Research Article

Remarks on the Periodic Conformable Sturm–Liouville Problems

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Received 5 March 2023; Revised 10 April 2023; Accepted 21 April 2023; Published 10 May 2023

1. Introduction

In 2014, Khalil et al. [1] introduced a new local derivative, called the conformable fractional derivative $D^\alpha_x$. This concept was quickly adopted by Abdeljawad in [2] where he claimed to have developed some tools of fractional calculus. Recently, Abdelhakim and Machado [3, 4] showed that a function $f$ has a conformable fractional derivative of order $\alpha$ at $x$ if and only if it is differentiable at $x$ and

$$D^\alpha_x f(x) = x^{1-\alpha} \dot{f}(x), \quad (1)$$

holds. From the above, the “conformability” of $D^\alpha_x$ comes precisely from the integer-order derivative, the factor $x^{1-\alpha}$, in (1). Also, Zhao and Luo [5] gave physical and geometrical interpretations of the conformable derivative. They generalized the definition of the conformable derivative to general conformable derivative by means of linear extended Gâteaux derivative and used this definition to demonstrate that the physical interpretation of the conformable derivative is a modification of classical derivative in direction and magnitude. Indeed, $D^\alpha_x$ is a weighted derivative but not a fractional one. Hence, we shall call $D^\alpha_x$ the conformable derivative in this paper.

The purpose of this work is to investigate some eigenvalue properties related to the conformable Sturm–Liouville (CSL) equation

$$-D^\alpha_x (p(x)D^\alpha_x y(x)) + q(x) y(x) = \lambda p(x) y(x), \quad (2)$$

where $D^\alpha_x$ is the conformable derivative of order $\alpha \in (0, 1]$. From the above discussion, (2) is equivalent to

$$-x^{1-\alpha} \left( p(x)x^{1-\alpha} \dot{y}(x) \right)' + q(x)y(x) = \lambda p(x)y(x). \quad (3)$$

Such an equation has been studied in a variety of contexts subject to separated boundary conditions of the form

$$y(0) \cos \beta = p(0)D^\alpha_x y(0) \sin \beta, \quad y(\pi) \cos \gamma = p(\pi)D^\alpha_x y(\pi) \sin \gamma, \quad (4)$$

with $\beta, \gamma \in [0, \pi]$. The basic eigenvalue existence and eigenfunction oscillation theory can be found in many results (see e.g., [6–11]). For the case of coupled (or nonseparated) boundary conditions, there is few work done so far on the existence of eigenvalues and some related properties. Here, we shall consider the periodic/antiperiodic conditions

$$y(0) = \pm y(\pi), \quad D^\alpha_x y(0) = \pm D^\alpha_x y(\pi). \quad (5)$$

Following Hill’s studies of planetary motion in the later part of the 19th century, Sturm–Liouville equations (as $\alpha = 1$) with periodic (or antiperiodic) conditions became of interest, and one can remark that such boundary conditions also appear in the study of wave motion, separation of
variables in classical boundary value problems, etc. A fascinating and interesting idea proposed by Binding and Volkmer [12, 13] demonstrated a method to reduce the periodic or antiperiodic Sturm–Liouville problems to an analysis of the Prüfer angle. This provides a simple and flexible alternative to the usual approaches via operator theory or the Hill discriminant. The Prüfer treatment is a simple and efficient method. It depends on elementary analysis of initial value problems, builds on standard ideas from the case of separated boundary conditions, and is less intricate (and shorter) than the Floquet/Hill theory. Motivated by the above, we intend to employ the method of Prüfer transformations to deduce the existence of periodic/antiperiodic eigenvalues and connect some basic definitions and properties of conformable calculus for the reader's convenience.

**Definition 1 (cf. [1, 2]).** Let $0 < \alpha \leq 1$ and $f:(0,\infty) \to \mathbb{R}$.

(i) The conformable derivative of $f$ of order $a \alpha x > 0$ is defined by

$$D_{x}^{a}\alpha f(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon x^{1-a}) - f(x)}{\epsilon}.$$  

and the conformable derivative at 0 is defined as $D_{x}^{a}\alpha f(0) = \lim_{x \to 0^+} D_{x}^{a}\alpha f(x)$. If $D_{x}^{a}\alpha f(x_0)$ exists, one can say that $f$ is $a$-differentiable at $x_0$.

(ii) The conformable integral of $f$ of order $\alpha$ is defined by

$$I_{n}f(x) = \int_{0}^{x} f(t)d_{n}t = \int_{0}^{x} t^{\alpha-1} f(t)dt,$$  

for $x > 0$.

Note that the space $L_{1}^{\alpha}(0,\pi)$ consists of all functions $f$ satisfying $\int_{0}^{\pi} |f(t)|d_{n}t < \infty$.

**Proposition 1 (cf. [1, 2, 10])**

(i) Let $f:(0,\infty) \to \mathbb{R}$ be any continuous function. Then, for all $x > 0$, we have

$$D_{x}^{a}\alpha I_{n}f(x) = f(x).$$

(ii) Let $f:(0,b) \to \mathbb{R}$ be differentiable. Then, for $x > 0$, we have

$$I_{n}D_{x}^{a}\alpha f(x) = f(x) - f(0).$$

(iii) For all $p \in \mathbb{R}$, $D_{x}^{a}\alpha (x^{p}) = px^{p-\alpha}$.

(iv) Let $f$, $g$: $(0,\infty) \to \mathbb{R}$ be $a$-differentiable. Then,

$$D_{x}^{a}\alpha (fg) = (D_{x}^{a}\alpha f)g + f(D_{x}^{a}\alpha g),$$

$$D_{x}^{a}\alpha \left( \frac{f}{g} \right) = \frac{(D_{x}^{a}\alpha f)g - f(D_{x}^{a}\alpha g)}{g^{2}} \quad \text{with} \quad g \neq 0.$$  

(v) ($\alpha$-chain rule). Let $f$, $g$: $(0,\infty) \to \mathbb{R}$ be $a$-differentiable and $h(x) = f(g(x))$. Then, $h(x)$ is $a$-differentiable, and for all $x$ with $x \neq 0$ and $g(x) \neq 0$, we have

$$D_{x}^{a}\alpha h(x) = D_{x}^{a}\alpha f(g(x)) \cdot D_{x}^{a}\alpha g(x) \cdot g(x)^{\alpha-1}.$$  

(vi) ($\alpha$-integration by parts). Let $f$, $g$: $[a,b] \to \mathbb{R}$ be two functions such that $fg$ is differentiable. Then,

$$\int_{a}^{b} x^{\alpha-1} f(x)D_{x}^{a}\alpha g(x)dx = f(x)g(x)|_{a}^{b} - \int_{a}^{b} x^{\alpha-1} g(x)D_{x}^{a}\alpha f(x)dx.$$

2. Some Preliminaries and Prüfer Substitution

The conformable calculus [1–4, 6, 14–17] is defined and well-studied from 2014. In this section, we first recall the elementary definitions and properties of conformable calculus for the reader’s convenience.
Next, we apply the Prüfer substitution to deduce the existence of various eigenvalues for (2). First consider (2) coupled with the separated boundary conditions

\[
y(0) \cos \beta = p(0)D_y^n y(0) \sin \beta, \quad y(\pi) \cos y = p(\pi)D_y^n y(\pi) \sin y,
\]

here employ the substitution for a nonzero solution \( y \) of (2) taking the form

\[
y = r \sin \theta, \\
pD_y^n y = r \cos \theta.
\]

Similar manipulation as in the classical case \( \alpha = 1 \) gives

\[
\theta' = x^{a-1} \left[ \frac{\cos^2 \theta}{p} + (\lambda \rho - q) \sin^2 \theta \right], \\
r' = x^{a-1} \left( \frac{1}{p} - \lambda \rho + q \right) \sin \theta \cos \theta,
\]

with the initial conditions

\[
\theta(0, \beta, \lambda) = \beta \in \mathbb{R}, \\
r(0, \beta, \lambda) = 1,
\]

(16)

(19)

Next, we quote a result.

Lemma 1 (cf. [9], Lemma 2.4). For the phase function \( \theta(\cdot, \beta, \lambda) \) of (2), the following is valid.

(i) The function \( \theta(\cdot, \beta, \lambda) \) is continuous and strictly increasing in \( \lambda \).

(ii) If \( \theta(x_n, \beta, \lambda) = n\pi \), then \( \theta(x, \beta, \lambda) > n\pi \) for all \( x > x_n \).

(iii) For any \( a \in (0, \pi] \),

\[
\lim_{\lambda \to \infty} \theta(a, \beta, \lambda) = \infty, \\
\lim_{\lambda \to -\infty} \theta(a, \beta, \lambda) = 0.
\]

The above suffices to give existence of a unique \( \lambda_k = \lambda_k(\beta, y) \) with oscillation count \( k \) for each \( k \geq 0 \) (except, that \( k \geq 1 \) when \( y = 0 \)). Now for (15) and the given \( \beta \), the eigenvalue condition gives

\[
\theta(\pi, \beta, \lambda) = y + k\pi, \text{ for } \beta, y \in (0, \pi).
\]

In this paper, we define \( k \) as the number of zeros in \((0, \pi]\) for the eigenfunction corresponding to \( \lambda_k \). From Lemma 1, for fixed \( \beta \) and \( y \), the unique existence of \( \lambda_k(\beta, y) \) with oscillation count \( k \) for \( k \geq 0 \) is valid.

3. Periodic/Antiperiodic Eigenvalues and Connections to Other Approaches

In this section, we will prove the existence of periodic/antiperiodic eigenvalues and connect the approach with a version of conformable Floquet/Hill theory. Now, we prepare some groundwork for this issue. For any fixed \( \lambda \), \( \theta(x, \beta, \lambda) \) is \( C^1 \) in \( \beta \), and \( \theta_\beta = \partial \theta / \partial \beta \) satisfies

\[
\theta'_\beta = -2x^{a-1} \theta_p \left( \frac{1}{p} - \lambda \rho + q \right) \sin \theta \cos \theta \text{ with } \theta_\beta(0, \beta, \lambda) = 1,
\]

by (17) and (19). Applying (18) and (22), one can obtain

\[
\left( r^2 \theta_\beta \right)' = 2rr' \theta_\beta + r^2 \theta'_\beta = 0.
\]

Hence,

\[
r^2 \theta_\beta = 1,
\]

holds whenever the solutions of (17) and (18) exist. By applying (16) and (19) (where \( \beta \) is as yet undetermined), (5) can be written as

\[
r(\pi, \beta, \lambda) = 1,
\]

(25)

where \( k \) is even (resp. odd) for a periodic (resp. antiperiodic) condition. Then, (24) and (25) yield

\[
\theta_\beta(\pi, \beta, \lambda) = 1,
\]

(27)

so the eigenvalues can be found by studying the Prüfer angle \( \theta \) without the radius \( r \). (This means that (25) holds if the Prüfer angle satisfies the right endpoint condition (26).) Indeed, it is useful to define a function \( \delta(\cdot, \lambda) : \beta \longrightarrow \delta(\beta, \lambda) \) by

\[
\delta(\beta, \lambda) = (\theta(\pi, \beta, \lambda) - \beta).
\]

Then, the periodic (or antiperiodic) eigenvalue condition implies that

\[
\delta(\beta, \lambda) = k\pi,
\]

(29)

That is, a real number \( \lambda \) is a periodic (or antiperiodic) eigenvalue if and only if \( k\pi \) is a critical value of the continuously differentiable function \( \delta(\cdot, \lambda) \). Besides, it is obvious that

\[
\theta(x, \beta + \pi, \lambda) = \theta(x, \beta, \lambda) + \pi.
\]

Now, one can extend the definition of \( \delta \) from \( \beta \in (0, \pi] \) to \( \beta \in \mathbb{R} \), and then, \( \delta \) is \( \pi \)-periodic in \( \beta \). The above suggests the introduction of
\[ m(\lambda) = \min_{\beta \in \mathbb{R}} \delta(\beta, \lambda), \tag{31} \]
\[ M(\lambda) = \max_{\beta \in \mathbb{R}} \delta(\beta, \lambda). \tag{32} \]

Then, we can derive the following.

**Lemma 2.** For the functions \( m \) and \( M \) defined as in (31) and (32), the following is valid.

(i) \( m \) and \( M \) are continuous and strictly increasing in \( \lambda \).

(ii) For each \( \lambda \),
\[ -\pi < m(\lambda) \leq M(\lambda) < m(\lambda) + \pi. \tag{33} \]

(iii) \( \lim_{\lambda \to -\infty} m(\lambda) = -\pi \), \( \lim_{\lambda \to -\infty} M(\lambda) = 0 \), and \( \lim_{\lambda \to +\infty} m(\lambda) = +\infty \).

**Proof**

(i) This follows from Lemma 1 (i) directly.

(ii) For \( \beta \in [0, \pi] \), it is obvious that
\[ \delta(\beta, \lambda) = \theta(\pi, \beta, \lambda) - \beta \geq -\beta > -\pi. \tag{34} \]
so \( m(\lambda) > -\pi \). For a fixed \( \lambda \), choose \( \beta_0 \) such that \( m(\lambda) = \delta(\beta_0, \lambda) \), and let \( \beta \in (\beta_0, \beta_0 + \pi] \). By (30) and \( \theta_j(\pi, \beta, \lambda) = 1 \), one gets
\[ \delta(\beta, \lambda) \leq \delta(\pi, \beta_0, \lambda) = \theta(\pi, \beta_0, \lambda) + \pi - \beta \]
\[ = \delta(\beta_0, \lambda) + \pi + \beta_0 - \beta < m(\lambda) + \pi. \tag{35} \]
This shows that \( M(\lambda) < m(\lambda) + \pi \).

(iii) Note that
\[ M(\lambda) \geq \theta(\pi, 0, \lambda) > 0. \tag{36} \]
For any \( \beta \in [0, \pi] \),
\[ -\pi < m(\lambda) \leq \theta(\pi, \beta, \lambda) - \beta, \tag{37} \]
so \( m(\lambda) \to -\pi \) as \( \lambda \to -\infty \) by (20). From (36) and (ii), the second limit follows. Also, (20) gives \( M(\lambda) \to +\infty \) as \( \lambda \to +\infty \) so the third limit follows from (ii). \( \square \)

Now, it suffices to deduce the existence of periodic (or antiperiodic) eigenvalues. By Lemma 2, \( m \) (resp. \( M \)) can attain each \( k\pi \) for \( k \geq 0 \) (resp. \( k \geq 1 \)) so one can define intervals
\[ I_k = \{ \lambda : \delta(\beta, \lambda) = k\pi \text{ for some } \beta \}, \tag{38} \]
\[ = \{ \lambda : m(\lambda) \leq k\pi \leq M(\lambda) \}. \]

Also, the end points of \( I_k \) are denoted by \( \lambda_k^- \leq \lambda_k^+ \) for each \( k \geq 0 \). Apart from \( \lambda_0^- = -\infty \), each \( \lambda_k^+ \) is finite, and
\[ m(\lambda_k^+) = 0, \]
\[ m(\lambda_k^+) = k\pi = M(\lambda_k^-), \quad \text{for all } k \geq 1. \tag{39} \]
Now, the end points \( \lambda_k^\pm \) represent the following properties.

**Theorem 2**

(i) Except for \( \lambda_0^- \), each \( \lambda_k^+ \) is an eigenvalue of (2) and (5), with oscillation count \( k \).

(ii) \( \lambda_k^+ \) in (i) are the only eigenvalues of (2) and (5) with oscillation count \( k \).

(iii) The interval \( I_k \) are disjoint, so \( \lambda_k^+ < \lambda_{k+1}^- \) for each \( k \geq 0 \).

**Proof**

(i) By (38) and (39), \( k\pi \) is an extreme value of \( \delta(\beta, \lambda_k^+) \), except for \( \lambda_0^- \), so (29) holds. Hence, the results in (i) are valid.

(ii) Suppose to the contrary that there exists some \( \lambda \) such that \( \delta(\cdot, \lambda) \) has three distinct critical values \( \delta_j \) corresponding to \( \beta_j \in [0, \pi] \), \( j = 0, 1, 2 \). Also assume that \( 0 < \beta_0 < \beta_1 < \beta_2 < \pi \). Let \( y_j \) be the solution of (2) with its Prüfer angle \( \theta(x, \beta_j, \lambda) \) and Prüfer radius \( r(x, \beta_j, \lambda) \) as in (16). For abbreviation, set \( \theta_j(x) = \theta(x, \beta_j, \lambda) \) and \( r_j(x) = r(x, \beta_j, \lambda) \). From the Prüfer substitution (16), consider the conformable Wronskian
\[ W_a(y_1, y_0)(x) = y_1(x) \cdot p(x)D_y^x y_0(x) - y_0(x) \cdot p(x)D_y^x y_1(x) \]
\[ = r_0(x)r_1(x)\sin(\theta_1(x) - \theta_0(x)). \tag{40} \]

Note that
\[ D_y^x W_a(y_1, y_0) = D_y^x y_1 \cdot pD_y^x y_0 + y_1 \cdot D_y^x (pD_y^x y_0) \]
\[ - D_y^x y_0 \cdot pD_y^x y_1 + y_0 \cdot D_y^x (pD_y^x y_1) = 0, \tag{41} \]
by using (2). Hence, \( W_a(y_1, y_0)(x) \) is constant. Comparing the values at \( x = 0 \) and \( x = \pi \) and noting that \( r_j(0) = r_j(\pi) = 1 \) by (19) and (24), one can obtain
\[ \sin(\theta_1(\pi) - \theta_0(\pi)) = \sin(\theta_1(0) - \theta_0(0)) = \sin(\beta_1 - \beta_0). \tag{42} \]
Now \( \beta_0 < \beta_1 < \beta_0 + \pi \), so
\[ \theta(\pi, \beta_0, \lambda) < \theta(\pi, \beta_1, \lambda) < \theta(\pi, \beta_0 + \pi, \lambda) \]
\[ = \theta(\pi, \beta_0, \lambda) + \pi. \tag{43} \]
by (30) and \( \partial\theta/\partial\beta|_{\beta=\pi} = 1 \). That is,
\[ \theta_0(\pi) < \theta_1(\pi) < \theta_0(\pi) + \pi. \tag{44} \]
implies

Complexity

From \( \delta_0 \neq \delta_1 \), one knows \( \theta_1 (\pi) - \theta_0 (\pi) \neq \beta_1 - \beta_0 \) by (28). Then, (36) yields

\[
\theta_1 (\pi) - \theta_0 (\pi) + \beta_1 - \beta_0 = \pi. \tag{45}
\]

Similarly,

\[
\begin{align*}
\theta_2 (\pi) - \theta_2 (\pi) + \beta_2 - \beta_1 &= \pi, \\
\theta_0 (\pi) - \theta_0 (\pi) + \beta_0 - \beta_2 &= -\pi,
\end{align*} \tag{46}
\]

and addition of the above three equations gives the contradiction 0 = \pi. This proves (ii).

(iii) Suppose not, let \( \lambda_k^+ \geq \lambda_{k+1}^- \) for some \( k \geq 0 \). Then, by the monotonicity of \( M \) and Lemma 2 (ii),

\[ M (\lambda_k^+) > M (\lambda_{k+1}^-) = k\pi + \pi = m (\lambda_k^+) + \pi > M (\lambda_k^+). \tag{47} \]

This reaches a contradiction. \( \square \)

Now, recall that \( \lambda = \lambda_k (\beta) \) satisfies \( \delta (\beta, \lambda) = k\pi \). This implies \( \lambda_k (\beta) \in I_0 \). Then, Theorem 2 yields Theorem 1.

Next, we will conclude by connecting our approach with a version of the conformable Floquet/Hill theory. Here, we seek nontrivial solutions of (2) that satisfy

\[
y (0) = \omega y (\pi),
\]

\[
D_y y (0) = \omega D_y y (\pi),
\]

which evidently generalizes (5), for some complex “Floquet multipliers” \( \omega \). Now assume that \( y_1 (x, \lambda) \) and \( y_2 (x, \lambda) \) are solutions of (2) satisfying

\[
y_1 (0, \lambda) = D_x y_1 (0, \lambda) = 0,
\]

\[
y_2 (0, \lambda) = D_x y_2 (0, \lambda) = 1. \tag{48}
\]

To determine the multipliers, the solution \( C_1 y_1 + C_2 y_2 \) is considered to satisfy (48), which yields

\[
\begin{align*}
C_1 y_1 (\pi, \lambda) + C_2 y_2 (\pi, \lambda) - \omega &= 0, \\
C_1 (D_x y_1 (\pi, \lambda) - \omega) + C_2 D_x y_2 (\pi, \lambda) &= 0. \tag{50}
\end{align*}
\]

For a nontrivial solution \( (C_1, C_2) \) to exist, the determinant of the coefficients must vanish. That is,

\[
\omega^2 - [y_2 (\pi, \lambda) + D_x y_1 (\pi, \lambda)] \omega + (D_x y_1 (\pi, \lambda) - y_2 (\pi, \lambda) - y_1 (\pi, \lambda) - D_x y_2 (\pi, \lambda)) = 0. \tag{51}
\]

From constancy of \( W_a (y_1, y_2) \) with \( p (0) = p (\pi) \), one can obtain the quadratic equation

\[
\omega^2 - d (\lambda) \omega + 1 = 0, \tag{52}
\]

where \( d (\lambda) = y_2 (\pi, \lambda) + D_x y_1 (\pi, \lambda) \) is Hill’s discriminant for (2). The roots \( \omega_1 \) and \( \omega_2 \) of (52) are distinct complex conjugates of magnitude 1 if

\[
|d (\lambda)| < 2. \tag{53}
\]

Now recall \( D_x (e^{i\alpha x}) = ce^{i\alpha x} \). In this case, two independent solutions exist, \( y_1 = ut_1 (x)e^{i\alpha x} \) and \( y_2 = ut_2 (x)e^{i\alpha x} \), where \( u_i (i = 1, 2) \) are periodic of period \( \pi \) and \( e^{i\alpha x} = \omega_i \). Thus, if (53) holds, all solutions of (2) are uniformly bounded on \((-\infty, \infty)\). Therefore, the value of \( \lambda \) for which (53) holds will be called the stability interval. Conversely, those \( \lambda \) for \( |d (\lambda)| > 2 \) can be called the instability interval. Next, from (52), we intend to distinguish the possibilities of \( \omega \) via the Prüfer angle. By (38), define intervals \( CI_k \) complementary to \( I_k \) by

\[
CI_k = (\lambda_k^+, \lambda_{k+1}^-) = \{ \lambda : k\pi < \delta (\beta, \lambda) < (k + 1)\pi, \quad \quad \text{for } \beta \in [0, \pi] \}. \tag{54}
\]

The following results are related to the Floquet multipliers connecting with \( I_k \) and \( CI_k \).

**Theorem 3.** The Floquet multipliers \( \omega_1 \) and \( \omega_2 \) corresponding \( \lambda \) are real (resp. nonreal) if and only if there exists an integer \( k \geq 0 \) such that \( \lambda \in I_k \) (resp. \( \lambda \in CI_k \)).

**Proof.** For some \( \lambda \), let \( y \) be a solution of (2) satisfying (48) with \( \omega \in \mathbb{R} \). Without loss of generality, one can assume that \( y (0) = \sin \beta \) and \( p (0) D_x y (0) = \cos \beta \) for some \( \beta \in [0, \pi] \). If (48) holds, then one gets \( \theta (\pi, \beta, \lambda) = \beta + k\pi \) for some integer \( k \geq 0 \). This means that \( \lambda \in I_k \). Conversely, if \( \lambda \in I_k \), there then exists \( \beta \) such that \( \theta (\pi, \beta, \lambda) = \beta + k\pi \). By the initial condition (19), one can see that (48) is satisfied with \( \omega \cdot r (\pi, \beta, \lambda) = (-1)^k \). This shows that \( \omega \in \mathbb{R} \). \( \square \)

**Theorem 4.** The stability and instability intervals correspond to \( CI_k \) and the interiors of \( I_k \), respectively.

**Proof.** By the above discussion of \( d (\lambda), \lambda \) for \( |d (\lambda)| < 2 \) (resp. \( |d (\lambda)| > 2 \)) lies in the stability interval (resp. instability interval). Besides, \( \omega_1 = \omega_2 \) equals \( \pm 1 \) in the case \( |d (\lambda)| = 2 \). So (5) holds, and \( \lambda \) is one of the endpoints as in Theorem 2. Putting these cases together with Theorem 3, one can complete the proof. \( \square \)

### 4. Conclusion

In this paper, we consider the periodic/antiperiodic conformable Sturm–Liouville problem (CSLP). We employ the Prüfer transformation to reduce the periodic or antiperiodic (CSLP) to an analysis of the Prüfer angle. By the efficiency of this method, we give the inequalities which interlace the eigenvalues corresponding to separated and coupled boundary conditions. We also conclude by connecting our approach with a version of conformable Floquet/Hill theory and characterize the so-called stability and instability intervals in terms of the Prüfer angle [18–20].

### Data Availability

All the data used to support the findings of this study are included within the article.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

References
