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Research Article

An Approach for a Multi-Period Portfolio Selection Problem by considering Transaction Costs and Prediction on the Stock Market

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This paper addresses a method to solve a multi-period portfolio selection on the stock market. The portfolio problem seeks an investor to trade stocks with a finite budget and a given integer number of stocks to hold in a portfolio. The trade must be performed through a stockbroker that charges its respective transaction cost and has its minimum required trade amount. A mathematical model has been proposed to deal with the constrained problem. The objective function is to find the best risk-return rate; thus, Sharpe Ratio and Treynor Ratio are used as objective functions. The returns are the same for these ratios, but the risks are not Sharpe considering covariance and Treynor systematical risk. The returns are predicted using a Neural Net with Long-Short-Term Memory (LSTM). This neural net is compared with simple forecasting methods through Mean Absolute Percentage Error (MAPE). Computational experiments show the quality prediction performed by LSTM. The heteroskedastic risk is estimated by Generalized Autoregressive Conditional Heteroskedasticity (GARCH), adjusting the variance for every period; this risk measure is used in Sharpe Ratio. The experiment contemplates a weekly portfolio selection with 5 and 10 stocks in 122 weekly periods for each Chilean market ratio. The best portfolio is Sharpe Ratio with ten stocks, performing a 62.28% real return beating the market, represented by the Selective Stock Price Index (IPSA). Even the worst portfolio, Treynor Ratio, overcomes the IPSA cumulative yield with ten stocks.

1. Introduction

Stocks are a capitalization investment instrument whose profitability depends on the company's results in its business environment, which is reflected in the purchase or sale of stock prices [1, 2]. Negotiating and exchanging financial products is carried out internationally; therefore, any investor has access to products from all over the world [3]. Individuals who want to buy stocks must decide which and how much they must acquire. This problem, without considering risk, as was considered by [4] and introduced by [5], is a combinatorial optimization problem known as "The Knapsack Problem," which is cataloged as NP-Hard. Markowitz [5] introduced the modern portfolio theory and expresses the expected return of a financial asset as the mean of its historical returns, the risk explained by the variance of the historical returns, and the concepts of diversification. Investors who want to buy securities should decide which and how many assets to include in the portfolio without considering the risk.

The sale and purchase of stocks must seek to increase the investor's wealth, where the recommended proportions to invest in the current period compared to those of the previous period reflect the decision to sell, buy, or stay in the current position. An optimization model must be aligned with an econometric prediction method or artificial intelligence scheme of expected returns to obtain the best relationship between risk and return.

The context of the problem is to consider the requirements of an investor with a finite budget to invest in the stock market, maximizing an expected return while minimizing the risk. These stocks must be traded by a stockbroker charging their respective transaction costs. Every stock is assumed as an integer asset. A stock portfolio must be rebalanced weekly. Once the stock portfolio is performed, the investor maintains the acquired position until it is time to rebalance the proportions. Therefore, the next week, the investor must decide which stocks to sell, buy, or hold. The asset must be selected due to the investor's risk aversion and return expectations to find the optimal proportion. This optimal combination must reach the best risk-return ratio [6].

This paper addresses a method to solve the stock market's multi-period portfolio selection problem. A mathematical model has been proposed to deal with the constrained problem. The objective function is to find the best risk-return rate; thus, Sharpe Ratio and Treynor Ratio are used as objective functions. The returns are the same for these ratios, but the risks are not Sharpe considering covariance and Treynor systematical risk. The returns are predicted using a Recurrent Neural Network with Long-Short-Term Memory (RNN-LSTM); this neural net is compared to simple forecasting methods through Mean Absolute Percentage Error (MAPE). A pilot test demonstrated the superior quality prediction performed with Long-Short-Term Memory (LSTM).

The heteroskedastic risk is estimated by Generalized Autoregressive Conditional Heteroskedasticity (GARCH), adjusting the variance for every period; this risk measure is used in Sharpe Ratio. The proposed approach is novel and could apply to a rich portfolio problem. The main contribution of this paper is to design a model that allows an investor to allocate money to the stock market, considering constraints such as transaction cost, integrality, minimum trade amount, and budget, among others. The mathematical model analyzes the return and risk to find the best combination. The proposed approach considers multiple periods without simplifying the problem within a single period, as usually considered in previously published works. Note that we solved a real problem using mathematical models and found the optimal solution for the considered subproblems (Initial and Rebalancing portfolio). However, approximate methods are not necessarily due to the prominent obtained results. Besides, we have considered transaction costs, real, local market conditions, and other real characteristics in solving a "Rich Multi-Period Portfolio Selection Problem." Finally, the application of the proposed approach for an emerging market such as Chilean allows for supporting the investment decision on quantitative methods.

The paper is organized as follows. Section 2 describes the portfolio problem's literature review with the considered problem's main characteristics. Section 3 details the proposed approach. Section 4 shows the computation results of the proposed methodology on the Chilean Market. Finally, Section 5 shows the concluding remarks and the future work section.

2. Literature Review

The problem of the stock portfolio is based mainly on a collection of financial assets, which are updated by selling the current positions and buying new stocks. This situation increases the portfolio's total available budget or selling securities to decrease the portfolio size [7].

Volatility is an inherent characteristic of the financial time series. Generally, its behavior is not constant, and traditional approaches consider time series models assuming homoscedastic variance unsuitable for modeling financial time series [8]. According to [9, 10], most authors have focused on the computational aspects of the stock portfolio problem. They have ignored the aspects and characteristics of the financial problem, such as the time series modeling. In this case, multiple financial assets, transaction costs for each rebalanced period, and the search for the prediction of returns are considered. However, this is very difficult to solve, and only some papers consider this problem [11].

Adebiyi et al. [12] applied the modern portfolio model [5], maximizing the Sharpe Ratio and considering stock lots, investor budget, transaction costs, and minimum and maximum amounts acquired per stock. This work considered a single period. The authors proposed a Particle Swarm Optimization to solve large instances within short computing times. Markowitz [5] stated that the expected return is calculated as the average of the historical returns, and their respective variance explains the risk. An investor wishes to obtain a return r_p, investing in financial assets minimizing the risk σ_p^2 given a value of r_p. The other approach is to set a risk parameter λ, which is the risk the investor is willing to accept. The objective function must seek to maximize return r_p given risk aversion λ. The proposed model by [5] has received various criticisms for its assumptions [13]. One is that the risk is constant over time for a financial time series (homoscedasticity). This assumption is unreal because the volatility has systematical changes, called heteroskedasticity (the variance of the returns has systematic changes over time) [4, 14]. The mean-variance model proposed by [5] is described in Appendix Section. This model includes diversification by choosing the least correlated stocks, minimizing the risk of the portfolio by choosing stocks with a lower variance, and finally maximizing profitability by looking for stocks with higher average historical returns [15].

However, several authors proposed other different risk metrics (different from the work proposed by [5]), such as Value at Risk (VaR) [16], Conditional Value at Risk (CVaR) [17], semi-variance [18], Generalized Autoregressive Conditional Heteroskedasticity (GARCH) and Dynamic Conditional Correlation Generalized Autoregressive Conditional Heteroskedasticity (DCC-GARCH) [19], and fuzzy-logic adjusted risk [20]. Besides, several authors proposed metrics for measuring returns, such as based on expert opinions [18], Autoregressive Integrated Moving Average (ARIMA) [12], RNN-LSTM [17], and Fuzzy Logic Adjusted Return [20].

There are two metrics to find the optimal combination of risk and return. The first metric is the Sharpe Ratio (SR), a measure to analyze an investment's excess return, considering the involved risk. SR is calculated by subtracting the return of a risk-free asset from the expected return and dividing this result by the risk [21]. This ratio incorporates diversification as part of the risk. Another stock market ratio is called the Treynor Ratio (TR). Peiro [22] explained that TR measures the excess return per unit of systematic risk. The calculation of these metrics is detailed in Appendix Section.

In the literature, all the current works generally focus on single-period optimization (SPO) for index tracking portfolio design [23]. However, in the financial markets, the methods may lead to frequent portfolio rebalances, resulting in high transaction costs. Huang et al. [23] proposed a novel multi-period optimization (MPO) approach to index tracking portfolio design, which can account for transaction costs and holding costs. Moghadam et al. [24] proposed a multi-period portfolio selection model considering investors' dependence, risk aversion, and diminishing sensitivity. A robust optimization approach was considered, and three metaheuristic algorithms were developed for solving large-size problems. Yang et al. [25] addressed the multi-period portfolio problem with short selling under a fuzzy environment. Three types of short-selling constraints, i.e., total short-selling proportion constraint, short-selling cardinality constraint, and lower and upper bound constraint, were considered. Li et al. [26] proposed a predictive control model for a multi-period portfolio optimization problem. Additional to the mean-variance objective, the authors constructed a portfolio whose allocation is given by model predictive control with a risk-parity objective. Finally, Jiang and Wang [27] considered a multi-period multiobjective portfolio selection problem with uncertainty. A weighted-sum approach was introduced to obtain the Pareto front of the problem.

A multi-period portfolio selection problem where the future security return rates are given by experts' estimations instead of historical data was proposed by [28]. A new mental account concept was introduced to reflect the conflicting risk attitudes for different goals. Also, realistic constraints such as background risk, liquidity risk, transaction cost, and cardinality constraint were considered. García et al. [29] extend the stochastic mean-semivariance model to a fuzzy multiobjective model to measure the performance of a portfolio. Uncertainty of future return and liquidity of each asset is modeled using LR-type fuzzy numbers belonging to the power reference function family. The main novelty of the work is the consideration of realistic constraints by investors.

Nokhandan et al. [30] proposed a Nash bargaining model to solve a novel multi-period competitive portfolio optimization problem for large investors in the stock market. The Competitive Portfolio Model (CPM) was developed following the Cournot competition principle for a static, non-cooperative, and non-zero-sum game with complete information. Real-world conditions such as transaction costs, risk-free assets, and cash were also included to match real-world problems. Three criteria control the model's investment risk: the average value at risk, the mean absolute semi-deviation, and entropy. Dymova et al. [31] proposed an approach to the bi-criteria multi-period fuzzy portfolio selection based on observing the variance as a measure of portfolio risk. Simple criteria of portfolio risk and return are proposed. A fuzzy portfolio selection one-period model was developed to solve the considered problem. Besides, a new two-stage bi-criteria optimization approach to portfolio selection was developed, tested, and used as the main component of the proposed multi-period portfolio selection model.

Generally, transaction costs are neglected in the decision-making process of the online stock portfolio problem. However, some authors included this concept for making decisions, such as [32, 33]. In [32], the authors proposed an adaptive online portfolio selection problem with transaction costs. An online moving average method (AOLMA) was used to predict future returns by incorporating an adaptive decaying factor into the moving average method. Moon and Yoon [33] considered the portfolio selection (OLPS) considering transaction costs and proposed a hybrid genetic reversion strategy evolving a population of portfolio vectors.

However, building efficient multi-period portfolios is a challenging problem, including defining the risk and return metrics and evaluating each share's position for the period. Early, Merton [34] proposed a policy in which an investor must continually seek to balance the invested proportions for each asset. However, continuously rebalancing the portfolio implies high transaction costs. Dynamic programming could solve a multi-period problem by choosing the best consecutive decisions; these decisions are affected by the number of stocks, market information, liquidity of assets, and short sales [35].

Brandt and Santa-Clara [36] suggested that a linear function could estimate the investment proportion. Bodnar et al. [37] solved the multi-period problem by assuming a unique repeating period. He et al. [38] proposed a method where a high-order model monthly estimates the risk and return. This approach has been tested in the US and Chinese markets. Skaf and Boyd [39] addressed the multi-period problem with stochastic considerations to increase wealth using dynamic programming and reality constraints. The authors found suboptimal policies to get feasible solutions. Indeed, some policies that help stock trading to handle transaction costs are the "No-Trade Region Policy" and the "Rolling Optimize-and-Hold Policy" proposed by [11]. Babazadeh and Esfahanipour [16] proposed a multi-period optimization considering the Value at Risk (VaR) as a measure of risk, generating an Average VaR model, which includes transaction costs, budget, and maximum and minimum purchases.

According to [40], Recurrent Neural Networks (RNN) were considered a methodology for processing sequential data, such as time series [41]. A neural network with long and short-term memory is one of the most successful RNN architectures [42]. LSTMs include memory cells, a computing unit that replaces traditional artificial neurons in the hidden layer of a network. With these memory cells, the networks can efficiently associate remote memory and inputs over time; thus, they are suitable for capturing the structure of data dynamically over time with high predictability [41].

Peng et al. [43] analyzed the factor zoo from a machine learning perspective, which has theoretical and empirical implications for finance. The authors discussed feature selection in the context of deep neural network models to predict the stock price direction. This work considered a set of

124 technical analysis indicators used as explanatory variables in the recent literature. It specialized trading websites-various classification metrics, accounting for profitability and transaction cost levels to analyze economic gains.

Muncharaz [44] showed the application of neural networks in creating predictive models. The work considered an RNN with LSTM instead of classic time series models such as the Exponential Smooth Time Series (ETS) and the Arima model (ARIMA). These models were estimated for 284 stocks from the S&P 500 stock market index, comparing the MAE obtained from their predictions. Rather [45] proposed a new method of predicting time-series-based stock prices considering the investment portfolio problem. A new regression scheme was implemented on a long-short-term memory-based deep neural network.

A novel portfolio construction approach using a hybrid model based on machine learning for stock prediction and the well-known mean-variance (MV) model for portfolio selection was proposed by [46]. Two stages were involved in the proposed model: stock prediction and portfolio selection. In the first stage, a hybrid model combining eXtreme Gradient Boosting (XGBoost) with an improved firefly algorithm (IFA) was proposed to predict stock prices for the next period. Zhao et al. [47] considered a multi-period investment portfolio selection problem. First, the portfolio selection model fits the extreme cases of 0% or 100% confidence views. The authors established a new programming problem based on the optimization approach and identified explicit solutions. Second, the author extended the model to multi-period form and discretized the results with a scenario tree, which solves the multi-period problems.

In the literature review, it has been found that multi-period optimization is widely studied, and the series prediction is part of the success of this technique. Many authors mention that the difficulty of the multi-period optimization problem is complex. Therefore, it is necessary to find ways to simplify the problem away from the real conditions of the stock market. The complexity is given by the consecutive decisions that must be performed, where the current decision affects the portfolio within the horizon time. Many authors considered the multi-period problem as a single-period problem repeated several times, helping reduce the computational complexity.

We have proposed a methodology for the stock portfolio problem by considering multiple periods without simplification (a single period repeated several times), seeking a better estimate of the returns differently from the average of the historical values and the risk measures considered. We have considered Sharpe's and Treynor's ratios' objective function values to find the best risk-return ratio. In this way, it is achieved that the proposed methodology can be applied adequately in fundamental emerging markets.

3. Materials and Methods

The main scientific concepts are explained in this section to approach the portfolio selection problem with sophisticated methods, allowing an investor to allocate money in the best risk-return combination of assets.

3.1. Initial Optimization Model. This approach uses an econometric method to estimate the volatility of a financial time series. Three different prediction metrics (Moving Average-MA, Exponential Smoothing-ES, and Long Short-Term Memory-LSTM) have been compared to estimate each asset's expected logarithmic returns and find the best one. The portfolio selection problem is solved by a mathematical formulation using the above information, finding the best risk-return ratio of assets, testing two different financial ratios, and comparing them with the index stock market. Finally, a mathematical model for rebalancing the portfolio is applied. We have considered the following main aspects:

(1) Short sell is not allowed

(2) Budget one-time financed at the beginning

(3) Money is not withdrawn

(4) The close price is known

The proposed methodology considers several steps:

3.1.1. Managing Returns Heteroskedasticity. The first step is to measure the volatility inherent to the financial time series, whose behavior is not constant. Consequently, homoscedastic variance methods, such as Markowitz, are unsuitable for modeling financial time series for the proposed approach. We have considered GARCH models widely used in finance to solve this issue. GARCH models stand for the General Autoregressive Conditional Heteroskedasticity model, and its mission is to capture the changing values of risk. In this case, the risk is expressed as a variance.

$$\sigma_t^2 = \omega + \alpha \sum_{p=1}^{p} \varepsilon_{t-p}^2 + \beta \sum_{q=1}^{q} \sigma_{t-q}^2, \quad (1)$$

where, ε_{t-p}^2 are the square of the perturbations of a time period $(t-p)$, σ_{t-q}^2 is the historic variance corresponding to the period $(t-q)$. ω, α, β could be estimated by maximum likelihood method. In Julia language [48] is available GARCH by using the package *ARCHModels.jl* [49].

3.1.2. Return Prediction by LSTM. The second step is analyzing the predictive data of logarithmic returns by the RNN. The input data must be considered from a previous exploration time step of the neuron as part of the incoming information. LSTM networks pass more information across the recurrent connection than the traditional RNN. The components of an LSTM unit are input, forget and output gate, block input memory cell, output, activation function, and peephole connections. The input gate protects the unit from irrelevant input events. The forget gate helps the unit forget previous memory contents. The output gate exposes the memory cell's contents at the LSTM unit's output. The output of the LSTM block is recurrently connected back to the block input and the gates of the LSTM block. The input, forget, and output gates in an LSTM unit have sigmoid activation functions for the [0, 1] constraint. The LSTM block input and output activation function (usually) is a Tanh Activation Function [50].

Forecasting with simple methods should be less effective in portfolio selection than LSTM. The moving average is calibrated by choosing n, representing the number of periods to look back, splitting the closing price data into 70% to find which $n \in \{2, 3, 4\}$ minimizes MAPE and the last 30% to contrast with LSTM. Exponential Smoothing follows the same procedure as MA, but instead of n, α is calibrated $\alpha \in]0, 1[$.

3.1.3. Experimental Design. The proposed algorithm is described in the flowchart shown in Figure 1. It starts at $t = 0$ by inputting the Risk-Free Rate rf, the budget β. The size of the portfolio M, so is input to the model the covariance matrix covariance(t) the expected logarithmic returns $E[r(t)]$ and prices(t) vectors, corresponding to the first optimization problem (please see Section 3.1.5). It is going to output the transaction cost $cTr(t)$, the stocks and their quantity portfolio (t), the surplus Surplus(t), the total Wealth (including the Surplus), Wealth(t), and the portfolio return expected minus transaction costs Rmax(t). After the above steps, it has inputted the same matrix and vectors However, in $t = 1$, this time including the previous portfolio and surplus, so are calculated yield(t), to evaluate the real obtained returns and NoOptRp(t) that show the Portfolio expected return if the portfolio is held. Afterward, in the Nth optimization, it will come out the updated variables $cTr(t)$,Portfolio(t),Wealth(t), Surplus(t), and Rmax(t). It is decided if the expected return by trading Rmax(t) is less than the expected without doing it (NoOptRp(t)). If it is true, then the previous portfolio is held.

3.1.4. Simplified Problem. The simplified problem could find an optimal solution due to the absence of complex constraints; moreover, the solution could be found within a short computing time with the *Ipopt*® solver. The objective function (2) shows the best risk-return ratio. Therefore, any constraint added to the problem would have suboptimal performance.

$$\text{Max}SR_p, TR_p. \tag{2}$$

Subject to

$$\sum_{i=1}^{n} w_i = 1. \tag{3}$$

3.1.5. Initial Portfolio: Sharpe Ratio. This portfolio selection is different from the next because there is only one decision to make how many stocks to buy. The objective is to maximize the Sharpe Ratio, choosing a given number of shares. The sets and parameters and decision variables are the following: (See Table1).

The objective function is calculated by.

$$\text{Max}SR_p = \frac{R_p - r_f}{\sigma_p}. \tag{4}$$

Subject to

$$R_p = \sum_{i=1}^{n} w_i * E[r_i], \tag{5}$$

$$\left(\frac{B}{\text{Min}(p)}\right) * z_i \geq q_i \quad \forall i \in N, \tag{6}$$

$$E[r_i] = ln\left(\frac{pf_i}{p_i}\right) \quad \forall i \in N, \tag{7}$$

$$\text{invest} * (1 + cTr) \leq B, \tag{8}$$

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} * w_i * w_j + \sigma_i^2 * w_i^2, \tag{9}$$

$$\text{invest} \geq \text{min}T, \tag{10}$$

$$w_i * \text{invest} = q_i * p_i \quad \forall i \in N, \tag{11}$$

$$z_i \leq q_i \quad \forall i \in N, \tag{12}$$

$$\text{invest} = \sum_{i=1}^{n} q_i * p_i, \tag{13}$$

$$w_i \leq W\text{max} * z_i \quad \forall i \in N, \tag{14}$$

$$\sum_{i=1}^{n} z_i = M, \tag{15}$$

$$w_i \geq W\text{min} * z_i \quad \forall i \in N, \tag{16}$$

$$\sum_{i=1}^{n} w_i = 1, \tag{17}$$

$$w_i, q_i \geq 0 \quad \forall i \in N. \tag{18}$$

The objective function (4) shows the maximization of the Sharpe Ratio, which considers the excess return per unit of portfolio risk. Equation (5) shows that the portfolio's return is given by the proportion invested in stock i of the total invested multiplied by its expected return. Equations (6) relate the number of stocks i with the variable possibility of including a stock in the portfolio. These equations limit the number of stocks i in the portfolio. A maximum amount equal to the division between the budget and the stock with the lowest price could be acquired. Constraints (7) show that the expected return is equal to the percentage change of the previous price of stock i concerning its current price. Equation (8) shows that the total amount to be invested plus the transaction cost of this investment must be less than or equal to the initial available budget.

The portfolio risk is measured through equation (9). The risk is calculated as its variance. The variance is equal to the sum of the return covariances weighted by the

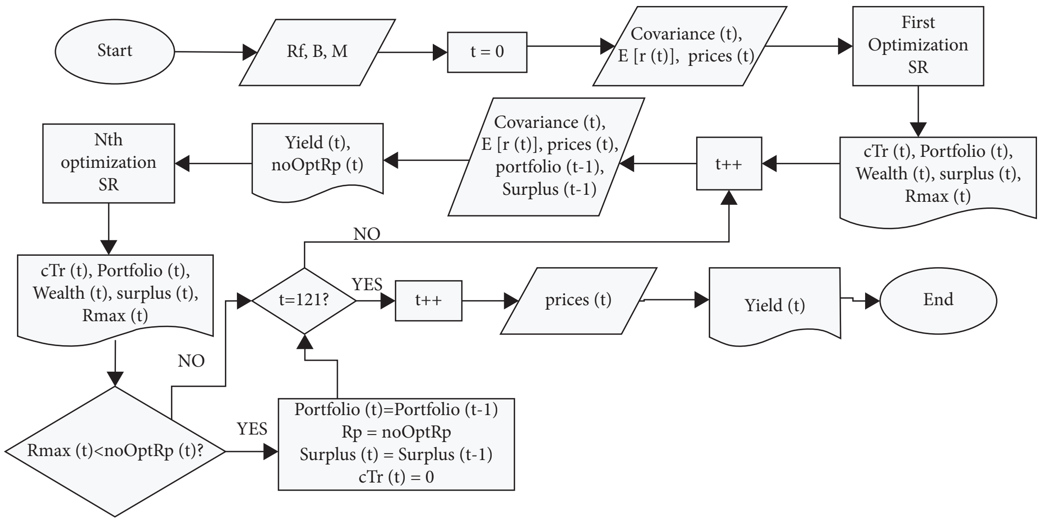

Figure 1: Flowchart of the proposed approach for sharpe ratio.

Table 1: Sets, parameters, and variables for the first proposed model.

	Description
Set	
N	Available stocks to be included in portfolio. $N = \{1, \dots, n\}$
Parameter	
n	Number of companies
σ_i	Standard deviation of the return's stock i. $\forall i \in N$
σ_{ij}	Covariance between stock i and stock j.$\forall i \in N, j \in N$
pf_i	Future predicted price of the stock for the next week. $\forall i \in N$
p_i	Current price of Stock i. $\forall i \in N$
$E[r_i]$	Expected return for stock i. $\forall i \in N$
M	Number of stocks to be included
r_f	Risk free rate
B	Available budget
β_i	Beta value of stock i. $\forall i \in N$
σ_{iM}	Covariance between stock i and the market. $\forall i \in N$
σ_M^2	Market return variance
$\min T$	Minimum trade amount
cTr	Transaction cost of the trade amount
$W\min_i$	Minimum weight to invest on stock i
$W\max_i$	Maximum weight to invest on stock i
Variable	
SR_p	Portfolio Sharpe ratio
TR_p	Portfolio Treynor ratio
q_i	Stock i to hold until the next week. $\forall i \in N$
w_i	Percentage to be invested in each stock i. $\forall i \in N$
z_i	$\begin{Bmatrix} 1, & \text{if stock } i \text{ is included,} \\ 0, & \text{in other case,} \end{Bmatrix}, \forall i \in N$
R_p	Portfolio expected return
σ_p	Portfolio standard deviation
β_p	Systematic risk
Invest	Total investment on the portfolio

proportion of the stock, plus the sum of the variance of the returns weighted by the square of the total invested proportion. Equation (10) shows that the total investment amount must be greater than the minimum transaction amount accepted in some markets. Constraint (9) shows that the total proportion of investment is equal to the number of shares i on own multiplied by their respective price i, which is divided by the total investment amount.

Equations (12) determine that if a stock i is purchased, the minimum purchase amount corresponds to the entire unit. Equation (13) shows that the total investment is the sum of the number of stocks i multiplied by their respective prices. Equations (14) and (16) determine the maximum and minimum proportion of the portfolio to invest for each stock, respectively. Equation (17) determines that the number of stocks to be included must be equal to M, where M is an arbitrary integer defined by the investor. The sum of the proportions must be equal to 1 to ensure that all capital invested in the portfolio is distributed (equation (16)) Finally, (18) determine the nature and integrality of the variables.

3.1.6. Initial Optimization: Treynor Ratio. The objective function is calculated by.

$$\text{Max}TR_p = \frac{R_p - r_f}{\beta_p}. \tag{19}$$

Subject to.

Constraints (5)–(8) and (10)–(18) plus the following constraints:

$$\beta_p = \sum_{i=1}^{N} w_i * \beta_i, \tag{20}$$

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} \quad \forall i \in N. \tag{21}$$

The objective function (19) considers the excess return per unit of systematic risk. Equation (20) calculates the portfolio's systematic risk, equivalent to the sum of the systematic risk of stock i weighted by the proportion of the total invested money. Finally, equations (21) calculate the systematic risk of stock i equal to the covariance of the returns of stock i concerning the market over the variance of the market returns.

3.2. Rebalancing Optimization: Sharpe Ratio. This formulation is repeated for all remaining periods. The decisions to make are which stocks buy, sell, or hold. The set N and some variables are shared with the first optimization: $SR_p, TR_p, w_i, z_i, R_p$, invest. Moreover, some parameters of the initial model as considered: $n, \sigma_i, \sigma_{ij}, p_i, E[r_i]$, $M, r_f, B, \beta_i, \sigma_{iM}, \sigma_M^2, cTr, \text{min}T$. The additional parameters and decision variables are the following. (See Table2).

The objective function is calculated by.

$$\text{Max}SR_p = \frac{R\text{max}_p - r_f}{\sigma_p}. \tag{22}$$

Subject to.

Constraints (5), (7), (9), (14)–(17) with the following additional constraints:

$$R\text{max}_p = R_p - Rctr, \tag{23}$$

$$m\text{Transac} \geq \text{min}T, \tag{24}$$

$$w_i = \frac{f_i * p_i}{\text{invest}} \quad \forall i \in N, \tag{25}$$

$$\text{patBef} = \sum_{i=1}^{n} s_i * p_i, \tag{26}$$

$$z_i \leq f_i \quad \forall i \in N, \tag{27}$$

$$y_i \leq s_i \quad \forall i \in N, \tag{28}$$

$$\text{invest} = \sum_{i=1}^{n} f_i * p_i, \tag{29}$$

$$m\text{Buy} \leq m\text{Sell} + \text{rest} - Rctr * \text{invest}, \tag{30}$$

$$m\text{Buy} = \sum_{i=1}^{n} x_i * p_i, \tag{31}$$

$$\left[\frac{(\text{patBef} + \text{rest})}{\text{Min}(p)}\right] * z_i \geq f_i \quad \forall i \in N, \tag{32}$$

$$m\text{Sell} = \sum_{i=1}^{n} y_i * p_i, \tag{33}$$

$$\left[\frac{(\text{patBef} + \text{rest})}{\text{Min}(p)}\right] * LX_i \geq x_i \quad \forall i \in N, \tag{34}$$

$$m\text{Transac} = m\text{Buy} + m\text{Sell}, \tag{35}$$

$$\left[\frac{(\text{patBef} + \text{rest})}{\text{Min}(p)}\right] * LY_i \geq y_i \quad \forall i \in N, \tag{36}$$

$$f_i = s_i + x_i - y_i \quad \forall i \in N, \tag{37}$$

$$1 \geq LY_i + LX_i \quad \forall i \in N, \tag{38}$$

$$cTr * m\text{Transac} = Rctr * \text{invest}, \tag{39}$$

$$w_i, x_i, y_i, f_i \geq 0 \quad \forall i \in N. \tag{40}$$

Equation (22) maximizes the Sharpe Ratio. Equation (23) calculates the portfolio's profitability, which is diminished by the transaction cost. Equation (24) restricts that the transaction amount must be more significant than that required by the broker. Equations (25) show that the proportion of the total investment is equal to the number of shares i to own multiplied by their respective price i, which is divided by the total investment amount.

The portfolio's value before optimizing equals the sum of the previous stocks at their respective current prices (equation (26)). Equations (27) and (28) restrict the

Table 2: Parameters and variables for the second proposed model.

	Description
Parameters	
s_i	Held stock i since last week. $\forall i \in N$
rest	Surplus, unused budget in the previous week
patBef	Portfolio actual value
Variable	
f_i	Stocks i to be held until the next period. $\forall i \in N$
x_i	Stocks i to buy. $i \in N$
y_i	Stocks i to sell. $\forall i \in N$
Rctr	Portfolio portion for transaction cost
mTransac	Trade amount
mBuy	Total amount to buy
mSell	Total amount to sell
$R\max$	Expected return minus the transaction cost
LX_i	$\left\{\begin{array}{ll} 1, & \text{if at least one stock i is bought,} \\ 0, & \text{en otro caso,} \end{array}\right\}, \forall i \in N$
LY_i	$\left\{\begin{array}{ll} 1, & \text{if at least one stock i is sold,} \\ 0, & \text{en otro caso,} \end{array}\right\}, \forall i \in N$

minimum number of stocks and the number of shares to be sold, respectively. Equation (29) considers the total amount, equivalent to the sum of the number of shares to own with their respective price. Equation (30) restricts that the total amount to be purchased must be less than or equal to the total sold amount, plus the money left over from the previous period, and less cost associated with the transaction. Equation (31) determines that the total purchase amount equals the sum of the number of stocks to be purchased multiplied by their respective prices.

Equations (32) show the relationship between the number of stocks with the maximum number to have. This maximum number is equivalent to the division between the initial value of the portfolio plus the rest and the cheapest share. Equation (33) indicates that the total sale amount equals the sum of the number of shares to be sold multiplied by their respective price. Constrains (34) show the relationship between the number of stocks to buy with the maximum amount equal to the division between the initial value of the portfolio plus the rest and the cheapest share.

Equation (35) determines that the transaction amount equals the total purchase and sale amount. The relationship between the number of stocks to be sold and included in the portfolio is determined by (36). Equations (37) show that the number of stocks to be held until the following week must be equal to the number of stocks already owned plus the number of bought stocks minus the number of sold stocks. Equations (38) indicate that only the purchase or sale of stocks could be performed, not both transactions simultaneously. Equation (39) indicates that it is possible to buy neither nor sell stock. Finally, (40) determine the nature and integrality of the variables.

3.3. Rebalancing Optimization: Treynor Ratio. The objective function is the following:

$$\text{Max}TR_p = \frac{R\text{max}_p - r_f}{\beta_p}. \quad (41)$$

Subject to constrains (5), (7), (14), (16), (18), (20), (21), and (23)–(40).

4. Results and Discussion

The multi-period optimization problem is applied using the Julia Mathematical Programming package (JuMP.jl), solving it through Gurobi® solver v9.1.2. We have performed the experimental computation on returns of the Chilean market's high and medium liquidity stocks from June 16th, 2017, to November 18th, 2019. First, the best prediction method must be performed. Next, we found the most profitable scenario: one must perform better than the IPSA index (Chilean Index).

4.1. Data Source and Data Pre-Processing. The experiment begins by downloading, from the platform "Santiago Stock Exchange," the daily open, high, low, and close prices and traded volume (OHLCV) of 100 Chilean stocks from 2012 to 2019. These are the most relevant traded stocks for that period. Then, for each stock, the number of days traded between 2012 and 2019. Thus, the initial number of stocks is reduced because only 30 stocks accomplish the minimum trading days.

The daily OHLCV data is transformed into a weekly one, finding the highest and lowest price, the open and close price, and the summary of the traded volume within a week. This transformation aims to have a longer time horizon than a daily one and significant changes in stock prices. Furthermore, it chose the data from January 6th, 2012, to October 19th, 2019, leaving the data set with 407 weeks of OHLCV data. In the same period, the OHLCV data of the IPSA index is downloaded from Santiago Stock Exchange, representing the most traded 30 stocks in the Chilean Stock Market. The period after the "social outburst" of 2019 and the SARS-CoV-2 pandemic is excluded.

4.2. Prediction Methods. A comparison is performed between the three methods where exponential smoothing seems to reach a better MAPE, followed by LSTM and then the moving average of the two periods. Table 3 shows the prediction comparison. The first column of Table 3 shows the used method: MA-2 (move average), ES (exponential smoothing), and LSTM (Neural Net with Long-Short-Term Memory). The second column shows the average obtained values of the MAPE from the forecast method. Finally, Table 4 shows the Standard Deviation of the MAPE.

In addition, to test the quality of the prediction, a pilot test has been performed with the three best methods (Figure 2), choosing Sharpe Ratio as the objective function and using the simplified formulation to reach an optimal portfolio.

The performance of Exponential Smoothing with an alpha near 1 makes the prediction almost equal to the current one. The expected returns are near zero, making it difficult to decide which risk-return ratio is the best because all expected returns are nearly identical. The moving average has a similar issue. In this case, the prediction not only depends on the current price, but the previous value does not help to improve the prediction either. Consequently, LSTM offers the best quality prediction.

We need to find the values of p and q that minimize the mean and variance of the error, in this case "Akaike (AIC)." Using an exhaustive search for values of p and q from 1 to 3. Table 5 shows the parameter calibration.

Hence, GARCH {2, 2} shows the best performance, with the lowest Akaike, that GARCH to adjust the risk is looking the perturbations (p) and volatility (q) from two-time steps back. Then, GARCH {2,2} is used in the predict function, which is also provided by Archmodels.jl package and works considering consecutive batches with the size of 52 weeks of the close price of a stock to adjust a volatility level for the next period, more specifically, the volatility is expressed as variance and with a year of data is predicted the risk level for the next week, using the described function. The risk of CHILE, the stock of "Banco de Chile," is shown in Figure 3. Thus, this method can adjust the heteroskedastic risk value for each period.

The input data are OHLCV of stock and IPSA to predict the stock's close price for the next week. This case represents one of the 30 stocks considered in the study. The data is split into two sets; 70% goes to a training one to let the model learn. Testing one (30%) to calibrate parameters like Number of Neurons (NA), Iteration (EPCOH), and Batch Size (BS), combinations are made following the current literature. A command is included that picks chronologically random data in the time series to avoid overfitting, making a synthetic database; thus, the calibration is not made over the same data used for the optimization. MAPE is an error parameter, then the mean of the 30 stock MAPES for each parameter's combination is calculated, choosing the minimum value corresponding to N°18.

As an example, Figure 4 shows the LSTM based prediction v/s, the actual price of "Banco de Chile (CHILE)" Stock for the last 122 periods.

Table 3: Predict comparison.

Method	Average (%)	Standard deviation (%)
MA-2	2.73	0.88
ES ($\alpha = 0.96$)	2.41	0.76
LSTM	2.52	0.78

Table 4: Parameter calibration for LSTM.

N°	BS	NA	EPOCH	Average MAPE
1	4	25	100	3.4247
2	4	25	500	3.1192
3	4	25	1000	3.0290
4	4	125	100	2.9559
5	4	125	500	3.1134
6	4	125	1000	2.9860
7	4	250	100	2.8270
8	4	250	500	2.7990
9	4	250	1000	2.7927
10	16	25	100	3.8156
11	16	25	500	2.9863
12	16	25	1000	2.8814
13	16	125	100	3.4495
14	16	125	500	2.8858
15	16	125	1000	2.8162
16	16	250	100	3.3553
17	16	250	500	2.8014
18	16	250	1000	2.7804
19	64	25	100	6.9081
20	64	25	500	3.5130
21	64	25	1000	2.9649
22	64	125	100	6.0745
23	64	125	500	3.1709
24	64	125	1000	2.9115
25	64	250	100	6.2232
26	64	250	500	3.1655
27	64	250	1000	2.9037

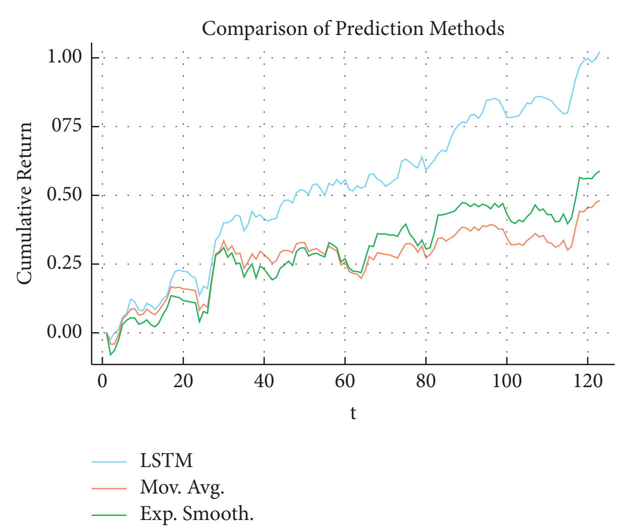

Figure 2: Pilot test for prediction methods.

Table 5: Parameter calibration (p, q).

(p, q)	$q = 1$	$q = 2$	$q = 3$
$p = 1$	−98614800	−96176000	−99884200
$p = 2$	−100737000	−101577000	−100090000
$p = 3$	−100234000	−100451000	−100433000

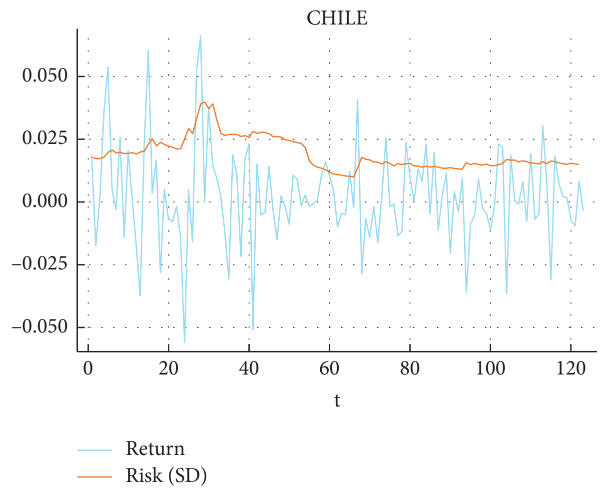

Figure 3: Adjusted standard deviation by period.

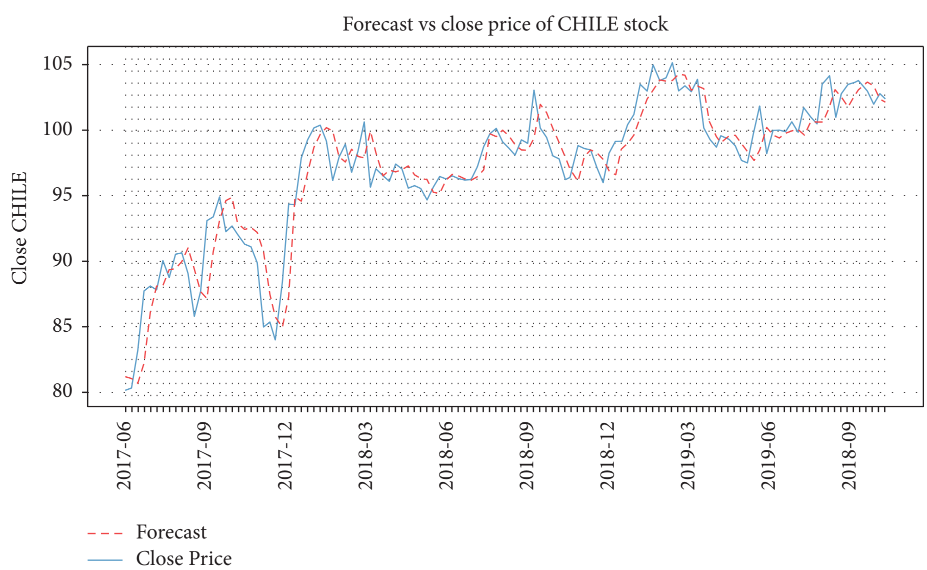

Figure 4: Stock close price prediction.

4.3. Portfolio Comparison of 5 and 10 Stocks. Each ratio is selected between 5 and 10 shares, with a computing time of 1800 seconds, 1,000,000 CLP budget, and the expected returns are LSTM based. Figure 5 shows the portfolio information for the first and last two periods for Sharpe Ratio, $E[R_p]$ stands for portfolio weekly expected return, "Return" expresses the real yield of the week, "Real Rmax" is the weekly return minus the transaction cost. Table 6 shows the portfolio performance comparison.

Figure 5 shows the cumulative returns for different configurations. SR5 and SR10 show the performance by considering 5 and 10 stocks using the Sharpe Ratio, respectively. TR5 and TR10 show the returns' performance by considering 5 and 10 stocks using the Taylor Ratio. Finally, IPSA describes the performance of the Index Chilean Market. Note that the best portfolio selection performed is Sharpe Ratio with ten stocks (SR10), and the worst one is Treynor Ratio with ten stocks.

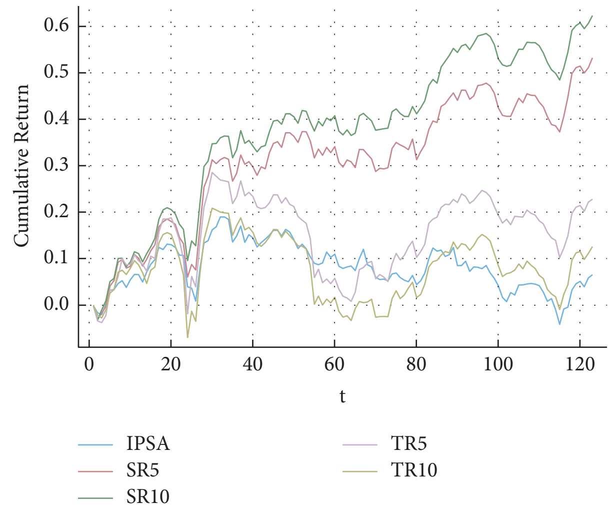

Figure 5: Portfolio yields comparison.

Table 6: Portfolio performance comparison.

t	Real Rmax		Cumulative Rmax		Real Rmax		Cumulative Rmax	
	SR 5 (%)	SR 10 (%)	SR 5 (%)	SR 10 (%)	TR 5 (%)	TR 10 (%)	TR 5 (%)	TR 10 (%)
0	−0.15	−0.15	−0.15	−0.15	−0.15	−0.15	−0.15	−0.15
26	9.17	9.02	16.62	21.84	8.03	10.12	6.15	6.63
52	−0.03	−0.01	37.15	41.71	−0.70	−0.58	17.55	12.04
78	2.29	2.16	35.58	42.73	1.28	1.21	13.74	4.88
104	1.92	1.73	44.36	55.06	−0.11	1.32	18.96	8.82
121	1.16	1.15	50.92	60.61	1.96	1.30	22.12	11.09
	μ_{week}		Final value		μ_{week}		Final value	
	0.43	0.50	53.04	62.28	0.18	0.10	22.74	12.56

4.4. Simplified Formulation v/s, SR10 and TR5. The simplified formulation is applied using the Sharpe Ratio as the objective function. Table 7 shows the cumulative returns for the simplified in the first column without considering the transaction cost; the remaining columns show SR10 and TR5 portfolio returns reduced by the transaction cost. The cumulative return has been increasing at a quarterly average of 11.02%, 6.70%, and 2.35% for simplified problems, the cumulative return of SR10 and TR5 with transaction cost, respectively.

Even if the simplified portfolio does not consider the transaction cost, this could be estimated assuming the same budget. Table 8 shows the comparison of the different methods for the considered problem. The first column of Table 8 describes the cumulative return and the cumulative return by considering transaction costs. We have compared the simplified portfolio, the SR10, and TR5. The new value of the cumulative return for the simplified portfolio is reduced by 24.57%, reaching 78.04% as Cumulative Rmax.

Figure 6 shows how the stock selection is close to the border of the cluster; this resembles an efficient frontier, validating the portfolio selection.

4.5. Discussion. The research addresses the design of a portfolio selection method and begins forecasting risk using an econometric procedure and returns using artificial intelligence. These predictions feed an exact method that optimizes the multi-period portfolio problem.

An RNN algorithm is compared to simple forecasting methods for time series. Considering MAPE, a simple method that performs better than LSTM, the MAPE criteria are discarded. However, it showed the superior quality prediction of LSTM by doing a pilot test. Indeed, Recurrent Neural Network with Long Short-Term Memory performs better in portfolio selection than simple forecasting methods.

Markowitz's method is the main concept for this approach, adjusting the best risk-return ratio, where Sharpe Ratio performs better than Treynor Ratio in both 5 and 10 stock portfolios. This situation is a heteroskedastic risk created by GARCH, considered for Sharpe Ratio, unlike Treynor Ratio, which has a constant systematic risk for 122 periods. Furthermore, it is concluded that the ten-stock Sharpe Ratio Portfolio (SR10) has the most significant cumulative return due to the diversification expressed as the correlation between the assets. Although the Sharpe simplified formulation overcomes every portfolio selection, it does not deal with the realistic problem, so this optimal solution could be used only as an upper bound to reach the proposed method.

This approach is like a multiple of Markowitz's formulation, with a weekly horizon. The expected returns are

Table 7: Cumulative yields of simplified, SR10 and TR5 portfolios.

Date	Cumulative ret (simplified) (%)	Cumulative *R*max (SR10) (%)	Cumulative *R*max (TR5) (%)
16-06-2017	0.00	−0.15	−0.15
15-09-2017	10.05	11.11	7.25
15-12-2017	25.19	21.84	13.24
16-03-2018	41.45	34.21	23.28
15-06-2018	54.24	41.72	17.55
14-09-2018	57.59	40.75	7.61
14-12-2018	64.15	42.74	13.74
15-03-2019	79.55	56.11	23.34
14-06-2019	83.74	55.06	18.96
17-09-2019	99.17	60.13	21.04

Table 8: Transaction cost of simplified v/s SR10 & TR5.

Value	Simplified portfolio w/cTr	SR10 w/cTr	TR5 w/cTr
Cumulative yield (%)	78.04	62.28	22.74
Cumulative cTr (CLP)	497.920	236.699	197.944

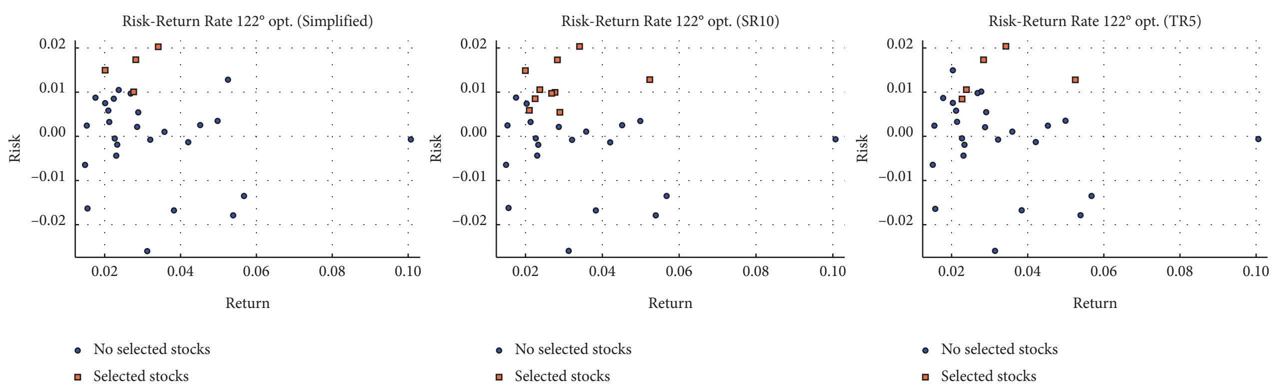

Figure 6: Graphic simplified v/s SR10 & TR5 portfolio selection comparison.

predicted through LSTM instead of the mean of the historical prices, and the variance for every week is estimated with GARCH, which is used in Sharpe Ratio. Treynor Ratio uses constant risk; this makes Treynor Ratio perform worse than Sharpe Ratio, but even the worst portfolio selected (TR10) beats the IPSA index, this is to say, the market.

5. Concluding Remarks

The portfolio selection problem is widely studied in the literature but always focuses on the computational dimension of the problem. This paper deals with the complexity of a realistic problem, proposing a mathematical formulation fed by an econometric and machine learning method. Therefore, an investor guided by this method must not consider any subjective variable, like opinion, news, or sentiment. Thus, this approach to the portfolio selection problem allows an investor to decide where to allocate the money through a complex method that helps to find the best combination of assets, considering transaction cost, the stock integrality, and minimum trade amount, and ensuring not to overcome the budget.

However, the model has some limitations; for instance, the close price is known, and this price is used for prediction, also is the bid and ask price, so, hypothetically, the prediction and the trade are made at the same time, just when the market closes, and this is virtually impossible. Finally, in practice, the price is constantly changing; this proposes a challenge to reduce the compute time, using heuristics and metaheuristics to quickly get a good solution, thus decreasing the price change since it is input in the model until it gets a solution.

In future work, we propose using extended multi-objective methods for logistic problems such as those proposed by [51–53]. Moreover, we could propose metaheuristic algorithms based on the granular tabu concept [54–56] for a large number of stocks. Besides, multicriteria techniques such as those proposed by [2] could solve real problems.

Appendix

A. Markowitz Model

The model of mean-variance proposed by [5] considers the following risk (A.1) and return (A.2) functions:

$$\text{Min}\sigma_p^2 = \sum_{i,j=1}^{N} w_i w_j \sigma_{ij}, \quad (A.1)$$

$$\text{Max}r_p = \sum_{i=1}^{N} w_i \mu_i. \quad (A.2)$$

Subject to

$$\sum w_i = 1, \quad (A.3)$$

where w_i and w_j determine the percentage of the invested budget in stocks i and j and σ_{ij} is the covariance of returns between stocks. Finally, the average of the historical returns is μ_i. The model is subject to the proportion to be invested must be equal to the available budget of the investor, as expressed in equation (A.3).

B. Sharpe Ratio

The Sharpe Ratio[57] is defined as the relationship between the additional benefit of an investment fund (difference between the return of the fund on the asset without risk) and its volatility, measured as its standard deviation. The Sharpe Ratio is calculated with.

$$SR = \frac{r_p - r_f}{\sigma_p}, \quad (B.1)$$

where r_p is the average return of the financial assets (normally a stock or fund), r_f is the average return of the asset with free risk, and σ_p is the deviation of the returns of the asset [21].

C. Taylor Ratio

The Taylor Ratio is calculated as the excess return per unit of systematic risk, unlike the Sharpe Ratio, which considers only the portfolio's systematic risk [58].

$$TR = \frac{r_p - r_f}{\beta_p}, \quad (C.1)$$

β_p is a systematic risk measure that considers the variation of the portfolio concerning the market. β_p is the weighted average of the individual betas of the portfolio assets.

$$\beta_p = \sum_{i=1}^{N} \beta_i * w_i, \quad (C.2)$$

β_i corresponds to the relationship between the risk of the asset concerning the market risk. This value measures the sensitivity of a change in the average return of an individual investment to the change in the market's return. The market risk is equal to 1. If an investment shows a β_i greater than 1, this asset is riskier concerning the market risk. An investment with a β_i less than 1 means the asset is less risky than the market risk. An investment with β_i equal to zero is risk-free, such as treasury bonds [1].

$$\beta_i = \frac{\text{cov}(R_i, R_m)}{\text{var}(R_m)}, \quad (C.3)$$

where R_i is the return of the asset i and R_m is the market return.

Data Availability

The data could be provided by an e-mail of the corresponding author.

Conflicts of Interest

The author(s) declare(s) that there are no conflicts of interest regarding the publication of this paper.

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References

[1] J. W. Escobar, "Metodología para la toma de decisiones de inversión en portafolio de acciones utilizando la técnica multicriterio AHP," *Contaduría y Administración*, vol. 60, no. 2, pp. 346–366, 2015.

[2] J. A. Vásquez, J. W. Escobar, and D. F. Manotas, "AHP–TOPSIS methodology for stock portfolio investments," *Risks*, vol. 10, no. 1, pp. 1–20, 2021.

[3] J. D. Río Miño, "Análisis y comparativa de los sistemas automáticos de trading frente al trading discrecional," 2015, http://hdl.handle.net/11531/3703.

[4] M. I. Gutiérrez Urzúa, P. Galvez Galvez, B. Eltit, and H. Reinoso, "Resolución del problema de carteras de inversion utilizando la heurística de colonia artificial de abejas," *Estudios Gerenciales*, vol. 33, no. 145, pp. 391–399, 2017.

[5] H. Markowitz, "Portfolio selection," *The Journal of Finance*, vol. 7, no. 1, pp. 77–91, 1952.

[6] Z. Bodie, K. Alex, and J. M. Alan, *Essentials of investments/Zvi Bodie*, McGraw-Hill Higher Education, New York, NY, USA, 2004.

[7] M. Grinblatt and S. Titman, *Financial Markets & Corporate Strategy*, Irwin/McGraw-Hill, New York, NY, USA, 2016.

[8] M. Casas Monsegny and E. Cepeda Cuervo, "Modelos ARCH, GARCH y EGARCH: aplicaciones a series financieras," *Cuadernos de Economía*, vol. 27, no. 48, pp. 287–319, 2008.

[9] K. Metaxiotis and K. Liagkouras, "Multiobjective evolutionary algorithms for portfolio management: a comprehensive literature review," *Expert Systems with Applications*, vol. 39, no. 14, Article ID 11685, 2012.

[10] J. W. Escobar, "Metodología para la conformación de portafolio de acciones utilizando la técnica Multicriterio de Borda," *INGE CUC*, vol. 10, no. 2, pp. 60–66, 2014.

[11] X. Mei and F. J. Nogales, "Portfolio selection with proportional transaction costs and predictability," *Journal of Banking & Finance*, vol. 94, pp. 131–151, 2018.

[12] A. A. Adebiyi, A. O. Adewumi, and C. K. Ayo, "Comparison of ARIMA and artificial neural networks models for stock price prediction," *Journal of Applied Mathematics*, vol. 2014, Article ID 614342, 7 pages, 2014.

[13] M. Sanabria and A. Miguel, "Análisis de ventanas temporales de la optimización por enjambre de partículas, aplicado a la selección de portafolio bajo el enfoque de media-varianza-simetría," master thesis, Ingeniería Industrial, Universidad Nacional, Tesis de Pregrado, Colombia, 2017.

[14] G. W. Schwert, "Why does stock market volatility change over time?" *The Journal of Finance*, vol. 44, no. 5, pp. 1115–1153, 1989.

[15] F. León, "Teoría de Portafolio de Markowitz," 2020, https://www.rankia.cl/blog/analisis-ipsa/3500963-teoria-portafolio-markowitz-concepto-ejemplos.

[16] H. Babazadeh and A. Esfahanipour, "A novel multi period mean-VaR portfolio optimization model considering practical constraints and transaction cost," *Journal of Computational and Applied Mathematics*, vol. 361, pp. 313–342, 2019.

[17] A. Fu and B. Wang, "Portfolio optimization based on LSTM neural network prediction," in *Proceedings of the 2020 IEEE International Conference on Networking, Sensing and Control (ICNSC)*, pp. 1–5, IEEE, Nanjing, China, October, 2020.

[18] W. Chen, D. Li, S. Lu, and W. Liu, "Multi-period mean–semivariance portfolio optimization based on uncertain measure," *Soft Computing*, vol. 23, no. 15, pp. 6231–6247, 2019.

[19] R. Siaw, E. Ofosu-Hene, and E. Tee, "Investment portfolio optimization with GARCH models," *ELK Asia Pacific Journal of Finance and Risk Management*, vol. 8, no. 2, 2017.

[20] W. G. Zhang, Y. J. Liu, and W. J. Xu, "A new fuzzy programming approach for multi-period portfolio optimization with return demand and risk control," *Fuzzy Sets and Systems*, vol. 246, pp. 107–126, 2014.

[21] A. Sevilla, "Economiapedia. Obtenido de Ratio de Sharpe," 2015, https://economiapedia.com/definiciones/ratio-de-sharpe.html.

[22] A. Peiro, "Definiciones, ratio de treynor. Obtenido de Economiapedia," 2015, https://economiapedia.com/definiciones/ratio-de-treynor.html.

[23] X. Huang, Z. Zhang, and Z. Zhao, "Multi-Period portfolio optimization for index tracking in finance," in *Proceedings of the2021 55th Asilomar Conference on Signals, Systems, and Computers*, pp. 1383–1387, IEEE, Pacific Grove, CA, USA, November, 2021.

[24] M. A. Moghadam, S. B. Ebrahimi, and D. Rahmani, "A constrained multi-period robust portfolio model with behavioral factors and an interval semi-absolute deviation," *Journal of Computational and Applied Mathematics*, vol. 374, Article ID 112742, 2020.

[25] X. Y. Yang, S. D. Chen, W. L. Liu, and Y. Zhang, "A Multi-period fuzzy portfolio optimization model with short selling constraints," *International Journal of Fuzzy Systems*, vol. 24, no. 6, pp. 2798–2812, 2022.

[26] X. Li, A. S. Uysal, and J. M. Mulvey, "Multi-period portfolio optimization using model predictive control with mean-variance and risk parity frameworks," *European Journal of Operational Research*, vol. 299, no. 3, pp. 1158–1176, 2022.

[27] L. Jiang and S. Wang, "Robust multi-period and multi-objective portfolio selection," *Journal of Industrial and Management Optimization*, vol. 17, no. 2, pp. 695–709, 2021.

[28] J. Chang, L. Sun, B. Zhang, and J. Peng, "Multi-period portfolio selection with mental accounts and realistic constraints based on uncertainty theory," *Journal of Computational and Applied Mathematics*, vol. 377, Article ID 112892, 2020.

[29] F. García, J. González-Bueno, F. Guijarro, and J. Oliver, "A multiobjective credibilistic portfolio selection model. Empirical study in the Latin American integrated market," *Entrepreneurship and Sustainability Issues*, vol. 8, no. 2, pp. 1027–1046, 2020.

[30] B. P. Nokhandan, K. Khalili-Damghani, A. Hafezalkotob, and H. Didehkhani, "A Nash bargaining solution for a multi period competitive portfolio optimization problem: Co-evolutionary approach," *Expert Systems with Applications*, vol. 184, Article ID 115509, 2021.

[31] L. Dymova, K. Kaczmarek, and P. Sevastjanov, "A new approach to the bi-criteria multi-period fuzzy portfolio selection," *Knowledge-Based Systems*, vol. 234, Article ID 107582, 2021.

[32] S. Guo, J. W. Gu, and W. K. Ching, "Adaptive online portfolio selection with transaction costs," *European Journal of Operational Research*, vol. 295, no. 3, pp. 1074–1086, 2021.

[33] S. H. Moon and Y. Yoon, "Genetic mean reversion strategy for online portfolio selection with transaction costs," *Mathematics*, vol. 10, no. 7, p. 1073, 2022.

[34] R. C. Merton, "Optimum consumption and portfolio rules in a continuous-time model," in *Stochastic Optimization Models in Finance*, pp. 621–661, Academic Press, Cambridge, MA, USA, 1975.

[35] R. H. S. Al-Halaseh, M. A. Islam, and R. Bakar, "Dynamic portfolio selection: a literature revisit," *International Business and Management*, vol. 10, no. 2, pp. 67–77, 2016.

[36] M. W. Brandt and P. Santa-Clara, "Dynamic portfolio selection by augmenting the asset space," *The Journal of Finance*, vol. 61, no. 5, pp. 2187–2217, 2006.

[37] T. Bodnar, N. Parolya, and W. Schmid, "A closed-form solution of the multi-period portfolio choice problem for a quadratic utility function," *Annals of Operations Research*, vol. 229, no. 1, pp. 121–158, 2015.

[38] J. He, Q. G. Wang, P. Cheng, J. Chen, and Y. Sun, "Multi-period mean-variance portfolio optimization with high-order coupled asset dynamics," *IEEE Transactions on Automatic Control*, vol. 60, no. 5, pp. 1320–1335, 2015.

[39] J. Skaf and S. Boyd, *Multi-period Portfolio Optimization with Constraints and Transaction Costs*, Citeseer, Pennsylvania, PA, USA, 2009.

[40] Z. C. Lipton, J. Berkowitz, and C. Elkan, "A critical review of recurrent neural networks for sequence learning," 2015, https://arxiv.org/abs/1506.00019.

[41] K. Chen, Y. Zhou, and F. Dai, "A LSTM-based method for stock returns prediction: a case study of China stock market," in *Proceedings of the 2015 IEEE International Conference on Big Data (Big Data)*, pp. 2823-2824, IEEE, Washington, DC, USA, November, 2015.

[42] S. Hochreiter and J. Schmidhuber, "Long short-term memory," *Neural Computation*, vol. 9, no. 8, pp. 1735–1780, 1997.

[43] Y. Peng, P. H. M. Albuquerque, H. Kimura, and C. A. P. B. Saavedra, "Feature selection and deep neural networks for stock price direction forecasting using technical analysis indicators," *Machine Learning with Applications*, vol. 5, Article ID 100060, 2021.

[44] J. O. Muncharaz, "Comparing classic time series models and the LSTM recurrent neural network: an application to S&P 500 stocks," *Finance, Markets and Valuation*, vol. 6, no. 2, pp. 137–148, 2020.

[45] A. M. Rather, "LSTM-Based deep learning model for stock prediction and predictive optimization model," *EURO Journal on Decision Processes*, vol. 9, Article ID 100001, 2021.
[46] W. Chen, H. Zhang, M. K. Mehlawat, and L. Jia, "Mean-–variance portfolio optimization using machine learning-based stock price prediction," *Applied Soft Computing*, vol. 100, Article ID 106943, 2021.
[47] D. Zhao, L. Bai, Y. Fang, and S. Wang, "Multi-period portfolio selection with investor views based on scenario tree," *Applied Mathematics and Computation*, vol. 418, Article ID 126813, 2022.
[48] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah, "Julia: a fresh approach to numerical computing," *SIAM Review*, vol. 59, no. 1, pp. 65–98, 2017.
[49] S. A. Broda and M. S. Paolella, "ARCHModels. jl: estimating ARCH Models in julia," *Jl: Estimating Arch Models in Julia*, vol. 1, 2020.
[50] J. Patterson and A. Gibson, *Deep Learning: A Practitioner's Approach*, O'Reilly Media, Inc, Sebastopol, CA, USA, 2017.
[51] J. W. Escobar, A. A. Marin, and J. D. Lince, "Multi-objective mathematical model for the redesign of supply chains considering financial criteria optimisation and scenarios," *International Journal of Mathematics in Operational Research*, vol. 16, no. 2, pp. 238–256, 2020.
[52] A. L. Velez, J. L. Ramirez, J. W. Escobar, and M. Aguilar, "A multi-objective mathematical model for the design of a closed cycle green distribution network of mass consumption products," *International Journal of Services and Operations Management*, vol. 1, no. 1, pp. 1–141, 2020.
[53] N. S. Rosas, J. W. Escobar, and J. C. Paz, "Optimisation of multi-objective supply chain networks considering cost minimisation and environmental criteria," *International Journal of Industrial and Systems Engineering*, vol. 40, no. 1, pp. 126–146, 2022.
[54] J. Bernal, J. W. Escobar, and R. Linfati, "A granular tabu search algorithm for a real case study of a vehicle routing problem with a heterogeneous fleet and time windows," *Journal of Industrial Engineering and Management*, vol. 10, no. 4, pp. 646–662, 2017.
[55] J. Bernal, J. W. Escobar, J. C. Paz, R. Linfati, and G. Gatica, "A probabilistic granular tabu search for the distance constrained capacitated vehicle routing problem," *International Journal of Industrial and Systems Engineering*, vol. 29, no. 4, pp. 453–477, 2018.
[56] J. Bernal, J. W. Escobar, and R. Linfati, "A simulated annealing-based approach for a real case study of vehicle routing problem with a heterogeneous fleet and time windows," *International Journal of Shipping and Transport Logistics*, vol. 13, no. 1/2, pp. 185–204, 2021.
[57] J. Trullols and F. People, "La importancia del ratio de Sharpe a la hora de seleccionar fondos," *Boletín Mensual*, vol. 52, 2013.
[58] H. Scholz and M. Wilkens, "Investor-specific performance measurement: a justification of Sharpe ratio and treynor ratio," *International Journal of Finance*, vol. 17, no. 4, 2005.