Research Article

Optimal Embedding of Graphs with Nonconcurrent Longest Paths in Archimedean Tessellations

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Optimal graph embeddings represent graphs in a lower dimensional space in a way that preserves the structure and properties of the original graph. These techniques have wide applications in fields such as machine learning, data mining, and network analysis. Do we have small (if possible minimal) $k$-connected graphs with the property that for any $j$ vertices there is a longest path avoiding all of them? This question of Zamfirescu (1972) was the first variant of Gallai’s question (1966): Do all longest paths in a connected graph share a common vertex? Several good examples answering Zamfirescu’s question are known. In 2001, he asked to investigate the family of geometrical lattices with respect to this property. In 2017, Chang and Yuan proved the existence of such graphs in Archimedean tiling. Here, we prove that the graphs presented by Chang and Yuan are not optimal by constructing such graphs of sufficiently smaller orders. The problem of finding nonconcurrent longest paths in Archimedean tessellations refers to finding paths in a lattice such that the paths do not overlap or intersect with each other and are as long as possible. The complexity of embedding graph is still unknown. This problem can be challenging because it requires finding paths that are both long and do not intersect, which can be difficult due to the constraints of the lattice structure.

1. Introduction

Graph embedding refers to the process of representing vertices (nodes) in a graph in a low-dimensional space, such that the structure and relationships between the nodes are preserved in the embedding. One way to do this is to use a graph layout algorithm, which is a type of algorithm that positions the nodes of a graph in a two-dimensional or three-dimensional space in a way that reflects the structure of the graph. Graph layout algorithms can be used to construct embeddings of graphs by placing the nodes in the low-dimensional space according to the relationships between nodes in the graph. Another way to construct embeddings manually is to use domain knowledge to design features that capture the relationships between nodes in the graph and then use these features to embed the nodes in a low-dimensional space. This approach requires a deep understanding of the structure and properties of the graph and can be useful for constructing embeddings for specialized applications or for small graphs. For more details and applications of embedding, we refer to [1, 2].

A path in a graph $G$ is called a longest path if in $G$ we do not have any other path that is strictly longer. In 1966, Gallai [3] asked: Do all longest paths in a connected graph have a vertex in common? The first answer is negative, given by Walther [4] who constructed a 1-connected planar graph of order 25 in which no vertex was in all longest paths. The best negative answer for Gallai’s problem is a nonplanar graph of order 12, independently found by Walther and Voss [5] and Zamfirescu [6], as shown in Figure 1(a). The optimality of this graph was verified (using computers) by Brinkmann and Van Cleemput [7]. Nadeem et al. proved the existence of graphs with the empty intersection of their longest cycles as subgraphs of Archimedean lattices [8]. For more results, we refer to [9–15].
In 1972, Zamfirescu [16] asked an alternative question: Are there \( k \)-connected graphs of small (if possible minimal) order such that, for any choice of \( i \) vertices, there is a longest path missing all of them? The same question was raised for planar graphs. Graphs up to connectivity 3 and for \( i = 1, 2 \) are known. The best known planar graph with nonconcurrent (no vertex in common) longest paths is due to Schmitz [17]. It is a 1-connected graph of order 17, as shown in Figure 1(a). The smallest known path such as a 2-connected graph found by Skupien [18] is of order 26, and the planar one of order 32 is due to Zamfirescu [6].

In 2001, Zamfirescu [19] asked to investigate the equilateral triangular lattice for such graphs, and the first answer is due to Zamfirescu [6]. Skupien [18] is of order 26, and the planar one of order 32 is due to Zamfirescu [6].

In their 2017 paper, Chang and Yuan [23] demonstrated the existence of 1-connected and 2-connected graphs in semiregular tessellations, showing nonconcurrent longest paths. However, our research advances beyond their findings by establishing that the graphs presented in [23] could be more optimal. We achieve this by successfully embedding graphs with smaller orders, surpassing the results of Chang and Yuan.

In this section, we describe 1-connected graphs having empty intersection of their longest paths in Archimedean tilings. Graphs presented here are of smaller order than the ones given in [23]. To get our desired results, we use the following lemma of [23].

Let \( P \) be a graph homeomorphic to the graph \( P' \) given in Figure 3. For each edge of \( P \), the labels in Figure 3 indicate the number of vertices of degrees one and two in the corresponding path of \( P \).

**Lemma 1** [23]. The longest paths of \( P \) are nonconcurrent if \( 0 \leq u \leq \min(y, z), \ 2x \geq y + 2z + 1, w \geq y + 2z - u + 1, w \geq x + z + 1, w \geq y + u + 1, \) and \( v = x + w - z. \)

The order of such graph is \( 9 + v + 2(w + y + z) + 4(x + u). \)

Our first result is as follows:

**Theorem 2.** There exists a graph of order 32 with nonconcurrent longest paths embeddable in \((3^4.6), (3^3.4^2),\) and \((3^2.4.3.4). \)

**Proof.** For \( x = 2, y = 0, z = 1, w = 4, v = 5, \) and \( u = 0, \) the conditions of Lemma 1 are satisfied. The resulting graph \( P \) is of order 32 and is embeddable in \((3^4.6), (3^3.4^2),\) and \((3^2.4.3.4). \) For embeddings of \( P \) in \((3^4.6), (3^3.4^2),\) and \((3^2.4.3.4), \) see Figures 4–6, respectively.

**Theorem 3.** There exists a graph embeddable in \((3.6.3.6)\) of order 56 having nonconcurrent longest paths.

**Proof.** The graph \( P \) obtained for \( x = 4, y = 2, z = 2, w = 7, \) \( v = 9, \) and \( u = 0 \) under conditions of Lemma 1 consists of 56 vertices and is embeddable in \((3.6.3.6), \) as depicted in Figure 7.

**Theorem 4.** The Archimedean tessellation \((3.4.6.4)\) contains a graph of order 52 such that there is no vertex belonging to all its longest paths.

**Proof.** The constraints given in Lemma 1 are also verified for \( x = 4, y = 0, z = 2, w = 7, v = 9, \) and \( u = 0, \) and the resulting

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**Figure 1:** Smallest known nonplanar and planar graphs with nonconcurrent longest paths.
graph $P$ is of order 52. We show an embedding of $P$ in $(3.4.6.4)$ in Figure 8.

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**Theorem 5.** There exists a graph of order 114 having empty intersection of its longest paths in $(4.8^2)$.

*Proof.* Here, we use another particular case of Lemma 1. The graph $P$ with the desired property is obtained by using $x = 9, y = 3, z = 6, w = 16, v = 19$, and $u = 0$ and consists of 114 vertices. An embedding of $P$ in $(4.8^2)$ is illustrated in Figure 9.

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**Theorem 6.** In $(4.6.12)$, we have a graph with nonconcurrent longest paths consisting of 112 vertices.
Proof. An embedding of the desired graph is presented in Figure 10, which is obtained by setting $x = 8, y = 5, z = 5, w = 16, v = 19,$ and $u = 0$ in Lemma 1. □

Theorem 7. In the Archimedean tiling $(3.12^2)$, we have a graph of order 152 whose longest paths share no vertex in common.

Proof. Again, by considering Lemma 1 for $x = 12, y = 4, z = 8, w = 21, v = 25,$ and $u = 1$, we obtained an embeddable graph $P$ in $(3.12^2)$ with the desired property and is shown in Figure 11. □

3. Embeddings of 2-Connected Graphs with Nonconcurrent Longest Paths in Archimedean Tessellations

This section is devoted to 2-connected graphs in which longest paths share no vertex and are embeddable in Archimedean tilings. We start with the following lemma of [21].

Let $Q$ be a graph homeomorphic to the graph $Q'$, as shown in Figure 12, where the variables $x, y, z, t,$ and $s$ represent the number of vertices of degree 2 on the corresponding paths to edges mentioned on the respective figures as well.

Lemma 8 [21]. The longest paths of $Q$ are nonconcurrent if $x \geq t \geq y + 2z + 1$ and $x + t = y + 2z + 1 + s$.

The order of such graph is $2(10 + s + 4z + 4x + 2y + t)$.

Theorem 9. There exists a graph of order 144 in $(3^4.6)$ with nonconcurrent longest paths.

Proof. The values $x = 8, y = 1, z = 3, s = 8,$ and $t = 8$ satisfy constraints of Lemma 8, and we get the graph $Q$ of order 144. Figure 13 depicts an embedding of $Q$ in $(3^4.6)$. □
Theorem 10. There exists a graph of order 98 in \((3^3.4^2)\) with longest paths sharing no vertex.

Proof. Under the conditions of Lemma 8, the graph \(Q\) obtained by setting \(x = 6, y = 0, z = 1, s = 7,\) and \(t = 4\) is of order 98. An embedding of \(Q\) is illustrated in Figure 14.

Theorem 11. There exists a graph of order 88 in \((3^3.4.3.4)\) with nonconcurrent longest paths.

Proof. We obtain the desired graph by setting \(x = 5, y = 0, z = 1, s = 6,\) and \(t = 4\) in Lemma 8, with an embedding given in Figure 15.

Theorem 12. There exists a graph of order 180 in \((3.6.3.6)\) with nonconcurrent longest paths.

Proof. If we take \(x = 11, y = 2, z = 3, s = 11,\) and \(t = 9\) in Lemma 8, then the resulting graph \(Q\) with the desired property is of order 180 and embeddable in \((3.6.3.6)\), see Figure 16.

Theorem 13. In \((3.4.6.4)\), we have a graph without concurrent longest paths of order 182.

Proof. Again, by considering Lemma 8 for \(x = 11, y = 1, z = 3, s = 13,\) and \(t = 10\), we obtain a desired graph \(Q\) of order 182. An embedding of \(Q\) in \((3.4.6.4)\) is illustrated in Figure 17.
The graph $Q'$ of Figure 12 is not embeddable in the cubic regular tilings $(4.8^2)$, $(4.6.12)$, and $(3.12^2)$ as it contains vertices of degree 4. To achieve results in the mentioned tilings, we considered the following lemma of [23].

Let $R$ be a graph homeomorphic to the graph $R'$ of Figure 18(a), which contains three isomorphic subgraphs homeomorphic to the graph $P'$ given in Figure 18(b), where $x, y, z, w, v,$ and $u$ represent the number of vertices of degree 2 on the corresponding paths.

**Lemma 14** [23]. The longest paths of $R$ are nonconcurrent if $y \geq 1$, $u \geq 0$, and $x = y + z - u \geq v = y + 2w + 1 - u$.

The order of such graph is $28 + 3(v + w) + 12(x + u) + 6(y + z)$.

**Theorem 15.** There exists an embeddable graph of order 367 in $(4.8^2)$ whose longest paths are nonconcurrent.

**Proof.** If we take $x = 16, y = 7, z = 9, w = 3, v = 14$, and $u = 0$ in Lemma 14, the resulting graph $R$ of order 367 has the desired property. An embedding of $R$ is presented in Figure 19.

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The order of such graph is $28 + 3(v + w) + 12(x + u) + 6(y + z)$.

**Theorem 15.** There exists an embeddable graph of order 367 in $(4.8^2)$ whose longest paths are nonconcurrent.

**Proof.** If we take $x = 16, y = 7, z = 9, w = 3, v = 14$, and $u = 0$ in Lemma 14, the resulting graph $R$ of order 367 has the desired property. An embedding of $R$ is presented in Figure 19.

**Theorem 16.** There exists a graph of order 424 with non-concurrent longest paths embedded in $(4.6.12)$.

**Proof.** An embedding of the graph $R$ of order 424 obtained under the conditions of Lemma 14 by setting $x = 18, y = 5, z = 13, w = 6, v = 18$, and $u = 0$ is shown in Figure 20.

**Theorem 17.** The Archimedean tiling $(3.12^2)$ contains a graph of order 481 without concurrent longest paths.

**Proof.** The conditions of Lemma 14 are also satisfied if we take $x = 20, y = 10, z = 11, w = 5, v = 20$, and $u = 1$. The order of the obtained graph $R$ is 481. Figure 21 reveals an embedding for $R$ in $(3.12^2)$.
4. Conclusion

In 2001, Zamfirescu [19] raised a question: Do we have small (if possible minimal) \( k \)-connected graphs with the property that for any \( j \) vertices there is a longest path avoiding all of them in lattices? To answer this question, in 2017, Chang and Yuan proved the existence of such graphs in Archimedean tiling [23]. Here, we proved that the graphs presented by Chang and Yuan are not optimal by constructing such graphs of sufficiently smaller orders. For comparison, see Table 1 given below.

We conclude this paper with the following problems:

Open Problem 18. Find embeddings of smaller order than those presented in Theorems 2–7 and 9–17. Moreover, we find lattices of smaller order than those presented in Theorems 2–7 and 9–17. Moreover, we find lattices of smaller order than those presented in Sections 2 and 3, admitting the desired embeddings.

Open Problem 19. Find embeddings of 2-connected graphs with the property that for any \( j \) vertices there is a longest path/cycle avoiding all of them.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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