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Research Article

Practical Input-to-State Stability of Nonlinear Systems with Time Delay

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An investigation of the input-to-state practical stability (ISpS) and the integral input-to-state practical stability (iISpS) of nonlinear systems with time delays (NSWTDs) is presented in this paper. The ISpS and iISpS of the systems are obtained by using a continuously differentiable Lyapunov–Krasovskii functional (LKF) within-definite derivative, which generalizes the classical LKF with a positive definite derivative. As a result, the uniform practical asymptotic stability (UPAS) criteria of NSWTD were also given by using the proposed LKF.

1. Introduction

Input-to-state stability (ISS) [1] ensures the asymptotic stability for zero input systems. It was proposed in [1] that a negative derivative of a Lyapunov function (LF) was sufficient for ISS, which was characterized later by Wang et al. [2]. Also, Wang et al. and Khalil [2, 3] showed that uniform asymptotic gain and ISS are equivalent. However, Sontag [4] introduced the notion of integral input-to-state stability (iISS), which is a nonlinear extension of $\mathscr{L}^2$ stability. As a result, it was strictly weaker than ISS. As we know, a time-invariant system is iISS if it has an LF with a derivative that is negative definite along the system [5]. There is a widespread phenomenon of time delay in practical control systems. Recent years have seen intensive studies of NSWTD's ISS property. For the ISS property of NSWTD, the paper [6] showed that the nonlinear small-gain theorem is equivalent to the Razumikhin-type theorem. For impulsive systems in [7], hybrid delayed systems in [8], and NSWTD in [9], LKF methods were used. Authors in [8] explain in detail how delay bounds and dwell times are related to hybrid delayed systems. It is difficult to choose a suitable LKF with a negative definite derivative to verify the ISS property based on the existing results in [7–11]. In many cases, a closed-loop system's ISS behavior cannot be ensured via feedback because it is too costly or impossible. According to [12], input-to-state practical stability (ISpS) has been proposed as a relaxation of the ISS concept for such applications. There is in fact an extension of the earlier technique of practical asymptotic stability of nonlinear differential equations in the ISpS, as shown in [13–16]. The characterizations of ISpS have been developed for a class of ordinary differential equations [17]. Nonlinear systems are usually stabilized using LFs [14, 18, 19]. In addition, they provide tools for designing a more robust controller and/or verifying its robustness.

In this paper, the ISpS and the iISpS properties of NSWTD are established. This paper introduces a new practical LKF to address the issue raised by time delays and stable scalar functions [14]. We establish some sufficient conditions for ISpS and iISpS by constructing a more general $\mathscr{K}\mathscr{L}$ function over [20]. We also present some new sufficient conditions for UPAS.

The following is the organization of this paper. We will discuss the main notations and definitions in the next section. By utilizing a new LKF, Section 3 provides the ISpS and iISpS criteria and provides some sufficient conditions for UPAS. The paper concludes with Section 4.

2. Basic Results

This work assumes $J = [0, +\infty]$, $\|.\|$ refers to the usual Euclidean norm, and $\mathscr{PC}(\mathbb{R}_+, \mathbb{R})$ is the space of piecewise continuous function on $\mathbb{R}_+$ to $\mathbb{R}$. Also, we denote by $\mathscr{C}$ the space of continuous functions ϖ: $[-\varsigma, 0] \longrightarrow \mathbb{R}^n$ equipped with the norm as follows:

$$|\varpi|_\infty = \max_{\omega\in[-\varsigma,0]} \|\varpi(\omega)\|, \text{with } \varsigma > 0. \tag{1}$$

Consider the following NSWTD:

$$\dot{\zeta}(t) = F(t, \zeta_t, u(t)), \tag{2}$$

where $t \in \mathbb{R}_+$ is the time, $\zeta(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control which is assumed to be measurable and locally essentially bounded, ζ_t is the function segment defined by $\zeta_t(\omega) = \zeta(t + \omega)$, $\omega \in [-\varsigma, 0]$, with $\varsigma > 0$ is the time delay, and F: $\mathbb{R}_+ \times \mathscr{C} \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is piecewise continuous with respect to t, locally Lipschitz in ζ_t, and uniformly continuous in u. Then, for a given initial state $\zeta_{l_0} \in \mathscr{C}$, and initial time l_0, system (2) exists in a unique solution $\zeta(t, \zeta_{l_0}, u)$, which satisfies $\zeta(t, \zeta_{l_0}, u) = \zeta_{l_0} = \varpi$, for all $t \in [l_0 - \varsigma, l_0]$.

Definition 1. System (2) is said to be ISpS if there exist a class $\mathscr{KL}$ function μ, a class $\mathscr{K}$ function γ, and $r > 0$, such that, for any initial state ζ_{l_0} and any measurable, locally essentially bounded input $u(t)$, the solution exists for all $t \geq l_0$ and satisfies the following:

$$\left\|\zeta\left(t, \zeta_{l_0}, u\right)\right\| \leq \mu\left(\left\|\zeta_{l_0}\right\|, t - l_0\right) + r + \gamma\left(\sup_{[l_0,t]} \|u\|\right). \tag{3}$$

Definition 2. System (2) is said to be iISpS if there exist a class $\mathscr{KL}$ function μ, a class $\mathscr{K}_\infty$ function α, and a class $\mathscr{K}$ function γ such that, for any initial state ζ_{l_0}, any measurable, locally essentially bounded input $u(t)$, the solution exists for all $t \geq l_0$ and satisfies the following:

$$\left\|\alpha\left(\zeta\left(t, \zeta_{l_0}, u\right)\right)\right\| \leq \mu\left(\left\|\zeta_{l_0}\right\|, t - l_0\right) + r + \int_{l_0}^{t} \gamma(\|u(s)\|)\mathrm{d}s. \tag{4}$$

Definition 3. System (2) is said to be globally uniformly practically asymptotically stable (GUPAS), if there exist $\mu \in \mathscr{KL}$ and $r > 0$, such that, for any initial state ζ_{l_0}, any measurable, locally essentially bounded input $u(t)$, the solution exists for all $t \geq l_0$ and satisfies the following:

$$\left\|\zeta\left(t, \zeta_{l_0}, u\right)\right\| \leq \mu(\|\zeta\|, s - t) + r. \tag{5}$$

Remark 4. The inequality (4) implies that $\|\zeta(t, \zeta_{l_0}, u)\|$ will be bounded by a small bound $r > 0$, that is, $\|\zeta(t, \zeta_{l_0}, u)\|$ will be small for sufficiently large s, by taking an initial condition outside the Ball $\mathscr{B}_r$. In this situation, a robustness result can be obtained if we suppose that r depends on a small parameter $\varepsilon > 0$; in this case, if $r = r(\varepsilon)$ approaches to zero as ε tends to zero, then, $\|\zeta(t, \zeta_{l_0}, u)\|$ approaches the origin exponentially as s goes to infinity.

Also, we need the following definition.

Definition 5. The derivative of the continuous function V: $J \times \mathscr{C} \longrightarrow J$ along the solutions of (2) is defined as follows:

$$D^+V(s, \varpi) = \limsup_{l\longrightarrow 0^+} \left\{\frac{1}{l}\left[V\left(s + l, \zeta_{s+l}(s, \varpi) - V(s, \varpi)\right)\right]\right\}. \tag{6}$$

Here is the definition that is given in [14].

Definition 6. Let $\varrho, \xi \in \mathscr{PC}(\mathbb{R}_+, \mathbb{R})$. ϱ is $\xi-$ globally uniformly practically exponentially stable (GUPES) if there is $\omega > 0$, $\upsilon \geq 0$, and $\theta > 0$, such that for every $s \geq t$,

$$\begin{aligned} \int_t^s \varrho(\varsigma)d\varsigma &\leq -(s - t)\omega + \upsilon, \\ \int_t^s |\xi(\varsigma)|\psi(s, \varsigma)d\varsigma &\leq \theta, \end{aligned} \tag{7}$$

with $\psi(s, t) = \exp(\int_t^s \varrho(\varsigma)d\varsigma)$.

Remark 7. Indeed, the notion of stable scalar functions was first given in [21, 22] after the author in [14] has extended these definitions to the practical case. It is worth noting that, if the function ϱ is $\xi-$ GUPES, then the system

$$\dot{y} = \varrho(t)y(t) + \xi(t), \tag{8}$$

is GUPES.

In the sequel, we will use the following auxiliary result called the generalized Gronwall–Bellman inequality, which is taken from [22].

Lemma 8. *Let $\upsilon, \psi \in \mathscr{PC}(\mathbb{R}_+, \mathbb{R})$, and ϖ: $\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a differentiable function, such that, for all $s \geq t$,*

$$\dot{\varpi}(s) \leq \upsilon(s)\varpi(s) + \psi(s). \tag{9}$$

Then, for every $s \geq t$, we have

$$\varpi(s) \leq \varpi(t)\exp\left(\int_t^s \upsilon(\varsigma)d\varsigma\right) + \int_t^s \exp\left(\int_u^s \upsilon(\varsigma)d\varsigma\right)\psi(u)du. \tag{10}$$

3. Main Results

Now, we are ready to give the main lemma.

Lemma 9. *Assume that y: $J \longrightarrow J$ is an absolutely continuous function, $u \in L_\infty(J)$, $\kappa \in \mathscr{K}$, and $\varrho, \xi \in \mathscr{PC}(J, \mathbb{R})$. If for almost all $t \geq l_0$,*

$$\dot{y}(s) \leq \varrho(s)y(s) + \xi(s), \quad \text{for all } y(t) \geq \kappa(\|y\|), \tag{11}$$

with an initial value $y(l_0)$, and ϱ is $\xi-$ GUPES, then, for all $t \geq l_0$, we have

$$y(t) \leq y(l_0)e^{-\omega(s-t)}e^{\upsilon} + \kappa\left(\sup_{[l_0,t]} \|u\|\right)e^{\upsilon} + \theta. \tag{12}$$

Proof. For the inequality

$$y(s) \geq \kappa(\|u(s)\|). \tag{13}$$

Let us consider the following two cases:

Case 1: (13) holds for almost all $l_0 \leq s \leq t$

Case 2: (13) does not hold for all $l_0 \leq s \leq t$

For Case 1, the generalized Gronwall–Bellman inequality gives us

$$y(t) \leq y(l_0)e^{\int_{t_0}^{t} \varrho(s)ds} + \int_{t_0}^{t} \xi(s)\psi(t,s)ds, \tag{14}$$

where $\psi(z,t) = \exp(\int_t^z \varrho(\varsigma)d\varsigma)$. Since, ϱ is $\xi-$ GUPES, then, there exist $\omega > 0$, $\upsilon \geq 0$, and $\theta > 0$, such that for all $t \geq l_0$,

$$\begin{aligned} \int_{l_0}^{t} \varrho(s)ds &\leq -(t-l_0)\omega + \upsilon, \\ \int_{l_0}^{t} |\xi(s)|\psi(t,s)ds &\leq \theta. \end{aligned} \tag{15}$$

Thus,

$$y(t) \leq y(l_0)e^{-(t-l_0)\omega}e^{\upsilon} + \theta. \tag{16}$$

For Case 2, we have the measure of $\mathscr{A} = \{l_0 \leq s \leq t; y(s) \leq \kappa(\|u(s)\|)\}$ is greater than zero. Let $t^* = \sup \mathscr{A}$. Then, either $t^* < t$ or $t^* = t$. If $t^* < t$, so for almost $t^* < s < t$, we obtain

$$\dot{y}(s) \leq \varrho(s)y(s) + \xi(s). \tag{17}$$

Thus, the estimation (14) holds for all $t^* < s < t$. Now, by the continuity of y and the fact that ϱ is $\xi-$ GUPES, one obtains

$$y(t) \leq y(t^*)e^{-(t-l_0)\omega}e^{\upsilon} + \theta. \tag{18}$$

From the definition of t^*, the last inequality gives

$$\begin{aligned} y(t) &\leq y(t^*)e^{-(t-l_0)\omega}e^{\upsilon} + \theta \\ &\leq \kappa\left(\sup_{[l_0,t]} \|u\|\right)e^{\upsilon} + \theta. \end{aligned} \tag{19}$$

If $t^* = t$, then there is a constant δ for every $t - \delta < s < t$,

$$y(s) \leq \kappa(\|u\|) \leq \kappa\left(\sup_{[l_0,t]} \|u\|\right)e^{\upsilon} + \theta. \tag{20}$$

If we let $s \longrightarrow t$, and we use the continuity of y, we get, for every $t > l_0$,

$$y(t) \leq \kappa(\|u\|) \leq \kappa\left(\sup_{[l_0,t]} \|u\|\right)e^{\upsilon} + \theta. \tag{21}$$

Now, combining (16), (19), and (21), we conclude that for all $t \geq l_0$

$$y(t) \leq y(l_0)e^{-(s-t)\omega}e^{\upsilon} + \kappa\left(\sup_{[l_0,t]} \|u\|\right)e^{\upsilon} + \theta. \tag{22}$$

□

Remark 10. A new comparison principle is presented in Lemma 9 to determine the upper bounds for NSWTD. It has the advantage that its derivative can be positive during some interval of time in J. In contrast, when $\xi(t) = 0$, the derivative of the LKF requires that the derivative of the differential inequality be negative definite such as [8, 9].

A useful property of a function is presented in the following lemma in $\mathscr{K}$.

Lemma 11. *Let g be a class $\mathscr{K}$ function. We have for all $t, m \geq 0$,*

$$\alpha(t+m) \leq \alpha(2t) + \alpha(2m). \tag{23}$$

Applying Lemmas 9 and 11, we obtain the following theorem for the ISpS of system (2) for an indefinite LF.

Theorem 12. *Suppose that there is a C^1 function $V: J \times \mathscr{C} \longrightarrow J$ functions $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$, a function $\kappa \in \mathscr{K}$, a positive constant a, and $\varrho, \xi \in \mathscr{PC}(J, \mathbb{R})$ such that for every $t \in J$, $\zeta_t \in \mathscr{C}$, and $u \in \mathbb{R}^m$,*

$$\alpha_1\left(\left\|\zeta_t(0)\right\|\right) \leq V(t,\zeta_t) \leq \alpha_2\left(\left\|\zeta_t\right\|_\infty\right) + a, \tag{24}$$

$$\dot{V}(t,\zeta_t) \leq \varrho(t)V(t,\zeta_t) + \xi(t), \quad \text{if } V(t,\zeta_t) \geq \kappa(\|u\|). \tag{25}$$

Then, system (2) is ISpS if ϱ is $\xi-$ GUPES.

Proof. Since ϱ is $\xi-$ GUPES, there is $\omega > 0$, $\upsilon \geq 0$, and $\theta > 0$, such that for every $t \geq l_0$,

$$\begin{aligned} \int_{l_0}^{t} \varrho(s)ds &\leq -(t-l_0)\omega + \upsilon, \\ \int_{l_0}^{t} |\xi(s)|\psi(t,s)ds &\leq \theta. \end{aligned} \tag{26}$$

Lemma 9 and the inequality (25) give for all $t \geq l_0$

$$V(t,\zeta_t) \leq V(l_0,\zeta_{l_0})e^{-(t-l_0)\omega}e^{\upsilon} + \kappa\left(\sup_{[l_0,t]} \|u\|\right)e^{\upsilon} + \theta. \tag{27}$$

Using the inequalities given in (24), we obtain for all $t \geq l_0$

$$\begin{aligned} \alpha_1\left(\left\|\zeta(t,\zeta_{l_0},u)\right\|\right) &\leq \left(\alpha_2\left(\left\|\zeta_{l_0}\right\|\right) + a\right)e^{-\omega(t-l_0)}e^{\upsilon} + \kappa\left(\sup_{[l_0,t]} \|u\|\right)e^{\upsilon} + \theta \\ &\leq \alpha_2\left(\left\|\zeta_{l_0}\right\|\right)e^{-(t-l_0)\omega}e^{\upsilon} + \kappa\left(\sup_{[l_0,t]} \|u\|\right)e^{\upsilon} + ae^{\upsilon} + \theta. \end{aligned} \tag{28}$$

Thus, for all $t \geq l_0$, one has

$$\left\|\zeta\left(t, \zeta_{l_0}, u\right)\right\| \leq \alpha_1^{-1}\left(\alpha_2\left(\left\|\zeta_{l_0}\right\|\right)e^{-\left(t-l_0\right)\omega}e^{v} + \kappa\left(\sup_{[l_0,t]}\|u\|\right)e^{v} + ae^{v} + \theta\right). \tag{29}$$

Now, using Lemma (14), we obtain for all $t \geq l_0$,

$$\left\|\zeta\left(t, \zeta_{l_0}, u\right)\right\| \leq \alpha_1^{-1}\left(\alpha_2\left(\left\|\zeta_{l_0}\right\|\right)e^{-\left(t-l_0\right)\omega}e^{v}\right) + \alpha_1^{-1}\left(\kappa\left(\sup_{[l_0,t]}\|u\|\right)e^{v}\right) + \alpha_1^{-1}\left(ae^{v} + \theta\right). \tag{30}$$

Hence, for all $t \geq l_0$, we have

$$\left\|\zeta\left(t, \zeta_t, u\right)\right\| \leq \mu\left(\left\|\zeta_{l_0}\right\|, t - l_0\right) + r + \gamma\left(\sup_{[l_0,t]}\|u\|\right), \tag{31}$$

with $\mu(r,s) = \alpha_1^{-1}\left(\alpha_2(r)e^{-\omega s}e^{v}\right)$, $r = \alpha_1^{-1}\left(ae^{v} + \theta\right)$, and $\gamma(s) = \alpha_1^{-1}\left(\kappa(s)e^{v}\right)$.

Based on the Definition 2, the system is ISpS.

ISS implies iISpS, as we know. Here, we give an iISpS result for the NSWTD (2) by using the indefinite derivative LKF. □

Theorem 13. *Suppose that there is a C^1 function $V: J \times \mathscr{C} \longrightarrow J$ functions $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$, a function $\kappa \in \mathscr{K}$, a positive constant a, and $\varrho, \xi \in \mathscr{PC}(J, \mathbb{R})$ such that for $u \in \mathbb{R}^m$ and $t \in J$, $\zeta_t \in \mathscr{C}$*

$$\alpha_1\left(\left\|\zeta_t(0)\right\|\right) \leq V\left(t, \zeta_t\right) \leq \alpha_2\left(\left\|\zeta_t\right\|_\infty\right) + a, \tag{32}$$

$$\dot{V}\left(t, \zeta_t\right) \leq \varrho(t)V\left(t, \zeta_t\right) + \xi(t) + \kappa(\|u\|). \tag{33}$$

Then, system (2) is iISpS if ϱ is $\xi-$ GUPES.

Proof. By Lemma 9, we obtain from (33) that for every $t \geq l_0$,

$$V\left(t, \zeta_t\right) \leq V\left(l_0, \zeta_{l_0}\right)e^{\int_{l_0}^{t}\varrho(\varsigma)d\varsigma} + \int_{l_0}^{t}\left[\kappa(\|u(s)\|) + \xi(s)\right]\psi(t,s)ds. \tag{34}$$

Since ϱ is $\xi-$ GUPES, there is $\omega > 0$, $v \geq 0$, and $\theta > 0$, such that for every $t \geq l_0$,

$$\int_{l_0}^{t}\varrho(s)ds \leq -\left(t - l_0\right)\omega + v, \qquad \int_{l_0}^{t}|\xi(s)|\psi(t,s)ds \leq \theta. \tag{35}$$

We further obtain for all $t \geq l_0$,

$$V\left(t, \zeta_t\right) \leq \alpha_2\left(\left\|\zeta_{l_0}\right\|\right)e^{-\left(t-l_0\right)\omega}e^{v} + \int_{l_0}^{t} e^{v}\kappa(\|u(\varsigma)\|)d\varsigma + ae^{v} + \theta. \tag{36}$$

From condition (32), we can see that

$$\alpha_1\left(\left\|\zeta\left(t, \zeta_{l_0}, u\right)\right\|\right) \leq \mu\left(\left\|\zeta_{l_0}\right\|, t - l_0\right) + \gamma\left(\int_{l_0}^{t}\kappa(\|u(\varsigma)\|d\varsigma\right) + r. \tag{37}$$

$\mu(r,s) = \alpha_2(r)e^{-\omega s}e^{v}$, $r = ae^{v} + \theta$ and $\gamma(s) = \kappa(s)e^{v}$. The inequality (37) implies that system (2) is iISpS. □

Remark 14. Condition (25) in Theorem 12 implies condition (25) in Theorem 13, On the other hand, the converse might not be true. The following two cases illustrate this statement:

(i) Case 1: If $V\left(t, \zeta_t\right) \geq \kappa(\|u\|)$, it follows from (25) in Theorem 12 that

$$\dot{V}\left(t, \zeta_t\right) \leq \varrho(t)V\left(t, \zeta_t\right) + \xi(t). \tag{38}$$

Following is an estimation that can easily be verified as follows:

$$\dot{V}\left(t, \zeta_t\right) \leq \varrho(t)V\left(t, \zeta_t\right) + \xi(t) + \kappa(\|u\|). \tag{39}$$

(ii) Case 2: If $V\left(t, \zeta_t\right) \leq \kappa(\|u\|)$, let the function $\tilde{\chi}$ be as follows:

$$\tilde{\chi}(s) = \sup\left\{\frac{\partial V}{\partial t}\left(t, \zeta_t\right) + \frac{\partial V}{\partial x}(t, \varpi)F\left(t, \zeta_t, u\right) - \varrho(t)V\left(t, \zeta_t\right) - \xi(t), \|u\| \leq s, V\left(t, \zeta_t\right) \leq \kappa(s)\right\}, \tag{40}$$

for $s \geq l_0 \geq 0$. We consider any $\mathscr{K}_\infty$ function χ with $\chi(s) \geq \tilde{\chi}(s)$, for every $s \geq l_0$. Therefore

$$\frac{\partial V}{\partial t}\left(t, \zeta_t\right) + \frac{\partial V}{\partial x}\left(t, \zeta_t\right)F\left(t, \zeta_t, u\right) \leq \varrho(t)V\left(t, \zeta_t\right) + \xi(t) + \chi(\|u\|). \tag{41}$$

When the two cases are combined, the statement is true.

Theorem 15. *Suppose that there is a C^1 function $V: J \times \mathscr{C} \longrightarrow J$ functions $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$, a function $\kappa_1, \kappa_2 \in \mathscr{K}$, a positive constant a, and $\varrho, \xi \in \mathscr{PC}(J, \mathbb{R})$ such that for every $t \in J$, $\zeta_t \in \mathscr{C}$, and $u \in \mathbb{R}^m$,*

$$\alpha_1\left(\left\|\zeta_t(0)\right\|\right) \leq V\left(t, \zeta_t\right) \leq \alpha_2\left(\left\|\zeta_t\right\|_\infty\right) + a, \tag{42}$$

$$\dot{V}\left(t, \zeta_t\right) \leq \left(\varrho(t) + \kappa_1(\|u\|)\right)V\left(t, \zeta_t\right) + \xi(t) + \kappa_2(\|u\|). \tag{43}$$

Then, system (2) is iISpS if $\varrho is \xi-$ GUPES.

Proof. Setting $\kappa = \zeta(\kappa_1, \kappa_2)$, then $\kappa_i \le \kappa, i = 1, 2$. As in the proof of 3.3, we can calculate

$$\begin{aligned} V(t,\zeta_t) &\le V(l_0,\zeta_{l_0}) e^{\int_{l_0}^{t} (\kappa_1(\|u(s)\|)+\varrho(s))ds} + \int_{l_0}^{t} [\kappa_2(\|u(s)\|) + \xi(s)]\psi(t,s) e^{\int_{s}^{t}\kappa_1(\|u(\varsigma)\|)d\varsigma} ds \\ &\le V(l_0,\zeta_{l_0}) e^{\int_{l_0}^{t}\kappa(\|u(s)\|)ds} e^{-(t-l_0)\omega} e^{\upsilon} + e^{\int_{l_0}^{t}\kappa(\|u(\varsigma)\|)d\varsigma} \int_{l_0}^{t} [\kappa(\|u(s)\|) + \xi(s)]\psi(t,s)ds. \end{aligned} \tag{44}$$

We obtain

$$\begin{aligned} V(l_0,\zeta_{l_0}) e^{\int_{l_0}^{t}\kappa(\|u(s)\|)ds} e^{-(t-l_0)\omega} e^{\upsilon} &= V(l_0,\zeta_{l_0}) e^{-(t-l_0)\omega} e^{\upsilon} + V(l_0,\zeta_{l_0}) e^{-(t-l_0)\omega} e^{\upsilon} \left[e^{\int_{l_0}^{t}\kappa(\|u(s)\|)ds} - 1 \right] \\ &\le V(l_0,\zeta_{l_0}) e^{-(t-l_0)\omega} e^{\upsilon} + \frac{1}{2}\left[V(l_0,\zeta_{l_0}) e^{-(t-l_0)\omega} e^{\upsilon} \right]^2 \\ &\quad + \frac{1}{2}\left[e^{\int_{l_0}^{t}\kappa(\|u(s)\|)ds} - 1 \right]^2. \end{aligned} \tag{45}$$

From condition (42), one obtains

$$\begin{aligned} V(l_0,\zeta_{l_0}) e^{\int_{l_0}^{t}\kappa(\|u(s)\|)ds} e^{-(t-l_0)\omega} e^{\upsilon} &\le \alpha_2\left(\left\|\zeta_{l_0}\right\|\right) e^{\upsilon} e^{-(t-l_0)\omega} + \frac{1}{2}\alpha_2\left(\left\|\zeta_{l_0}\right\|\right)^2 e^{2\upsilon} e^{-(t-l_0)\omega} \\ &\quad + \frac{1}{2}\alpha_2\left(\left\|\zeta_{l_0}\right\|\right) e^{2\upsilon} a e^{-(t-l_0)\omega} + a e^{\upsilon} + \frac{a^2}{2} e^{2\upsilon} + \frac{1}{2}\left[e^{\int_{l_0}^{t}\kappa(\|u(s)\|)ds} - 1 \right]^2. \end{aligned} \tag{46}$$

Therefore, for all $t \ge l_0$,

$$\begin{aligned} \alpha_1\left(\left\|\zeta(t,\zeta_{l_0},u)\right\|\right) &\le \alpha_2\left(\left\|\zeta_{l_0}\right\|\right) e^{\upsilon} e^{-(t-l_0)\omega} + \frac{1}{2}\alpha_2\left(\left\|\zeta_{l_0}\right\|\right)^2 e^{2\upsilon} e^{-(t-l_0)\omega} \\ &\quad + \frac{1}{2}\alpha_2\left(\left\|\zeta_{l_0}\right\|\right) e^{2\upsilon} a e^{-(t-l_0)\omega} + a e^{\upsilon} + \frac{a^2}{2} e^{2\upsilon} + \frac{1}{2}\left[e^{\int_{l_0}^{t}\kappa(\|u(s)\|)ds} - 1 \right]^2 \\ &\quad + e^{\int_{l_0}^{t}\kappa(\|u(\varsigma)\|)d\varsigma} \int_{l_0}^{t} \kappa(\|u(s)\|) e^{\upsilon} ds + \theta e^{\int_{l_0}^{t}\kappa(\|u(\varsigma)\|)d\varsigma} \\ &\le \mu\left(\left\|\zeta_{l_0}\right\|, t - l_0\right) + \gamma\left(\int_{l_0}^{t} \kappa(\|u(\varsigma)\| d\varsigma\right) + r, \end{aligned} \tag{47}$$

where $\mu(r,s) = \alpha_2(r)e^{-\omega s}e^{\upsilon}(1 + e^{\upsilon}/2\alpha_2(r) + ae^{\upsilon}/2)$, $r = ae^{\upsilon} + a^2e^{2\upsilon}\theta/2$, and $\gamma(s) = 1/2(e^s - 1)^2 + (se^{\upsilon} + \theta)e^s$. Following Definition 2, system (2) is iISpS. This completes the proof.

UPAS is implied by ISpS and iISpS properties. UPAS for NSWTD (2) with $u = 0$ is given as a by-product of Theorem 12.

With $u = 0$, we give the following uniformly practically stable result for (2). □

Corollary 16. *Suppose that there is a C^1 function $V: J \times \mathscr{C} \longrightarrow J$ functions $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$, a function $\kappa \in \mathscr{K}$, a positive constant a, and $\varrho, \xi \in \mathscr{PC}(J, \mathbb{R})$ such that for every $t \in J$, $\zeta_t \in \mathscr{C}$, and $u \in \mathbb{R}^m$,*

$$\alpha_1(\|\zeta_t(0)\|) \leq V(t, \zeta_t) \leq \alpha_2(\|\zeta_t\|_\infty) + a, \tag{48}$$

$$\dot{V}(t, \zeta_t) \leq \varrho(t)V(t, \zeta_t) + \xi(t). \tag{49}$$

Then, system (2) with $u = 0$ is GUPAS if ϱ is $\xi-$ GUPES.

Proof. Similar to Theorem 12 we obtain for every $t \geq l_0$,

$$V(t, \zeta_t) \leq V(l_0, \zeta_{l_0})e^{\int_{l_0}^t \varrho(\varsigma)d\varsigma} + \int_{l_0}^t \xi(s)\psi(t,s)ds. \tag{50}$$

Thus, for all $t \geq l_0$,

$$V(t, \zeta_t) \leq \alpha_1^{-1}\left(\alpha_2\left(\|\zeta_{l_0}\|\right)e^{-(t-l_0)\omega}e^{\upsilon}\right) + \alpha_1^{-1}(ae^{\upsilon} + \theta). \tag{51}$$

Hence, for all $t \geq l_0$, we have

$$\|\zeta(t, \zeta_t, 0)\| \leq \mu\left(\|\zeta_{l_0}\|, t - l_0\right) + r, \tag{52}$$

with $\mu(r,s) = \alpha_1^{-1}(\alpha_2(r)e^{-\omega s}e^{\upsilon})$ and $r = \alpha_1^{-1}(ae^{\upsilon} + \theta)$.

Now, we conclude that the system is GUPAS following Definition 5. □

Example 1. We consider the following NSWTD:

$$\dot{\zeta}(t) = A(t)\zeta + B(t)\zeta(t - \varsigma) + f(t, \zeta(t - \varsigma)), \tag{53}$$

where $A, B \in \mathscr{PC}(J, \mathbb{R}^{n\times n})$, $\varsigma > 0$ is a constant, and $f: J \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Now, let us consider the following hypothesis:

(i) There is $P(t) = P^T(t) \in \mathscr{C}^1(J, \mathbb{R}^{n\times n})$, two constants $p_2 > p_1 > 0$, and $\alpha in \mathscr{PC}(\mathbb{R}_+, \mathbb{R})$, such that for every $t \in J$, we have

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) \leq \alpha(t)P(t), \tag{54}$$

$$p_1 I \leq P(t) \leq p_2 I. \tag{55}$$

(ii) There exists a function $\xi \in \mathscr{PC}(J, J) \cap L^1(J)$, such that for every $(t, \zeta) \in \mathbb{R}_+ \times \mathbb{R}^n$,

$$\|\zeta^T(t)P(t)f(t, \zeta(t - \varsigma))\| \leq \frac{1}{2}\xi(t). \tag{56}$$

(ii) There exists $\delta > 0$ such that for all $s \geq t$,

$$\int_t^s \|B(u)\|^2 du < \delta. \tag{57}$$

We suppose that α is $\xi-$ globally uniformly practically stable and then, let us consider the following LKF:

$$V(t, \zeta_t) = \zeta^T P(t)\zeta + \int_{t-\varsigma}^t f_0(t,u)\|B(u + \varsigma)\zeta(u)\|^2 du, \tag{58}$$

where $f_0(t,u) = p_2/\sigma\eta + u - t/\varsigma(p_2/\sigma\eta - p_2/\eta)$, $\sigma \in [0,1]$, $\eta = \sigma - 1/\varsigma - \alpha_0 \geq 0$, and $\alpha_0 = \inf_J \alpha$, whose time-derivative satisfies (see [23]) the following:

$$\dot{V}(t, \zeta_t) \leq \varrho(t)V(t, \zeta_t) + \xi(t), \tag{59}$$

where $\varrho(t) = \alpha(t) + \eta + p_2/\sigma\eta p_2\|B(t + \varsigma)\|^2$. Using almost the same techniques as in [23], one can check that ϱ is $\xi-$ GUPES. Hence, the NSWTD (53) is GUPAS.

Consider a class of a SDOF system with a time delay described by the following system [24, 25]:

$$\ddot{q} + r(t)\dot{q} + c(t)q = u - g(t, q(t - \varsigma), \dot{q}(t - \varsigma)), \tag{60}$$

where both the damping coefficient, $r(t)$, and the elastic constant, $s(t)$, are time-varying ([22]), the variable $q \in \mathbb{R}$ represents the position of the mass with respect to its rest position, we use the notation $\dot{q}$ to denote the derivative of q with respect to time (i.e., the velocity of the mass) and $\ddot{q}$ to represent the second derivative (acceleration), $g: J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ stands for control gain according to displacement and velocity (called displacement gain and velocity gain for the sake of simplicity), u is external excitation, and ς is the time delay.

Let $\dot{q} = x$. Then, system (60) can be rewritten as follows:

$$\dot{\zeta} = A(t)\zeta(t) + u(t, \zeta(t - \varsigma)) - g(t, \zeta(t - \varsigma)), \tag{61}$$

with $\zeta^T = (q, x)$, $A(t) = \begin{bmatrix} 0 & 1 \\ -r(t) & -s(t) \end{bmatrix}$, and $u: J \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a continuously differentiable function such that

$$u(t, \zeta(t - \varsigma)) = B(t)\zeta(t - \varsigma), \tag{62}$$

with $B \in \mathscr{PC}(J, \mathbb{R}^{2\times 2})$ and satisfies (57).

Let $r(t) = 2 - \gamma\sin(t)$ and $s(t) = 2 - \gamma\cos(t)$, where the scalar γ is a constant parameter that accounts for the variability, we take $\gamma = 2.125$.

Consider the following LKF:

$$V(t, \zeta_t) = \zeta^T P(t)\zeta + \int_{t-\varsigma}^t f_0(t,u)\|B(u + \varsigma)\zeta(u)\|^2 du, \tag{63}$$

where $P(t) = \begin{bmatrix} 1 & 1/5 \\ 1/5 & 17/50 \end{bmatrix}$, which satisfies inequalities (54) and (55) with $\alpha(t) = 1/2\lambda_{\max}((A(t)^T P(t) + P(t)A(t))P^{-1}(t))$, $p_1 = \lambda_{\min}(P)$, $p_2 = \lambda_{\max}(P)$, $f_0(t,u) = p_2/\sigma\eta + u - t/\varsigma(p_2/\sigma\eta - p_2/\eta)$, $\sigma \in [0,1]$, $\eta = \sigma - 1/\varsigma - \alpha_0 \geq 0$, and

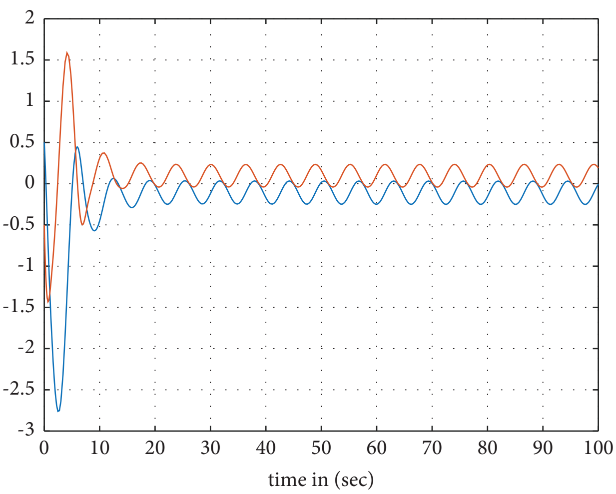

Figure 1: Time evolution of the solution $(q(t), x(t))$ of system (61).

$\alpha_0 = \inf_J \alpha$. ($\lambda_{\min}, \lambda_{\max}$ are, respectively, the smallest and the largest eigenvalue of the matrix). Now, if $f = -g$ satisfies (56) such that α is $\xi-$ globally uniformly practically stable, one obtains

$$\dot{V}(t, \zeta_t) \leq \varrho(t) V(t, \zeta_t) + \xi(t), \tag{64}$$

where $\varrho(t) = \alpha(t) + \eta + p_2/\sigma\eta p_2 \|B(t + \varsigma)\|^2$. Therefore, the NSWTD (61) is GUPAS.

For simulation, we choose $\varsigma = 1$,

$$B(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & \dfrac{1}{1+t} \end{bmatrix} \text{and } g(t, \zeta(t-\varsigma)) = \begin{bmatrix} \dfrac{2\exp(-t) + 0.1}{\|\zeta(t-\varsigma)\|^2 + 1} \\ \dfrac{2\exp(-t) + 0.1}{\|\zeta(t-\varsigma)\|^2 + 1} \end{bmatrix}. \tag{65}$$

Figure 1 shows the convergence of the solutions toward a neighborhood of the origin with initial condition $(q(0), x(0)) = (0.5, -0.5)$.

4. Conclusion

A nonlinear system with delay employing an indefinite LKF has been investigated for the ISpS and iISpS criteria. Indefinite negative derivatives have been used to generalize the classical LKF. Through the use of the indefinite derivative LKF, some sufficient conditions for verifying the ISpS, iISpS, and UPAS are established.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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