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Research Article

Techniques for Finding Analytical Solution of Generalized Fuzzy Differential Equations with Applications

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Engineering and applied mathematics disciplines that involve differential equations include classical mechanics, thermodynamics, electrodynamics, and general relativity. Modelling a wide range of real-world situations sometimes comprises ambiguous, imprecise, or insufficient situational information, as well as multiindex, uncertainty, or restriction dynamics. As a result, intuitionistic fuzzy set models are significantly more useful and versatile in dealing with this type of data than fuzzy set models, triangular, or trapezoidal fuzzy set models. In this research, we looked at differential equations in a generalized intuitionistic fuzzy environment. We used the modified Adomian decomposition technique to solve generalized intuitionistic fuzzy initial value problems. The generalized modified Adomian decomposition technique is used to solve various higher-order generalized trapezoidal intuitionistic fuzzy initial value problems, circuit analysis problems, mass-spring systems, steam supply control sliding value problems, and some other problems in physical science. The outcomes of numerical test applications were compared to exact technique solutions, and it was shown that our generalized modified Adomian decomposition method is efficient, robotic, and reliable, as well as simple to implement.

1. Introduction

Differential equation plays a vital role in every field of science and engineering. Differential equations are used to study bacterial growth, the motion of an object that moves to and fro, to calculate the flow of electricity, to explain the concept of thermodynamics, and to observe the propagation of heat or sound, fluid flow, temperature problems etc., [1–4].

In 1965, Zadeh [5] introduced the concept of fuzzy set theory as the extension of classical set theory. In classical set theory, we deal only with true or false values and do not analyze any values between them, whereas in fuzzy set theory, we deal with uncertain environmental conditions as membership values. The concept of fuzzy set theory has been applied to various fields of science and engineering to handle vagueness and uncertainty. In 1987, Kandel and Byatt introduced the fuzzy differential equations [6]. The fuzzy differential equations have been applied in numerous daily life problems [7]. Vasavi et al. [8] discussed fuzzy differential for cooling problems. Devi and Ganesan used fuzzy differential equations in modelling electric circuit problem [9]. Ahmad et al. [10] studied a mathematical method to find the solution of fuzzy integro-differential equations. Sadeghi et al. [11] studied the system of fuzzy differential equation. In 1986, Atanassov [12] introduced an extension of fuzzy set theory known as intuitionistic fuzzy set. The intuitionistic fuzzy set not only provides the information about membership values but also the nonmembership values, respectively, and so that the sum of both values is less than one. Intuitionistic fuzzy differential equations are being studied widely and being used in various fields of Physics, Chemistry, Biology as well as among other fields of Science and Engineering. Melliani and Chadli obtained the approximate and numerical solutions of intuitionistic fuzzy differential equations with linear differential operators [13, 14]. Akin and Bayeg [15, 16] studied a method to find general solution of second-order intuitionistic fuzzy differential equation. Prasad Mondal and Kumar Roy [17–19] studied the generalized intuitionistic fuzzy Laplace transform method. Shams et al. used generalized trapezoidal intuitionistic fuzzy numbers and triangular intuitionistic fuzzy numbers in [20, 21] to solve a system of fuzzy initial value problems. The Adomian decomposition method (ADM) which is a semianalytical method was first presented by George Adomian in 1980s [22, 23]. This method is very efficient in finding the solutions of differential equations, algebraic equations as well as integral equations. In this article, we will propose the generalized modified Adomian decomposition method (GMADM) to find the solutions of generalized intuitionistic fuzzy differential equations with linear differential operators. This modification was proposed by Wazwaz [24]. He presented a reliable modification to the ADM. In this modification, Wazwaz divides the original function into two parts, one part assigned to the initial term of the series and the other to the second term. This modification results in a different series being generated. The efficiency of this method depends only on the choice of the parts into which the original function is to be divided.

Motivated by the aforesaid work, in this article, we use a GMADM to solve generalized trapezoidal intuitionistic fuzzy initial value problems (FIVPs).

The main contributions of this research work are summarized as follows:

(i) Using initial conditions as a generalized trapezoidal intuitionistic fuzzy number, a differential equation is solved using GMADM

(ii) The computational complexity of the proposed GMADM is described in order to solve problematic generalized trapezoidal intuitionistic fuzzy differential equations

(iii) Applications of generalized trapezoidal intuitionistic fuzzy differential equations in mechanical and electrical engineering are considered in a generalized trapezoidal intuitionistic fuzzy environment

(iv) The efficiency and applicability of the modified technique are assessed using computational tools

This paper is organized as follows: in Section 2, we recall some basic definitions which we will use in further sections. In Section 3, we introduced our proposed method and the efficiency of this method has been illustrated by applications. In the last section, we give conclusions.

2. Preliminaries

In this section, the fundamental definitions of fuzzy set and intuitionistic fuzzy set are presented.

Definition 1 [25]. Let X be the largest set under consideration, then $\overset{*}{A}$ be a subset of X is said to be a fuzzy set if it is defined as follows:

$$\mu_{\overset{*}{A}}(x) = X \longrightarrow [0,1], \tag{1}$$

defines the degree of membership of an element $x\epsilon X$ to the set $\overset{*}{A}$ which is a subset of X.

Definition 2 [25]. α-cut of a fuzzy set $\overset{*}{A}$ is a crisp set $\overset{*}{A}_\alpha$ which is defined as follows:

$$\overset{*}{A}_\alpha = \left\{ \left(x \mid \mu_{\overset{*}{A}}(x) \geq \alpha \colon x\epsilon X \right\}. \tag{2}$$

Definition 3 [19]. If $\overset{*}{A}$ is a fuzzy set, then height of a fuzzy set is denoted by $h(\overset{*}{A})$ and is defined as the largest membership function obtained by any element in that set, i.e.,

$$h(\overset{*}{A}) = \sup \mu \overset{*}{A}(x). \tag{3}$$

Definition 4 [15]. Let X be a nonempty finite set of real numbers, then an intuitionistic fuzzy set $\overset{*}{F}$ on X is

$$\overset{*}{F} = \left\{ \left(x, \mu_{\overset{*}{F}}(x), \upsilon_{\overset{*}{F}}(x) \right) \colon x\epsilon X \right\}, \tag{4}$$

where the functions

$$\begin{aligned} \mu_{\overset{*}{F}}(x) &= X \longrightarrow [0,1], \\ \upsilon_{\overset{*}{F}}(x) &= X \longrightarrow [0,1], \end{aligned} \tag{5}$$

define the degree of membership and degree of nonmembership, respectively, of an element $x\epsilon X$ to the set $\overset{*}{F}$ which is a subset of X, and for every $x\epsilon X$, the $0 \leq \mu_{\overset{*}{F}}(x) + \upsilon_{\overset{*}{F}}(x) \leq 1$, condition must be satisfied.

Definition 5 [19]. An intuitionistic fuzzy set $\overset{*}{F}$ is said to be normal if there exists an $x_0 \epsilon X$, such that $\mu_{\overset{*}{F}}(x_0) = 1$ so $\upsilon_{\overset{*}{F}}(x_0) = 0$.

Definition 6 [19]. An intuitionistic fuzzy set $\overset{*}{F}$ is said to be convex set for the membership function if it satisfies the following condition:

$$\mu_{\overset{*}{F}}(x)(\eta x + (1-\eta)y) \geq \min\left(\mu_{\overset{*}{F}}(x), \mu_{\overset{*}{F}}(y) \right); \forall x, y\epsilon X, \eta\epsilon[0,1]. \tag{6}$$

Definition 7 [19]. An intuitionistic fuzzy set $\overset{*}{F}$ is said to be concave set for the nonmembership function if it satisfies the following condition:

$$\upsilon_{\overset{*}{F}}(x)(\eta x + (1-\eta)y) \geq \max\left(\upsilon_{\overset{*}{F}}(x), \upsilon_{\overset{*}{F}}(y) \right); \forall x, y\epsilon X, \eta\epsilon[0,1]. \tag{7}$$

2.1. Generalized Intuitionistic Fuzzy Number

Definition 8 [26]. A generalized intuitionistic fuzzy number

$$\overset{*}{N} = < (a_1, a_2, a_3, a_4; \nu_A); (b_1, b_2, b_3, b_4; \nu_B) >, \tag{8}$$

is said to be generalized trapezoidal intuitionistic fuzzy number (GTIFN) if its membership and nonmembership functions are defined as follows:

$$\mu_{\overset{*}{N}}(x) = \begin{cases} \nu_A\left(\dfrac{x - a_1}{a_2 - a_1}\right), & a_1 \le x \le a_2, \\ \nu_A, & a_2 \le x \le a_3, \\ \nu_A\left(\dfrac{a_4 - x}{a_4 - a_3}\right), & a_3 \le x \le a_4, \\ 0, & \text{otherwise}, \end{cases}$$

$$\upsilon_{\overset{*}{N}}(x) = \begin{cases} \dfrac{(b_2 - x) + \nu_B(x - b_1)}{b_2 - b_1}, & b_1 \le x \le b_2, \\ \nu_B, & b_2 \le x \le b_3, \\ \dfrac{(x - b_3) + \nu_B(b_4 - x)}{b_4 - b_3}, & b_3 \le x \le b_4, \\ 1, & \text{otherwise}, \end{cases} \tag{9}$$

where $b_1 \le a_1 \le b_2 \le a_2 \le a_3 \le b_3 \le a_4 \le b_4, 0 \le \nu_A, \nu_B \le 1$ and $0 < \nu_A + \nu_B \le 1$.

Definition 9. (α, β)-Cut set [27] of GTIFN $\overset{*}{N} = < (a_1, a_2, a_3, a_4; \nu_A); (b_1, b_2, b_3, b_4; \nu_B) >$ is a crisp subset of X which is defined as follows:

$$\begin{aligned}\overset{*}{N}(\alpha, \beta) &= \left\{\left(x, \mu_{\overset{*}{N}}(x), \upsilon_{\overset{*}{N}}(x)\right): x\epsilon X, \mu_{\overset{*}{N}}(x) \ge \alpha, \upsilon_{\overset{*}{N}}(x) \le \beta\right\}, \\ &= \left\{\left[\overset{*}{N}_1(\alpha), \overset{*}{N}_2(\alpha)\right]; \left[\overset{*}{N}_1(\beta), \overset{*}{N}_2(\beta)\right]\right\},\end{aligned} \tag{10}$$

where $\alpha + \beta < 1, \alpha\epsilon[0, \nu_A]$ and $\beta\epsilon[\nu_B, 1]$ as shown in Figure 1.

Figure 1 shows generalized trapezoidal intuitionistic fuzzy number.

Arithmetic operations on GTIFNs are defined as follows.

Definition 10 [28–30]. Let $\overset{*}{N}_1 = < (a_1, a_2, a_3, a_4; \nu_{A_1}); (b_1, b_2, b_3, b_4; \nu_{A_2}) >$ and $\overset{*}{N}_2 = < (c_1, c_2, c_3, c_4; \nu_{B_1}); (d_1, d_2, d_3, d_4; \nu_{B_2}) >$ be two GTIFNs and ω be a real number. Then,

(i) $\overset{*}{N}_1 + \overset{*}{N}_2 = < (a_1 + c_1, a_2 + c_2, a_3 + c_3, a_4 + c_4;$ min $\{\nu_{A_1}, \nu_{B_1}\}); (b_1 + d_1, b_2 + d_2, b_3 + d_3, b_4 + d_4;$ max $\{\nu_{A_2}, \nu_{B_2}\}) >$,

(ii) $\overset{*}{N}_1 - \overset{*}{N}_2 = < (a_1 - c_4, a_2 - c_3, a_3 - c_2, a_4 - c_1;$ min $\{\nu_{A_1}, \nu_{B_1}\}); (b_1 - d_4, b_2 - d_3, b_3 - d_2, b_4 - d_1;$ max $\{\nu_{A_2}, \nu_{B_2}\}) >$,

(iii) $\overset{*}{N}_1 \times \overset{*}{N}_2 = < (a_1c_1, a_2c_2, a_3c_3, a_4c_4;$ min $\{\nu_{A_1}, \nu_{B_1}\});$ $(b_1d_1, b_2d_2, b_3d_3, b_4d_4;$ max $\{\nu_{A_2}, \nu_{B_2}\}) >$, where $\overset{*}{N}_1 > 0, \overset{*}{N}_2 > 0$.

(iv) $\overset{*}{N}_1 \times \overset{*}{N}_2 = < (a_1c_4, a_2c_3, a_3c_2, a_4c_1;$ min $\{\nu_{A_1}, \nu_{B_1}\}); (b_1d_4, b_2d_3, b_3d_2, b_4d_1;$ max $\{\nu_{A_2}, \nu_{B_2}\}) >$, where $\overset{*}{N}_1 < 0, \overset{*}{N}_2 > 0$.

(v) $\overset{*}{N}_1 \times \overset{*}{N}_2 = < (a_4c_4, a_3c_3, a_2c_2, a_1c_1;$ min $\{\nu_{A_1}, \nu_{B_1}\}); (b_4d_4, b_3d_3, b_2d_2, b_1d_1;$ max $\{\nu_{A_2}, \nu_{B_2}\}) >$, where $\overset{*}{N}_1 < 0, \overset{*}{N}_2 < 0$.

(vi) $\overset{*}{N}_1 \div \overset{*}{N}_2 = < (a_1/c_4, a_2/c_3, a_3/c_2, a_4/c_1;$ min $\{\nu_{A_1}, \nu_{B_1}\}); (b_1/d_4, b_2/d_3, b_3/d_2, b_4/d_4;$ max $\{\nu_{A_2}, \nu_{B_2}\}) >$, where $\overset{*}{N}_2 > 0$.

(vii) $\omega\overset{*}{N}_1 = < (\omega a_1, \omega a_2, \omega a_3, \omega a_4;$ min $\{\nu_{A_1}, \nu_{B_1}\}); (\omega b_1, \omega b_2, \omega b_3, \omega b_4;$ max $\{\nu_{A_2}, \nu_{B_2}\}) >$, where $\omega > 0$.

(viii) $\omega\overset{*}{N}_1 = < (\omega a_4, \omega a_3, \omega a_2, \omega a_1;$ min $\{\nu_{A_1}, \nu_{B_1}\}); (\omega b_4, \omega b_3, \omega b_4, \omega b_5;$ max $\{\nu_{A_2}, \nu_{B_2}\}) >$, where $\omega < 0$.

3. The Generalized Modified Adomian Decomposition Method

A key concept is that the Adomian decomposition series expansion about the initial solution component function permits solution by recursion, in which the aforesaid rearrangement is accomplished through the choice of the recursion scheme. The modified ADM yields a rapidly convergent sequence of analytic functions as the approximate solutions of the original mathematical model. Thus, the modified Adomian decomposition method subsumes even the classic power series method while extending the class of amenable nonlinearity to include any analytic nonlinearity. Here, we generalized the MADM to GMADM to solve system of intuitionistic triangular fuzzy differential equation.

Let us consider the intuitionistic fuzzy differential equation with linear differential operator as follows [25]:

$$L\overset{*}{y}(t) + R\overset{*}{y}(t) + N(t, \overset{*}{y}(t)) = \overset{*}{f}(t), \tag{11}$$

where L is the highest order linear differential operator, R is the remaining part of the linear differential operator, N may be linear or nonlinear function of t and $\overset{*}{y}(t)$, and $f(t)$ is an intuitionistic fuzzy function. Here, in this case, we take N as a linear function of $\overset{*}{y}(t)$ and t.

Taking (α, β)-cut of (11), we get

$$\begin{aligned}&L\left(\left[\overset{*}{y}_1(t,\alpha); \overset{*}{y}_2(t,\alpha)\right]; \left[\overset{*}{y}_1(t,\beta); \overset{*}{y}_1(t,\beta)\right]\right) + R\left(\left[\overset{*}{y}_1(t,\alpha); \overset{*}{y}_1(t,\alpha)\right]; \left[\overset{*}{y}_1(t,\beta); \overset{*}{y}_2(t,\beta)\right]\right) \\ &+\left(\left[N_1\left(t, \overset{*}{y}_1(t,\alpha)\right); N_2\left(t, \overset{*}{y}_2(t,\alpha)\right)\right]; \left[N_1\left(t, \overset{*}{y}_1(t,\beta)\right); N_2\left(t, \overset{*}{y}_2(t,\beta)\right)\right]\right) = \left[\overset{*}{f}_1(t,\alpha); \overset{*}{f}_2(t,\alpha)\right]; \left[\overset{*}{f}_1(t,\beta); \overset{*}{f}_2(t,\beta)\right].\end{aligned} \tag{12}$$

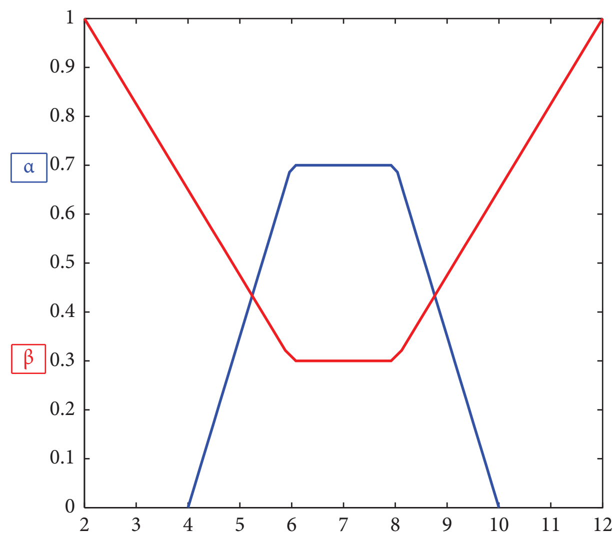


FIGURE 1: Generalized trapezoidal intuitionistic fuzzy number.

From (12), we obtain the following equations:

$$L\overset{*}{y}_1(t,\alpha) + R\overset{*}{y}_1(t,\alpha) + N_1\left(t, \overset{*}{y}_1(t,\alpha)\right) = \overset{*}{f}_1(t,\alpha), \tag{13}$$

$$L\overset{*}{y}_2(t,\alpha) + R\overset{*}{y}_2(t,\alpha) + N_2\left(t, \overset{*}{y}_2(t,\alpha)\right) = \overset{*}{f}_2(t,\alpha), \tag{14}$$

$$L\overset{*}{y}_1(t,\beta) + R\overset{*}{y}_1(t,\beta) + N_1\left(t, \overset{*}{y}_1(t,\beta)\right) = \overset{*}{f}_1(t,\beta), \tag{15}$$

$$L\overset{*}{y}_2(t,\beta) + R\overset{*}{y}_2(t,\beta) + N_1\left(t, \overset{*}{y}_2(t,\beta)\right) = \overset{*}{f}_2(t,\beta). \tag{16}$$

By taking L^{-1} operator on both sides of equations (13)–(16), we get

$$\overset{*}{y}_1(t,\alpha) = \Psi_1(t,\alpha) - L^{-1}\left(R\overset{*}{y}_1(t,\alpha)\right) - L^{-1}\left(N_1\left(t, \overset{*}{y}_1(t,\alpha)\right) + L^{-1}\left(\overset{*}{f}_1(t,\alpha)\right)\right), \tag{17}$$

$$\overset{*}{y}_2(t,\alpha) = \Psi_2(t,\alpha) - L^{-1}\left(R\overset{*}{y}_2(t,\alpha)\right) - L^{-1}\left(N_2\left(t, \overset{*}{y}_2(t,\alpha)\right) + L^{-1}\left(\overset{*}{f}_2(t,\alpha)\right)\right), \tag{18}$$

$$\overset{*}{y}_1(t,\beta) = \Psi_1(t,\beta) - L^{-1}\left(R\overset{*}{y}_1(t,\beta)\right) - L^{-1}\left(N_1\left(t, \overset{*}{y}_1(t,\beta)\right) + L^{-1}\left(\overset{*}{f}_1(t,\beta)\right)\right), \tag{19}$$

$$\overset{*}{y}_2(t,\beta) = \Psi_2(t,\beta) - L^{-1}\left(R\overset{*}{y}_2(t,\beta)\right) - L^{-1}\left(N_2\left(t, \overset{*}{y}_2(t,\beta)\right) + L^{-1}\left(\overset{*}{f}_2(t,\beta)\right)\right), \tag{20}$$

where

$$\begin{aligned} \Psi_i(t,\alpha) &= L\Psi_i(t,\alpha) \\ &= 0; i = 1,2, \\ \Psi_i(t,\beta) &= L\Psi_i(t,\beta) \\ &= 0; i = 1,2, \end{aligned} \tag{21}$$

the above functions are found by using the initial conditions.

Now by using the GMADM, the solutions of the (17)–(20) can be expressed in the form of an infinite series for the unknown functions as follows:

$$\overset{*}{y}_1(t,\alpha) = \sum_{n=0}^{\infty} \overset{*}{y}_{1_n}(t,\alpha), \tag{22}$$

$$\overset{*}{y}_2(t,\alpha) = \sum_{n=0}^{\infty} \overset{*}{y}_{2_n}(t,\alpha), \tag{23}$$

$$\overset{*}{y}_1(t,\beta)=\sum_{n=0}^{\infty}\overset{*}{y}_{1_n}(t,\beta), \tag{24}$$

$$\overset{*}{y}_2(t,\beta)=\sum_{n=0}^{\infty}\overset{*}{y}_{2_n}(t,\beta). \tag{25}$$

Using (22)–(25), in (17)–(20), we have

$$\sum_{n=0}^{\infty}\overset{*}{y}_{1_n}(t,\alpha)=\Psi_1(t,\alpha)-L^{-1}\left(R\sum_{n=0}^{\infty}\overset{*}{y}_{1_n}(t,\alpha)\right)-L^{-1}\left(N_1\left(t,\sum_{n=0}^{\infty}\overset{*}{y}_{1_n}(t,\alpha)\right)\right)+L^{-1}\left(\overset{*}{f}_1(t,\alpha)\right), \tag{26}$$

$$\sum_{n=0}^{\infty}\overset{*}{y}_{2_n}(t,\alpha)=\Psi_2(t,\alpha)-L^{-1}\left(R\sum_{n=0}^{\infty}\overset{*}{y}_{2_n}(t,\alpha)\right)-L^{-1}\left(N_2\left(t,\sum_{n=0}^{\infty}\overset{*}{y}_{2_n}(t,\alpha)\right)\right)+L^{-1}\left(\overset{*}{f}_2(t,\alpha)\right), \tag{27}$$

$$\sum_{n=0}^{\infty}\overset{*}{y}_{1_n}(t,\beta)=\Psi_1(t,\beta)-L^{-1}\left(R\sum_{n=0}^{\infty}\overset{*}{y}_{1_n}(t,\beta)\right)-L^{-1}\left(N_1\left(t,\sum_{n=0}^{\infty}\overset{*}{y}_{1_n}(t,\beta)\right)\right)+L^{-1}\left(\overset{*}{f}_1(t,\beta)\right), \tag{28}$$

$$\sum_{n=0}^{\infty}\overset{*}{y}_{2_n}(t,\beta)=\Psi_2(t,\beta)-L^{-1}\left(R\sum_{n=0}^{\infty}\overset{*}{y}_{2_n}(t,\beta)\right)-L^{-1}\left(N_2\left(t,\sum_{n=0}^{\infty}\overset{*}{y}_{2_n}(t,\beta)\right)\right)+L^{-1}\left(\overset{*}{f}_2(t,\beta)\right). \tag{29}$$

According to the GMADM, the recursive relation for the (26)–(29) is as follows:

$$\begin{aligned}
\overset{*}{y}_{1_0}(t,\alpha)&=\Psi_1(t,\alpha),\\
\overset{*}{y}_{1_1}(t,\alpha)&=L^{-1}\left(\overset{*}{f}_1(t,\alpha)\right)-L^{-1}\left(R\overset{*}{y}_{1_0}(t,\alpha)\right)-L^{-1}\left(N_1\left(t,\overset{*}{y}_{1_0}(t,\alpha)\right)\right),\\
\overset{*}{y}_{1_{k+1}}(t,\alpha)&=-L^{-1}\left(R\overset{*}{y}_{1_k}(t,\alpha)\right)-L^{-1}\left(N_1\left(t,\overset{*}{y}_{1_k}(t,\alpha)\right)\right),k\geq 1,\\
\overset{*}{y}_{2_0}(t,\alpha)&=\Psi_2(t,\alpha),\\
\overset{*}{y}_{2_1}(t,\alpha)&=L^{-1}\left(\overset{*}{f}_2(t,\alpha)\right)-L^{-1}\left(R\overset{*}{y}_{2_0}(t,\alpha)\right)-L^{-1}\left(N_2\left(t,\overset{*}{y}_{2_0}(t,\alpha)\right)\right),\\
\overset{*}{y}_{2_{k+1}}(t,\alpha)&=-L^{-1}\left(R\overset{*}{y}_{2_k}(t,\alpha)\right)-L^{-1}\left(N_2\left(t,\overset{*}{y}_{2_k}(t,\alpha)\right)\right),k\geq 1,\\
\overset{*}{y}_{1_0}(t,\beta)&=\Psi_1(t,\beta),\\
\overset{*}{y}_{1_1}(t,\beta)&=L^{-1}\left(\overset{*}{f}_1(t,\beta)\right)-L^{-1}\left(R\overset{*}{y}_{1_0}(t,\beta)\right)-L^{-1}\left(N_1\left(t,\overset{*}{y}_{1_0}(t,\beta)\right)\right),\\
\overset{*}{y}_{1_{k+1}}(t,\beta)&=-L^{-1}\left(R\overset{*}{y}_{1_k}(t,\beta)\right)-L^{-1}\left(N_1\left(t,\overset{*}{y}_{1_k}(t,\beta)\right)\right),k\geq 1,\\
\overset{*}{y}_{2_0}(t,\beta)&=\Psi_2(t,\beta),\\
\overset{*}{y}_{2_1}(t,\beta)&=L^{-1}\left(\overset{*}{f}_2(t,\beta)\right)-L^{-1}\left(R\overset{*}{y}_{2_0}(t,\beta)\right)-L^{-1}\left(N_2\left(t,\overset{*}{y}_{2_0}(t,\beta)\right)\right),\\
\overset{*}{y}_{2_{k+1}}(t,\beta)&=-L^{-1}\left(R\overset{*}{y}_{2_k}(t,\beta)\right)-L^{-1}\left(N_2\left(t,\overset{*}{y}_{2_k}(t,\beta)\right)\right),k\geq 1.
\end{aligned} \tag{30}$$

The nth term approximation to the solution is defined as follows:

$$\begin{cases} \phi_{1n}(t,\alpha) = \sum_{i=0}^{n-1} * \; y_{1_i}(t,\alpha), \\ \phi_{2n}(t,\alpha) = \sum_{i=0}^{n-1} * \; y_{2_i}(t,\alpha), \\ \phi_{1n}(t,\beta) = \sum_{i=0}^{n-1} * \; y_{1_i}(t,\beta), \\ \phi_{2n}(t,\beta) = \sum_{i=0}^{n-1} * \; y_{2_i}(t,\beta). \end{cases} \quad (31)$$

Hence,

$$\begin{cases} \lim_{n\longrightarrow\infty}(\phi_{1n}(t,\alpha)) = \overset{*}{y}_1\,(t,\alpha), \\ \lim_{n\longrightarrow\infty}(\phi_{2n}(t,\alpha)) = \overset{*}{y}_2\,(t,\alpha), \\ \lim_{n\longrightarrow\infty}(\phi_{1n}(t,\beta)) = \overset{*}{y}_1\,(t,\beta), \\ \lim_{n\longrightarrow\infty}(\phi_{2n}(t,\beta)) = \overset{*}{y}_2\,(t,\beta). \end{cases} \quad (32)$$

The complexity of the GMADM algorithm is approximately $O(N\log(N))$, according to numerical results using two or more fuzzy systems of intuitionistic differential equations.

3.1. Stability of the Proposed Scheme. When the solution produced by a technique is unaffected by small changes in the inputs and parameters and when it is expected that changes in the parameters carried on by impacts in equations and conditions, the method is said to be stable. By giving examples and analyzing the stability of the GMADM in this study, we suggested contrasting the GMADM with other existing methods, i.e., ADM and Taylor series method (TSM).

3.1.1. Applications. Here, we discuss some engineering applications [31–33] of generalized intuitionistic fuzzy environments.

Example 1. Series Circuit Problem (An Electrical Engineering Application).

Consider $\sigma_3 = <(11,12,13,14;0.8);(10,12,13,15;0.2)>$ volt battery is connected to a series circuit in which inductance is $\sigma_1 = <(0.4,0.5,0.6,0.7;0.8);(0.3,0.5,0.6,0.8;0.2)>$ henry, and the resistance is $\sigma_2 = <(8,9,10,11;0.8);(7,9,10,12;0.2)>$ ohms. Determine the current $\overset{*}{i}$ if the initial current is $\overset{*}{i}(0) = <(1,2,3,4;0.8);(0,2,3,5;0.2)>$.

The intuitionistic fuzzy differential equation related to above problem is as follows:

$$\begin{cases} \sigma_1 \dfrac{d\overset{*}{i}(t)}{dt} + \sigma_2 * \overset{*}{i}(t) = \sigma_3, \\ \overset{*}{i}(0) = \sigma_4. \end{cases} \quad (33)$$

In standard form, (33) can be written as follows:

$$\begin{cases} \dfrac{d\overset{*}{i}(t)}{dt} + \dfrac{\sigma_2}{\sigma_1} * \overset{*}{i}(t) = \dfrac{\sigma_3}{\sigma_1}, \\ \overset{*}{i}(0) = \sigma_4. \end{cases} \quad (34)$$

where $\sigma_2/\sigma_1 = <(11,15,20,28;0.8);(9,15,20,40;0.2)>$ and $\sigma_3/\sigma_1 = <(16,20,26,35;0.8);(13,20,26,50;0.2)>$.

By taking (α,β)-cut of (34), we obtain the following equations:

$$\frac{d\overset{*}{i}_1(t,\alpha)}{dt} + (5\alpha+11)\overset{*}{i}_1(t,\alpha) = 5\alpha+16, \qquad \overset{*}{i}_1(0,\alpha) = 1.25\alpha+1, \quad (35)$$

$$\frac{d\overset{*}{i}_2(t,\alpha)}{dt} + (28-10\alpha)\overset{*}{i}_2(t,\alpha) = 35-11.25\alpha, \qquad \overset{*}{i}_2(0,\alpha) = 4-1.25\alpha, \quad (36)$$

$$\frac{d\overset{*}{i}_1(t,\beta)}{dt} + (16.5-7.5\beta)\overset{*}{i}_1(t,\beta) = 21.75-8.75\beta, \qquad \overset{*}{i}_1(0,\beta) = -2.5\beta+2.5, \quad (37)$$

$$\frac{d\overset{*}{i}_2(t,\beta)}{dt} + (25\beta+15)\overset{*}{i}_2(t,\beta) = 30\beta+20, \qquad \overset{*}{i}_2(0,\beta) = 2.5\beta+2.5. \quad (38)$$

Here, $L = d/dt$ and by taking $L^{-1}(.) = \int_0^t (.)dt$ on both sides of (35)–(38), and using the initial conditions, we obtain

$$\overset{*}{i}_1(t,\alpha) = -5\int_0^t\int_0^t (\alpha+2.2)\overset{*}{i}_1(x,\alpha)dx + 5\alpha t + 1.25\alpha + 16t + 1, \quad (39)$$

$$\overset{*}{i}_2(t,\alpha) = 10\int_0^t (\alpha-2.8)\overset{*}{i}_2(x,\alpha)dx - 11.25\alpha t - 1.25\alpha + 35t + 4, \quad (40)$$

$$\overset{*}{i}_1(t,\beta) = 7.5\int_0^t (\alpha-2.2)\overset{*}{i}_1(x,\beta)dx - 8.75\beta t + 21.75t - 2.5\beta + 2.5, \quad (41)$$

$$\overset{*}{i}_2(t,\beta) = -25\int_0^t (\alpha+0.6)\overset{*}{i}_2(x,\beta)\mathrm{d}x + 30\beta t + 2.5\beta + 20t + 2.5. \tag{42}$$

Now by using GMADM, the solution of (39)–(42) can be expressed as follows:

$$\begin{cases} \overset{*}{i}_{1_0}(t,\alpha) = 1.25\alpha + 1, \\ \overset{*}{i}_{1_1}(t,\alpha) = \{(5\alpha+16) - (5\alpha+11)(1.25\alpha+1)\}t, \\ \overset{*}{i}_{1_{k+1}}(t,\alpha) = -5\int_0^t (\alpha+2.2)\overset{*}{i}_k(x,\alpha)dx, k \geq 1, \end{cases} \tag{43}$$

$$\begin{cases} \overset{*}{i}_{2_0}(t,\alpha) = -1.25\alpha + 4, \\ \overset{*}{i}_{2_1}(t,\alpha) = \{(-11.25\alpha+35) + (10\alpha-28)(-1.25\alpha+4)\}t, \\ \overset{*}{i}_{2_{k+1}}(t,\alpha) = 10\int_0^t (\alpha-2.8)\overset{*}{i}_{2_k}(x,\alpha)dx, k \geq 1, \end{cases} \tag{44}$$

$$\begin{cases} \overset{*}{i}_{1_0}(t,\beta) = -2.5\beta + 2.5, \\ \overset{*}{i}_{1_2}(t,\beta) = \{(-8.75\beta+21.75) + (7.5\beta-16.5)(-2.5\beta+2.5)\}t, \\ \overset{*}{i}_{1_{k+1}}(t,\beta) = 7.5\int_0^t (\alpha-2.2)\overset{*}{i}_{1_k}(x,\beta)dx, k \geq 1, \end{cases} \tag{45}$$

$$\begin{cases} \overset{*}{i}_{2_0}(t,\beta) = 2.5\beta + 2.5, \\ \overset{*}{i}_{2_1}(t,\beta) = \{(30\beta+20) - (25\beta+15)(2.5\beta+2.5)\}t, \\ \overset{*}{i}_{2_{k+1}}(t,\beta) = -25\int_0^t (\alpha+0.6)\overset{*}{i}_{2_k}(x,\beta)\mathrm{d}x, k \geq 1. \end{cases} \tag{46}$$

By solving (43)–(46), we get the closed form solution after four iterations as follows using GMADM:

$$\begin{aligned} \overset{*}{i}_1(t,\alpha) &= 1.25\alpha + 1 + \frac{1}{5\alpha+11}[(5\alpha+16) - (5\alpha+11)(1.25\alpha+1)]\left[1 - e^{-(5\alpha+11)t}\right], \\ \overset{*}{i}_2(t,\alpha) &= 4 - 1.25\alpha + \frac{1}{28-10\alpha}[(-11.25\alpha+35) - (-10\alpha+28)(-1.25\alpha+4)]\left[1 - e^{-(-10\alpha+28)t}\right], \\ \overset{*}{i}_1(t,\beta) &= 2.5 - 2.5\beta + \frac{1}{16.5-7.5\beta}[(-8.75\beta+21.75) - (-7.5\beta+16.5)(-2.5\beta+2.5)]\left[1 - e^{-(-7.5\beta+16.5)t}\right], \\ \overset{*}{i}_2(t,\beta) &= 2.5 + 2.5\beta + \frac{1}{25\beta+15}[(30\beta+20) - (25\beta+15)(2.5\beta+2.5)]\left[1 - e^{-(25\beta+15)t}\right]. \end{aligned} \tag{47}$$

The exact and approximate solutions of the IVP-I used in Example 1 are shown in Tables 1 and 2, respectively.

Figures 2(a) and 2(b) shows an approximate solution of the membership and nonmembership function for IVP-I used in Example 1.

The exact solution of Example 1 by classical method is given as follows:

$$\begin{aligned}
\overset{*}{i}_1(t,\alpha) &= e^{-(5\alpha+11)t}\left(\frac{5}{4}\alpha + 1 - \frac{(5\alpha+16)}{(5\alpha+11)}\right) + \frac{(5\alpha+16)}{(5\alpha+11)},\\
\overset{*}{i}_2(t,\alpha) &= e^{2(5\alpha-14)t}\left(4 - \frac{5}{4}\alpha - \frac{5}{8}\frac{(-28+9\alpha)}{(5\alpha-14)}\right) + \frac{5}{8}\frac{(-28+9\alpha)}{(5\alpha-14)},\\
\overset{*}{i}_1(t,\beta) &= e^{(3/2)(5\beta-11)t}\left(\frac{5}{2} - \frac{5}{2}\beta - \frac{1}{6}\frac{(-87+35\beta)}{(-11+5\beta)}\right) + \frac{1}{6}\frac{(-87+35\beta)}{(-11+5\beta)},\\
\overset{*}{i}_2(t,\beta) &= e^{-5(5\beta+3)t}\left(\frac{5}{2}\beta + \frac{5}{2} - \frac{2(3\beta+2)}{(5\beta+3)}\right) + \frac{2(3\beta+2)}{(5\beta+3)}.
\end{aligned} \tag{48}$$

Figure 3(a) depicts the exact solution of the membership function of the generalized intuitionistic fuzzy IVP-I, 3(b) depicts the exact solution of the nonmembership function, and 3(c) depicts the exact solution of the generalized intuitionistic fuzzy IVP-I as described in Example 1.

In Table 3, n represents the number of iterations, Err represents the residual error, and CPU time represents the computational time in seconds for finding the approximate solution of the generalized intuitionistic fuzzy IVP-I described in Example 1. Figures 2(a) and 2(b) show an approximate solution of the membership and nonmembership function for IVP-I used in Example 1. The exact solutions to the membership and nonmembership functions for IVP-I used in Example 1 are depicted in Figures 3(a)–3(c).

Example 2. Mass Spring System (Physical Engineering Application).

Consider a spring with a mass of $\overset{*}{m} = <(2,3,4,5;0.6);(1,3,4,6;0.3)>$ kg is held stretched $\overset{*}{s} = <(0.5,0.6,0.7,0.8;0.6);(0.4,0.6,0.7,0.9;0.3)>$ m beyond its natural length by a force of $\overset{*}{F} = <(19,20,21,23;0.6);(18,20,21,24;0.3)>$ N. If the spring begins at its equilibrium position, then a push gives it an initial velocity of $\overset{*}{x}_0 = <(11,12,13,14;0.6);(10,12,13,15;0.3)>\ \mathrm{ms}^{-1}$.

The intuitionistic fuzzy initial value problem related to this problem is given as follows:

$$\overset{*}{m}\frac{d^2x(t)}{dt^2} + \overset{*}{k}\overset{*}{x}(t) = 0, \tag{49}$$

with initial conditions:

$$\begin{cases} \overset{*}{x}(0) = <(5,6,7,8;0.6);(4,6,7,9;0.3)>,\\ \overset{*}{x}'(0) = <(11,12,13,14;0.6);(10,12,13,15;0.3)>. \end{cases} \tag{50}$$

Here, $\overset{*}{k}$ is spring constant and its value is calculated as follows:

$$\begin{aligned} \overset{*}{k} &= \frac{\overset{*}{F}}{\overset{*}{s}} \\ &= <(24,29,35,46;0.8);(20,29,35,60;0.3)>. \end{aligned} \tag{51}$$

Using the values of $\overset{*}{m}$ and $\overset{*}{k}$ in (49), we get the following intuitionistic fuzzy differential equation in standard form as follows:

$$\frac{d^2\overset{*}{x}(t)}{dt^2} + <(5,7,12,23;0.6);(3,7,12,60;0.3)>\overset{*}{x}(t) = 0. \tag{52}$$

By taking (α,β)-cut of (50) and (52), we get

$$\begin{aligned} &\frac{d_1^2\overset{*}{x}(t,\alpha)}{dt^2} + (3\alpha+5)\overset{*}{x}_1(t,\alpha) = 0,\\ &\overset{*}{x}_1(0,\alpha) = 2\alpha+5,\\ &\overset{*}{x}_1'(0,\alpha) = 2\alpha+11, \end{aligned} \tag{53}$$

$$\begin{aligned} &\frac{d_2^2\overset{*}{x}(t,\alpha)}{dt^2} + (23-18\alpha)\overset{*}{x}_2(t,\alpha) = 0,\\ &x_2(0,\alpha) = 8-2\alpha,\\ &\overset{*}{x}_2'(0,\alpha) = 14-2\alpha, \end{aligned} \tag{54}$$

$$\begin{aligned} &\frac{d_1^2\overset{*}{x}(t,\beta)}{dt^2} + (9-6\beta)\overset{*}{x}_1(t,\beta) = 0,\\ &\overset{*}{x}_1(0,\beta) = 7-3\beta,\\ &\overset{*}{x}_1'(0,\beta) = 13-3\beta, \end{aligned} \tag{55}$$

$$\begin{aligned} &\frac{d_2^2\overset{*}{x}(t,\beta)}{dt^2} + (69\beta-9)\overset{*}{x}_2(t,\beta) = 0,\\ &\overset{*}{x}_2(0,\beta) = 3\beta+6,\\ &\overset{*}{x}_2'(0,\beta) = 3\beta+12. \end{aligned} \tag{56}$$

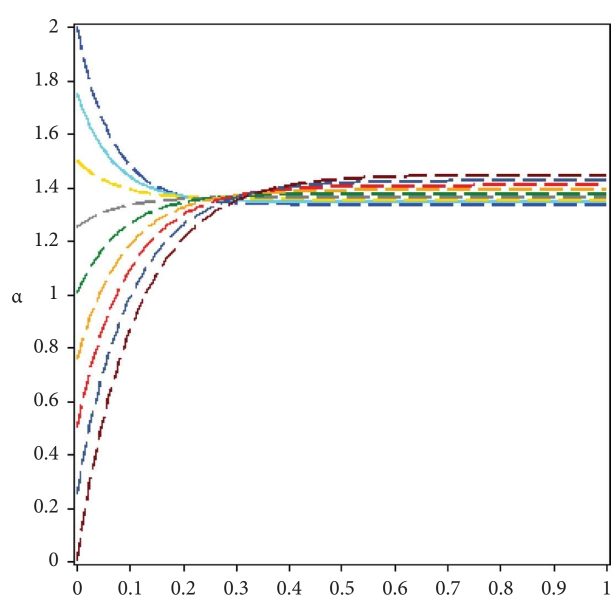
Here, $L = d^2/dt^2$ and by taking $L^{-1}(.) = \int_0^t\int_0^t (.)dt\,dt$ on both sides of (53)–(56), and using the initial conditions, we obtain

Table 1: Approximate solution of Example 1 at $t = 1$.

α	$\overset{*}{i}_1(t,\alpha)$	$\overset{*}{i}_2(t,\alpha)$	β	$\overset{*}{i}_1(t,\beta)$	$\overset{*}{i}_2(t,\beta)$
0	1.4546	1.2500	0.2	1.3333	1.3000
0.1	1.4348	1.2546	0.3	1.3421	1.2889
0.2	1.4167	1.2596	0.4	1.3519	1.2800
0.3	1.4000	1.2650	0.5	1.3627	1.2727
0.4	1.3846	1.2708	0.6	1.3750	1.2667
0.5	1.3704	1.2772	0.7	1.3889	1.2615
0.6	1.3571	1.28409	0.8	1.4048	1.2571
0.7	1.3448	1.2917	0.9	1.4231	1.2533
0.8	1.3333	1.3000	1.0	1.4444	1.2500

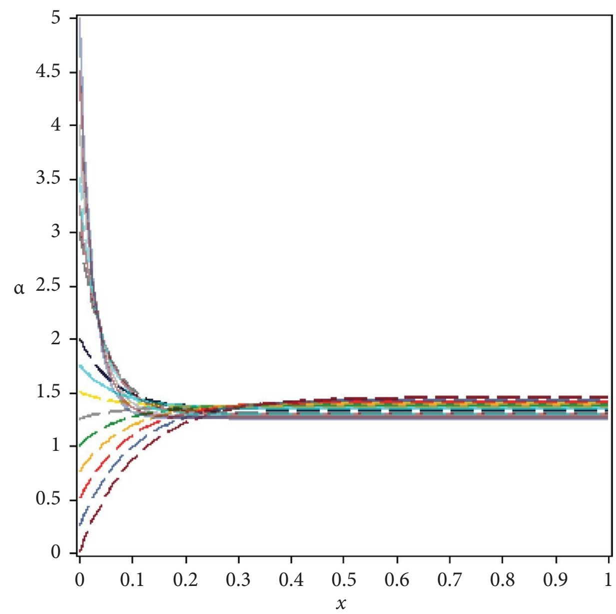
Table 2: Exact solution of Example 1 at $t = 1$.

α	$\overset{*}{i}_1(t,\alpha)$	$\overset{*}{i}_2(t,\alpha)$	β	$\overset{*}{i}_1(t,\beta)$	$\overset{*}{i}_2(t,\beta)$
0	1.4546	1.2500	0.2	1.3333	1.3000
0.1	1.4348	1.2546	0.3	1.3421	1.2889
0.2	1.4167	1.2596	0.4	1.3519	1.2800
0.3	1.4000	1.2650	0.5	1.3627	1.2727
0.4	1.3846	1.2708	0.6	1.3750	1.2667
0.5	1.3704	1.2772	0.7	1.3889	1.2615
0.6	1.3571	1.28409	0.8	1.4048	1.2571
0.7	1.3448	1.2917	0.9	1.4231	1.2533
0.8	1.3333	1.3000	1.0	1.4444	1.2500



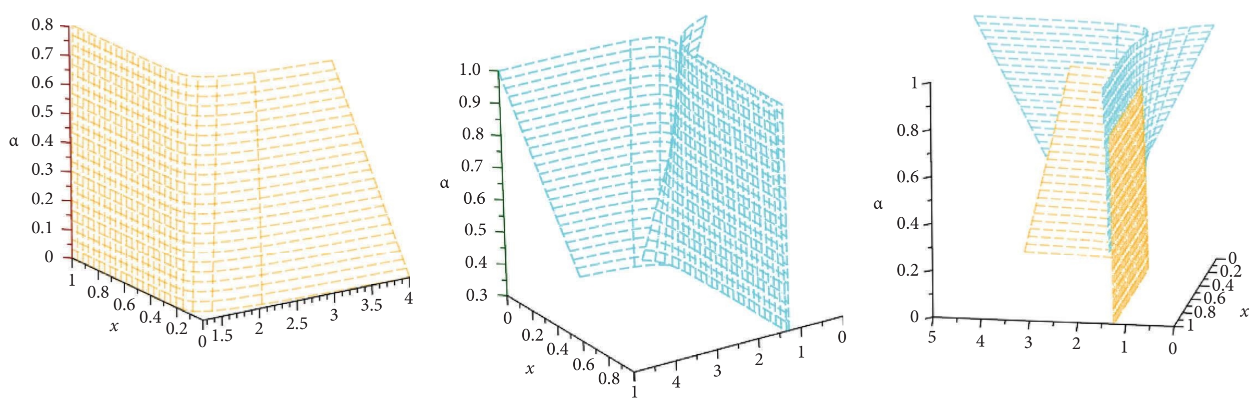
(a)

Figure 2: Continued.



(b)

Figure 2: (a) Membership function and (b) nonmembership function.



(a) (b) (c)

Figure 3: (a) Exact solution of membership function for GITF-IVP-I, (b) exact solution of nonmembership function for GITF-IVP-I, and (c) exact solution of GTIF-IVP-I.

Table 3: Error comparison of GMADM, ADM, and TSM for solving GTIF-IVP-I used in Example 1.

Methods	GMADM	ADM	TSM
n	04	17	08
Err	$1.2e-32$	$6.2e-15$	$7.1e-5$
CPU time	0.0123	0.2413	0.1452

$$\begin{cases} \overset{*}{x}_1(t,\alpha) = \int_0^t (-t+y)(3\alpha+5)\overset{*}{x}_1(y,\alpha)\mathrm{d}y + 2\alpha t + 2\alpha + 11t + 5, \\ \overset{*}{x}_2(t,\alpha) = \int_0^t \Big(-(-t+y)(-23+18\alpha)\overset{*}{x}_2(y,\alpha)\Big)\mathrm{d}y - 2\alpha t - 2\alpha + 14t + 8, \\ \overset{*}{x}_1(t,\beta) = \int_0^t \Big(-3(-t+y)(-3+2\beta)\overset{*}{x}_1(y,\beta)\Big)\mathrm{d}y - 3\beta t - 3\beta + 13t + 7, \\ \overset{*}{x}_2(t,\beta) = \int_0^t 3(-t+y)(-3+23\beta)\overset{*}{x}_2(y,\beta)\mathrm{d}y + 3\beta t + 3\beta + 12t + 6. \end{cases} \tag{57}$$

Now, by using GMADM, we get

$$\begin{cases} \overset{*}{x}_{1_0}(t,\alpha) = 5 + (2\alpha+11)t + 2\alpha, \\ \overset{*}{x}_{1_1}(t,\alpha) = -\alpha^2 t^3 - \frac{43}{6}\alpha t^3 - \frac{55}{6}t^3 - 3\alpha^2 t^2 - \frac{25}{2}\alpha t^2 - \frac{25}{2}t^2, \\ \overset{*}{x}_{1_{k+1}}(t,\alpha) = \int_0^t (-t+y)(3\alpha+5) * x_{1_k}(y,\alpha)\mathrm{d}y, k \geq 1. \end{cases} \tag{58}$$

$$\begin{cases} \overset{*}{x}_{2_0}(t,\alpha) = 8 + (14-2\alpha)t - 2\alpha, \\ \overset{*}{x}_{2_1}(t,\alpha) = -6\alpha^2 t^3 - \frac{149}{3}\alpha t^3 - \frac{161}{3}t^3 - 18\alpha^2 t^2 + 95\alpha t^2 - 92t^2, \\ \overset{*}{x}_{2_{k+1}}(t,\alpha) = \int_0^t \Big(-(-t+y)(-23+18\alpha) * x_{2_k}(y,\alpha)\mathrm{d}y, k \geq 1. \end{cases} \tag{59}$$

$$\begin{cases} \overset{*}{x}_{1_0}(t,\beta) = 7 + (13-3\beta)t - 3\beta, \\ \overset{*}{x}_{1_2}(t,\beta) = -3\beta^2 t^3 + \frac{35}{2}\beta t^3 - \frac{39}{2}t^3 - 9\beta^2 t^2 + \frac{69}{2}\beta t^2 - \frac{63}{2}t^2, \\ \overset{*}{x}_{1_{k+1}}(t,\beta) = \int_0^t \int_0^t \Big(-3(-t+y)(-3+2\beta) * x_{1_k}(y,\beta)\mathrm{d}y, k \geq 1. \end{cases} \tag{60}$$

$$\begin{cases} \overset{*}{x}_{2_0}(t,\beta) = 6 + (3\beta+12) + 3\beta, \\ \overset{*}{x}_{2_1}(t,\beta) = -\frac{69}{2}\beta^2 t^3 - \frac{267}{2}\beta t^3 + 18t^3 - \frac{207}{2}\beta^2 t^2 - \frac{387}{2}\beta t^2 + 27t^2, \\ \overset{*}{x}_{2_{k+1}}(t,\beta) = \int_0^t 3(-t+y)(-3+23\beta) * x_{2_k}(y,\beta)\mathrm{d}y, k \geq 1. \end{cases} \tag{61}$$

By solving the (58)–(61), we get the approximate solution after four iterations as follows:

$$\overset{*}{x}_2\,(t,\alpha) = 8 - 2\alpha + 14t - 2\alpha t + \frac{279841}{25920}t^9 + \frac{625}{8064}t^8 - \frac{161}{3}t^3 - 92t^2 + \frac{3703}{60}t^5 + \frac{529}{3}t^4 - \frac{12167}{360}t^7 - \frac{12167}{90}t^6 - \frac{81}{140}t^9\alpha^5 + \frac{981}{140}t^9\alpha^4$$
$$- \frac{7383}{280}t^9\alpha^3 + \frac{28037}{630}t^9\alpha^2 - \frac{6412009}{181440}t^9\alpha - \frac{729}{140}t^8\alpha^5 + \frac{3321}{70}t^8\alpha^4 - \frac{44091}{280}t^8\alpha^3 + \frac{69299}{280}t^8\alpha^2 - \frac{3783937}{20160}t^8\alpha - \frac{81}{35}t^7\alpha^4 + \frac{351}{14}t^7\alpha^3$$
$$- \frac{10281}{140}t^7\alpha^2 + \frac{212129}{2520}t^7\alpha - \frac{81}{5}t^6\alpha^4 + \frac{1269}{10}t^6\alpha^3 - \frac{1311}{4}t^6\alpha^2 + \frac{126431}{360}t^6\alpha - \frac{27}{5}t^5\alpha^3 + \frac{258}{5}t^5\alpha^2 - \frac{1265}{12}t^5\alpha - 27t^4\alpha^3 + 177t^4\alpha^2$$
$$- \frac{3841}{12}t^4\alpha - 6t^3\alpha^2 + \frac{149}{3}t^3\alpha - 18t^2\alpha^2 + 95t^2\alpha,$$
$$\overset{*}{x}_1\,(t,\beta) = 7 - 3\beta + 13t - 3\beta t + \frac{1053}{4480}t^9 + \frac{729}{640}t^8 - \frac{39}{2}t^3 - \frac{63}{2}t^2 + \frac{351}{40}t^5 + \frac{189}{8}t^4 - \frac{1053}{560}t^7 - \frac{567}{80}t^6 - \frac{3}{280}t^9\beta^5 + \frac{31}{80}t^9\beta^4 - \frac{237}{560}t^9\beta^3$$
$$+ \frac{27}{35}t^9\beta^2 - \frac{3051}{4480}t^9\beta - \frac{27}{280}t^8\beta^5 + \frac{45}{56}t^8\beta^4 - \frac{297}{112}t^8\beta^3 + \frac{243}{56}t^8\beta^2 - \frac{3159}{896}t^8\beta - \frac{9}{70}t^7\beta^4 + \frac{159}{140}t^7\beta^3 - \frac{27}{8}t^7\beta^2 + \frac{2349}{560}t^7\beta - \frac{9}{10}t^6\beta^4$$
$$+ \frac{123}{20}t^6\beta^3 - \frac{621}{40}t^6\beta^2 + \frac{1377}{80}t^6\beta - \frac{9}{10}t^5\beta^3 + \frac{33}{5}t^5\beta^2 - \frac{549}{40}t^5\beta - \frac{9}{2}t^4\beta^3 + 24t^4\beta^2 - \frac{333}{8}t^4\beta - 3t^3\beta^2 + \frac{35}{2}t^3\beta - 9t^2\beta^2 + \frac{69}{2}t^2\beta. \tag{62}$$

The exact and approximate solutions of the IVP-II used in Example 2 are shown in Tables 4 and 5, respectively.

Figures 4(a) and 4(b) show an approximate solution of the membership and nonmembership functions for IVP-II used in Example 2.

The exact solution of Example 2 given by classical method is as follows:

$$\begin{aligned}
x_1\,(t,\alpha) &= \frac{(2\alpha + 11)\sin\sqrt{3\alpha + 5}t}{\sqrt{3\alpha + 5}} + (5 + 2\alpha)\cos\sqrt{3\alpha + 5}t,\\
x_2\,(t,\alpha) &= \frac{2\sqrt{-18\alpha + 23}\,(\alpha - 7)\sin\sqrt{-18\alpha + 23}t}{-18\alpha + 23} + (8 - 2\alpha)\cos\sqrt{-18\alpha + 23}t,\\
x_1\,(t,\beta) &= \frac{1}{3}\,\frac{2\sqrt{-6\beta + 9}\,(3\beta - 13)\sin\sqrt{-6\beta + 9}t}{2\beta - 3} + (7 - 3\beta)\cos\sqrt{-6\beta + 9}t,\\
x_2\,(t,\beta) &= \frac{\sqrt{69\beta - 9}\,(\beta + 4)\sin\sqrt{69\beta - 9}t}{23\beta - 3} + (6 = 3\beta)\cos\sqrt{69\beta - 9}t.
\end{aligned} \tag{63}$$

Figure 5(a) depicts the exact solution of the membership function of the generalized intuitionistic fuzzy IVP-II, 5(b) depicts the exact solution of the nonmembership function, and 5(c) depicts the exact solution of the generalized intuitionistic fuzzy IVP-I1 described in Example 2.

In Table 6, n represents the number of iterations, Err represents the residual error, and CPU time represents the computational time in seconds for finding the approximate solution of the generalized intuitionistic fuzzy IVP-II described in Example 2. Figures 4(a) and 4(b) show an approximate solution of the membership and nonmembership function for IVP-II used in Example 2. The exact solutions to the membership and nonmembership functions for IVP-II used in Example 2 are depicted in Figures 5(a)–5(c).

Example 3. Steam Supply Control Slide Value Problem (Mechanical Engineering Application).

The motion $\overset{*}{y}(t)$ of a steam supply control slide valve is governed by the third-order differential equation as follows:

$$m\frac{d^3\overset{*}{y}}{dt^3} + f\frac{d^2\overset{*}{y}}{dt^2} + k\frac{d\overset{*}{y}}{dt} + h\frac{\alpha}{I}\overset{*}{y} = 0, \tag{64}$$

where m is mass of the valve, f is friction, k is a constant characterizing the properties of the slide valve spring, h is a constant depending on the dimensions of the equipment, α is a proportionality constant relating the motion and the acceleration of the control valve, and I is the moment of inertia of the turbine. If we neglect the friction and take $m = 50$kg, $k = 25$ $h = 1, \alpha = 6$, and $I = 0.8$, then the (64) becomes

$$50\frac{d^3\overset{*}{y}}{dt^3} + 25\frac{d\overset{*}{y}}{dt} + \frac{6}{0.8}\overset{*}{y} = 0, \tag{65}$$

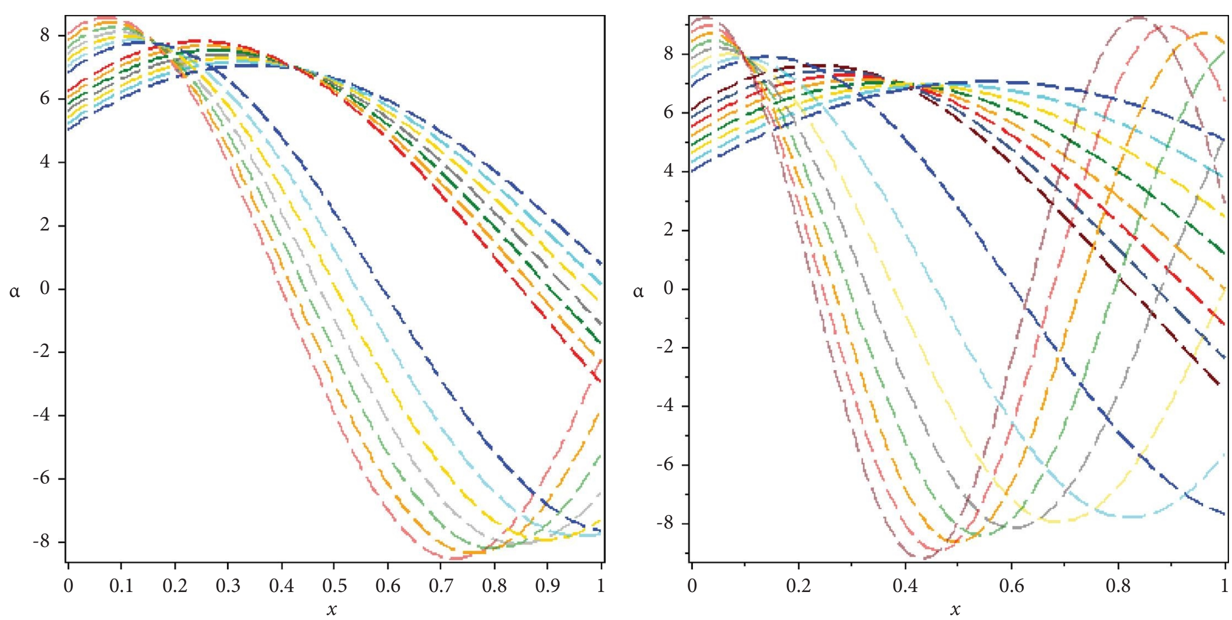
with initial conditions

Table 4: Approximate solution of Example 2 at $t = 1$.

α	$x_1(t, \alpha)$	$x_2(t, \alpha)$	β	$x_1(t, \beta)$	$x_2(t, \beta)$
0	6.6109	−3.8972	0.3	5.7772	2.7760
0.1	6.5617	−2.9696	0.4	6.0440	−1.4241
0.2	6.5010	−1.9884	0.5	6.2821	−4.5497
0.3	6.4290	−0.9561	0.6	6.4899	−6.6794
0.4	6.3461	0.1248	0.7	6.6661	−7.8787
0.5	6.2525	1.2514	0.8	6.8090	−8.1914
0.6	6.1483	2.4206	0.9	6.9174	−7.6319
			1.0	6.9895	−6.1758

Table 5: Exact solution of Example 2 at $t = 1$.

α	$x_1(t, \alpha)$	$x_2(t, \alpha)$	β	$x_1(t, \beta)$	$x_2(t, \beta)$
0	6.6109	−3.8972	0.3	5.7772	2.7760
0.1	6.5617	−2.9696	0.4	6.0440	−1.4241
0.2	6.5010	−1.9884	0.5	6.2821	−4.5497
0.3	6.4290	−0.9561	0.6	6.4899	−6.6794
0.4	6.3461	0.1248	0.7	6.6661	−7.8787
0.5	6.2525	1.2514	0.8	6.8090	−8.1914
0.6	6.1483	2.4206	0.9	6.9174	−7.6319
			1.0	6.9895	−6.1758



(a) (b)

Figure 4: (a) Membership function and (b) nonmembership function.

$$\overset{*}{y}(0) = <(1,2,3,4;0.5);(0,2,3,5;0.3)>,$$
$$\overset{*}{y}{}'(0) = <(7,8,9,10;0.5);(6,8,9,11;0.3)>,$$
$$\overset{*}{y}{}''(0) = <(21,22,23,24;0.5);(20,22,23,25;0.3)>. \quad (66)$$

In standard form, (65) can be written as follows:

$$\frac{d^3\overset{*}{y}}{dt^3} + \frac{1}{5}\frac{d\overset{*}{y}}{dt} + \frac{3}{20}\overset{*}{y} = 0. \quad (67)$$

By taking (α, β)-cut of (67) and (66), we get the following equations:

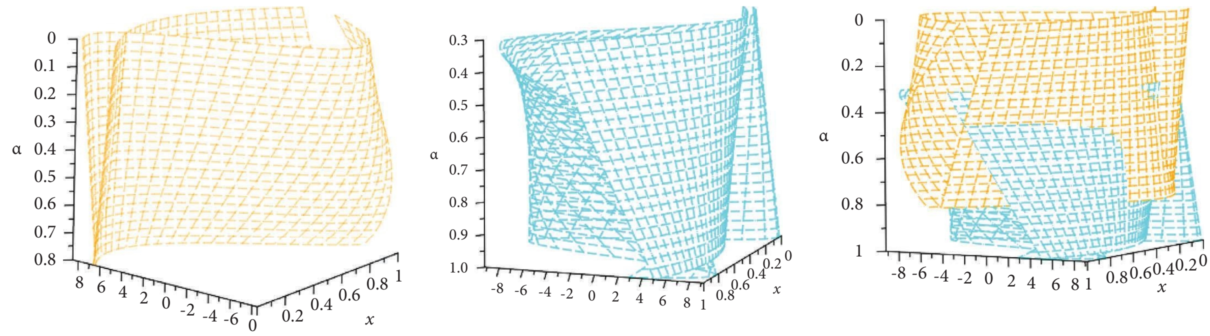


FIGURE 5: (a) Exact solution of membership function for GITF-IVP-II, (b) exact solution of nonmembership function for GITF-IVP-II, and (c) exact solution of GTIF-IVP-II.

TABLE 6: Error comparison of GMADM, ADM, and TSM for solving GTIF-IVP-II used in Example 2.

Methods	GMADM	ADM	TSM
n	04	14	06
Err	$4.2e-45$	$1.2e-19$	$0.1e-7$
CPU time	0.0143	0.2113	0.1872

$$\begin{cases} \dfrac{d^3 \overset{*}{y}_1(t,\alpha)}{dt^3} + \dfrac{1}{5}\dfrac{d\overset{*}{y}_1(t,\alpha)}{dt} + \dfrac{3}{20}\overset{*}{y}_1(t,\alpha) = 0, \\ \overset{*}{y}_1(0,\alpha) = 1 + 2\alpha, \\ \overset{*}{y}_1'(0,\alpha) = 7 + 2\alpha, \\ \overset{*}{y}_1''(0,\alpha) = 21 + 2\alpha, \end{cases} \quad (68)$$

$$\begin{cases} \dfrac{d^3 \overset{*}{y}_2(t,\alpha)}{dt^3} + \dfrac{1}{5}\dfrac{d\overset{*}{y}_2(t,\alpha)}{dt} + \dfrac{3}{20}\overset{*}{y}_2(t,\alpha) = 0, \overset{*}{y}_2(0,\alpha) = 4 - 2\alpha, \\ \overset{*}{y}_2'(0,\alpha) = 10 - 2\alpha, \overset{*}{y}_2''(0,\alpha) = 24 - 2\alpha, \end{cases} \quad (69)$$

$$\begin{cases} \dfrac{d^3 \overset{*}{y}_1(t,\beta)}{dt^3} + \dfrac{1}{5}\dfrac{d\overset{*}{y}_1(t,\beta)}{dt} + \dfrac{3}{20}\overset{*}{y}_1(t,\beta) = 0, \\ \overset{*}{y}_1(0,\beta) = 3 - 3\beta, \\ \overset{*}{y}_1'(0,\beta) = 9 - 3\beta, \\ \overset{*}{y}_1''(0,\beta) = 23 - 3\beta, \end{cases} \quad (70)$$

$$\begin{cases} \dfrac{d^3 \overset{*}{y}_2(t,\beta)}{dt^3} + \dfrac{1}{5}\dfrac{d\overset{*}{y}_2(t,\beta)}{dt} + \dfrac{3}{20}\overset{*}{y}_2(t,\beta) = 0, \\ \overset{*}{y}_2(0,\beta) = 2 + 3\beta, \\ \overset{*}{y}_2'(0,\beta) = 8 + 3\beta, \\ \overset{*}{y}_2''(0,\beta) = 22 + 3\beta. \end{cases} \quad (71)$$

Here, $L = \mathrm{d}^3/\mathrm{d}t^3$ and by taking $L^{-1}(.) = \int_0^t \int_0^t \int_0^t (.)\mathrm{d}t\,\mathrm{d}t\,\mathrm{d}t$ on both sides of (68)–(71), and using the initial conditions, we obtain

$$\begin{aligned}
\overset{*}{y}_1(t,\alpha) &= \int_0^t -\frac{1}{40}\overset{*}{y}_1(x,\alpha)(x-t)(3x-3t-8)\mathrm{d}x + 1 + \frac{53}{5}t^2 + \frac{6}{5}t^2\alpha + 2\alpha t + 2\alpha + 7t,\\
\overset{*}{y}_2(t,\alpha) &= \int_0^t -\frac{1}{40}\overset{*}{y}_2(x,\alpha)(x-t)(3x-3t-8)\mathrm{d}x + 4 + \frac{62}{5}t^2 - \frac{6}{5}t^2\alpha - 2\alpha t - 2\alpha + 10t,\\
\overset{*}{y}_1(t,\beta) &= \int_0^t -\frac{1}{40}\overset{*}{y}_1(x,\beta)(x-t)(3x-3t-8)\mathrm{d}x + 3 + \frac{59}{5}t^2 - \frac{9}{5}t^2\beta - 3\beta t - 3\beta + 9t,\\
\overset{*}{y}_2(t,\beta) &= \int_0^t -\frac{1}{40}\overset{*}{y}_2(x,\beta)(x-t)(3x-3t-8)\mathrm{d}x + 2 + \frac{56}{5}t^2 + \frac{9}{5}t^2\beta + 3\beta t + 3\beta + 8t.
\end{aligned} \tag{72}$$

Now, by using GMADM, we get

$$\begin{cases}
\overset{*}{y}_{1_0}(t,\alpha) = 1 + \dfrac{1}{10}t^2(1+2\alpha) + \dfrac{1}{2}(21+2\alpha)t^2 + (7+2\alpha)t + 2\alpha,\\
\overset{*}{y}_{1_1}(t,\alpha) = -\dfrac{53}{2000}t^5 - \dfrac{3}{1000}t^5\alpha - \dfrac{529}{2400}t^4 - \dfrac{13}{400}t^4\alpha - \dfrac{31}{120}t^3 - \dfrac{7}{60}t^3\alpha - \dfrac{1}{10}t^2 - \dfrac{1}{5}t^2\alpha,\\
\overset{*}{y}_{1_{k+1}}(t,\alpha) = \displaystyle\int_0^t -\frac{1}{40}(x-t)(3x-3t-8)\overset{*}{y}_{1_k}(x,\alpha)\mathrm{d}x, k \geq 1.
\end{cases} \tag{73}$$

$$\begin{cases}
\overset{*}{y}_{2_0}(t,\alpha) = 4 + \dfrac{1}{10}t^2(4-2\alpha) + \dfrac{1}{2}(24-2\alpha)t^2 + (10-2\alpha)t - 2\alpha,\\
\overset{*}{y}_{2_1}(t,\alpha) = -\dfrac{31}{1000}t^5 + \dfrac{3}{1000}t^5\alpha - \dfrac{323}{1200}t^4 + \dfrac{13}{400}t^4\alpha - \dfrac{31}{30}t^3 + \dfrac{7}{60}t^3\alpha - \dfrac{2}{5}t^2 + \dfrac{1}{5}t^2\alpha,\\
\overset{*}{y}_{2_{k+1}}(t,\alpha) = \displaystyle\int_0^t -\frac{1}{40}(x-t)(3x-3t-8)\overset{*}{y}_{2_k}(x,\alpha)\mathrm{d}x, k \geq 1.
\end{cases} \tag{74}$$

$$\begin{cases}
\overset{*}{y}_{1_0}(t,\beta) = 3 + \dfrac{1}{10}t^2(3-2\beta) + \dfrac{1}{2}(23-3\beta)t^2 + (9-3\beta)t - 3\beta,\\
\overset{*}{y}_{1_1}(t,\beta) = -\dfrac{59}{2000}t^5 + \dfrac{9}{2000}t^5\beta - \dfrac{607}{2400}t^4 + \dfrac{39}{800}t^4\beta - \dfrac{3}{8}t^3 + \dfrac{7}{40}t^3\beta - \dfrac{3}{10}t^2 + \dfrac{3}{10}t^2\beta,\\
\overset{*}{y}_{1_{k+1}}(t,\beta) = \displaystyle\int_0^t -\frac{1}{40}(x-t)(3x-3t-8)\overset{*}{y}_{1_k}(x,\beta)\mathrm{d}x, k \geq 1.
\end{cases} \tag{75}$$

$$\begin{cases}
\overset{*}{y}_{2_0}(t,\beta) = 2 + \dfrac{1}{10}t^2(2+3\beta) + \dfrac{1}{2}(22+3\beta)t^2 + (8+3\beta)t + 3\beta,\\
\overset{*}{y}_{2_1}(t,\beta) = -\dfrac{59}{2000}t^5 + \dfrac{9}{2000}t^5\beta - \dfrac{607}{2400}t^4 + \dfrac{39}{800}t^4\beta - \dfrac{3}{8}t^3 + \dfrac{7}{40}t^3\beta - \dfrac{3}{10}t^2 + \dfrac{3}{10}t^2\beta,\\
\overset{*}{y}_{2_{k+1}}(t,\beta) = \displaystyle\int_0^t -\frac{1}{40}(x-t)(3x-3t-8)\overset{*}{y}_{2_k}(x,\beta)\mathrm{d}x, k \geq 1.
\end{cases} \tag{76}$$

By solving the (73)–(76), we get the approximate solution after four iterations as follows:

$$\begin{aligned}
\overset{*}{y}_1(t,\alpha) = {} & 1 + t^2\alpha + 7t + \frac{1}{71751680000000}t^{14}\alpha + \frac{37}{30750720000000}t^{13}\alpha + \frac{41}{1013760000000}t^{12}\alpha - \frac{47381}{71751680000000}t^{10} - \frac{1}{750}t^5\alpha - \frac{7}{240}t^4\alpha \\
& - \frac{7}{60}t^3\alpha - \frac{71}{3000}t^5 - \frac{7}{32}t^4 - \frac{31}{120}t^3 + \frac{2357}{1330560000000}t^{11} + \frac{57}{32000}t^6 - \frac{18539}{14515200000}t^9 + \frac{113}{26880000}t^8 + \frac{2711}{10080000}t^7 + \frac{137}{5040000}t^7\alpha \\
& + \frac{53}{430510080000000}t^{14} + \frac{1801}{184504320000000}t^{13} + \frac{3967}{1330560000000}t^{12} + 2\alpha + \frac{109}{221760000000}t^{11}\alpha - \frac{71}{17280000000}t^{10}\alpha - \frac{1283}{7257600000}t^9\alpha \\
& + \frac{21}{2}t^2 - \frac{1}{1612800}t^8\alpha + \frac{49}{144000}t^6\alpha 2t\alpha, \\
\overset{*}{y}_2(t,\alpha) = {} & 4 - t^2\alpha + 10t - \frac{1}{71751680000000}t^{14}\alpha - \frac{37}{30750720000000}t^{13}\alpha + \frac{1}{750}t^5\alpha - \frac{41}{1013760000000}t^{12}\alpha + \frac{7}{240}t^4\alpha + \frac{7}{60}t^3\alpha - \frac{77}{3000}t^5 - \frac{21}{80}t^4 \\
& - \frac{13}{30}t^3 + \frac{1669}{665280000000}t^{11} - \frac{25927}{362880000000}t^{10} - \frac{5597}{3628800000}t^9 + \frac{11}{3360000}t^8 + \frac{223}{720000}t^7 + \frac{11}{4800}t^6 + \frac{31}{215255040000000}t^{14} - 2\alpha \\
& + \frac{97}{8386560000000}t^{13} + \frac{1207}{3548160000000}t^{12} - \frac{109}{221760000000}t^{11}\alpha + \frac{71}{17280000000}t^{10}\alpha + \frac{1283}{7257600000}t^9\alpha + 12t^2 + \frac{1}{1612800}t^8\alpha \\
& - \frac{137}{5040000}t^7\alpha - \frac{49}{144000}t^6\alpha - 2t\alpha, \\
\overset{*}{y}_2(t,\beta) = {} & 2 + \frac{3}{2}t^2\beta + 8t + \frac{3}{143503360000000}t^{14}\beta + \frac{37}{20500480000000}t^{13}\beta + \frac{41}{675840000000}t^{12}\beta - \frac{1}{500}t^5\beta - \frac{7}{160}t^4\beta - \frac{7}{40}t^3\beta - \frac{73}{3000}t^5 - \frac{7}{30}t^4 \\
& - \frac{19}{60}t^3 + \frac{61}{30240000000}t^{11} - \frac{6109}{90720000000}t^{10} - \frac{9911}{7257600000}t^9 + \frac{157}{40320000}t^8 + \frac{89}{315000}t^7 + \frac{281}{144000}t^6 + \frac{1}{7687680000000}t^{14} \\
& + \frac{239}{23063040000000}t^{13} + \frac{709}{2365440000000}t^{12} + 3\beta + \frac{109}{147840000000}t^{11}\beta - \frac{71}{11520000000}t^{10}\beta - \frac{1283}{4838400000}t^9\beta + 11t^2 - \frac{1}{1075200}t^8\beta \\
& + \frac{137}{3360000}t^7\beta + \frac{49}{96000}t^6\beta + 3t\beta.
\end{aligned} \tag{77}$$

The exact and approximate solutions of the IVP-III used in Example 3 are shown in Tables 7 and 8, respectively.

Figures 6(a) and 6(b) show an approximate solution of the membership and nonmembership functions for IVP-III used in Example 3.

The exact solution of Example 3 given by classical method is as follows:

$$
\begin{aligned}
\overset{*}{y}_1(t,\alpha) &= \frac{1}{9}\frac{1}{(\xi)^{2/3}-60)(\chi)}\begin{pmatrix}
\begin{pmatrix}
168(\xi)^{7/3}+128887200000+7938000(\xi)^{2/3}\alpha+97200(\xi)^{2/3}\sqrt{19185}\,\alpha-818424000(\xi)^{1/3}\alpha-5832000(\xi)^{1/3}\\
\sqrt{19185}\,\alpha+48(\xi)^{7/3}\alpha-47871000(\xi)^{2/3}+48600(\xi)^{2/3}\sqrt{19185}-2864484000(\xi)^{1/3}-20412000(\xi)^{1/3}\sqrt{19185}+13258080000\alpha\\
+95904000\alpha\sqrt{19185}+10080(\xi)^{5/3}+2880(\xi)^{5/3}\alpha+930528000\sqrt{19185}
\end{pmatrix}\\
e^{(\sigma_2 t)}+\frac{2}{9}\frac{1}{\chi}\left(\sqrt{3}\left(84(\xi)^{5/3}+24(\xi)^{5/3}\alpha+7608(\xi)^{4/3}+302400(\xi)^{1/3}-456480(\xi)^{2/3}+816(\xi)^{4/3}\alpha+86400(\xi)^{1/3}\alpha-48960(\xi)^{2/3}\alpha\right)e^{\sigma_1 t}\sin(\sigma_3 t)\right)\\
-\frac{4}{9}\frac{1}{\left((\xi)^{2/3}-60\right)(\chi)}\begin{pmatrix}
\left(42(\xi)^{7/3}-8375400(\xi)^{2/3}\alpha-48600(\xi)^{2/3}\sqrt{19185}\,\alpha-204606000(\xi)^{1/3}\alpha-1458000(\xi)^{1/3}\sqrt{19185}\,\alpha+32532597000\right.\\
+12(\xi)^{7/3}\alpha-17147700(\xi)^{2/3}-24300(\xi)^{2/3}\sqrt{19185}-716121000(\xi)^{1/3}-5103000(\xi)^{1/3}\sqrt{19185}+3936114000\alpha+28350000\alpha\\
\sqrt{19185}+2520(\xi)^{5/3}+720(\xi)^{5/3}\alpha+234819000\sqrt{19185}
\end{pmatrix}e^{\sigma_1 t}\cos(\sigma_3 t)
\end{pmatrix},\\
\overset{*}{y}_2(t,\alpha) &= -\frac{2}{9}\frac{1}{\left((\xi)^{2/3}-60\right)(\chi)}\left(\begin{pmatrix}
24(\xi)^{7/3}\alpha+47952000\sqrt{19185}\,\alpha+6629040000\alpha+1440(\xi)^{5/3}\alpha-74387160000-409212000(\xi)^{1/3}\alpha-2916000(\xi)^{1/3}\\
\sqrt{19185}\,\alpha+48600(\xi)^{2/3}\sqrt{19185}\,\alpha+3969000(\xi)^{2/3}\alpha-120(\xi)^{7/3}-97200(\xi)^{2/3}\sqrt{19185}+17982000(\xi)^{2/3}+2046060000(\xi)^{1/3}+14580000(\xi)^{1/3}\sqrt{19185}-7200(\xi)^{5/3}-537192000\sqrt{19185}
\end{pmatrix}e^{\sigma_2 t}\right)\\
&\quad-\frac{4}{9}\frac{1}{\chi}\left(\sqrt{3}\left(12(\xi)^{5/3}\alpha+408(\xi)^{4/3}\alpha+43200(\xi)^{1/3}\alpha-24480(\xi)^{2/3}\alpha-60(\xi)^{5/3}-4416(\xi)^{4/3}-216000(\xi)^{1/3}+264960(\xi)^{2/3}\right)e^{\sigma_1 t}\sin(\sigma_3 t)\right)\\
&\quad+\frac{8}{9}\frac{1}{\left((\xi)^{2/3}-60\right)(\chi)}\left(\begin{pmatrix}
6(\xi)^{7/3}\alpha+1417500\sqrt{19185}\,\alpha+1968057000\alpha+360(\xi)^{5/3}\alpha-102303000(\xi)^{1/3}\alpha-729000(\xi)^{1/3}\sqrt{19185}\,\alpha-19218384000-4187700(\xi)^{2/3}\alpha\\
-243000(\xi)^{2/3}\sqrt{19185}\,\alpha-30\left(2025+15\sqrt{19185}\right)^{7/3}+14855400(\xi)^{2/3}+48600(\xi)^{2/3}\sqrt{19185}+511515000(\xi)^{1/3}+364500(\xi)^{1/3}\sqrt{19185}-1800(\xi)^{5/3}-138672000\sqrt{19185}
\end{pmatrix}e^{\sigma_1 t}\cos(\sigma_3 t)\right),\\
\overset{*}{y}_1(t,\beta) &= -\frac{1}{3}\frac{1}{\left((\xi)^{2/3}-60\right)(\chi)}\left(\begin{pmatrix}
24(\xi)^{7/3}\beta+47952000\beta\sqrt{19185}+662904000\beta+1440(\xi)^{5/3}\beta-409212000(\xi)^{1/3}\beta-2916000(\xi)^{1/3}\sqrt{19185}\,\beta\\
+3969000(\xi)^{2/3}\beta+48600(\xi)^{2/3}\sqrt{19185}\,\beta-72(\xi)^{7/3}+13311000(\xi)^{2/3}-48600(\xi)^{2/3}\sqrt{19185}+1227636000(\xi)^{1/3}+8748000(\xi)^{1/3}\\
\sqrt{19185}-4320(\xi)^{5/3}-47381760000-342144000\sqrt{19185}
\end{pmatrix}e^{\sigma_2 t}\right)\\
&\quad-\frac{2}{3}\frac{1}{(\chi)}\left(\sqrt{3}\left(12(\xi)^{5/3}\beta+408(\xi)^{4/3}\beta+43200(\xi)^{1/3}\beta-24480(\xi)^{2/3}\beta-36(\xi)^{5/3}-2808(\xi)^{4/3}-129600(\xi)^{1/3}+168480(\xi)^{2/3}\right)e^{\sigma_1 t}\sin(\sigma_3 t)\right)\\
&\quad+\frac{4}{3}\frac{1}{\left((\xi)^{2/3}-60\right)(\chi)}\left(\begin{pmatrix}
6(\xi)^{7/3}\beta+14175000\beta\sqrt{19185}+1968057000\beta+360(\xi)^{5/3}\beta-102303000(\xi)^{1/3}\beta-729000(\xi)^{1/3}\beta\sqrt{19185}-4187700(\xi)^{2/3}\beta+24300(\xi)^{2/3}\sqrt{19185}\,\beta-18(\xi)^{7/3}\\
8507700(\xi)^{2/3}+24300(\xi)^{2/3}\sqrt{19185}+306909000(\xi)^{1/3}+2187000(\xi)^{1/3}\sqrt{19185}-1080(\xi)^{5/3}-12156237000-87723000\sqrt{19185}
\end{pmatrix}e^{\sigma_1 t}\cos(\sigma_3 t)\right),\\
\overset{*}{y}_2(t,\beta) &= \frac{1}{9}\frac{1}{\left((\xi)^{2/3}-60\right)(\chi)}\left(\begin{pmatrix}
72(\xi)^{7/3}\beta+143856000\sqrt{19185}\,\beta+19887120000\beta+4320(\xi)^{5/3}\beta-1227636000(\xi)^{1/3}\beta-8748000(\xi)^{1/3}\sqrt{19185}\,\beta+11907000(\xi)^{2/3}\beta+145800(\xi)^{2/3}\sqrt{19185}\,\beta\\
+192(\xi)^{7/3}+135516240000-43902000(\xi)^{2/3}+97200(\xi)^{2/3}\sqrt{19185}-3273696000(\xi)^{1/3}-23328000(\xi)^{1/3}\sqrt{19185}+11520(\xi)^{5/3}+978480000\sqrt{19185}
\end{pmatrix}e^{\sigma_2 t}\right)\\
&\quad+\frac{2}{9}\frac{1}{(\chi)}\left(\sqrt{3}\left(36(\xi)^{5/3}\beta+1224(\xi)^{4/3}\beta+129600(\xi)^{1/3}\beta-73440(\xi)^{2/3}\beta+96(\xi)^{5/3}+8016(\xi)^{4/3}+345600(\xi)^{1/3}-480960(\xi)^{2/3}\right)e^{\sigma_1 t}\sin(\sigma_3 t)\right)\\
&\quad-\frac{4}{9}\frac{1}{\left((\xi)^{2/3}-60\right)(\chi)}\left(\begin{pmatrix}
18(\xi)^{7/3}\beta+42525000\beta\sqrt{19185}+5904171000\beta+1080(\xi)^{5/3}\beta-306909000(\xi)^{1/3}\beta-2187000(\xi)^{1/3}\sqrt{19185}\,\beta-1256310(\xi)^{2/3}\beta-72900(\xi)^{2/3}\sqrt{19185}\,\beta\\
+48(\xi)^{7/3}+34500654000-21335400(\xi)^{2/3}-48600(\xi)^{2/3}\sqrt{19185}-818424000(\xi)^{1/3}-5832000(\xi)^{1/3}\sqrt{19185}+2880(\xi)^{5/3}+248994000\sqrt{19185}
\end{pmatrix}e^{\sigma_1 t}\cos(\sigma_3 t)\right),
\end{aligned}
\tag{78}
$$

where

$$
\begin{aligned}
\sigma_1 &= \frac{1/60\left(2025+15\sqrt{19185}\right)^{2/3}-1}{\left(2025+15\sqrt{19185}\right)^{1/3}},\\
\sigma_2 &= \frac{-1/30\left(2025+15\sqrt{19185}\right)^{2/3}-60}{\left(2025+15\sqrt{19185}\right)^{1/3}},\\
\sigma_3 &= \frac{1/60\left(2025+15\sqrt{19185}\right)^{2/3}\sqrt{3}+60\sqrt{3}}{\left(2025+15\sqrt{19185}\right)^{1/3}},\\
\chi &= \frac{1}{2302200+16200\sqrt{19185}},\\
\xi &= 2025+15\sqrt{19185}.
\end{aligned}
\tag{79}
$$

Figure 7(a) depicts the exact solution of the membership function of the generalized intuitionistic fuzzy IVP-III, 7(b) depicts the exact solution of the nonmembership function, and 7(c) depicts the exact solution of the generalized intuitionistic fuzzy IVP-III described in Example 3.

In Table 9, n represents the number of iterations, Err represents the residual error, and CPU time represents the computational time in seconds for finding the approximate solution of the generalized intuitionistic fuzzy IVP-III described in Example 3. Figures 6(a) and 6(b) show an approximate solution of the membership and nonmembership functions for IVP-III used in Example 3. The exact solutions to the membership and nonmembership functions for IVP-III used in Example 3 are depicted in Figures 7(a) and 7(c).

Example 4. An Embedded Beam (A Physical Application). A beam of length l is embedded at its left end and free at its right end. Find the deflection of the beam if a load $w_0 = 24EI$ is distributed along its length.

The intuitionistic fuzzy initial value problem related to above problem is given as follows:

Table 7: Approximate solution of Example 3 at $t=1$.

α	$\overset{*}{y}_1(t,\alpha)$	$\overset{*}{y}_2(t,\alpha)$	β	$\overset{*}{y}_1(t,\beta)$	$\overset{*}{y}_2(t,\beta)$
0	18.00	25.28	0.3	20.67	22.61
0.1	18.49	24.80	0.4	19.94	23.34
0.2	18.97	24.31	0.5	19.21	24.07
0.3	19.46	23.83	0.6	18.49	24.80
0.4	19.94	23.35	0.7	17.76	25.52
0.5	20.43	22.85	0.8	17.03	26.25
			0.9	16.30	26.98
			1.0	15.57	27.71

Table 8: Exact solution of Example 3 at $t=1$.

α	$\overset{*}{y}_1(t,\alpha)$	$\overset{*}{y}_2(t,\alpha)$	β	$\overset{*}{y}_1(t,\beta)$	$\overset{*}{y}_2(t,\beta)$
0	18.00	25.28	0.3	20.67	22.61
0.1	18.49	24.80	0.4	19.94	23.34
0.2	18.97	24.31	0.5	19.21	24.07
0.3	19.46	23.83	0.6	18.49	24.80
0.4	19.94	23.35	0.7	17.76	25.52
0.5	20.43	22.85	0.8	17.03	26.25
			0.9	16.30	26.98
			1.0	15.57	27.71

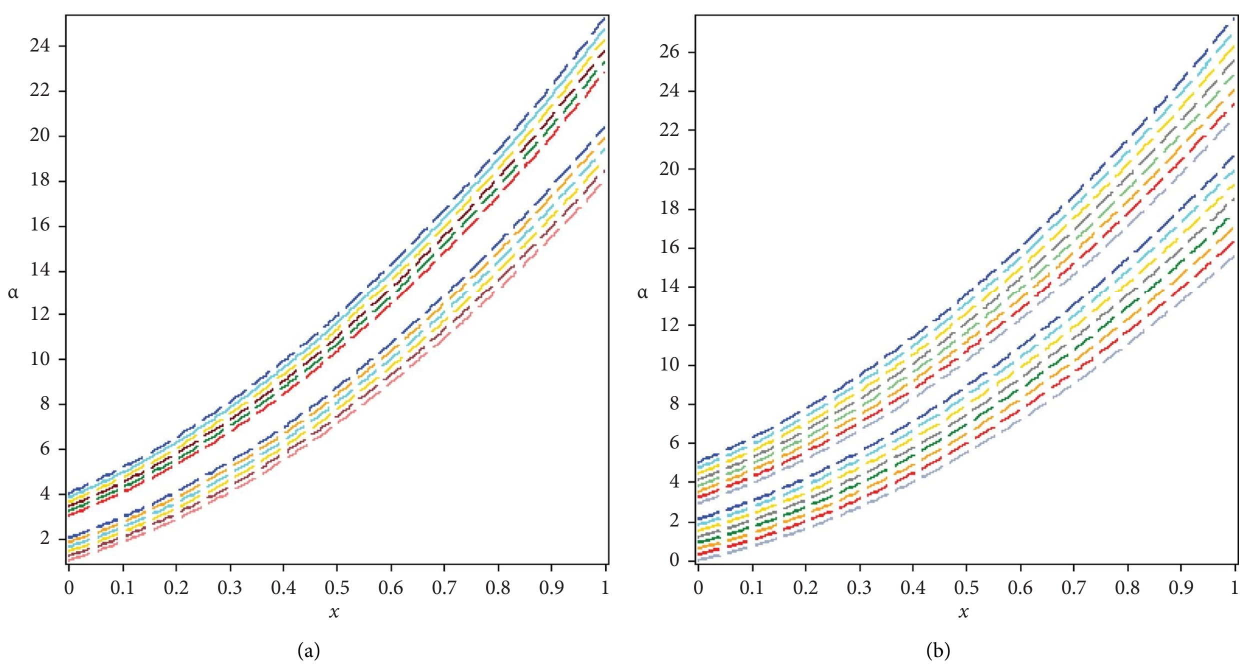


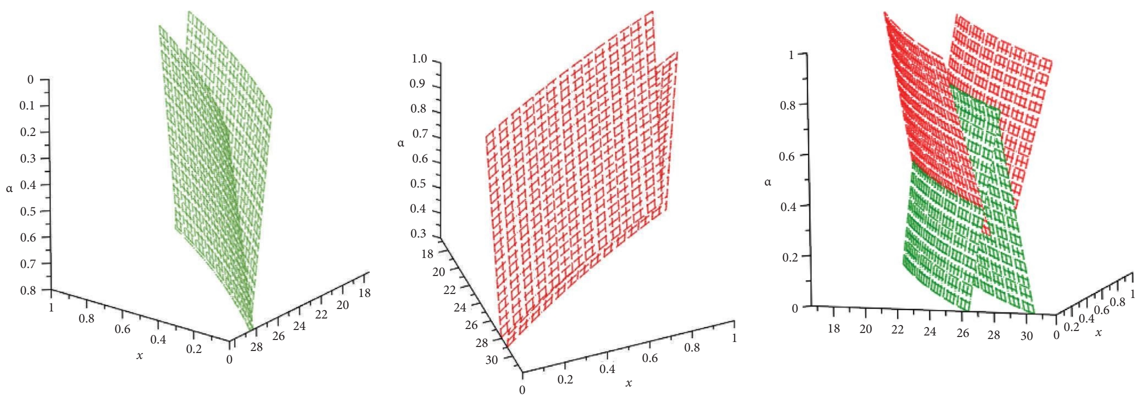
Figure 6: (a) Membership function and (b) nonmembership function.

$$EI\frac{d^4\overset{*}{y}}{dx^4} = w_0, \tag{80}$$

Using the above value of w_0 in (80), and writing in standard form, we have

$$\frac{d^4\overset{*}{y}}{dx^4} = 24, \tag{81}$$

with initial conditions:



(a) (b) (c)

Figure 7: (a) Exact solution of membership function for GITF-IVP-III, (b) exact solution of nonmembership function for GITF-IVP-III, and (c) exact solution of GTIF-IVP-III.

Table 9: Error comparison of GMADM, ADM, and TSM for solving GTIF-IVP-III used in Example 3.

Methods	GMADM	ADM	TSM
n	04	13	06
Err	$5.2e-37$	$6.2e-28$	$1.0e-8$
CPU time	0.1123	0.2013	0.0452

$$\begin{cases} \overset{*}{y}(0) = <(1,2,3,4;0.5);\ (0,2,3,5;0.3)>, \\ \overset{*}{y}'(0) = <(7,8,9,10;0.5);\ (6,8,9,11;0.3)>, \\ \overset{*}{y}''(0) = <(21,22,23,24;0.5);\ (20,22,23,25;0.3)>, \\ \overset{*}{y}'''(0) = <(32,33,34,35;0.5);\ (31,33,34,36;0.3)>. \end{cases} \tag{82}$$

By taking (α,β)-cut of (81) and (82), we get

$$\begin{cases} \dfrac{d^4\overset{*}{y}_1(x,\alpha)}{dx^4} = 24, \\ \overset{*}{y}_1(0,\alpha) = 2\alpha+1, \\ \overset{*}{y}_1'(0,\alpha) = 2\alpha+7, \\ \overset{*}{y}_1''(0,\alpha) = 2\alpha+21, \\ \overset{*}{y}_1'''(0,\alpha) = 2\alpha+32, \end{cases} \tag{83}$$

$$\begin{cases} \dfrac{d^4\overset{*}{y}_1(x,\alpha)}{dx^4} = 24, \\ \overset{*}{y}_2(0,\alpha) = -2\alpha+4, \\ \overset{*}{y}_2'(0,\alpha) = -2\alpha+10, \\ \overset{*}{y}_2''(0,\alpha) = -2\alpha+24, \\ \overset{*}{y}_2'''(0,\alpha) = -2\alpha+35, \end{cases} \tag{84}$$

$$\begin{cases} \dfrac{d^4 \overset{*}{y}_1\ (x,\alpha)}{dx^4} = 24, \\ \overset{*}{y}_1\ (0,\beta) = -3\beta + 3, \\ \overset{*}{y}_1'(0,\beta) = -3\beta + 9, \\ \overset{*}{y}_1''(0,\beta) = -3\beta + 23, \\ \overset{*}{y}_1'''(0,\beta) = -3\beta + 34, \end{cases} \tag{85}$$

$$\begin{cases} \dfrac{d^4 \overset{*}{y}_1\ (x,\alpha)}{dx^4} = 24, \\ \overset{*}{y}_1\ (0,\beta) = -3\beta + 3, \\ \overset{*}{y}_1'(0,\beta) = -3\beta + 9, \\ \overset{*}{y}_1''(0,\beta) = -3\beta + 23, \\ \overset{*}{y}_1'''(0,\beta) = -3\beta + 34. \end{cases} \tag{86}$$

Here, $L = d^4/dx^4$, and by taking $L^{-1}(.) = \int_0^x \int_0^x \int_0^x \int_0^x (.)dx\,dx\,dx\,dx$ on both sides of (83)–(86), and using the initial conditions, we obtain

$$\begin{aligned} \overset{*}{y}_1\ (x,\alpha) &= x^4 + 2\alpha x + 2\alpha + 7x + 1 + x^2\alpha + \frac{21}{2}x^2 + \frac{1}{3}x^3\alpha + \frac{16}{3}x^3, \\ \overset{*}{y}_2\ (x,\alpha) &= x^4 + 4 - 2\alpha x - 2\alpha + 10x - x^2\alpha + 12x^2 - \frac{1}{3}x^3\alpha + \frac{35}{6}x^3, \\ \overset{*}{y}_1\ (x,\beta) &= x^4 + 3 - 3\beta x - 3\beta + 9x - \frac{3}{2}x^2\beta + \frac{23}{2}x^2 - \frac{1}{2}x^3\beta + \frac{17}{3}x^3, \\ \overset{*}{y}_2\ (x,\beta) &= x^4 + 3\beta x + 3\beta + 8x + 2 + \frac{3}{2}x^2\beta + 11x^2 + \frac{1}{2}x^3\beta + \frac{11}{2}x^3. \end{aligned} \tag{87}$$

Now, by using GMADM, we get

$$\begin{cases} \overset{*}{y}_{1_0}(x,\alpha) = 1 + 2\alpha x + 2\alpha + 7x + x^2\alpha + \dfrac{21}{2}x^2 + \dfrac{1}{3}x^3\alpha + \dfrac{16}{3}x^3, \\ \overset{*}{y}_{1_1}(x,\alpha) = x^4, \\ \overset{*}{y}_{1_{k+1}}(x,\alpha) = 0, k \geq 1. \end{cases} \tag{88}$$

$$\begin{cases} \overset{*}{y}_{2_0}(x,\alpha) = 4 - 2\alpha x - 2\alpha + 10x - x^2\alpha + 12x^2 - \dfrac{1}{3}x^3\alpha + \dfrac{35}{6}x^3, \\ \overset{*}{y}_{2_1}(x,\alpha) = x^4, \\ \overset{*}{y}_{2_{k+1}}(x,\alpha) = 0, k \geq 1. \end{cases} \tag{89}$$

$$\begin{cases} \overset{*}{y}_{1_0}(x,\beta) = 3 - 3\beta x - 3\beta + 9x - \frac{3}{2}x^2\beta + \frac{23}{2}x^2 - \frac{1}{2}x^3\beta + \frac{17}{3}x^3, \\ \overset{*}{y}_{1_1}(x,\beta) = x^4, \\ \overset{*}{y}_{1_{k+1}}(x,\beta) = 0, k \geq 1. \end{cases} \tag{90}$$

$$\begin{cases} \overset{*}{y}_{2_0}(x,\beta) = 2 + 3\beta x + 3\beta + 8x + \frac{3}{2}x^2\beta + 11x^2 + \frac{1}{2}x^3\beta + \frac{11}{2}x^3, \\ \overset{*}{y}_{2_1}(x,\beta) = x^4, \\ \overset{*}{y}_{2_{k+1}}(x,\beta) = 0, k \geq 1. \end{cases} \tag{91}$$

By solving the (88)–(91), we get the approximate solution after four iterations as follows:

$$\begin{aligned} \overset{*}{y}_{1_0}(x,\alpha) &= x^4 + 2\alpha x + 2\alpha + 7x + 1 + x^2\alpha + \frac{21}{2}x^2 + \frac{1}{3}x^3\alpha + \frac{16}{3}x^3, \\ \overset{*}{y}_{2_0}(x,\alpha) &= x^4 + 4 - 2\alpha x - 2\alpha + 10x - x^2\alpha + 12x^2 - \frac{1}{3}x^3\alpha + \frac{35}{6}x^3, \\ \overset{*}{y}_{1_0}(x,\beta) &= x^4 + 3 - 3\beta x - 3\beta + 9x - \frac{3}{2}x^2\beta + \frac{23}{2}x^2 - \frac{1}{2}x^3\beta + \frac{17}{3}x^3, \\ \overset{*}{y}_{2_0}(x,\beta) &= x^4 + 2 + 3\beta x + 3\beta + 8x + \frac{3}{2}x^2\beta + 11x^2 + \frac{1}{2}x^3\beta + \frac{11}{2}x^3. \end{aligned} \tag{92}$$

The exact and approximate solutions of the IVP-IV used in Example 4 are shown in Tables 10 and 11, respectively.

Figures 8(a) and 8(b) show an approximate solution of the membership and nonmembership functions for IVP-IV used in Example 4.

The exact solution of Example 4 determined by classical method is as follows:

$$\begin{aligned} \overset{*}{y}_{1_0}(x,\alpha) &= x^4 + 2\alpha x + 2\alpha + 7x + 1 + x^2\alpha + \frac{21}{2}x^2 + \frac{1}{3}x^3\alpha + \frac{16}{3}x^3, \\ \overset{*}{y}_{2_0}(x,\alpha) &= x^4 + 4 - 2\alpha x - 2\alpha + 10x - x^2\alpha + 12x^2 - \frac{1}{3}x^3\alpha + \frac{35}{6}x^3, \\ \overset{*}{y}_{1_0}(x,\beta) &= x^4 + 3 - 3\beta x - 3\beta + 9x - \frac{3}{2}x^2\beta + \frac{23}{2}x^2 - \frac{1}{2}x^3\beta + \frac{17}{3}x^3, \\ \overset{*}{y}_{2_0}(x,\beta) &= x^4 + 2 + 3\beta x + 3\beta + 8x + \frac{3}{2}x^2\beta + 11x^2 + \frac{1}{2}x^3\beta + \frac{11}{2}x^3. \end{aligned} \tag{93}$$

Figure 9(a) depicts the exact solution of the membership function of the generalized intuitionistic fuzzy IVP-IV, 9(b) depicts the exact solution of the nonmembership function, and 9(c) depicts the exact solution of the generalized intuitionistic fuzzy IVP-IV described in Example 4.

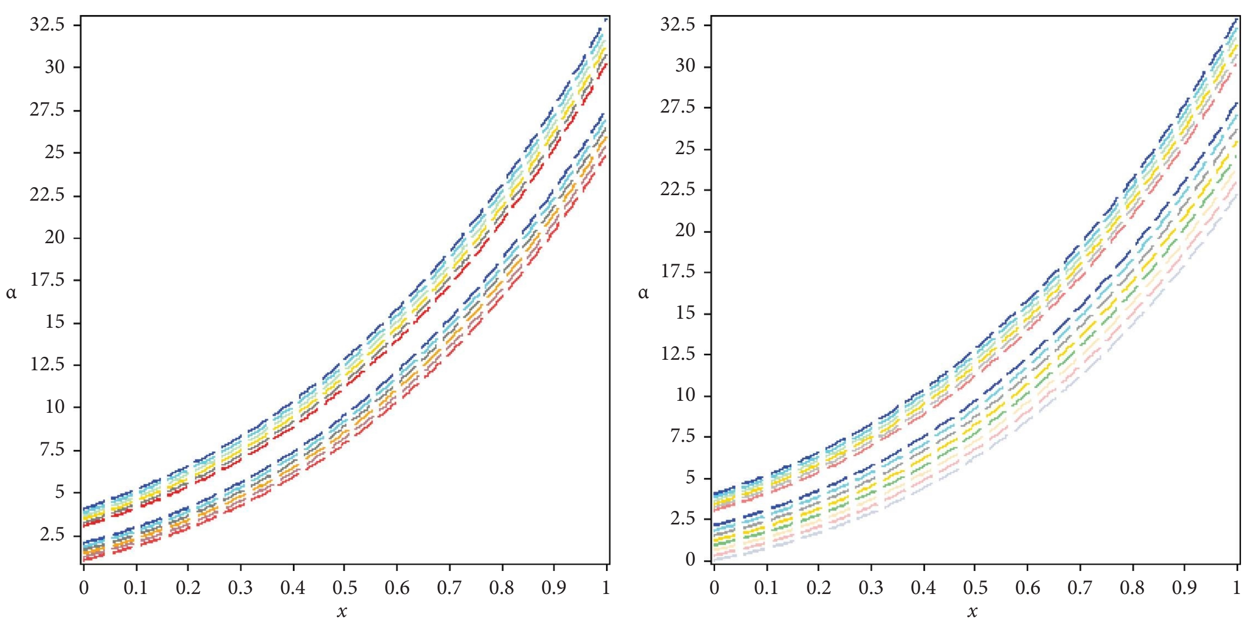
In Table 12, n represents the number of iterations, Err represents the residual error, and CPU time represents the computational time in seconds for finding the approximate solution of the generalized intuitionistic fuzzy IVP-IV described in Example 4. Figures 8(a) and 8(b) show an approximate solution of the membership and nonmembership functions for IVP-IV used in Example 4. The exact solutions to the membership and nonmembership functions for IVP-IV used in Example 4 are depicted in Figures 9(a)–9(c).

Table 10: Approximate solution of Example 4 at $x = 1$.

α	$\overset{*}{y}_1(x,\alpha)$	$\overset{*}{y}_2(x,\alpha)$	β	$\overset{*}{y}_1(x,\beta)$	$\overset{*}{y}_2(x,\beta)$
0	24.83	32.63	0.3	27.77	29.90
0.1	25.37	32.30	0.4	26.97	30.70
0.2	25.90	31.77	0.5	26.17	31.50
0.3	26.43	31.23	0.6	25.37	32.30
0.4	26.97	30.70	0.7	24.57	33.10
0.5	27.50	30.16	0.8	23.77	33.90
			0.9	22.97	34.70
			1.0	22.17	35.50

Table 11: Exact solution of Example 4 at $x = 1$.

α	$\overset{*}{y}_1(x,\alpha)$	$\overset{*}{y}_2(x,\alpha)$	β	$\overset{*}{y}_1(x,\beta)$	$\overset{*}{y}_2(x,\beta)$
0	24.83	32.83	0.3	27.77	29.90
0.1	25.37	32.30	0.4	26.97	30.70
0.2	25.90	31.77	0.5	26.17	31.50
0.3	26.43	31.23	0.6	25.37	32.30
0.4	26.97	30.70	0.7	24.57	33.10
0.5	27.50	30.16	0.8	23.77	33.90
			0.9	22.97	34.70
			1.0	22.17	35.50



(a) (b)

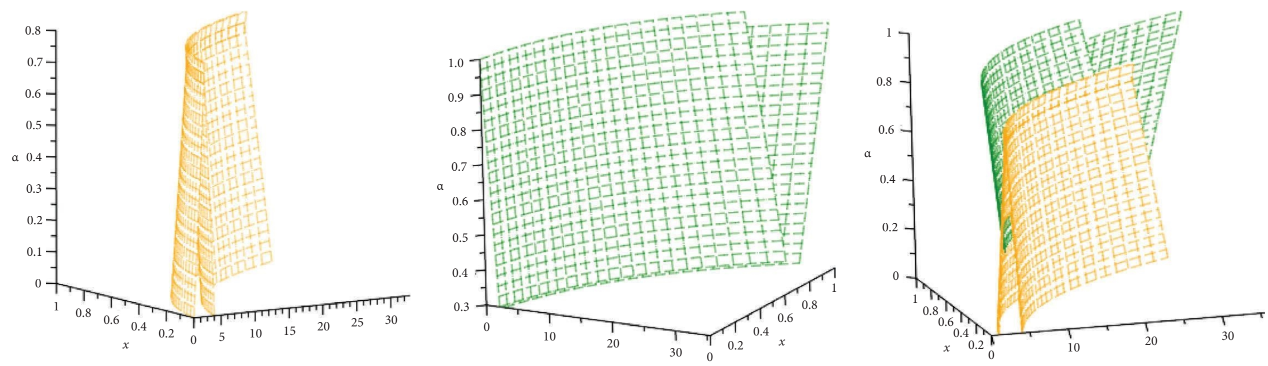
Figure 8: (a) Membership function and (b) nonmembership function.

Example 5. Higher-Order Generalized Intuitionistic Fuzzy Differential Equation.

Let us consider a generalized intuitionistic fuzzy initial value problem in which the coefficients are real crisp values and the initial conditions are generalized trapezoidal fuzzy numbers as follows:

$$\frac{d^5\overset{*}{y}}{dx^4} + 4\frac{d\overset{*}{y}}{dx^3} + 4\frac{d\overset{*}{y}}{dx} = 0, \quad (94)$$

with initial conditions:



(a) (b) (c)

Figure 9: (a) Exact solution of membership function for GITF-IVP-IV, (b) exact solution of nonmembership function for GITF-IVP-IV, and (c) exact solution of GTIF-IVP-IV.

Table 12: Error comparison of GMADM, ADM, and TSM for solving GTIF-IVP-IV used in Example 4.

Methods	GMADM	ADM	TSM
n	04	13	05
Err	$7.2e-29$	$6.2e-11$	$7.1e-4$
CPU time	0.0163	0.0293	0.1352

$$\begin{cases} \overset{*}{y}(0) = <(1,2,3,4;0.5);(0,2,3,5;0.3)>, \\ \overset{*}{y}'(0) = <(7,8,9,10;0.5);(6,8,9,11;0.3)>, \\ \overset{*}{y}''(0) = <(21,22,23,24;0.5);(20,22,23,25;0.3)>, \\ \overset{*}{y}'''(0) = <(32,33,34,35;0.5);(31,33,34,36;0.3)>, \\ \overset{*}{y}^{(\overset{*}{i}v)} = <(16,17,18,19;0.5);(15,17,18,20;0.3)>. \end{cases} \tag{95}$$

By taking (α,β)-cut of (94) and (95), we get

$$\begin{cases} \frac{d^5 \overset{*}{y}_1(x,\alpha)}{dx^4} + 4\frac{d^3 \overset{*}{y}_1(x,\alpha)}{dx^3} + 4\frac{d\overset{*}{y}_1(x,\alpha)}{dx} = 0, \\ \overset{*}{y}_1(0,\alpha) = 2\alpha + 1, \overset{*}{y}_1'(0,\alpha) = 2\alpha + 7, \\ \overset{*}{y}_1''(0,\alpha) = 2\alpha + 21, \overset{*}{y}_1'''(0,\alpha) = 2\alpha + 32, \\ \overset{*}{y}_1^{(\overset{*}{i}v)}(0,\alpha) = 2\alpha + 16, \end{cases}$$

$$\begin{cases} \dfrac{d^5 \overset{*}{y}_1(x,\alpha)}{dx^4} + 4\dfrac{d^3 \overset{*}{y}_1(x,\alpha)}{dx^3} + 4\dfrac{d\overset{*}{y}_1(x,\alpha)}{dx} = 0, \\ \overset{*}{y}_2(0,\alpha) = -2\alpha + 4, \overset{*}{y}_2'(0,\alpha) = -2\alpha + 10, \\ \overset{*}{y}_2''(0,\alpha) = -2\alpha + 24, \overset{*}{y}_2'''(0,\alpha) = -2\alpha + 35, \\ \overset{*}{y}_2^{(iv)}(0,\alpha) = -2\alpha + 19, \end{cases} \tag{97}$$

$$\begin{cases} \dfrac{d^5 \overset{*}{y}_1(x,\alpha)}{dx^4} + 4\dfrac{d^3 \overset{*}{y}_1(x,\alpha)}{dx^3} + 4\dfrac{d\overset{*}{y}_1(x,\alpha)}{dx} = 0, \\ \overset{*}{y}_1(0,\beta) = -3\beta + 3, \overset{*}{y}_1'(0,\beta) = -3\beta + 9, \\ \overset{*}{y}_1''(0,\beta) = -3\beta + 23, \overset{*}{y}_1'''(0,\beta) = -3\beta + 34, \\ \overset{*}{y}_1^{(iv)}(0,\beta) = -3\beta + 19, \end{cases} \tag{98}$$

$$\begin{cases} \dfrac{d^5 \overset{*}{y}_1(x,\alpha)}{dx^4} + 4\dfrac{d^3 \overset{*}{y}_1(x,\alpha)}{dx^3} + 4\dfrac{d\overset{*}{y}_1(x,\alpha)}{dx} = 0, \\ \overset{*}{y}_2(0,\beta) = 3\beta + 2, \overset{*}{y}_2'(0,\beta) = 3\beta + 8, \\ \overset{*}{y}_2''(0,\beta) = 3\beta + 22, \overset{*}{y}_2'''(0,\beta) = 3\beta + 33, \\ \overset{*}{y}_2^{(iv)}(0,\beta) = 3\beta + 17. \end{cases} \tag{99}$$

Here, $L = d^5/dx^5$ and by taking $L^{-1}(.) = \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x (.)dx\,dx\,dx\,dx\,dx$ on both sides of (96)–(99), and using the initial conditions, we obtain

$$\begin{aligned}
\overset{*}{y}_1(x,\alpha) &= \frac{2}{3}\int_0^x \overset{*}{y}_1(t,\alpha)(-x+t)\left(t^2-2tx+x^2+6\right)dt + 1 + \frac{3}{4}x^4\alpha \\
&\quad + \frac{13}{3}x^4 + 5x^2\alpha + 2x\alpha + 2\alpha + 7x + \frac{25}{2}x^2 + \frac{5}{3}x^3\alpha + 10x^3, \\
\overset{*}{y}_2(x,\alpha) &= \frac{2}{3}\int_0^x \overset{*}{y}_2(t,\alpha)(-x+t)\left(t^2-2tx+x^2+6\right)dt + 4 - \frac{3}{4}x^4\alpha \\
&\quad + \frac{131}{24}x^4 - 5x^2\alpha - 2x\alpha - 2\alpha + 10x + 20x^2 - \frac{5}{3}x^3\alpha + \frac{25}{2}x^3, \\
\overset{*}{y}_1(x,\beta) &= \frac{2}{3}\int_0^x \overset{*}{y}_1(t,\beta)(-x+t)\left(t^2-2tx+x^2+6\right)dt + 3 - \frac{9}{8}x^4\beta \\
&\quad + \frac{41}{8}x^4 - \frac{15}{2}x^2\beta - 3x\beta - 3\beta + 9x + \frac{35}{2}x^2 - \frac{5}{2}x^3\beta + \frac{35}{3}x^3, \\
\overset{*}{y}_2(x,\beta) &= \frac{2}{3}\int_0^x \overset{*}{y}_2(t,\beta)(-x+t)\left(t^2-2tx+x^2+6\right)dt + 2 + \frac{9}{8}x^4\beta \\
&\quad + \frac{113}{24}x^4 + \frac{15}{2}x^2\beta + 3x\beta + 3\beta + 8x + 15x^2 + \frac{5}{2}x^3\beta + \frac{65}{6}x^3.
\end{aligned} \tag{100}$$

Now, by using GMADM, we get

$$\begin{cases}
\overset{*}{y}_{1_0}(x,\alpha) = 1 + \frac{3}{4}x^4\alpha + \frac{13}{3}x^4 + 5x^2\alpha + 2x\alpha + 2\alpha + 7x + \frac{25}{2}x^2 \\
+ \frac{5}{3}x^3\alpha + 10x^3, \\
\overset{*}{y}_{1_1}(x,\alpha) = -2x^2 - \frac{1}{560}x^8\alpha - \frac{1}{26}x^7\alpha - \frac{7}{45}x^6\alpha - \frac{2}{5}x^5\alpha - \frac{14}{3}x^3 \\
-\frac{4}{3}x^3\alpha - 2x^4\alpha - 4x^2\alpha - \frac{13}{1260}x^8 - \frac{1}{121}x^7 - \frac{43}{60}x^6 - \frac{67}{30}x^5 - \frac{13}{3}x^4, \\
\overset{*}{y}_{1_{k+1}}(x,\alpha) = \frac{2}{3}\int_0^x (-x+t)\left(t^2-2tx+x^2+6\right)\overset{*}{y}_{1_k}(t,\alpha)dt, k \geq 1.
\end{cases} \tag{101}$$

$$\begin{cases}
\overset{*}{y}_{2_0}(x,\alpha) = 4 - \frac{3}{4}x^4\alpha + \frac{131}{24}x^4 - 5x^2\alpha - 2x\alpha - 2\alpha + 10x + 20x^2 \\
- \frac{5}{3}x^3\alpha + \frac{25}{2}x^3, \\
\overset{*}{y}_{2_1}(x,\alpha) = -8x^2 + \frac{1}{560}x^8\alpha + \frac{1}{26}x^7\alpha + \frac{7}{45}x^6\alpha + \frac{2}{5}x^5\alpha - \frac{20}{3}x^3 \\
+ \frac{4}{3}x^3\alpha + 2x^4\alpha + 4x^2\alpha - \frac{131}{10080}x^8 - \frac{5}{84}x^7 - \frac{19}{20}x^6 - \frac{17}{6}x^5 - \frac{22}{3}x^4, \\
\overset{*}{y}_{2_{k+1}}(x,\alpha) = \frac{2}{3}\int_0^x (-x+t)\left(t^2-2tx+x^2+6\right)\overset{*}{y}_{2_k}(t,\alpha)dt, k \geq 1.
\end{cases} \tag{102}$$

$$\begin{cases} \overset{*}{y}_{1_0}(x,\beta) = 3 - \frac{9}{8}x^4\beta + \frac{41}{8}x^4 - \frac{15}{2}x^2\beta - 3x\beta - 3\beta + 9x + \frac{35}{2}x^2 \\ -\frac{5}{2}x^3\beta + \frac{35}{3}x^3, \\ \overset{*}{y}_{1_1}(x,\beta) = -6x^2 + \frac{3}{1120}x^8\beta + \frac{1}{84}x^7\beta + \frac{7}{30}x^6\beta + \frac{3}{5}x^5\beta - 6x^3 \\ +2x^3\beta + 3x^4\beta + 6x^2\beta - \frac{41}{3360}x^8 - \frac{1}{18}x^7 - \frac{79}{90}x^6 - \frac{79}{30}x^5 - \frac{19}{3}x^4, \\ \overset{*}{y}_{1_{k+1}}(x,\beta) = \frac{2}{3}\int_0^x (-x+t)\left(t^2 - 2tx + x^2 + 6\right)\overset{*}{y}_{1_k}(t,\beta)dt, k \geq 1. \end{cases} \tag{103}$$

$$\begin{cases} \overset{*}{y}_{2_0}(x,\beta) = 2 + \frac{9}{8}x^4\beta + \frac{113}{24}x^4 + \frac{15}{2}x^2\beta + 3x\beta + 3\beta + 8x + 15x^2 \\ +\frac{5}{2}x^3\beta + \frac{65}{6}x^3, \\ \overset{*}{y}_{2_1}(x,\beta) = -4x^2 - \frac{3}{1120}x^8\beta - \frac{1}{84}x^7\beta - \frac{7}{30}x^6\beta - \frac{3}{5}x^5\beta - \frac{16}{3}x^3 \\ -2x^3\beta - 3x^4\beta - 6x^2\beta - \frac{113}{10080}x^8 - \frac{13}{252}x^7 - \frac{143}{180}x^6 - \frac{73}{30}x^5 - \frac{16}{3}x^4, \\ \overset{*}{y}_{2_{k+1}}(x,\beta) = \frac{2}{3}\int_0^x (-x+t)\left(t^2 - 2tx + x^2 + 6\right)\overset{*}{y}_{2_k}(t,\beta)dt, k \geq 1. \end{cases} \tag{104}$$

By solving the (101)–(104), we get the approximate solution after three iterations as follows:

$$\begin{aligned} \overset{*}{y}_1\ (x,\alpha) = {} & 1 + 7x + 2\alpha x - \frac{17}{22680}x^{10}\alpha - \frac{1}{540}x^9\alpha - \frac{163}{14968800}x^{12}\alpha \\ & - \frac{67}{1247400}x^{11}\alpha - \frac{2}{42567525}x^{14}\alpha - \frac{1}{18162144000}x^{16}\alpha \\ & - \frac{1}{2043241200}x^{15}\alpha - \frac{2}{6081075}x^{13}\alpha - \frac{197}{22680}x^9 - \frac{181}{3742200}x^{12} \\ & - \frac{31}{103950}x^{11} - \frac{11}{5040}x^{10} - \frac{17}{8845200}x^{13} - \frac{1}{3143448000}x^{16} \\ & - \frac{1}{340540200}x^{15} - \frac{337}{1362160800}x^{14} - \frac{37}{180}x^6 - \frac{13}{10}x^5 + \frac{17}{2520}x^8 \\ & + \frac{31}{315}x^7 + \frac{2}{3}x^4 + \frac{16}{3}x^3 + \frac{21}{2}x^2 + \frac{1}{3}x^3\alpha + \frac{1}{12}x^4\alpha + x^2\alpha + \frac{1}{90} \\ & \cdot x^7\alpha - \frac{1}{45}x^6\alpha - \frac{2}{15}x^5\alpha - \frac{19}{1680}x^8\alpha + 2\alpha, \end{aligned}$$

$$\begin{aligned}
y_2^*(x,\alpha) &= 4 + 10x - 2\alpha x + \frac{17}{22680}x^{10}\alpha + \frac{1}{540}x^9\alpha + \frac{163}{14968800}x^{12}\alpha \\
&\quad + \frac{67}{1247400}x^{11}\alpha + \frac{2}{42567525}x^{14}\alpha + \frac{1}{18162144000}x^{16}\alpha \\
&\quad + \frac{1}{2043241200}x^{15}\alpha + \frac{2}{6081075}x^{13}\alpha - \frac{13}{1134}x^9 - \frac{1937}{29937600}x^{12} \\
&\quad - \frac{1}{2640}x^{11} - \frac{5}{1512}x^{10} - \frac{47}{19459440}x^{13} - \frac{131}{326918592000}x^{16} \\
&\quad - \frac{1}{272432160}x^{15} - \frac{433}{1362160800}x^{14} - \frac{43}{180}x^6 - \frac{3}{2}x^5 - \frac{103}{10080}x^8 \\
&\quad + \frac{29}{252}x^7 + \frac{19}{24}x^4 + \frac{35}{6}x^3 + 12x^2 - \frac{1}{3}x^3\alpha - \frac{1}{12}x^4\alpha \\
&\quad - x^2\alpha - \frac{1}{90}x^7\alpha + \frac{1}{45}x^6\alpha + \frac{2}{15}x^5\alpha + \frac{19}{1680}x^8\alpha - 2\alpha, \\
y_1^*(x,\beta) &= 3 + 9x - 3\beta x + \frac{17}{15120}x^{10}\beta + \frac{1}{360}x^9\beta + \frac{163}{9979200}x^{12}\beta \\
&\quad + \frac{67}{831600}x^{11}\beta + \frac{1}{14189175}x^{14}\beta + \frac{1}{12108096000}x^{16}\beta \\
&\quad + \frac{1}{1362160800}x^{15}\beta + \frac{1}{2027025}x^{13}\beta - \frac{239}{22680}x^9 - \frac{17}{285120}x^{12} \\
&\quad - \frac{439}{1247400}x^{11} - \frac{667}{226800}x^{10} - \frac{73}{32432400}x^{13} - \frac{41}{108972864000}x^{16} \\
&\quad - \frac{1}{291891600}x^{15} - \frac{101}{340540200}x^{14} - \frac{7}{30}x^6 - \frac{43}{30}x^5 + \frac{43}{10080}x^8 \\
&\quad + \frac{23}{210}x^7 + \frac{19}{24}x^4 + \frac{17}{3}x^3 + \frac{23}{2}x^2 - \frac{1}{2}x^3\beta - \frac{1}{8}x^4\beta - \frac{3}{2}x^2\beta \\
&\quad - \frac{1}{60}x^7\beta + \frac{1}{30}x^6\beta + \frac{1}{5}x^5\beta + \frac{19}{1120}x^8\beta - 3\beta, \\
y_2^*(x,\beta) &= 2 + 8x + 3\beta x - \frac{17}{15120}x^{10}\beta - \frac{1}{360}x^9\beta - \frac{163}{9979200}x^{12}\beta \\
&\quad - \frac{67}{831600}x^{11}\beta - \frac{1}{14189175}x^{14}\beta - \frac{1}{12108096000}x^{16}\beta \\
&\quad - \frac{1}{1362160800}x^{15}\beta - \frac{1}{2027025}x^{13}\beta - \frac{109}{11340}x^9 - \frac{179}{3326400}x^{12} \\
&\quad - \frac{811}{2494800}x^{11} - \frac{29}{11340}x^{10} - \frac{29}{13899600}x^{13} - \frac{113}{326918592000}x^{16} \\
&\quad - \frac{1}{314344800}x^{15} - \frac{41}{151351200}x^{14} - \frac{13}{60}x^6 - \frac{41}{30}x^5 + \frac{11}{10080}x^8 \\
&\quad + \frac{131}{1260}x^7 + \frac{17}{24}x^4 + \frac{11}{2}x^3 + 11x^2 + \frac{1}{2}x^3\beta + \frac{1}{8}x^4\beta + \frac{3}{2}x^2\beta \\
&\quad + \frac{1}{60}x^7\beta - \frac{1}{30}x^6\beta - \frac{1}{5}x^5\beta - \frac{19}{1120}x^8\beta + 3\beta.
\end{aligned} \tag{105}$$

The exact and approximate solutions of the IVP-V used in Example 5 are shown in Tables 13 and 14, respectively.

Figures 10(a) and 10(b) show an approximate solution of the membership and nonmembership functions for IVP-V used in Example 5.

The exact solution of Example 5 found by classical method is given as follows:

$$\begin{aligned}
\overset{*}{y}_1(x,\alpha) &= \frac{9}{2}\alpha + 26 + \frac{7}{4}\sqrt{2}\sin(\sqrt{2}x)\alpha + \frac{37}{4}\sin(\sqrt{2}x)\sqrt{2} - \frac{5}{2}\sqrt{2}\cos(\sqrt{2}x)\alpha \\
&\quad - v25\cos(\sqrt{2}x) - \frac{3}{4}\sqrt{2}\sin(\sqrt{2}x)x\alpha - \frac{29}{4}\sqrt{2}\sin(\sqrt{2}x)x \\
&\quad - \frac{3}{2}\cos(\sqrt{2}x)x\alpha - \frac{23}{2}\cos(\sqrt{2}x)x, \\
\overset{*}{y}_2(x,\alpha) &= -\frac{9}{2}\alpha + \frac{131}{4} - \frac{7}{4}\sqrt{2}\sin(\sqrt{2}x)\alpha + \frac{95}{8}\sin(\sqrt{2}x)\sqrt{2} + \frac{5}{2}\sqrt{2}\cos \\
&\quad \cdot(\sqrt{2}x)\alpha - \frac{115}{4}\cos(\sqrt{2}x) + \frac{3}{4}\sqrt{2}s\overset{*}{i}n(\sqrt{2}x)x\alpha - \frac{67}{8}\sqrt{2}\sin \\
&\quad \cdot(\sqrt{2}x)x + \frac{3}{2}\cos(\sqrt{2}x)x\alpha - \frac{55}{4}\cos(\sqrt{2}x)x, \\
\overset{*}{y}_1(x,\beta) &= -\frac{27}{4}\beta + \frac{123}{4} - \frac{21}{8}\sqrt{2}\sin(\sqrt{2}x)\beta + 11\sin(\sqrt{2}x)\sqrt{2} + \frac{15}{4}\sqrt{2}\cos \\
&\quad \cdot(\sqrt{2}x)\beta - \frac{111}{4}\cos(\sqrt{2}x) + \frac{9}{8}\sqrt{2}\sin(\sqrt{2}x)x\beta - \frac{65}{8}\sqrt{2}\sin \\
&\quad \cdot(\sqrt{2}x)x + \frac{9}{4}\cos(\sqrt{2}x)x\beta - 13\cos(\sqrt{2}x)x, \\
\overset{*}{y}_2(x,\beta) &= \frac{27}{4}\beta + \frac{113}{4} + \frac{21}{8}\sqrt{2}\sin(\sqrt{2}x)\beta + \frac{81}{8}\sin(\sqrt{2}x)\sqrt{2} - \frac{15}{4}\sqrt{2}\cos \\
&\quad \cdot(\sqrt{2}x)\beta - \frac{105}{4}\cos(\sqrt{2}x) - \frac{9}{8}\sqrt{2}\sin(\sqrt{2}x)x\beta - \frac{61}{8}\sqrt{2}\sin \\
&\quad \cdot(\sqrt{2}x)x - \frac{9}{4}\cos(\sqrt{2}x)x\beta - \frac{49}{4}\cos(\sqrt{2}x)x.
\end{aligned} \tag{106}$$

Figure 11(a) depicts the exact solution of the membership function of the generalized intuitionistic fuzzy IVP-V, 11(b) depicts the exact solution of the nonmembership function, and 11(c) depicts the exact solution of the generalized intuitionistic fuzzy IVP-V described in Example 5.

In Table 15, n represents the number of iterations, Err represents the residual error, and CPU time represents the computational time in seconds for finding the approximate solution of the generalized intuitionistic fuzzy IVP-V described in Example 5. Figures 10(a) and 10(b) show an approximate solution of the membership and nonmembership functions for IVP-V used in Example 5. The exact solutions to the membership and nonmembership functions for IVP-V used in Example 5 are depicted in Figures 11(a)–11(c).

3.1.2. Advantages of the GMADM

(i) When solving generalized trapezoidal intuitionistic fuzzy differential equations, it is found that GMADM converges faster and more efficient as compared to exact techniques

(ii) The fundamental benefit of the GMADM algorithm is that it can solve all types of fuzzy differential equations using more generalized fuzzy numbers, namely, generalized trapezoidal intuitionistic fuzzy number

(iii) The GMADM also offers the useful benefit of lowering computation costs while keeping improved numerical solution precision

Table 13: Approximate solution of Example 5 at $t = 1$.

α	$\overset{*}{y}_1(x,\alpha)$	$\overset{*}{y}_2(x,\alpha)$	β	$\overset{*}{y}_1(x,\beta)$	$\overset{*}{y}_2(x,\beta)$
0	23.09	30.98	0.3	26.02	28.08
0.1	23.61	30.45	0.4	25.23	28.87
0.2	24.14	29.92	0.5	24.44	29.66
0.3	24.67	29.40	0.6	23.65	30.45
0.4	25.19	28.87	0.7	22.86	31.24
0.5	25.72	28.35	0.8	22.07	32.03
			0.9	21.28	32.82
			1.0	20.50	33.60

Table 14: Exact solution of Example 5 at $x = 1$.

α	$\overset{*}{y}_1(x,\alpha)$	$\overset{*}{y}_2(x,\alpha)$	β	$\overset{*}{y}_1(x,\beta)$	$\overset{*}{y}_2(x,\beta)$
0	23.10	31.01	0.3	26.04	28.11
0.1	23.63	30.48	0.4	25.25	28.90
0.2	24.16	29.96	0.5	24.46	29.69
0.3	24.68	29.43	0.6	23.67	30.48
0.4	25.21	28.90	0.7	22.87	31.28
0.5	25.74	28.38	0.8	22.08	32.07
			0.9	21.29	32.86
			1.0	20.50	33.65

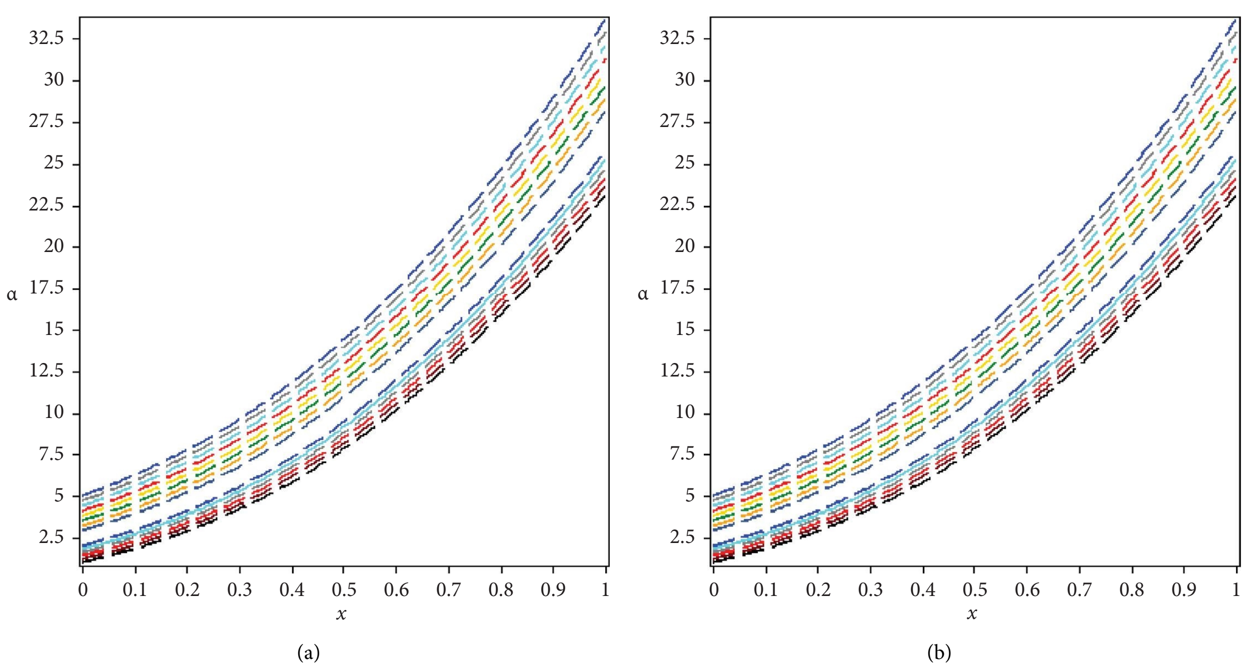


Figure 10: (a) Membership function and (b) nonmembership function.

(iv) GMADM can efficiently, rapidly, and accurately solve a large class of generalized trapezoidal intuitionistic fuzzy differential equations with closed form solutions that rapidly converge to exact solutions

(v) The GMADM has demonstrated to be very efficient and produces significant accuracy and computation time reductions, as illustrated in Figures 1–11 and Tables 1 to 15

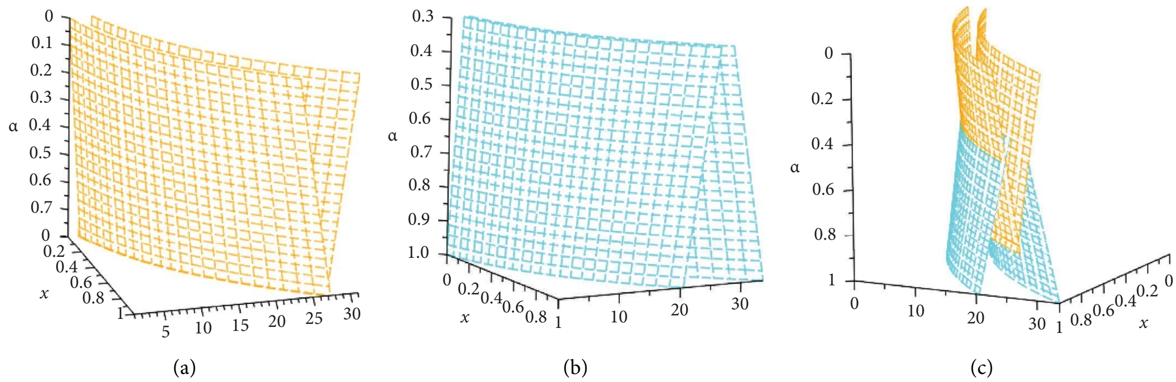


FIGURE 11: (a) Exact solution of membership function for GITF-IVP-V, (b) exact solution of nonmembership function for GITF-IVP-V, and (c) exact solution of GTIF-IVP-V.

TABLE 15: Error comparison of GMADM, ADM, and TSM for solving GTIF-IVP-V used in Example 5.

Methods	GMADM	ADM	TSM
n	03	19	07
Err	$0.2e-49$	$0.2e-21$	$7.9e-8$
CPU-time	0.0723	0.5403	0.4512

4. Conclusion

In this article, we solved the generalized trapezoidal intuitionistic fuzzy differential equations by applying the procedure of modified Adomian decomposition method with linear differential operator. We have applied this procedure to mechanical engineering problems. From all Tables 1 to 15 and Figures 1–11, it clearly shows the dominance efficiency of GMADM over exact technique in terms of computational time, number of iterations, and in errors. Moreover, we have shown that this method is more reliable by comparing the approximations with the exact solution. Future research will therefore focus on the solution of systems of linear and nonlinear first-order differential equations and their applications [34–36] in a generalized trapezoidal intuitionistic fuzzy environment, as well as systems of higher-order generalized trapezoidal intuitionistic fuzzy differential equations [37, 38].

Data Availability

The data supporting the current study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

All authors contributed equally to the preparation of this manuscript.

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