Spatiotemporal Tipping Induced by Turing Instability and Hopf Bifurcation in a Population Ecosystem Model with the Fear Factor

1. Introduction

It is proposed by Leslie [1] that predator-prey is the basic component in ecological competition systems and the most fundamental species relationship in nature. Predation plays an important role in ecological population systems. Predation controls the density of biological populations and directly affects the amount of natural resources. In view of predation, there has been tremendous progress in the dynamics of predator and prey [2–5]. Recently, many populations have been on the verge of extinction due to destructive human exploitation, pollution, and hunting that have had a huge impact on nature and species. The studies of ecological competition dynamic models have been pushed to the hot spot [6–9]. Predator-prey systems are an essential component to revealing the competition mechanism, growth law, and control strategy among populations, so it is important to establish a scientific predator-prey model.

Local stability of the system means that if the initial state is adjacent to the equilibrium state, the system will not vibrate, and its state trajectory will eventually fall at the equilibrium state. Specially, Hopf bifurcation is a dynamic bifurcation phenomenon, which shows that when the system parameter varies near the critical value, the stability of the equilibrium point will change and periodic solutions will be generated. From a dynamical point of view, the quantitative relationship between prey and predators sometimes has periodic fluctuations, and the limit cycle may arise due to the Hopf bifurcation, which is an equal amplitude oscillation caused by the critical state. In this sense, the Hopf bifurcation is an important issue for the study of the dynamics of population competition models. Therefore, most of the
previous work focused on the existence of coexisting equilibria and bifurcation as well as the dynamic behavior of nonlinear systems with periodic solutions and chaos (see [10–15]). However, these existing ecological competition models are characterized by ordinary differential equations (ODEs) or fractional ordinary differential equations (FODEs), which can only describe the evolution of species in the time dimension but cannot depict the migration in space. In the reaction-diffusion system proposed by Turing [16,17], the spatial heterogeneity caused by the internal diffusion characteristics of the system results in the loss of system symmetry and makes the system self-organization to produce some spatial patterns. The process of pattern formation is called Turing instability (Turing bifurcation). The symmetry of the system is broken, leading to the formation of Turing patterns. Therefore, we call this phenomenon Turing instability caused by diffusive reaction [18]. In fact, species often migrate from high-density areas to low-density areas in search of better survival conditions. The uneven spatial distribution of species has an impact on predation behaviors. Hence, more and more scholars have concentrated on predator-prey models with diffusion terms, which can be described by reaction-diffusion equations. The stability, diffusion-induced Turing instability, Hopf/Bogdanov-Takens bifurcations, and Turing pattern have been studied by many scholars [19–24].

The fear factor of predators also has a significant impact on interspecies predation. As an indirect effect, fear from predators can affect the population size of prey. Such fear can reduce reproductive capacity. An increase in the number of predators results in greater fear of prey, and then the growth rate of prey drops sharply. The reduction in prey density decreases the efficiency of predator feeding behavior. Therefore, the growth rate of the predator is declining [25,26]. Therefore, the fear factor of prey towards predators is also essential for the dynamic model of ecological competition. On the contrary, processes such as the gestation of prey, digestion of predators, spread of diseases, and human impact are not instantaneous behaviors. The introduction of time delays to predator-prey systems is more meaningful [27–29]. The fear of time delay can have a significant effect on the dynamics of ecological population systems. This article considers primarily the effect of fear delay on the dynamic behaviors of population competition systems.

It is worth noting that despite considerable advances in the dynamics of population-competitive ecosystems, there are still many issues that deserve further study. Little is known about the tipping phenomena of predator-prey systems. It is observed that a variety of real-world complex systems can exhibit abrupt transitions in dynamical patterns and states of behaviors, then they are called to tip from one emergent mode to another [30, 31]. Some such transitions that can influence the production and lives of all human beings are the shutdown of the thermohaline circulation, global changes in climate, extinction of species in ecosystems, the crashing of financial markets, and the surge of COVID-19 epidemics [32–37]. These tipplings occur mostly due to small perturbations to the threshold values of state variables or system parameters, giving rise to dramatic qualitative changes in the dynamics. They have often had a tremendous influence and are mostly irreversible.

Habitat transformations and connections between species can disappear in an ecological system. As the fraction of vanishing connections continues to increase beyond a threshold, the ecological system may arrive at a tipping point past which the whole system crumbles. Then, all species die at the same time [38]. Therefore, to understand the dynamical mechanism of the tipping point of predator-prey systems and to predict its occurrence are the overriding problems of broad interest and great importance. Research studies have shown that the primary deterministic mechanisms behind several of the tipping events are so-called bifurcations [31, 39]. This article aims to propose a population competition model having fear delays and diffusion terms, and then predicts and exploring the tipping points induced by Turing instability and Hopf bifurcations.

We organize the rest of the article as follows. A diffusive predator-prey model including fear factors is presented in Section 2. Section investigates Turing instability-induced tipping for the diffusive population competition model without time delays. The impact of diffusion terms on the tipping dynamics is discussed. Meantime, both time delay and diffusion factors are considered, and the tipping point induced by Hopf bifurcation is studied. The Hopf bifurcation criterion of the population evolution model is proven. Section 4 determines the Hopf bifurcation direction in order to gain insight into tipping mechanisms. Section 5 provides several examples to testify to the validity of the theoretical analysis. The conclusion is drawn in Section 6.

### 2. Model Formulation

We establish a diffusive predator-prey system including fear delay in this section.

In [40], a classical predator-prey system is summarized by

\[
\begin{align*}
\frac{d\rho}{dr} &= r\rho \left(1 - \frac{\rho}{K}\right) - \frac{m\rho}{A\theta + \rho}, \\
\frac{d\theta}{dr} &= \theta \left(1 - h\frac{\theta}{\rho}\right).
\end{align*}
\]

The basic assumptions of system (1) are described below:

(i) \(\rho\) and \(\theta\) stand for the densities of prey and predator, respectively. \(K\) and \(r\) represent the carrying capacity and the intrinsic growth rate of the logistical growth of prey population.

(ii) \(m\rho/(A\theta + \rho)\) describes a modified ratio-dependent functional response of Holling type II.

(iii) \(s\) stands for the inherent growth rate of predators. The carrying capacity of predators is \(\rho/h\), which is described by a function of prey population.

(iv) All the coefficients \(K, r, m, A, h, s > 0\).

Since the fear factor is essential for prey growth, the fear delay is naturally introduced for system (1) as follows:
and it becomes

\[
\begin{align*}
\frac{\partial \rho(t, \omega)}{\partial t} &= \frac{d_1}{r} \Delta \rho(t, \omega) + \frac{\rho(t, \omega)(1 - \rho(t, \omega))}{1 + c \theta(t - \tau, \omega)} \rho(t, \omega), \\
\frac{\partial \theta(t, \omega)}{\partial t} &= \frac{d_2}{r} \Delta \theta(t, \omega) + \delta \theta(t, \omega) \left( \beta \frac{\theta(t, \omega)}{\rho(t, \omega)} - \theta(t, \omega) - \theta^* \right),
\end{align*}
\]

where \( C > 0 \) denotes the fear coefficient and \( r' \) is the fear delay.

Considering the impact of populations on different spatial locations and the heterogeneous distribution of food resources, we put forward a novel diffusive predator-prey system, which can be described by

\[
\begin{align*}
\frac{d\rho}{dt} &= \frac{r}{1 + C\theta(t - \tau') \rho(t, \omega)} - \frac{m\rho}{A + \theta} \rho(t, \omega), \\
\frac{d\theta}{dt} &= \theta \left( 1 - \frac{\rho(t, \omega)}{s} \right),
\end{align*}
\]

with Neumann boundary and initial conditions

\[
\begin{align*}
\frac{\partial \rho(t, \omega)}{\partial \varepsilon} = \frac{\partial \theta(t, \omega)}{\partial \varepsilon} = 0, \omega \in \partial \Theta, t > 0, \\
\rho(t, \omega) = \rho_0(t, \omega) \geq 0, \theta(t, \omega) = \theta_0(t, \omega) \geq 0, \\
(t, \omega) \in [-\tau, 0] \times \Theta,
\end{align*}
\]

where \( \rho(t, \omega) \) and \( \theta(t, \omega) \) represent the population density of prey and predators at time \( t \) and spatial location \( \omega \), respectively. \( \Delta \) is the Laplacian operator and the positive diffusion coefficients \( d_1 \) and \( d_2 \) are related to \( \rho(t, \omega) \) and \( \theta(t, \omega) \), respectively. The bounded domain \( \Theta = (0, m) \times (0, n) \) has a smooth boundary \( \partial \Theta \). \( \varepsilon \) represents the outward unit normal vector on \( \partial \Theta \), \( \rho_0(t, \omega) \) and \( \theta_0(t, \omega) \) are the non-negative continuous functions.

The diffusive predator-prey system (3) can be conversed by the following transformation:

\[
rt \longrightarrow t, \frac{\rho}{K} \longrightarrow \rho, \frac{\theta m}{rK} \longrightarrow \theta,
\]

and it becomes

\[
\begin{align*}
\frac{\partial \rho(t, \omega)}{\partial t} &= \frac{d_1}{r} \Delta \rho(t, \omega) + \rho(t, \omega)(1 - \rho(t, \omega)) \frac{1}{1 + c \theta(t - \tau, \omega)} \rho(t, \omega), \\
\frac{\partial \theta(t, \omega)}{\partial t} &= \frac{d_2}{r} \Delta \theta(t, \omega) + \delta \theta(t, \omega) \left( \beta \frac{\theta(t, \omega)}{\rho(t, \omega)} - \theta(t, \omega) - \theta^* \right),
\end{align*}
\]

where

\[
\begin{align*}
d_1 &= \frac{d_1}{r} = c = \frac{rK}{m}, a = \frac{A}{m}, \tau = r \tau', \\
d_2 &= \frac{d_2}{r} = \delta = \frac{m}{hr}.
\end{align*}
\]

The dynamical behaviors of system (6) are equivalent to those of system (3). The following derivation revolves around system (6).

## 3. Tipping Dynamics Analysis

Given the practical implications of ecosystems, we focus on the coexisting equilibrium. If \( 1 + a\beta - \beta > 0 \), then there is unique positive equilibrium \( E^*(\rho^*, \theta^*) \) of system (6), where

\[
\rho^* = \frac{1 + a\beta - \beta}{c\beta^2 + a\beta + 1}, \quad \theta^* = \beta \rho^*.
\]

Next, this article will be based on \( 1 + a\beta - \beta > 0 \).

Remark 1. Assuming that there is no diffusion effects, system (6) may degenerate to the system described by ODEs. The emergence of diffusion terms does not change the equilibria.

Now, we make the transformation of variables:

\[
\tilde{\rho}(t, \omega) = \rho(t, \omega) - \rho^*, \tilde{\theta}(t, \omega) = \theta(t, \omega) - \theta^*.
\]

We drop the hats for the sake of simplified calculation and the linearized system of (6) is as follows:
functions are given by \( (\rho, \theta) \) on the phase space.

\[ \frac{\partial \rho(t, \omega)}{\partial t} = a_{11} \Delta \rho(t, \omega) + a_{12} \theta(t, \omega) + a_{13} \theta(t - \tau, \omega), \]
\[ \frac{\partial \theta(t, \omega)}{\partial t} = a_{21} \Delta \theta(t, \omega) + a_{22} \theta(t, \omega), \]

where
\[ a_{11} = 1 - 2 \rho^* + \frac{a \theta^*}{1 + c \theta^*}, a_{12} = -\frac{\rho^*}{(\rho^* + a \theta^*)^2}, \]
\[ a_{13} = \frac{c \rho^* (\rho^* - 1)}{1 + c \theta^*}, a_{21} = -\delta \beta^2, a_{22} = -\delta \beta^2. \]

Let \( D = \text{diag} \{ a_1, a_2 \} \). System (10) can be written as follows:
\[
\begin{bmatrix}
\frac{\partial \rho(t, \omega)}{\partial t} \\
\frac{\partial \theta(t, \omega)}{\partial t}
\end{bmatrix} = D \Delta \begin{bmatrix} \rho(t, \omega) \\ \theta(t, \omega) \end{bmatrix} + L \begin{bmatrix} \rho(t, \omega) \\ \theta(t, \omega) \end{bmatrix},
\]
in which
\[ L(\varphi) = \begin{bmatrix} a_{11} \phi_1(0) + a_{12} \phi_2(0) + a_{13} \phi_2(-\tau) \\ a_{21} \phi_1(0) + a_{22} \phi_2(0) \end{bmatrix}, \]
for \( \varphi = (\varphi_1, \varphi_2)^T \).

The characteristic equation of system (12) is as follows:
\[ \lambda \eta - \Delta \eta - L(e^{\lambda \eta}) = 0, \eta \in \text{do m}(\Delta)[0]. \]

As we know, the eigenvalues of \( \Delta \) are \(-k^2 \) (\( k \in \{0, 1, 2, \ldots \} \)) on the phase space \( X \) and the corresponding eigenfunctions are given by
\[ \beta_k^1 = \begin{pmatrix} y_k \\ 0 \end{pmatrix}, \beta_k^2 = \begin{pmatrix} 0 \\ y_k \end{pmatrix}, \]
where \( y_k = \cos(kx) \). \([\beta_k^1, \beta_k^2]_{k=0}^{\infty} \) construct a basis of \( X \), which can be described by
\[ X = \{ \zeta \in W^{2,2}(0, \pi): \zeta(\pi) = \zeta(0) = 0 \}. \]

So, equation (14) can be turned into the following form:
\[ \lambda^2 + p_1 \lambda + q_1 - a_{21}(a_{12} + a_{13} e^{-\lambda \tau}) = 0, \]
in which
\[ p_1 = d_1 k^2 + d_2 k^2 - a_{11} - a_{22}, \]
\[ q_1 = (d_1 k^2 - a_{11})(d_2 k^2 - a_{22}). \]

3.1. Tipping Induced by Turing Instability. We intend to probe into the impact of diffusion on the dynamical evolution of system (6) without time delays and study the tipping point caused by Turing instability in this subsection.

When \( \tau = 0 \), equation (17) becomes
\[ \lambda^2 + p_1 \lambda + q_1 - a_{21}(a_{12} + a_{13}) = 0. \]

If \( d_1 = d_2 = 0 \) holds, equation (19) can be reduced to the following diffusion-free form:
\[ \lambda^2 + p_2 \lambda + q_2 - a_{21}(a_{12} + a_{13}) = 0, \]
where
\[ p_2 = -a_{11} - a_{22}, q_2 = a_{11} a_{22}. \]

We make the following assumption:
\[ (H1) p_2 > 0, q_2 - a_{21}(a_{12} + a_{13}) > 0. \]

Lemma 1. We assume that \( d_1 = d_2 = 0 \) and \( \tau = 0 \). If (H1) is satisfied, then the trajectories of system (6) converge to the equilibrium \( E^* \).

The proof of Lemma 1 is straightforward on the basis of Routh–Hurwitz criterion.

If \( d_1 > 0, d_2 > 0 \) are true, then it is easy to see that \( p_1 > 0 \) when (H1) holds. Therefore, in order to ensure the occurrence of Turing instability in system (6), we need to ensure that
\[ (H2) q_1 - a_{21}(a_{12} + a_{13}) < 0. \]

Lemma 2. If \( d_1 > 0, d_2 > 0 \) and (H2) are satisfied, equation (19) has a positive root. Then, system (6) without time delay is unstable at \( E^* \) for some \( k \in N_0 \).

The following theorem is true.

Theorem 1 (Turing instability-induced tipping). The following results are true for system (6) without time delay.

(i) If \( d_1 = d_2 = 0 \) and (H1) hold, the trajectories of system (6) without time delay converge to \( E^* \).

(ii) If \( d_1 > 0 \) and \( d_2 > 0 \) and (H2) hold, there exists at least one \( k \in N_0 \) such that system (6) without time delay becomes unstable at \( E^* \), while a Turing instability occurs. Hence, this results in a Turing instability-induced tipping.

Remark 2. It should be pointed out that this article adopts the linearization method [23, 24, 29, 41] to deal with the dynamics analysis of systems, including the local stability, Turing instability and Hopf bifurcation. It is common knowledge that Lyapunov’s second method is important to stability theory of dynamical systems and control theory. However, this method is not suitable for investigating the dynamics of the ecological competitive system with delay and diffusion proposed in this article. The Lyapunov stability criterion can only give a sufficient condition for the stability.
of a system. In this article, not only the condition of the local stability is established but also the boundary of stability (the onset of Hopf bifurcation) is determined.

Remark 3. It is obvious from Theorem 1 that the predator-prey system (6) without spatial diffusion terms is stable at the endemic equilibrium, but the introduction of diffusions makes the system unstable. This phenomenon is called Turing instability. Meanwhile, a transition from a stable mode to an unstable one is called a tipping caused by Turing instability.

\[(H3) - a_{11} - a_{22} > 0, a_{11}d_2 + a_{22}d_1 < 0, a_{11}a_{22} - a_{21}(a_{12} + a_{13}) > 0. \] (24)

3.2. Tipping Induced by Hopf Bifurcation. The impacts of diffusion and time delays on the spatiotemporal evolution of system (6) forecast the tipping point induced by Hopf bifurcation in this section.

Equation (19) is the characteristic equation for system (6) without delay. We can give the stability condition of \( E^* \) of system (6) as follows:

\[
\omega^4 + P\omega^2 + Q = 0. \tag{30}
\]

Let \( z = \omega^2 \). We define

\[
h(z) = z^2 + Pz + Q. \tag{31}
\]

When \( k = 0 \), we let

\[
P_0 = p_2^2 - 2q_2 + 2a_{21}a_{12},
\]

\[
Q_0 = (q_2 - a_{21}a_{12})^2 - a_{21}^2a_{13}^2. \tag{32}
\]

Next, the following assumptions are made for \( k = 0 \):

(i) When (H4) is satisfied, then \( P_0^2 - 4Q_0 < 0 \) or \( P_0^2 > 0 \)

(ii) When (H5) is satisfied, then \( P_0 < 0 \), \( P_0^2 - 4Q_0 = 0 \) or \( Q_0 < 0 \)

(iii) When (H6) is satisfied, then \( P_0 < 0 \), \( Q_0 > 0 \), and \( P_0^2 - 4Q_0 > 0 \)

Lemma 4. Assume that \( k = 0 \). The following conclusions are true for equation (30).

(i) When (H4) is satisfied, then there are no positive roots for equation (30).

(ii) When (H5) is satisfied, then there is a positive root for equation (30).

(iii) When (H6) is satisfied, then there are two positive roots for equation (30).

We assume that when \( k = 0 \), there exist two positive roots for equation (30), defined by \( \omega_m > 0 \) \((m = 1, 2) \). It is obvious from equation (27) that

\[
\cos \omega_m \tau = \frac{-a_{21}^2 + a_{12}^2 - a_{21}a_{13}^2}{a_{21}a_{13}}. \tag{33}
\]

Thus,

\[
\tau_m^{(j)} = \frac{1}{\omega_m} \left[ \arccos \frac{-a_{21}^2 + a_{12}^2 - a_{21}a_{13}^2}{a_{21}a_{13}} + 2j\pi \right], \tag{34}
\]

where \( m = 1, 2 \) and \( j = 0, 1, \ldots \). Thus, equation (17) has a pair of purely imaginary roots \( \pm i\omega_m \) when \( \tau = \tau_m^{(j)} \). We define
\[ \tau_0 = \tau_{m_0}^{(0)} = \min_{m \in \{1, 2\}} \{ \tau_{m}^{(0)} \}, \omega_0 = \omega_{m_0}. \]  

We assume that equation (17) has the root \( \lambda(\tau) = \theta(\tau) + i\omega(\tau) \), which satisfies \( \theta(\tau_{m}^{(j)}) = 0 \) and \( \omega(\tau_{m}^{(j)}) = \omega_{m} \).

**Lemma 5.** Suppose that \( \tau_{m} = \omega_{m}^{2} \) and \( h'(\tau_{m}) \neq 0 \). Then,

\[
\Re\left( \frac{d\lambda}{d\tau} \right) = 0,
\]

and the sign of \( \Re(\frac{d\lambda}{d\tau})_{\tau = \tau_{m}^{(0)}} \) is accordant with that of \( h'(\tau_{m}) \).

**Proof.** We substitute \( \lambda(\tau) \) into equation (17) and differentiate the resulting equation in \( \tau \) and then have

\[
\frac{d\lambda}{d\tau}^{-1} = \frac{2\lambda + p_{1} + a_{12}a_{13}e^{-\beta t}}{a_{21}a_{13}e^{-\beta t}},
\]

\[
= \frac{[2\lambda + p_{1}]e^{-\beta t} + a_{12}a_{13}t}{a_{21}a_{13}e^{-\beta t}}.
\]

Then,

\[
\Re\left( \frac{d\lambda}{d\tau} \right)_{\tau = \tau_{m}^{(0)}} = \frac{\omega_{m}^{2}}{a_{21}a_{13}e^{-\beta t}}.
\]

Clearly, \( a_{21}a_{13}^{2} > 0 \). Therefore, the sign of \( \Re(\frac{d\lambda}{d\tau})_{\tau = \tau_{m}^{(0)}} \) is consistent with that of \( h'(\tau_{m}) \). This completes the proof.

For convenience, we further give the following assumption:

\[
(H7) h'(\tau_{m}) > 0.
\]

It is common knowledge that a Hopf bifurcation happens when \( \Re(\lambda) = 0 \) and \( \Im(\lambda) \neq 0 \) at \( k = 0 \). Suppose that equation (17) has a solution \( \omega_{0} \) (\( \omega_{0} > 0 \)) at \( k = 1 \); then, one obtains

\[
\omega_{1}^{2} + P\omega_{1}^{2} + Q = 0.
\]

Thus, if the following assumption holds:

\( (H8) P > 0, Q > 0 \) for \( k \geq 1 \); then, there are no purely imaginary roots for equation (17) at \( k \geq 1 \).

**Theorem 2.** Suppose \((H3)\) holds. We come to the following statements.

(i) If \((H4)\) and \((H8)\) are satisfied, then the trajectories of system (6) converge to the equilibrium point \( E^{*} \) for all \( \tau \geq 0 \).

(ii) If either \((H5)\) or \((H6), (H7), \) and \((H8)\) are satisfied, the trajectories of system (6) tend to \( E^{*} \) for \( \tau < \tau_{0} \) and is far from \( E^{*} \) for \( \tau > \tau_{0} \). System \( (iii) A \) Hopf bifurcation is produced at \( E^{*} \) when \( \tau = \tau_{0} \) and a periodic Oscillation appears. This leads to a Hopf bifurcation-induced tipping.

**Remark 4.** Theorem 2 illustrates that, as the bifurcation parameter-delay passes through the threshold \( \tau_{0} \) that is the tipping point, system (6) goes from the stable mode to the unstable state at \( E^{*} \) and a periodic oscillation occurs. Such a transition is called the tipping driven by Hopf bifurcation. It is worth noting that the tipping can be predicted by the expression of the tipping point \( \tau_{0} \).

### 4. Direction of Hopf Bifurcation

In order to understand the mechanism of tipping more deeply, this section will further explore a deeper understanding of Hopf bifurcation via the center manifold theorem for partial functional differential equations (PFDEs).

We denote \( \tau_{0} \) by \( \bar{\tau} \) and \( \omega_{0} \) by \( \omega_{k} \) and set \( \alpha = \tau - \bar{\tau} \). We make the time-scale transformation \( t \rightarrow \frac{t}{\tau} \). Thus, system (6) can be rewritten in \( \varphi_{k} = C([-1, 0], X) \) as follows:

\[
\dot{U}(t) = \bar{\tau} \Delta U(t) + L_{*}(\bar{\tau})(U_{1}) + F_{*}(U_{1}, a),
\]

in which \( L_{*} : \varphi \rightarrow R^{2} \) and \( F_{*} : \varphi \times R^{2} \rightarrow R^{2} \) are described by

\[
L_{*}(\bar{\tau})(\varphi) = \bar{\tau} \left( \begin{array}{c} a_{11}\varphi_{1}(0) + a_{12}\varphi_{2}(0) + a_{13}\varphi_{3}(0) \\ a_{21}\varphi_{1}(0) + a_{22}\varphi_{2}(0) \end{array} \right),
\]

\[
F_{*}(\varphi, a) = a\Delta\varphi_{0} + L^{*}(a)(\varphi) + (\bar{\tau} + a) \left( \begin{array}{c} f_{1*} \\ f_{2*} \end{array} \right),
\]

where

\[
L^{*}(a)(\varphi) = a \left( \begin{array}{c} a_{11}\varphi_{1}(0) + a_{12}\varphi_{2}(0) + a_{13}\varphi_{3}(0) \\ a_{21}\varphi_{1}(0) + a_{22}\varphi_{2}(0) \end{array} \right),
\]

for \( \varphi = (\varphi_{1}, \varphi_{2})^{T} \in \varphi_{k} \), and

\[
f_{1*} = h_{v11\varphi_{1}^{2}(0)} + h_{v12\varphi_{1}(0)\varphi_{2}(0)} + h_{v13\varphi_{1}(0)\varphi_{3}(0)} + h_{v21\varphi_{2}^{2}(0)} + h_{v22\varphi_{2}(0)\varphi_{3}(0)} + h_{v23\varphi_{3}^{2}(0)} + h_{v31\varphi_{1}(0)\varphi_{2}(0)} + h_{v32\varphi_{2}(0)\varphi_{3}(0)} + h_{v33\varphi_{3}^{2}(0)} - 1,
\]

\[
f_{2*} = f_{v11\varphi_{1}^{2}(0)} + f_{v12\varphi_{1}(0)\varphi_{2}(0)} + f_{v21\varphi_{2}^{2}(0)} + f_{v22\varphi_{2}(0)\varphi_{3}(0)} + f_{v23\varphi_{3}^{2}(0)} + f_{v31\varphi_{1}(0)\varphi_{2}(0)} + f_{v32\varphi_{2}(0)\varphi_{3}(0)} + f_{v33\varphi_{3}^{2}(0)} - 1.
\]
In here,

\[ h_{v1,v1} = \frac{-2}{1 + c\theta^2} + \frac{2a\theta^2}{(\rho^* + a\theta^*)^3}, h_{v1,v2} = -\frac{2a\rho^\ast}{(\rho^* + a\theta^*)^3}, \]
\[ h_{v1,v3} = \frac{c(2\rho^* - 1)}{(1 + c\theta^2)^2}, h_{v2,v2} = \frac{2a\rho^\ast}{(\rho^* + a\theta^*)^3}, h_{v3,v3} = -\frac{-2c\rho^\ast(\rho^* - 1)}{(1 + c\theta^*)^3}, h_{v1,v1,v1} = \frac{-6a\theta^2}{(\rho^* + a\theta^*)^4}, \]
\[ h_{v1,v2,v2} = \frac{4a\rho^* - 2a^2\theta^2}{(\rho^* + a\theta^*)^3}, h_{v1,v3,v3} = \frac{6c^3\rho^*(\rho^* - 1)}{(1 + c\theta^*)^4}. \]

From the previous discussions, we can see that system (41) has the equilibrium (0, 0) and the following equation

\[ U(t) = \tau(t) L^\ast \tau(U), \tag{46} \]

has a pair of purely imaginary eigenvalues \( \Lambda_0 = [i\omega_\mu \tilde{\tau}, i\omega_\mu \tilde{\tau}] \).

\( \alpha \) is a critical point of Hopf bifurcation in system (41).

It follows from Riesz representation theorem that there is a 2 × 2 matrix function \( \epsilon(\kappa, \tau) \) satisfying

\[ L_\ast(\tilde{\tau})(\phi) = \int_{-1}^{0} dx (\kappa, \tilde{\tau})\phi(x). \tag{47} \]

One can take

\[ \epsilon(\kappa, \tilde{\tau}) = -\tilde{\tau} \begin{pmatrix} a_{11} & a_{12} + a_{13}e^{-\lambda} \\ a_{21} & a_{22} \end{pmatrix} \zeta(\kappa + 1), \tag{48} \]

in which

\[ \zeta(\kappa) = \begin{cases} 0, -1 \leq \kappa < 0, \\ 1, \kappa = 0. \end{cases} \tag{49} \]

Suppose that \( A(\tilde{\tau}) \) is the infinitesimal generator of the semigroup and \( A^\ast \) is the formal adjoint of \( A(\tilde{\tau}) \) such that

\[ \text{for } \phi \in C, \psi \in C^\ast = C([0, 1], R^2). \]

We can give the definition of \( A(\tilde{\tau}) \) as follows:

\[ A(\tilde{\tau}) = \tilde{\tau} \begin{pmatrix} a_{11} & a_{12} + a_{13}e^{-\lambda} \\ a_{21} & a_{22} \end{pmatrix}. \tag{51} \]

One knows that there are a pair of simple purely imaginary eigenvalues \( \pm i \omega_\mu \tilde{\tau} \) for \( A(\tilde{\tau}) \). We can calculate the eigenfunction \( q_1(s) \) of \( A^\ast \) corresponding to \( i\omega_\mu \tilde{\tau} \) and the eigenfunction \( p_1(\kappa) \) of \( A(\tilde{\tau}) \) corresponding to \( i\omega_\mu \tilde{\tau} \). Suppose that \( p_1(\kappa) = e^{i\omega_\mu n\tilde{\tau} \kappa} (1, \mu)^T \) and \( q_1(s) = (1, \delta) e^{-i\omega_\mu n\tilde{\tau} s}. \)

From the definition of \( A(\tilde{\tau}) \), we derive

\[ (\lambda I - A)p_1(0) = 0|_{\lambda = i\omega_\mu \tilde{\tau}}, \]

\[ (\lambda I - A^\ast)q_1(0)^T = 0|_{\lambda = -i\omega_\mu \tilde{\tau}}. \tag{52} \]

Then,
\[ \mu = \frac{a_{21}}{i\omega_n - a_{22}} \delta = \frac{-i\omega_n - a_{12}}{a_{21}}, \]  
\[ (53) \]

To simplify the calculation, we let \( p_2(\kappa) = \overline{p_1}(\kappa) \) and \( q_2(s) = \overline{q_1}(s) \). Then, we can calculate the real and imaginary parts of \( p_1(\kappa) \) and \( q_1(s) \), respectively. We denote \( \phi = (\phi_1, \phi_2) \) and \( \psi^* = (\psi_1^*, \psi_2^*)^\top \), where

\[ \phi_1(\kappa) = \frac{1}{2} \left[ p_1(\kappa) + p_2(\kappa) \right] \]
\[ = \begin{pmatrix} \cos \omega_n \tau \kappa \\ \sigma_1 \cos \omega_n \tau \kappa + \sigma_2 \sin \omega_n \tau \kappa \end{pmatrix}, \]
\[ (54) \]
\[ \phi_2(\kappa) = \frac{1}{2i} \left[ p_1(\kappa) - p_2(\kappa) \right] \]
\[ = \begin{pmatrix} \sin \omega_n \tau \kappa \\ \sigma_3 \cos \omega_n \tau \kappa + \sigma_4 \sin \omega_n \tau \kappa \end{pmatrix}, \]

for \( \theta \in [-1, 0] \). In here,
\[ \sigma_1 = \frac{a_{21}a_{22}}{\omega_n^2 + a_{22}^2}, \]
\[ \sigma_2 = \frac{a_{21}\omega_n}{\omega_n^2 + a_{22}^2}, \]
\[ \sigma_3 = \frac{a_{21}a_{22}}{\omega_n^2 + a_{22}^2}, \]
\[ \sigma_4 = -\frac{a_{21}\omega_n}{\omega_n^2 + a_{22}^2}. \]

\[ (55) \]

\[ \psi_1^*(s) = \frac{1}{2} \left[ q_1(s) + q_2(s) \right] \]
\[ = \begin{pmatrix} \cos \omega_n \tau s, \frac{a_{11}}{a_{21}} \cos \omega_n \tau s - \frac{\omega_n}{a_{21}} \sin \omega_n \tau s \end{pmatrix}, \]
\[ (56) \]

\[ \psi_2^*(s) = \frac{1}{2i} \left[ q_1(s) - q_2(s) \right] \]
\[ = \begin{pmatrix} -\sin \omega_n \tau s, \frac{\omega_n}{a_{21}} \cos \omega_n \tau s + \frac{a_{11}}{a_{21}} \sin \omega_n \tau s \end{pmatrix}, \]

for \( 0 \leq s \leq 1 \). By (50), we can compute

\[ \psi_1^*(\psi_1) = \overline{\tau}a_{13} \frac{a_{21}}{\omega_n^2 + a_{22}^2} \left[-a_{22} \left( \frac{1}{2} \cos \omega_n \tau + \frac{\sin \omega_n \tau}{2\omega_n} \right) - \frac{1}{2} \omega_n \sin \omega_n \tau \right] + \frac{a_{11}a_{22}}{\omega_n^2 + a_{22}^2}, \]
\[ \psi_1^*(\psi_2) = -\overline{\tau}a_{13} \frac{a_{21}}{\omega_n^2 + a_{22}^2} \left[ \omega_n \left( \frac{1}{2} \cos \omega_n \tau + \frac{\sin \omega_n \tau}{2\omega_n} \right) - \frac{1}{2} a_{22} \sin \omega_n \tau \right] + \frac{a_{11}a_{22}}{\omega_n^2 + a_{22}^2}, \]
\[ \psi_2^*(\psi_1) = \overline{\tau}a_{13} \frac{a_{21}}{\omega_n^2 + a_{22}^2} \left[ \frac{1}{2} a_{22} \sin \omega_n \tau - \omega_n \left( \frac{1}{2} \cos \omega_n \tau - \frac{\sin \omega_n \tau}{2\omega_n} \right) \right] + \frac{a_{22} \omega_n}{\omega_n^2 + a_{22}^2}, \]
\[ \psi_2^*(\psi_2) = \overline{\tau}a_{13} \frac{a_{21}}{\omega_n^2 + a_{22}^2} \left[ \frac{1}{2} \omega_n \sin \omega_n \tau + a_{22} \left( \frac{1}{2} \cos \omega_n \tau - \frac{\sin \omega_n \tau}{2\omega_n} \right) \right] + \frac{\omega_n^2}{\omega_n^2 + a_{22}^2}. \]

\[ (57) \]

We define \( (\psi^*, \phi) = (\psi_{i*}, \phi_i) \) \( (i, n = 1, 2) \) and build a basis
\[ \psi = (\psi_1, \psi_2)^\top = (\psi^*, \phi)^{-1} \psi^*. \]
\[ (58) \]

Clearly, \( (\psi, \phi) = I_2 \) is a second-order identity matrix. Furthermore, we define \( u_k = (y_1^k, y_2^k) \) and \( v \cdot u_k = (v_1, v_2)^\top y_k \) for \( v = (v_1, v_2)^\top \). Thus, the center space of equation (46) is derived by \( P_{C}N \psi^* \), where

\[ P_{C}N \varphi = \psi(\psi^*, \varphi, u_k, \varphi) \cdot u_k, \varphi \in \psi^*, \]
\[ \langle \varphi, u_k \rangle = \left( \langle \varphi, y_1^k \rangle, \langle \varphi, y_2^k \rangle \right)^\top, \]
\[ \langle e, f \rangle = \frac{1}{\pi} \int_0^\pi e_1 f_2 dx + \frac{1}{\pi} \int_0^\pi e_2 f_2 dx, \]

for \( e = (e_1, e_2)^\top \) and \( f = (f_1, f_2)^\top \).
We assume that \( A(\bar{\tau}) \) is the infinitesimal generator induced by the solution of equation (46). Then, equation (41) turns into

\[
U_t = A_\tau U_t + X_\alpha U_t + \alpha_t (U_t, \alpha), \tag{61}
\]

where

\[
X_0 (\kappa) = \begin{cases} 0, & -1 \leq \kappa < 0, \\ 1, & \kappa = 0. \end{cases}
\]

Using equation (59) and the decomposition \( \mathcal{P}_x = P_{CN}\mathcal{P}_x \oplus Q \), the solutions of equation (61) is

\[
U_t = \phi \left( y_1 (t) \right) u_k + \rho (y_1, y_2, \alpha), \tag{63}
\]

where

\[
\begin{pmatrix} y_1 (t) \\ y_2 (t) \end{pmatrix} = (\psi, \langle U_t, u_k \rangle). \tag{64}
\]

The bifurcation direction is based on \( \alpha = 0 \), so the solution of equation (41) is as follows:

\[
U^*_t = \phi \left( y_1 (t) \right) u_k + \rho (y_1, y_2), \tag{65}
\]

where \( \rho (y_1, y_2) = \rho (y_1, y_2, 0) \).

We set \( v = y_1 - iy_2 \) and note that \( \Lambda_1 = \phi_1 + i\phi_2 \). Equation (65) becomes

\[
U^*_{1t} (0) = w_1 (0) \frac{\nu^2}{2} + w_1 (0) \nu \bar{v} + w_2 (0) \frac{\bar{v}^2}{2} + \frac{1}{2} (v + \bar{v}) y_k + o (|v, \bar{v}|^3),
\]

\[
U^*_{2t} (0) = w_1 (0) \frac{\nu^2}{2} + w_1 (0) \nu \bar{v} + w_2 (0) \frac{\bar{v}^2}{2} + \frac{1}{2} (\mu \nu + \bar{\rho} v) y_k + o (|v, \bar{v}|^3),
\]

\[
U^*_{1t} (-1) = w_1 (-1) \frac{\nu^2}{2} + w_1 (-1) \nu \bar{v} + w_2 (-1) \frac{\bar{v}^2}{2} + \frac{1}{2} (\nu e^{-i\omega t} + \bar{\nu} e^{i\omega t}) y_k + o (|v, \bar{v}|^3),
\]

\[
U^*_{2t} (-1) = w_1 (-1) \frac{\nu^2}{2} + w_1 (-1) \nu \bar{v} + w_2 (-1) \frac{\bar{v}^2}{2} + \frac{1}{2} (\mu e^{-i\omega t} + \bar{\mu} e^{i\omega t}) y_k + o (|v, \bar{v}|^3).
\]

This, together with equation (41), indicates that

\[
f_{1*} (U^*_t, 0) = Y_1 \left( \frac{\nu^2}{2} \chi_{20} + v \bar{v} \chi_{11} + \frac{\bar{v}^2}{2} \chi_{02} \right) + \frac{\nu^2 \bar{v}}{2} (\chi_1 y_k + \chi_2 y^3_k) + \cdots,
\]

\[
f_{2*} (U^*_t, 0) = Y_1 \left( \frac{\nu^2}{2} \chi_{20} + v \bar{v} \chi_{11} + \frac{\bar{v}^2}{2} \chi_{02} \right) + \frac{\nu^2 \bar{v}}{2} (\zeta_1 y_k + \zeta_2 y^3_k) + \cdots.
\]
and the corresponding coefficients are as follows:

\[
\chi_{20} = \frac{1}{2} \left( h_{\nu \gamma_1} + h_{\nu \gamma_2} \mu + h_{\nu \gamma_3} \mu e^{-i\omega \tau} + h_{\nu \gamma_3} \mu^2 + h_{\nu \gamma_3} \mu^2 e^{-2i\omega \tau} \right),
\]

\[
\chi_{11} = \frac{1}{4} \left[ 2h_{\nu \gamma_1} + h_{\nu \gamma_2} (\mu + \bar{\mu}) + h_{\nu \gamma_3} \left( \mu e^{-i\omega \tau} + \mu e^{i\omega \tau} \right) + 2h_{\nu \gamma_3} \mu \bar{\mu} + 2h_{\nu \gamma_3} \mu \bar{\mu} \right],
\]

\[
\chi_{02} = \bar{\chi}_{20},
\]

\[
\chi_1 = h_{\nu \gamma_1} \left[ w_{10}^{(1)} (0) + 2w_{11}^{(1)} (0) \right]
\]

\[
+ h_{\nu \gamma_1} \left[ \frac{1}{2} w_{10}^{(1)} (0) \bar{\mu} + w_{11}^{(1)} (0) \mu + \frac{1}{2} w_{20}^{(2)} (0) + w_{11}^{(2)} (0) \right]
\]

\[
+ h_{\nu \gamma_1} \left[ \frac{1}{2} w_{20}^{(1)} (0) \bar{\mu} e^{i\omega \tau} + w_{11}^{(1)} (0) \mu e^{-i\omega \tau} + \frac{1}{2} w_{20}^{(2)} (-1) + w_{11}^{(2)} (-1) \right]
\]

\[
+ h_{\nu \gamma_1} \left[ w_{20}^{(1)} (0) \bar{\mu} + 2w_{11}^{(2)} (0) \mu \right]
\]

\[
+ h_{\nu \gamma_1} \left[ w_{20}^{(1)} (-1) \bar{\mu} e^{i\omega \tau} + 2w_{11}^{(2)} (-1) \mu e^{-i\omega \tau} \right],
\]

\[
\chi_2 = \frac{3}{4} h_{\nu \gamma_1} + \frac{1}{4} h_{\nu \gamma_1 \gamma_2} (\bar{\mu} + 2\mu) + \frac{1}{4} h_{\nu \gamma_1 \gamma_3} \left( 2\mu \bar{\mu} + \mu^2 \right)
\]

\[
+ \frac{1}{4} h_{\nu \gamma_1 \gamma_3} \left( 2\mu e^{-i\omega \tau} + \mu e^{i\omega \tau} \right) + \frac{1}{4} h_{\nu \gamma_1 \gamma_3} \left( \mu^2 e^{-2i\omega \tau} + 2\mu \bar{\mu} \right)
\]

\[
+ \frac{3}{4} h_{\nu \gamma_1 \gamma_3} \mu^2 \bar{\mu} + \frac{3}{4} h_{\nu \gamma_1 \gamma_3} \mu^2 \bar{\mu} e^{-i\omega \tau}.
\]

The constants \( \xi_{20} \) are given by:

\[
\xi_{20} = \frac{1}{2} \left( f_{\nu \gamma_1} + f_{\nu \gamma_2} \mu + f_{\nu \gamma_3} \mu^2 \right),
\]

\[
\xi_{11} = \frac{1}{4} \left[ 2f_{\nu \gamma_1} + f_{\nu \gamma_2} (\mu + \bar{\mu}) + 2f_{\nu \gamma_3} \mu \bar{\mu} \right],
\]

\[
\xi_{02} = \xi_{20},
\]

\[
\xi_1 = f_{\nu \gamma_1} \left[ w_{10}^{(1)} (0) + 2w_{11}^{(1)} (0) \right] + f_{\nu \gamma_2} \left[ w_{20}^{(2)} (0) \bar{\mu} + 2w_{11}^{(2)} (0) \mu \right]
\]

\[
+ f_{\nu \gamma_3} \left[ \frac{1}{2} w_{20}^{(1)} (0) \bar{\mu} + w_{11}^{(1)} (0) \mu + \frac{1}{2} w_{20}^{(2)} (0) + w_{11}^{(2)} (0) \right],
\]

\[
\xi_2 = \frac{3}{4} f_{\nu \gamma_1} + \frac{1}{4} f_{\nu \gamma_1 \gamma_2} (\bar{\mu} + 2\mu) + \frac{1}{4} f_{\nu \gamma_1 \gamma_3} \left( 2\mu \bar{\mu} + \mu^2 \right).
\]
Let \((\rho_1, \rho_2) = \psi_1(0) - i\psi_2(0)\). Then, one can get from equations (40), (69), (70), and (72) that \(g_{20} = g_{11} = g_{02} = 0\), for \(k = 1, 2, 3, \ldots\). If \(k = 0\), then
\[
\begin{align*}
g_{20} &= \tilde{T}_{20}\rho_1 + \tilde{T}_{20}\rho_2, \\
g_{11} &= \tilde{T}_{11}\rho_1 + \tilde{T}_{11}\rho_2, \\
g_{02} &= \tilde{T}_{02}\rho_1 + \tilde{T}_{02}\rho_2, \\
g_{21} &= \tilde{T}_{21}\rho_1 + \tilde{T}_{21}\rho_2,
\end{align*}
\]
(74)
where
\[
\frac{dw}{d\tau} = w_{20}v + w_{11}v + w_{02}v + \cdots, A_\tau w(v, \psi) = A_{\tau} w_{20} + A_{\tau} w_{11} + A_{\tau} w_{02} + \cdots.
\]
Moreover, \(w(v, t, \psi(t))\) satisfies
\[
\frac{dw}{d\tau} = A_\tau w + H(v, \psi),
\]
(77)
in which
\[
H(v, \psi) = H_{20}v^2 + H_{11}v + H_{02}v + \cdots
\]
and
\[
= \Delta F_\psi(U^*_1, 0) - \phi(\psi, \langle\Delta F_\psi(U^*_1, 0), u_k\rangle) \cdot u_k.
\]
(78)
So far, we have given the expressions of \(g_{20}, g_{11}, \text{and} g_{02}\). The value of \(g_{21}\) depends on \(w_{20}(\kappa)\) and \(w_{11}(\kappa)\). From equations (70), one obtains
\[
\begin{align*}
\kappa_1 &= \frac{X_1}{\pi} \int_0^\pi \cos^2\kappa x dx + \frac{X_2}{\pi} \int_0^\pi \cos^4\kappa x dx, \\
\kappa_2 &= \frac{X_1}{\pi} \int_0^\pi \cos^2\kappa x dx + \frac{X_2}{\pi} \int_0^\pi \cos^4\kappa x dx.
\end{align*}
\]
(75)
We substitute equation (68) into the derivative of equation (70) and compare the coefficients with equations (77) and (78); then,
\[
\begin{align*}
\left(2i\omega - A_{\tau}\right)w_{20}(\kappa) &= H_{20}(\kappa), \\
A_{\tau}w_{11}(\kappa) &= -H_{11}(\kappa).
\end{align*}
\]
(79)
From equation (79), we can see that, for \(\kappa \in [-1, 0)\),
\[
H(v, \psi, \kappa) = -\phi(\kappa)v(0)\langle F_\psi(U^*_1, 0), u_k\rangle \cdot u_k
\]
\[
= -\left(\frac{p_1(\kappa) + p_2(\kappa)}{2}\right)\frac{p_1(\kappa) - p_2(\kappa)}{2i} \langle\psi_1(0), \psi_2(0)\rangle
\]
\[
\cdot \langle F_\psi(U^*_1, 0), u_k\rangle \cdot u_k
\]
\[
= \frac{1}{2} \cdot \langle F_\psi(U^*_1, 0), u_k\rangle \cdot u_k
\]
\[
\cdot [p_1(\theta)(\psi_1(0) - i\psi_2(0)) + p_2(\theta)(\psi_1(0) + i\psi_2(0))]
\]
\[
= \frac{1}{2} \left[g_{20}p_1(\kappa) + \bar{g}_{02}p_2(\kappa)\right] \frac{v^2}{2} \cdot u_k
\]
\[
= \frac{1}{2} \left[g_{11}p_1(\kappa) + \bar{g}_{11}p_2(\kappa)\right] v \cdot \psi \cdot u_k + \cdots.
\]
(80)
Therefore, for $\kappa \in [-1, 0)$,

$$
H_{20}(\kappa) = \begin{cases} 
0, & k \in N_0, \\
\frac{1}{2} \left[ g_{20} P_1(\kappa) + \overline{g}_{02} P_2(\kappa) \right] \cdot u_k, & k = 0,
\end{cases}
$$

$$
H_{11}(\kappa) = \begin{cases} 
0, & k \in N_0, \\
\frac{1}{2} \left[ g_{10} P_1(\kappa) + \overline{g}_{11} P_2(\kappa) \right] \cdot u_k, & k = 0,
\end{cases}
$$

By the definition of $A_7$, we get from equation (80) that

$$
\dot{w}_{20}(\kappa) = 2i \omega_n \tau w_{20}(\kappa) + \frac{1}{2} \left[ g_{20} P_1(\kappa) + \overline{g}_{02} P_2(\kappa) \right] \cdot u_k,
$$

$$
\dot{w}_{11}(\kappa) = -\frac{i}{2 \omega_n \tau} \left[ g_{10} P_1(\kappa) + \overline{g}_{11} P_2(\kappa) \right] \cdot u_k,
$$

for $-1 \leq \kappa < 0$. Notice that $P_1(\kappa) = P_1(0) e^{i \omega_n \tau \kappa}$; we, therefore, obtain

$$
\dot{w}_{20}(\kappa) = \frac{i}{2 \omega_n \tau} \left[ g_{20} P_1(\kappa) + \overline{g}_{02} P_2(\kappa) \right] \cdot u_k + E_1 e^{2i \omega_n \tau \kappa},
$$

$$
\dot{w}_{11}(\kappa) = \frac{i}{2 \omega_n \tau} \left[ -g_{10} P_1(\kappa) + \overline{g}_{11} P_2(\kappa) \right] \cdot u_k + E_2,
$$

where $1 \leq \kappa < 0$, and $E_1, E_2 \in \mathbb{R}^2$ denote the constant vectors which will be determined.

By the above discussion and equation (79), we obtain

$$
2i \omega_n E_1 - A_7 E_1 - L_\tau (\tau) E_1 e^{2i \omega_n \tau} = \tau \left( \begin{array}{c} X_{20} \\ \zeta_{20} \end{array} \right) \cos^2 kx,
$$

$$
- A_7 E_2 - L_\tau (\tau) E_2 = \tau \left( \begin{array}{c} X_{11} \\ \zeta_{11} \end{array} \right) \cos^2 kx.
$$

Hence, we let

$$
E_1 = \tau E_3 \left( \begin{array}{c} X_{20} \\ \zeta_{20} \end{array} \right) \cos^2 kx,
$$

$$
E_2 = \tau E_4 \left( \begin{array}{c} X_{11} \\ \zeta_{11} \end{array} \right) \cos^2 kx,
$$

where

$$
E_3 = \left( \begin{array}{cc} 2i \omega_n + d_1 k^2 & -a_{11} -a_{12} -a_{13} e^{-2i \omega_n \tau} \\ -a_{21} & 2i \omega_n + d_2 k^2 - a_{22} \end{array} \right)^{-1},
$$

$$
E_4 = \left( \begin{array}{cc} d_1 k^2 - a_{11} -a_{12} -a_{13} & -a_{21} \\ -a_{21} & d_2 k^2 - a_{22} \end{array} \right)^{-1}.
$$
Theorem 3. The following results are true:

(i) If \( \mu_2 < 0 \) (\( \mu_2 > 0 \)), the Hopf bifurcation is subcritical (supercritical)
(ii) If \( \beta_2 < 0 \) (\( \beta_2 > 0 \)), the bifurcating periodic solutions on the center manifold are orbitally asymptotically stable (unstable)
(iii) If \( T_2 < 0 \) (\( T_2 > 0 \)), the period of bifurcating periodic orbits decreases (increases)

Remark 5. The measures that characterize tipping phenomena are highly challenging subjects of current research. Theorem 3 gives a measure to evaluate the bifurcated oscillators induced by tipping events. The direction of the tipping and the stability and period of periodic solutions can be ascertained by three indicative parameters. A thorough understanding of the internal mechanisms of population evolution will allow us to more accurately predict the occurrence of tipping.

5. Numerical Simulations

We supply several numerical examples for system (3) to testify to the main results in Sections 3 and 4. In numerical simulations, \( t_0 \) of system (3) corresponds to \( t_0 \) of system (6).

First, we take into account the case of no time delay for system (3). In order to investigate the Turing instability of system (3), we set \( \bar{d}_1 = 0, \bar{d}_2 = 0, \tau = 0, r = 0.8, m = 0.8, K = 1, A = 0.5, C = 1, h = 0.7143, \) and \( s = 0.672 \) and the initial condition \((0.08, 0.1)\).

We change the diffusion coefficients to \( \bar{d}_1 = 0.0088 \) and \( \bar{d}_2 = 0.04 \) and other parameters are consistent with those in Figure 1. Assumption (H2) is not satisfied, and the trajectories of system (3) still tend to \( E^* \) as displayed in Figure 2. Furthermore, we set \( \bar{d}_1 = 0.0088 \) and \( \bar{d}_2 = 0.288 \). It can be verified that (H2) is reached. From Theorem 1, \( E^* \) begins to oscillate in the spatial region and the stripe patterns appear, which are depicted in Figures 3 and 4, respectively. This implies that system (3) transitions into an unstable state and the Turing instability-induced tipping occurs.

Next, we discuss the influence of time delays on tippings. We fix \( r = 0.8, m = 0.8, K = 1, A = 0.5, C = 1, h = 1.25, s = 0.384, \tau = 6.25, \bar{d}_1 = 1.6, \bar{d}_2 = 1.6, \) and \( l = 1 \) and ensure that (H4) is reached. The trajectories go into \( E^* \) for all time delay (see Figure 5).

We select \( r = 0.8, m = 0.8, K = 1, A = 0.5, C = 1, h = 0.7143, s = 0.672, \bar{d}_1 = 1.6, \bar{d}_2 = 1.6, \) and \( l = 1 \). It follows that \( t_0' = 22.1344 \) is the tipping point. The trajectories converge at the equilibrium point \( E^* \) when \( \tau' < t_0' \) (see Figure 6), while the trajectories go away from \( E^* \) and the Hopf bifurcation appears as \( \tau' \) passes through the tipping point \( t_0' \) (see Figure 7). This implies that the Hopf bifurcation-induced
Figure 2: Spatiotemporal behaviors of populations of system (3) with $d_1 = 0.0088$, $d_2 = 0.04$, $\tau' = 0$, $r = 0.8$, $m = 0.8$, $K = 1$, $A = 0.5$, $C = 1$, $h = 0.7143$, and $s = 0.672$.

Figure 3: Spatiotemporal behaviors of populations of system (3) with $d_1 = 0.0088$, $d_2 = 0.288$, $\tau' = 0$, $r = 0.8$, $m = 0.8$, $K = 1$, $A = 0.5$, $C = 1$, $h = 0.7143$, and $s = 0.672$.

Figure 4: Turing patterns.
tipping occurs. These numerical results agree well with Theorem 2.

Furthermore, we can calculate the indicative parameters $\mu_2 = 0.2353 > 0$, $\beta_2 = -9.3536 < 0$, and $T_2 = -0.8550 < 0$ in (66), when $r' = \tau_0'$. Hence, the bifurcation that induces the tipping is of the supercritical type, and the periodic oscillator driven by the Hopf bifurcation is orbitally stable.

In addition, we set $r = 0.8$, $m = 0.8$, $K = 1$, $A = 0.5$, $C = 1$, $h = 1.25$, and $s = 0.384$. We can compute two critical values, $4.0855$ and $11.7601$, which are two tipping points. When $r' < 4.0855$, the orbits converge to equilibrium $E^*$ as illustrated in Figure 8, and when $r'$ crosses the tipping point of 4.0855, the orbits turn away from $E^*$ and the Hopf bifurcation emerges as displayed in Figure 9. Furthermore, when $r'$ continues to increase and passes through the other tipping point 11.7601, $E^*$ returns to the stability as depicted in Figure 10. It is found that the spatiotemporal evolution of system (3) switches between stable focus and limit cycles several times with the increase in time delay. This reveals that there may be many tipping points in ecological competition systems, and the tipping may occur many times as the fear delay increases.

Finally, we explore the influence of fear factors on the densities of species. We fix $r = 0.8$, $m = 0.8$, $K = 1$, $A = 0.5$, $h = 0.7143$, and $s = 0.672$, and let the fear parameter $C$ vary. Figure 11 illustrates that both predator and prey densities are in decline with the increase of fear parameter $C$. 

\vspace{0.5cm}

Figure 5: Waveform plots of system (3) with $\bar{d}_1 = 1.6$, $\bar{d}_2 = 1.6$, $r = 0.8$, $m = 0.8$, $K = 1$, $A = 0.5$, $C = 1$, $h = 1.25$, and $s = 0.384$.

Figure 6: Waveform plots of system (3) with $\bar{d}_1 = 1.6$, $\bar{d}_2 = 1.6$, $r = 0.8$, $m = 0.8$, $K = 1$, $A = 0.5$, $C = 1$, $h = 0.7143$, and $s = 0.672$. $E^*$ is local asymptotical stable, where $r' = 14 < \tau_0' = 22.1344$. 

\vspace{0.5cm}

Complexity 15
Figure 7: Waveform plots of system (3) with $d_1 = 1.6$, $d_2 = 1.6$, $r = 0.8$, $m = 0.8$, $K = 1$, $A = 0.5$, $C = 1$, $h = 0.7143$, and $s = 0.672$. The periodic oscillations occur, where $\tau' = 24 > \tau_0' = 22.1344$.

Figure 8: Waveform plots of system (3) with $d_1 = 1.6$, $d_2 = 1.6$, $r = 0.8$, $m = 0.8$, $K = 1$, $A = 0.2$, $C = 1$, $h = 1.4286$, and $s = 0.336$. $E^*$ is locally asymptotically stable, where $\tau' = 2 < 4.0855$.

Figure 9: Waveform plots of system (3) with $d_1 = 1.6$, $d_2 = 1.6$, $r = 0.8$, $m = 0.8$, $K = 1$, $A = 0.2$, $C = 1$, $h = 1.4286$, and $s = 0.336$. The periodic oscillations occur, where $4.0855 < \tau' = 7 < 11.7601$. 

16 Complexity
6. Conclusion

Population ecosystems presenting a tipping point are widely circulated, and it is crucial to dive into the tipping mechanism and develop tools for predicting the occurrence of tipping points. To achieve these goals, this article puts forward a predator-prey competitive model with fear delays and spatial diffusion and focuses on the analysis of bifurcation-induced tipping events. By means of bifurcation theory, the tipping points of the evolution of population size can be predicted. Mechanisms, processes, and measures of tipping are explored. For the case without time delays, the Turing instability-induced tipping is studied, and the specific condition under which Turing instability occurs is derived. For the case of time delays, the Hopf bifurcation-induced tipping is analyzed, and the expression of tipping points is given clearly. This makes it possible to forecast the appearance of tipping points. In order to understand the internal mechanism of tipping more deeply, we can determine the direction of Hopf bifurcations by using the center manifold theorem for PFDEs. It is also found that many tipping points may exist in ecological competition systems, and the tipping occurs many times as the fear delay increases. Our future work will be devoted to the factor of defense and cross-diffusion in predator-prey systems.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (nos. 62073172 and 61573194), the Natural Science Foundation of Jiangsu Province of China (no. BK20221329), and the Open Research Project of the State Key Laboratory of Industrial Control Technology of Zhejiang University (no. ICT2022B43).

References


