(0.34199597)**	$\hat{\upsilon}_1 = \mathbf{0.1348798}$ **(0.09174458)**	$\hat{\upsilon}_2 = \mathbf{0.7459412}$ **(0.37361548)**	$\hat{\upsilon}_3 = \mathbf{2.0816461}$ **(0.31643097)**	$\hat{\sigma} = \mathbf{16.7708474}$ **(0.42220816)**	$\hat{a} = \mathbf{16.8583684}$ **(1.05932801)**	$\hat{b} = \mathbf{0.2395321}$ **(0.10259565)**	$\hat{\theta} = \mathbf{5.1302611}$ **(2.80967429)**	$\hat{\alpha} = \mathbf{1.5092304}$ **(0.11912349)**	$\hat{\beta} = \mathbf{3.7514208}$ **(0.24414985)**

IThe bold values represent the best results in the tables.

Table 14: The information criteria for the Stanford heart transplant data.

Distribution	AIC	CAIC	BIC	HQIC
LBXII	382.3919	404.2003	398.2003	388.7949
LTLF	390.6971	412.5055	406.5055	397.1
LTLGW	643.5983	669.0414	662.0414	651.0684
LNGEFW	367.781	404.1283	394.1283	378.4526

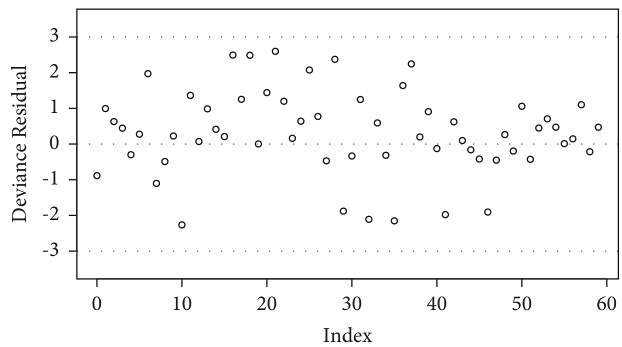


Figure 7: The index plot of the deviance residual for the LNGEFW regression model for the voltage data.

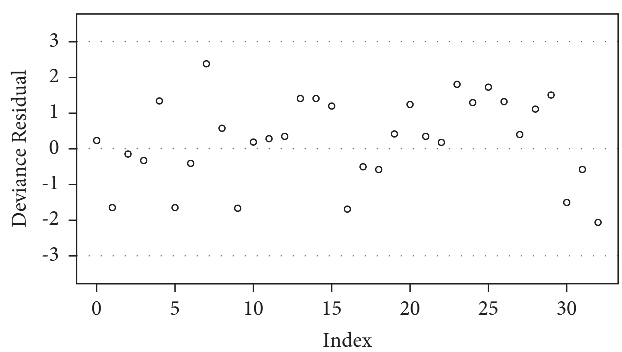


Figure 8: The index plot of the deviance residual for the LNGEFW regression model for the leukemia data.

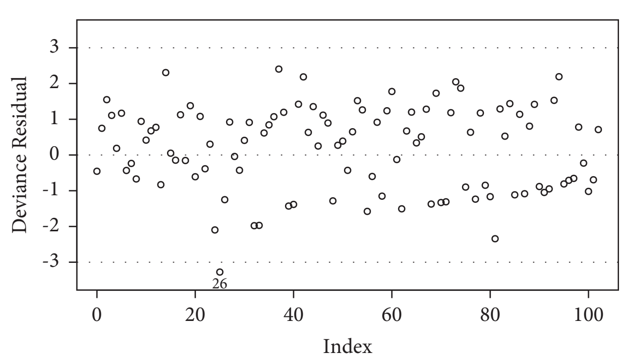


Figure 9: The index plot of the deviance residual for the LNGEFW regression model for the Stanford heart transplant data.

11. Conclusions

Introducing flexible distributions to real data models is of great importance to more accurately model different real datasets. In addition, many regression models must be developed to analyze the effect of covariates on the data in numerous practical applications. Thus, in this article, the NGEFWD is proposed in order to overcome the complexity of the pattern of some datasets. Some useful statistical properties of the new distribution are derived. The maximum likelihood method is applied to estimate the model's parameters. Additionally, in order to examine these MLEs, some Monte Carlo simulation studies are conducted for different cases for which the results indicate that the proposed estimators have a good performance, and it is quite clear from the results that as sample size increases, a better estimate is obtained. Thus, the consistency, normality, and maximum efficiency properties of the MLE are effective. The suggested distribution can be applied in different applications, such as engineering, reliability, and many other real-life data. Hence, the usefulness of the new distribution is examined by analyzing three real datasets. It has been observed that the NGEFWD distribution consistently provides a better and accurate fit than some other common competitive models. Moreover, based on the NGEFWD, the log-location-scale technique is applied to introduce the LNGEFW regression model. The maximum likelihood method for the right-censored data is considered to estimate the parameters of the LNGEFW regression model. Some simulation studies with various values of parameters, sample size, and censoring percentage are considered to demonstrate the new regression model's versatility. Based on the results of two Monte Carlo simulation studies conducted for the LNGEFW regression model, the MLEs provided consistently good results. The LNGEFW regression model performed very well when applied to three real-world datasets and provided the best fits among some other competitor regression models based on the information criteria. Therefore, it can be considered the most appropriate model among all the others. Hence, NGEFWD and its extension LNGEFW regression model are expected to attract the attention of various applied sciences due to their suitability and flexibility. Further studies could be conducted by using other methods of estimation, such as the moment estimation method, and different regression techniques, such as quantile regression.

Data Availability

The references for the data used to support the findings of this study are cited within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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