Research Article

A New Robust Adaptive Control Method for Complex Nontriangular Nonlinear Systems

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The existing research studies on adaptive control frequently introduce many parameter estimations and lead to a complicated controller. This paper investigates the robust regulation issue for high-order system and plants to raise a new approach for adaptive control. Specifically, the considered system has odd system power, nontriangular form, and external disturbance. By introducing the transformations of a parameter estimation, the studied system is transformed into a new dynamic system. By employing fuzzy systems and some inequality skills, the appropriate bounds of nonlinear terms are established. Based on the adaptive method and homogeneous control, a recursive control design algorithm is provided to construct a new adaptive controller, which dominates those uncertain bounds and guarantees that the closed-loop system is semiglobally uniformly ultimately bounded (SUUB). The constructed controller employs only one adaptive law and has a much simpler form. Simulation examples verify the validness of the presented method.

1. Introduction

In engineering, plenty of practical plants [1, 2] suffer from various disturbances, which produce bad influences to system responses and stability. For such kind of systems, robust control approaches played an important role in stabilizing the system [3]. On the other hand, multiple uncertainties exist in nonlinear systems and often add the obstacles to design the controllers [4, 5]. To overcome the difficulty, it is necessary to employ an effect tool, i.e., the adaptive technique, which mainly utilizes the idea of certainty equivalence [6]. In addition, some nonlinear systems may have nontriangular form [7, 8], which makes the traditional methods such as the backstepping method and the sliding mode approach difficult to apply. Since different nonlinear systems own their special structures, it is unable to present a universal control strategy to regulate all systems [9]. Up to now, there still exist many challenging control issues for nonlinear systems.

Controls of nonlinear systems have been a hot topic in recent years [10–12]. Scholars mainly employed the idea of state feedback or output feedback during the design procedure. Specifically, in control design steps, state feedback control utilizes all the system state signals, while output feedback control uses partial measurable signals and the signals obtained from the constructed state observer. For systems with uncertainties, many excellent strategies were reported over the past years. For instance, to achieve the asymptotic tracking control, the authors in [13] presented an adaptive \( \sigma \)-modification control method for systems that contained uncertain control directions and the authors in [14] raised a novel adaptive recursive RISE control for system with mismatched uncertainties. To obtain the asymptotic prescribed tracking performance, the authors in [15] proposed a RBFNN-based adaptive control algorithm for complicated systems with heavy modeling uncertainties. To study the finite-time control problem, the authors in [16] provided a fuzzy based event-triggered controller for uncertain systems, and the authors in [17] studied quantized systems and proposed an finite-time controller using the adaptive strategy. When the system involves unmeasurable states, the authors in [18] raised an adaptive neural output feedback scheme, and the authors in [19] gave a new extended-state-observer based adaptive control.
strategy. The authors in [20] further studied stochastic systems using the dynamic gain and the state observer. For systems with external disturbances, some robust control approaches were employed, see [21] and references therein.

When the system powers are odd numbers, the related control problems are more difficult. One reason lies in that many existing approaches are not applicable since the system is not feedback linearization and the upper bounds of nonlinear terms are difficult to identify. To tackle the control problems of systems with odd system powers, the authors in [22] proposed the adding a power integrator (API) scheme. Later, the domination idea and homogenous properties were applied to solve the regulation of homogeneous systems [23]. Recently, these methods have been extended to solve many nonlinear systems. Particularly, by employing API scheme and dynamic gain strategy, the authors in [24] designed a stable controller for time-delay systems. Utilizing homogeneous domination (HD) approach, the authors in [25] studied nonlinear systems and constructed a homogeneous controller to make the system globally stable. For some uncertain systems, the authors in [26, 27] considered systems with different structures and designed the associated controller. Actually, control issues of nontriangular systems are more challenging. So far, intelligent methods have been extended to solve many nonlinear systems. Particularly, by employing API scheme and dynamic system is obtained. By choosing an appropriate controller, a control input is designed without using the information of the unknown functions.

### 2. Problem Formulation

Consider nontriangular nonlinear system

\[
\begin{align*}
\dot{x}_1 &= c_1(t)x_1^p + f_1(\theta, x_1, \ldots, x_n) + w(t), \\
\dot{x}_2 &= c_2(t)x_2^p + f_2(\theta, x_1, \ldots, x_n) + w(t), \\
& \quad \vdots \\
\dot{x}_n &= c_n(t)x_n^p + f_n(\theta, x_1, \ldots, x_n) + w(t),
\end{align*}
\]

where \( x = [x_1, \ldots, x_n]^\top \in \mathcal{R} \) is the state vector, \( u \in \mathcal{R} \) denotes the system input, \( \theta \) is the unknown parameter vector, \( w(t) \) is the disturbance, \( f_i(\cdot) \) is the continuous function, and \( p \) is the odd system power. For \( 1 \leq i \leq n \), \( c_i(t) \) satisfies \( c_0 \leq c_i(t) \leq c_m \) where \( c_0 > 0 \) and \( c_m > 0 \) are known constants. \( |w(t)| \leq w_0 \), where \( w_0 > 0 \) is a constant.

**Remark 1.** It shows that system (1) has odd system power and is in a nontriangular form. It cannot be regulated with the conventional methods like the backstepping control and sliding control for triangular systems. The existing methods for odd system power such as the API scheme and the HD approach are also inapplicable since it is difficult to deal with those nontriangular uncertain nonlinear terms. In this work, by introducing new transformations, we will transform the system into a new one. Then, we propose a fuzzy approximation based HD method for control design.

The following Lemmas are needed later.

**Lemma 2** (See [25]). The inequality \(|\kappa G^p H^{|\frac{d+1}{c}} + \lambda/\delta + \lambda(\delta/c(\delta + \lambda))|^{\delta/\lambda}|\kappa^{\delta/\lambda}|H_{|\frac{d+1}{c}}| \) holds, where \( \kappa, G, H \in \mathcal{R} \) are any constants, \( \delta, \lambda, c \in \mathcal{R} \) are positive constants.

**Lemma 3** (See [27]). For any constant \( \delta_0 > 0 \) and a continuous function \( \pi_0(\theta, X) \) defined on the compact set \( \Lambda \), a fuzzy system \( \Psi^\top \mathcal{L}_1(X) \) exists and guarantees \( \sup_{x \in \Lambda} |\pi_0(\theta, X) - \Psi^\top \mathcal{L}_1(X)| \leq \delta_0 \).

### 3. Main Results

Introduce the transformations

\[
\eta_i = \frac{x_i}{H^r(\bar{\theta}(t))} - \frac{u(t)}{H^{r+1}(\bar{\theta}(t))}, \quad i = 1, \ldots, n,
\]

where \( r_1 = r_0 \geq 1 \), \( r_{i+1} = r_i + r_0 \). \( \bar{\theta}(t) \) is the estimation of parameter \( \theta \) to be defined later and satisfies \( \bar{\theta}(t) \geq 1 \), \( H(\bar{\theta}) \) is selected such that \( H(\bar{\theta}) \geq 1 \), \( dH(\bar{\theta})/d\bar{\theta} > 0 \). Then, we transform system (1) into

\[
\begin{align*}
\dot{\eta}_1 &= H^r(\bar{\theta})c_1(t)\eta_1^p - \frac{r_1}{H(\bar{\theta})} \frac{dH(\bar{\theta})}{d\bar{\theta}} \delta_{\eta_1} + F_1(\cdot) + W_1(\cdot), \\
\dot{\eta}_n &= H^r(\bar{\theta})c_n(t)\eta_n^p - \frac{r_n}{H(\bar{\theta})} \frac{dH(\bar{\theta})}{d\bar{\theta}} \delta_{\eta_n} + F_n(\cdot) + W_n(\cdot),
\end{align*}
\]
where $F_j(\cdot) = H^{-r_j}(\hat{\theta})f_j$, $W_j(\cdot) = H^{-r_j}(\hat{\theta})w(t)$, $1 \leq j \leq n$.

**Remark 4.** In (3), the adaptive law should be designed such that $\hat{\theta} \geq 0$, $H(\hat{\theta}) \geq 1$. In this way, it can be utilized to dominate some nonlinear terms raised during the control design. For details, see the following.

Next, we give the following results.

**Theorem 5.** System (1) has the adaptive controller

\[
\begin{align*}
\dot{\hat{\theta}} &= \frac{\lambda}{H(\hat{\theta})} \sum_{j=1}^{n} S_j^2, \quad \hat{\theta}(0) \geq 1, \\
\dot{u} &= -H^{-r}(\hat{\theta})\beta_p S_p,
\end{align*}
\]

where $r_m, q, \lambda$ and $\beta_j$ are constants to be specified, $\hat{\theta}$ is estimation of an unknown parameter $\theta$ defined later, $S_1 = \eta_1$, $S_j = \eta_j - \eta_{j-1}^*$, $\eta_j^* = -\beta_{j-1} S_{j-1}, j = 2, \ldots, n$. It ensures that all system solutions are semiglobally uniformly ultimately bounded (SUUB).

**Proof.** We first provide the control design procedure. Later, we perform the stability analysis. \qed

**Part 6.** Control design procedure.

**Initial Step 7.** Choosing the function $V_1 = 1/(p + 1)S_i^{p+1}$, we get

\[
V_1 = S_i^p \left( H^{r_i}(\hat{\theta})c_1(t)\eta_2^p - \frac{r_i}{H(\hat{\theta})} dH(\hat{\theta}) \right) \hat{\theta}_1 + F_1(\cdot) + W_1(\cdot)
\]

\[
= H^{r_i}(\hat{\theta})c_1(t)S_i^p (\eta_i^p - \eta_2^{*p}) + H^{r_i}(\hat{\theta})c_1(t)S_i^{p^2} \eta_2^{*p} - \frac{r_i}{H(\hat{\theta})} dH(\hat{\theta}) \hat{\theta}_1 S_i^{p+1} + S_i^p (F_1 + W_1).
\]

By Lemma 3, there exists a fuzzy system $\Psi^T L_1$ such that $f_1 = \Psi^T L_1 + \delta_1$, $|\delta_1| \leq \delta_0$. Then, we obtain

\[
S_i^p F_1 = H^{-r_i}(\hat{\theta})S_i^p (\Psi^T L_1 + \delta_1)
\]

\[
\leq H^{-r_i}(\hat{\theta})||S_i^p|| (||\Psi^T L_1|| + ||\delta_1||)
\]

\[
\leq H^{-r_i}(\hat{\theta}) \left( a_i \| \Psi_1^T 2S_i^{2p} + \frac{1}{4a_i} \right) L_i^2 + a_i S_i^{p^2} + \frac{1}{4a_i} \delta_0^2.
\]

(6)

\[
S_i^p W_1 = H^{-r_i}(\hat{\theta})S_i^p w_1 \leq H^{-r_i}(\hat{\theta}) \left( a_i S_i^{p^2} + \frac{1}{4a_i} w_0^2 \right).
\]

Choosing $\eta_2^p = -(na_1/c_0)^{1/p} S_i = -\beta_1 S_i$ and substituting (6) and (7) into (5), we have

\[
V_1 \leq -(n-k+2)p_{k-2} \sum_{j=1}^{k} H^{r_j}(\hat{\theta})a_j S_j^{p^2} - \sum_{j=1}^{k} h_{k-1} \frac{dH(\hat{\theta})}{d\hat{\theta}} S_j^{p+1} + H^{r_i}(\hat{\theta})c_1(t)S_i^p (\eta_i^p - \eta_2^{*p})
\]

\[
+ H^{r_i}(\hat{\theta})c_1(t)S_i^{p^2} \eta_2^{*p} - \frac{r_i}{H(\hat{\theta})} dH(\hat{\theta}) \hat{\theta}_1 S_i^{p+1} + S_i^p (F_1 + W_1).
\]

Choosing $\eta_2^p = -(na_1/c_0)^{1/p} S_i = -\beta_1 S_i$ and substituting (6) and (7) into (5), we have

\[
V_1 \leq -(n-k+2)p_{k-2} \sum_{j=1}^{k} H^{r_j}(\hat{\theta})a_j S_j^{p^2} - \sum_{j=1}^{k} h_{k-1} \frac{dH(\hat{\theta})}{d\hat{\theta}} S_j^{p+1} + H^{r_i}(\hat{\theta})c_1(t)S_i^p (\eta_i^p - \eta_2^{*p})
\]

\[
+ H^{r_i}(\hat{\theta})c_1(t)S_i^{p^2} \eta_2^{*p} - \frac{r_i}{H(\hat{\theta})} dH(\hat{\theta}) \hat{\theta}_1 S_i^{p+1} + S_i^p (F_1 + W_1).
\]

Step $k$ ($k \geq 2$). Suppose that for step $k - 1$ there exists a continuous function $V_{k-1}$ and transformations $S_j = \eta_j - \eta_j^*$, $\eta_j^* = -\beta_{j-1} S_{j-1}$, $j = 2, \ldots, k$ which satisfy

\[
\dot{V}_{k-1} \leq -(n-k+2)p_{k-2} \sum_{j=1}^{k} H^{r_j}(\hat{\theta})a_j S_j^{p^2} - \sum_{j=1}^{k} h_{k-1} \frac{dH(\hat{\theta})}{d\hat{\theta}} S_j^{p+1} + H^{r_i}(\hat{\theta})c_1(t)S_i^p (\eta_i^p - \eta_2^{*p})
\]

\[
+ H^{r_i}(\hat{\theta})c_1(t)S_i^{p^2} \eta_2^{*p} - \frac{r_i}{H(\hat{\theta})} dH(\hat{\theta}) \hat{\theta}_1 S_i^{p+1} + S_i^p (F_1 + W_1).
\]
In this step, choose \( V_k = \rho_{k-1} V_{k-1} + 1/(p + 1)S_k^{p+1} \), which leads to

\[
\dot{V}_k = \rho_{k-1} \dot{V}_{k-1} + S_k^p (\dot{\eta}_k - \dot{\eta}_k^*)
\leq -(n - k + 2)\rho_0 \cdots \rho_{k-1} \sum_{j=1}^{k-1} H^e(\bar{\theta}) a_j S_j^p - \sum_{j=1}^{k-1} \rho_{k-1} \dot{\eta}_k - \frac{1}{H(\bar{\theta})} \frac{dH}{d\bar{\theta}} \delta S_j^{p+1}
+ \rho_{k-1} c_k H^e(\bar{\theta}) S_k^{p-1} (\eta_k^p - \eta_k^*) + \sum_{i=1}^{k-1} \rho_{i} \cdots \rho_{k-1} \left( \sum_{j=1}^{i} H^{-r_j}(\bar{\theta}) \left( 2a_j + a_i \|\psi_j\|^2 \right) S_j^p \right)
+ \sum_{j=1}^{i} H^{-r_j}(\bar{\theta}) a_j \dot{D}_j + S_k^p \left( H^e(\bar{\theta}) c_k (\eta_k^p - \frac{r_k}{H(\bar{\theta})} \frac{dH}{d\bar{\theta}} \delta \eta_k + F_k + W_k - \sum_{j=1}^{k-1} \frac{\partial \eta_k^*}{\partial \eta_j} \dot{\eta}_j \right).
\]

By Lemma 2, we obtain

\[
\rho_{k-1} c_k H^e(\bar{\theta}) S_k^{p-1} (\eta_k^p - \eta_k^*) \leq H^e(\bar{\theta}) \phi_s \eta^p_k + H^e(\bar{\theta}) \frac{\rho_0 \cdots \rho_{k-1} \delta^2}{3},
\]

where \( \phi_{si} > 0 \) is a constant. Noting that \( H(\bar{\theta}) \geq 1 \), we obtain

\[
-S_k^p \frac{r_k}{H(\bar{\theta})} \frac{dH(\bar{\theta})}{d\bar{\theta}} \delta \eta_k \leq -r_k b_0 \frac{1}{H(\bar{\theta})} \frac{dH(\bar{\theta})}{d\bar{\theta}} \delta S_k^{p+1} + B_k \frac{1}{H(\bar{\theta})} \frac{dH(\bar{\theta})}{d\bar{\theta}} \delta S_k^{p+1},
\]

where \( 0 < b_0 < 1 \) and \( B_k > 0 \) are constants. By Lemma 3, there is a fuzzy system \( \Psi_k^T L_k \) such that \( f_k = \Psi_k^T L_k + \delta_k \), \( |\delta_k| \leq \delta_0 \). By Lemma 2, we get \( S_k^p \Psi_k^T L_k \leq a_k \|\Psi_k\|^2 S_k^p + \|

\[
S_k^p (F_k + W_k) \leq H^{-r_k}(\bar{\theta}) \left( 2a_k + a_i \|\psi_i\|^2 \right) S_k^p + H^{-r_k}(\bar{\theta}) \frac{D_k}{4a_k},
\]

where \( a_k \) is a positive constant, \( D_k = \|L_k\|^2 + \delta_0^2 + w_0^2 \). Similarly, using Lemma 3, it is deduced that

\[
-S_k^p \sum_{j=1}^{k-1} \frac{\partial \eta_j^*}{\partial \eta_j} \dot{\eta}_j \leq H^e(\bar{\theta}) \phi_{\ell} S_k^p + H^e(\bar{\theta}) \frac{2\rho_0 \cdots \rho_{k-1}}{3} \sum_{j=1}^{k-1} S_j^p + \frac{1}{H(\bar{\theta})} \frac{dH(\bar{\theta})}{d\bar{\theta}} \delta S_k^{p+1}
+ B_{\ell} \frac{1}{H(\bar{\theta})} \frac{dH(\bar{\theta})}{d\bar{\theta}} \sum_{j=1}^{k-1} S_j^{p+1} + \sum_{j=1}^{k-1} H^{-r_j}(\bar{\theta}) \left( 2a_j + a_i \|\psi_j\|^2 \right) S_j^p + \frac{1}{H(\bar{\theta})} \frac{dH(\bar{\theta})}{d\bar{\theta}} \delta S_k^{p+1},
\]

where \( \phi_{\ell} \) and \( B_{\ell} \) are positive constants, \( D_i = 1/4(\|L_i\|^2 + \delta_0^2 + w_0^2) \). Considering that \( \eta_{k+1}^p = \eta_{k+1}^p - \eta_{k+1} + \eta_{k+1} \) and choosing \( \eta_{k+1}^* = -((n - k + 1)a_k + \phi_{k+1} + \phi_{\ell}/c_0)^{1/p} S_k \), it follows that
\[ V_k \leq -(n - k + 1)\rho_0 \cdots \rho_{k-1} \sum_{j=1}^{k} H^e(\bar{\theta}) a_j S_j^2 - h_k \sum_{j=1}^{k} \frac{1}{H(\bar{\theta})} \frac{dH(\bar{\theta})}{d\bar{\theta}} S_j^{p+1} \]

\[ + c_k H^e(\bar{\theta}) S_j^p \left( \eta_{k+1}^p - \eta_{k+1}^* \right) + \sum_{j=1}^{k-1} \rho_i \cdots \rho_{k-1} \left( \sum_{j=1}^{i} H^{-r_j}(\bar{\theta}) \left( 2a_j + a_j \|\Psi\|_2^2 \right) S_j^p \right) \]

\[ + \sum_{j=1}^{i} H^{-r_j}(\bar{\theta}) a_j^{-1} D_j \]  

(15)

where \( h_k = \min \{ \rho_{k-1}, h_{k-1} - B_{k1} - B_{k2} \rho_{k-1}/2 \} \). This completes the recursive design.

In the last step, we choose \( V_n = \rho_{n-1} V_{n-1} + 1/p + 1 S_n^{p+1} + \rho/2\lambda\bar{\theta}^2 \), where \( \lambda > 0 \) is a constant, \( \rho = \min_{1 \leq j \leq n} \{ \rho_0 \cdots \rho_{n-1}a_j \} \), \( \bar{\theta} = \bar{\theta} - \theta \). Using the similar procedure, we obtain the control input

\[ V_n \leq - \rho_0 \cdots \rho_{n-1} \sum_{j=1}^{n} H^e(\bar{\theta}) a_j S_j^2 - h_n \sum_{j=1}^{n} \frac{1}{H(\bar{\theta})} \frac{dH(\bar{\theta})}{d\bar{\theta}} S_j^{p+1} \]

\[ + \sum_{j=1}^{n} \rho_{1} \cdots \rho_{n-1} \left( \sum_{j=1}^{i} H^{-r_j}(\bar{\theta}) \left( 2a_j + a_j \|\Psi\|_2^2 \right) S_j^p \right) \]

\[ + \sum_{j=1}^{n} H^{-r_j}(\bar{\theta}) a_j^{-1} D_j \]  

(17)

Noting that

\[ \sum_{j=1}^{n} \rho_{1} \cdots \rho_{n-1} \sum_{j=1}^{i} H^{-r_j}(\bar{\theta}) \left( 2a_j + a_j \|\Psi\|_2^2 \right) S_j^p \]

\[ = H^{-q}(\bar{\theta}) \sum_{j=1}^{n} \rho_{1} \cdots \rho_{n-1} \left( 2a_1 S_1^2 + \cdots + 2a_n S_n^2 \right) \]

\[ = H^{-q}(\bar{\theta}) \rho_1 \cdots \rho_{n-1} 2a_1 S_1^2 \]

\[ + H^{-q}(\bar{\theta}) \rho_1 \cdots \rho_{n-1} 2a_2 S_2^2 \]

\[ + \cdots + H^{-q}(\bar{\theta}) \rho_1 \cdots \rho_{n-1} 2a_n S_n^2 \]

\[ = H^{-q}(\bar{\theta}) \left( \sum_{j=1}^{n} \rho_{1} \cdots \rho_{n-1} \right) 2a_n S_n^2, \]

(18)

where \( q = \min_{1 \leq j \leq n} \{ r_j \} \). Defining \( A = \max_{1 \leq j \leq n} \{ \sum_{j=1}^{n} \rho_{1} \cdots \rho_{n-1} \} \), \( D = \max_{1 \leq j \leq n} \{ \sum_{j=1}^{n} \rho_{1} \cdots \rho_{n-1} a_j^{-1} D_j \} \), we have

\[ V_n \leq \rho \left( 1 - \varepsilon \right) \sum_{j=1}^{n} H^e(\bar{\theta}) S_j^2 \]

\[ - \rho \left( e H^e(\bar{\theta}) - \frac{\bar{\theta}}{H^q(\bar{\theta})} \right) \sum_{j=1}^{n} S_j^2 \]

\[ + \rho \left( \frac{\bar{\theta}}{H^q(\bar{\theta})} \right) \sum_{j=1}^{n} S_j^2 \]

(20)

where \( 0 < \varepsilon < 1 \) is a constant. Choosing \( H(\bar{\theta}) = (\bar{\theta}/a)^{1/r_{e+q}} \) and
\[
\dot{\hat{\theta}} = \frac{\lambda}{H^q(\hat{\theta})} \sum_{j=1}^{n} S_j^{2p}, \quad \hat{\theta}(0) \geq 1, \tag{21}
\]

we obtain
\[
\dot{V}_n \leq -\overline{\rho} \sum_{j=1}^{n} S_j^{2} + \overline{D}, \tag{22}
\]

where \(\overline{\rho} = \rho (1 - \epsilon)\) and \(\overline{D} = D\) are constants.

**Part 8. Stability analysis.** Define the set \(\Lambda = \{S \mid \sum_{j=1}^{n} S_j^{2p} \leq \overline{D}/\rho\}\), where \(S = [S_1, \ldots, S_n]^T\). From (22), it follows that \(S\) will stay in \(\Lambda\) for all \(S \in \Lambda\). For \(S\) which satisfies \(\sum_{j=1}^{n} S_j^{2p} \leq \overline{D}/\rho\), \(S\) will converge into the set \(\Lambda\). Hence, the vector \(S\) is bounded. Actually, \(\hat{\theta}(t)\) is also bounded on \([0, +\infty)\). We prove it by contradiction. Suppose that there is a finite time \(T\) which satisfies \(\lim_{t \to \infty} \hat{\theta}(t) = +\infty\). Then, integrate both sides of (21) and noting that \(H^q(\hat{\theta}) \geq 1\), we obtain
\[
\hat{\theta}(T) - \hat{\theta}(0) = \int_{0}^{T} \dot{\hat{\theta}}(l) dl = \int_{0}^{T} \frac{\lambda}{H^q(\hat{\theta}(l))} \sum_{j=1}^{n} S_j^{2p}(l) dl \leq \lambda \int_{0}^{T} \sum_{j=1}^{n} S_j^{2p}(l) dl < \lambda_0,
\]

where \(\lambda_0 > 0\) is a constant. Hence, \(\hat{\theta}(T) \leq \hat{\theta}(0) + \lambda_0\), which is in contradiction with \(\lim_{t \to \infty} \hat{\theta}(t) = +\infty\). Therefore, \(\hat{\theta}(t)\) is bounded. Utilizing \(S_1 = \eta_1, S_j = \eta_j - \beta_j - S_{j-1}, j = 2, \ldots, n\), it yields that \(\eta_j, 1 \leq i \leq n\) are bounded. By the definition of \(H^q(\hat{\theta})\) and the boundedness of \(\hat{\theta}\), we know that \(x_i, 1 \leq i \leq n\) are bounded. So, for any initial condition, all the system signals are bounded. Thus, the system is SUUB.

**Remark 6.** This paper presents an adaptive control strategy, which guarantees a SUUB stability result. The advantages of the result are as follows: (1) the result is applicable to systems that require less stronger conditions. In many existing results, the nonlinear conditions should satisfy some growing conditions, such as linear growing condition in [28], high-order growing condition [26], and homogeneous growing condition [23]. These conditions are somewhat stronger. Also, the systems should satisfy the triangular form. In this work, the nonlinear functions can be in a more general form which covers all those conditions. Besides, the considered system has a nontriangular structure. (2) The result is achieved via a much simpler adaptive controller. Different from the methods [2, 6, 24], this paper employs one adaptive law and does not introduce many complicated nonlinear terms in the virtual controllers to dominate the uncertainties in each steps. Instead, we utilize the adaptive law to dominate them in the last design steps. Hence, the presented adaptive controller has a much simpler form. (3) The result can be extended to asymptotic stability under some nonlinear growing conditions. For instance, suppose that the disturbance \(w = 0\) and the functions satisfy \(f_i \leq \hat{\theta}_{n+1} |x_i|^{p_i}\).

Following the similar deductions, we can obtain \(S_j^{2p} \leq H^{q-\alpha}(\hat{\theta}) S_j^{2p} + H^{q-\alpha}(\hat{\theta}) \overline{D} \sum_{i=1}^{n} S_i |x_i|^{p_i}\), where \(\alpha > 0, C_k > 0\) are constants. Following the similar design steps, we can design the adaptive controller such that the derivative of the Lyapunov function satisfies \(\dot{V}_n \leq -\overline{\rho} \sum_{j=1}^{n} S_j^{2p}\). By the Lyapunov stability theorem, we can show that the system is globally asymptotically stable.

### 4. Simulation Example

**Example 1.** We study the single-link robot system (see [29]):

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3 - Bx_2 - g \sin(x_1) + \mu(x_1), \\
\dot{x}_3 &= u - \frac{K}{M} x_2 - \frac{L}{M} x_3,
\end{align*}
\]

where \(B, K, L\) are all unknown parameters, \(\mu(x_1) = \sin(x_1)\). With the help of the method in Theorem 5, we obtain the adaptive controller

\[
\begin{align*}
\dot{u} &= -H^{q-\alpha}(\hat{\theta}) \beta_3 S_3, \\
\dot{\hat{\theta}} &= \frac{\lambda}{H^{q}(\hat{\theta})} \sum_{j=1}^{n} S_j^{2}, \quad \hat{\theta}(0) = 1,
\end{align*}
\]

where \(S_1 = x_1/H^{q}(\hat{\theta}), \quad S_2 = x_3/H^{q}(\hat{\theta}) + \beta_1 S_1, \quad S_3 = x_3/H^{q}(\hat{\theta}) + S_2\). In the simulation, \(H(\hat{\theta}) = (\hat{\theta}/e)^{1/\kappa}\), the system parameters are selected as \(M = 1, L = 5, B = 1, g = 10, \text{ and } K = 10\). The design parameters are \(q = 1, e = 2/3, r_0 = r_1 = 1, r_2 = 2, r_3 = 3, r_4 = 4, \beta_1 = 1.5, \beta_2 = 2, \beta_3 = 7, \text{ and } \lambda_0 = 10\). The initial conditions are given as \(\xi(0) = 0.5, \xi(0) = 0, \text{ and } \xi(0) = 0\). To make a comparison, the simulation is also performed using the fuzzy method in [30] under the same condition. The trajectories of system signals are provided in Figures 1–4. As can be seen, with the method of this paper, the states \(x_1, x_2, x_3\) have smaller amplitudes. In detail, under the presented method, the amplitudes of \(x_1, x_2, x_3\) are about 0.08, 1.1, and 2.5. Under the method in [30], the amplitudes of \(x_1, x_2, x_3\) are about 0.2, 3, and 18. The states \(x_1, x_2, x_3\) can converge to the neighborhood of the original faster. Specifically, under the presented method, the convergence time of the states \(x_1, x_2, x_3\) are about 2 s, 2 s, and 2 s. Under the method in [30], the convergence times of the states \(x_1, x_2, x_3\) are about 3 s, 3 s, and 3 s. Besides, the amplitude of the control input \(u\) in this paper is smaller. More exactly, under the presented method, the amplitude of \(u\) is about 20. Under the method in [30], the amplitude of \(u\) is about 500. Thus, the presented method is effective and has better system response.

**Example 2.** Consider the system

\[
\begin{align*}
\dot{x}_1 &= c_1 x_1^3 + \theta_1 x_1 x_2 + w(t), \\
\dot{x}_2 &= c_2 u^2 + \theta_2 x_2^2,
\end{align*}
\]
Figure 1: Trajectory of $x_1$ in system (24).

Figure 2: Trajectory of $x_2$ in system (24).

Figure 3: Trajectory of $x_3$ in system (24).
where $x_1, x_2$ are the states, $u$ is the input, $\theta_1, \theta_2$ are unknown functions, $\omega(t)$ is the disturbance. Let $H = (\bar{\theta}/\epsilon)^{9/14}$ and choose $\epsilon = 1/2$, $r_0 = r_1 = 1$, $q = 5/9$, $r_2 = 2/3$, and $r_3 = 5/9$. Following the presented algorithm, we can construct the adaptive controller

$\begin{align*}
\dot{u} &= -H^{+\epsilon} (\bar{\theta}) \beta_2 S_2,
\dot{\bar{\theta}} &= \frac{\lambda}{H^{3/2}(\bar{\theta})} (S_1^{2p} + S_2^{2p}), 
\bar{\theta}(0) = 1,
\end{align*}$

(27)

where $p = 3$, $S_1 = x_1/H^{\epsilon} (\bar{\theta})$, and $S_2 = x_2/H^{\epsilon} (\bar{\theta}) + \beta_1 S_1$. 

Figure 4: Trajectory of $u$ given in (25).

Figure 5: Trajectory of $x_1$ in system (26).

Figure 6: Trajectory of $x_2$ in system (26).

Figure 7: Trajectory of $u$ provided in (27).

Figure 8: Trajectory of $\bar{\theta}$ in (27).
In the simulation, we choose $c_1 = 1, c_2 = 1, \theta_1 = 1, \theta_2 = 1, \omega(t) = 0.1 \sin t, \beta_2 = 9, \beta_1 = 3$, and $\lambda = 500$. Figures 5–8 provide the trajectories of the system signals. As can be seen, the control effort is small. Besides, all system states are bounded. Hence, the provided strategy is valid.

5. Conclusion

In the research, the adaptive problem of the nontriangular system has been discussed. In the control design procedure, the function of parameter estimation is skillfully employed. By utilizing the idea of homogeneous control, fuzzy systems, and the adaptive scheme, a new robust adaptive controller has been constructed. Since homogeneous control is employed and only one parameter adaptive law is introduced, the obtained adaptive controller has a simpler form. Actually, future studies can focus on the following problems. For instance, if the plant contains time-delay, can we construct the similar stable controller? Can we extend the result to stochastic nonlinear system? If the system has unmodeled dynamics, can we construct the adaptive controller with similar form?

Data Availability

The data used to support the findings of this study are included within the article.

Disclosure

Anyone else who contributed to the manuscript but does not qualify for authorship has been acknowledged with their permission. There are no copyediting or translation services used for the preparation of the paper.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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References


