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Research Article

Homogeneity Test of Response Rate Functions in Bilateral Correlated Data under Dallal's Model

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In clinical studies, paired binary metadata are often encountered during collecting information from paired organs or parts. The risk difference, relative risk ratio, and odds ratios of response rates are widely used to measure cured effects for such data. Under Dallal's model, the paper extends these measures to the general case in g groups. We propose eight test statistics for various functions when the dependency measures $\gamma_i\,(i = 1, \ldots, g)$ are equal or not. Simulation results show that the Score and Rosner-type tests can produce robust empirical type I error rates, while Wald-type and likelihood ratio tests have better power in testing the risk difference, relative risk ratio, and odds ratios. Finally, two real examples illustrate our proposed method's practicability.

1. Introduction

Researchers often obtain bilateral data from patients' paired organs in clinical studies. For example, in ophthalmology, patients are randomly divided into g treatment groups. All clinical outcomes of patients' two eyes could be summarized as unilateral, bilateral, and no response(s). Since ignoring the correlation of paired data may bring about a series of misleading results, it is essential to consider this issue. On this account, many corresponding statistical models have been developed by statisticians. Rosner [1] proposed a classical intercorrelation model assuming that the dependence between two eyes is a constant R. That is to say, with one eye responding, the probability of another eye responding is R times that of an unconditional one. Until now, there have been valuable results on the homogeneity test under Rosner's model. Tang et al. [2] developed eight statistics to test the equality of response rates in two groups. Ulteriorly, Ma et al. [3] proposed asymptotic test statistics to analyze the abovementioned problems in the case of g groups ($g \geq 2$), which can be seen as the generalization of results obtained in [2]. Furthermore, the rate difference and ratio tests were investigated between two proportions of the bilateral data in [4, 5].

However, Dallal [6] designated that Rosner's model would give ill-fitting results if the bilateral response occurred almost certainly under the population-specific prevalence rate. Given this, Dallal proposed a substitutable model assuming that the probability of one organ responding is independent of the probability of another organ responding. In other words, $\Pr(Z_{ijk} = 1 \mid Z_{ij(3-k)} = 1) = \gamma_i$, where $Z_{ijk} = 1$ is the kth eye response of the jth patient in the i group, where $i = 1, \ldots, g$, $j = 1, 2, \ldots, m_i$, and $k = 1, 2$. Further research has shown that Dallal's model is more suitable for the correlated data in [7]. Therefore, the correlational research on Dallal's model has received considerable attention, and some interesting results have been reported in [8–10]. In the case of $\gamma_1 = \gamma_2 = \gamma$, MIan et al. [8] proposed three objective Bayesian methods to investigate the risk difference, relative risk ratio, and odds ratio of paired data. Chen et al. [9] put forward eight statistics to test whether the response rates of each treatment group were equal when the parameter $\gamma_i\,(i = 1, \ldots, g)$ is not equal. Moreover, Sun et al. [10] studied the homogeneity test of risk differences between response rates of different treatment groups.

To date, under Rosner's model, numerous results compare the effectiveness of different treatments (i.e., risk difference and relative risk ratio). Significantly, a general function relationship usually represents the response rate between multiple bilateral data. However, hypothesis testing is less considered under Dallal's model, let alone the homogeneity testing with general function relationship under Dallal's model. The above discussion arouses our research interest. For this, this paper proposes a novel hypothesis to test the homogeneity of response rate functions in bilateral data.

The structure of this article is as follows: Section 2 sorts out the data structure, obtains the likelihood function, and establishes the function hypothesis test under Dallal's model according to two cases: (i) the parameter $\gamma_i\,(i = 1, \ldots, g)$ is not equal; (ii) $\gamma_1 = \cdots = \gamma_g = \gamma$. In Section 3, the iterative algorithm calculates the maximum likelihood estimates (MLEs) of unknown parameters in these two cases. In Section 4, we derive the general expressions of eight test statistics in the function hypothesis and present three specific forms in combination with particular problems. In Section 5, simulation experiments are carried out in different cases to compare the performance of the statistics of Section 4 in terms of the empirical type I error rates (TIEs) and power. Section 6 uses two real examples to illuminate our proposed method. The summary conclusion and prospects of the article are given in Section 7.

2. Preliminaries

Assume that the N patients are randomly divided into g groups with different treatments. The number of patients in the ith group is assumed to be m_i $(i = 1, 2, \ldots, g)$; thus, $N = \sum_{i=1}^{g} m_i$. Let m_{li} be the number of patients with exact l $(l = 0,\ 1,\ 2)$ responses in the ith group, then $m_i = \sum_{l=0}^{2} m_{li}$. Moreover, S_l denotes the total number of patients with l responses, then obviously, $S_l = \sum_{i=1}^{g} m_{li}$. Table 1 summarizes the data structure for the patient frequencies.

Let a binary variable $Z_{ijk} = 1$ when the kth site (the kth body site) of the jth subject is disease-free in the ith treatment group, otherwise $Z_{ijk} = 0$ $(i = 0,\ 1,\ j = 1,\ \ldots, m_i,$ and $k = 1,\ 2)$. We take the ophthalmologic diseases under Dallal's model for example. We suppose that $\Pr(Z_{ijk} = 1) = \pi_i$ and $\Pr(Z_{ij1} = 1 \mid Z_{ij2} = 1) = \gamma_i$ for $i = 1, \ldots, g$. Let the probability of no response, one response, and both responses be p_{0i}, p_{1i}, and p_{2i}, then $p_{0i} + p_{1i} + p_{2i} = 1$ for any fixed i. Let $\mathbf{m}_i = (m_{0i}, m_{1i}, m_{2i})$, according to Table 1, the probability density function of the ith group can be written as

$$f\left(\mathbf{m}_i \mid p_{0i}, p_{1i}, p_{2i}\right) = \frac{m_i!}{m_{0i}!m_{1i}!m_{2i}!} p_{0i}^{m_{0i}} p_{1i}^{m_{1i}} p_{2i}^{m_{2i}}, \quad i = 1, 2, \ldots, g. \tag{1}$$

After derivation, we can get $p_{0i} = 1 - 2\pi_i + \pi_i\gamma_i$, $p_{1i} = 2\pi_i(1 - \gamma_i)$, and $p_{2i} = \pi_i\gamma_i$.

The observed data are represented by vector $\mathbf{m} = (m_{01}, m_{11}, m_{21}, \ldots, m_{0g}, m_{1g}, m_{2g})$, and its maximum likelihood function is as follows:

$$L(\boldsymbol{\pi}, \boldsymbol{\gamma} \mid \mathbf{m}) = \prod_{i=1}^{g} f\left(m_i \mid p_{0i}, p_{1i}, p_{2i}\right) = \prod_{i=1}^{g} \frac{m_i!}{m_{0i}!m_{1i}!m_{2i}!} p_{0i}^{m_{0i}} p_{1i}^{m_{1i}} p_{2i}^{m_{2i}}, \tag{2}$$

where $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_g)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_g)$. Then, the log-likelihood function is expressed by

$$l(\boldsymbol{\pi}, \boldsymbol{\gamma} \mid \mathbf{m}) = \sum_{i=1}^{g} \left[m_{0i} \log(1 - 2\pi_i + \pi_i\gamma_i) + m_{1i} \log(2\pi_i(1 - \gamma_i)) + m_{2i} \log(\pi_i\gamma_i)\right] + \log C, \tag{3}$$

where $C = \prod_{i=1}^{g} (m_i!/m_{0i}!m_{1i}!m_{2i}!)$ is a constant and $\pi_i \in [0, (1/2 - \gamma_i)]$.

Unlike previous research, we propose a function hypothesis test, including the risk difference, relative risk ratio, and odds ratio. Let $f_i(x)$ be continuously differentiable and let it have an inverse mapping $f_i^{-1}(x)$. To study the prevalence of ophthalmologic diseases among patients with different treatments, some relevant hypotheses are given as follows:

Case (i): $\gamma_i \neq \gamma_j$ for some $i \neq j \in \{1, \ldots, g\}$, the hypotheses are given by

H_{01}: $f_1(\pi_1) = f_2(\pi_2) = \cdots = f_g(\pi_g)$ versus H_{1a}: $f_i(\pi_i) \neq f_j(\pi_j)$, $i = 1, \ldots, g$.

Case (ii): $\gamma_1 = \gamma_2 = \cdots = \gamma_g = \gamma$, the hypotheses are given by

H_{02}: $f_1(\pi_1) = f_2(\pi_2) = \cdots = f_g(\pi_g)$ versus H_{1b}: $f_i(\pi_i) \neq f_j(\pi_j)$, $i = 1, \ldots, g$.

Table 1: Related data structure for the paired organ.

Responses (l)	Groups (i)				Total
	1	2	...	g	
0	m_{01}	m_{02}	...	m_{0g}	S_0
1	m_{11}	m_{12}	...	m_{1g}	S_1
2	m_{21}	m_{22}	...	m_{2g}	S_2
Total	m_1	m_2	...	m_g	N

3. Parameter Estimation

3.1. MLEs under H_{01} and H_{02}. Taking different function forms of $f_i(x)$ can study various problems in bilateral correlated data. To this end, the functional forms of $f_i(\pi_i)$ are set as follows:

(i) When $f_i(\pi_i) = \pi_i - \varrho_i$, it can be used to investigate the risk difference (RD) of bilateral correlated data. In this case, the null hypothesis is $\pi_1 - \varrho_1 = \pi_2 - \varrho_2 = \cdots = \pi_g - \varrho_g$.

(ii) When $f_i(\pi_i) = (1/\varrho_i)\pi_i$, it can be used to investigate the relative risk ratio (RR) of bilateral correlated data. In this case, the null hypothesis is $(1/\varrho_1)\pi_1 = (1/\varrho_2)\pi_2 = \cdots = (1/\varrho_g)\pi_g$.

(iii) When $f_i(\pi_i) = (1/\varrho_1)\pi_i/1 - \pi_i$, it can be used to investigate the odds ratio (OR) of bilateral correlated data. In this case, the null hypothesis is $(1/\varrho_1)\pi_1/1 - \pi_1 = (1/\varrho_2)\pi_2/1 - \pi_2 = \cdots = (1/\varrho_g)\pi_g /1 - \pi_g$.

Under the null hypothesis H_{01}, we have $f_1(\pi_1) = f_2(\pi_2) = \cdots = f_g(\pi_g) = f(\pi)$. Since $f_i(x)$ has an inverse mapping $f_i^{-1}(x)$, let $g_i(\pi) = f^{-1}f(\pi)$. Then, the log-likelihood function is

$$l_0^a(\pi, \gamma_i \mid \mathbf{m}) = \sum_{i=1}^{g} \left[m_{0i}\log\left(1 - 2g_i(\pi) + g_i(\pi)\gamma_i\right) + m_{1i}\log\left(2g_i(\pi)\left(1-\gamma_i\right)\right) + m_{2i}\log\left(g_i(\pi)\gamma_i\right)\right] + \log C, \quad (4)$$

where $C = \prod_{i=1}^{g}(m_i!/m_{0i}!m_{1i}!m_{2i}!)$ is a constant. To determine the MLEs of parameters, we set the partial differentiations equal to zero, i.e.,

$$\frac{\partial l_0^a}{\partial \pi} = 0, \qquad \frac{\partial l_0^a}{\partial \gamma_i} = 0, \quad (5)$$

where

$$\frac{\partial l_0^a}{\partial \pi} = \sum_{i=1}^{g} \frac{\partial g_i(\pi)}{\partial \pi}\left[\frac{m_{0i}(\gamma_i - 2)}{1 - 2g_i(\pi) + g_i(\pi)\gamma_i} + \frac{m_{1i} + m_{2i}}{g_i(\pi)}\right],$$
$$\frac{\partial l_0^a}{\partial \gamma_i} = \frac{m_{0i}g_i(\pi)}{1 - 2g_i(\pi) + g_i(\pi)\gamma_i} - \frac{m_{1i}}{1-\gamma_i} + \frac{m_{2i}}{\gamma_i}. \quad (6)$$

Since there is no explicit solution for the abovementioned equation, we use the Newton–Raphson algorithm to solve the equations. The solution to the abovementioned equations is obtained by using the two-step algorithm:

(1) We take the initial value $\pi^{(0)} = \widehat{\pi}_i^a$ and $\gamma^{(0)} = \widehat{\gamma}_i^a\,(i = 1, \ldots, g)$, where $(\partial l_0^a/\partial \gamma_i) = 0$ can be reduced to a second-order polynomial:

$$m_i g_i(\pi)\gamma_i^2 - \left[g_i(\pi)\left(m_i + m_{1i} + 2m_{2i}\right) - \left(m_{1i} + m_{2i}\right)\right]\gamma_i + m_{2i}\left(2g_i(\pi) - 1\right) = 0. \quad (7)$$

(2) The $(t+1)$ update $\pi^{(t+1)}$ of π is given by the Newton–Raphson algorithm as follows:

$$\pi^{(t+1)} = \pi^{(t)} - \left(\frac{\partial^2 l_0^a}{\partial \pi^2}\right)^{-1}\frac{\partial l_0^a}{\partial \pi}\bigg|_{\gamma_i=\gamma_i^{(t)},\pi=\pi^{(t)}}, \quad (8)$$

where

$$\frac{\partial^2 l_0^a}{\partial \pi^2} = -\sum_{i=1}^{g}\left(\frac{\partial g_i(\pi)}{\partial \pi}\right)^2\left[\frac{m_{0i}(\gamma_i-2)^2}{(g_i(\pi)\gamma_i-2g_i(\pi)+1)^2}+\frac{m_{1i}+m_{2i}}{g_i^2(\pi)}\right] + \sum_{i=1}^{g}\frac{\partial^2 g_i(\pi)}{\partial \pi^2}\left[\frac{m_{0i}(\gamma_i-2)}{g_i(\pi)\gamma_i-2g_i(\pi)+1}+\frac{m_{1i}+m_{2i}}{g_i(\pi)}\right]. \tag{9}$$

For H_{02}, we have $f_1(\pi_1) = f_2(\pi_2) = \cdots = f_g(\pi_g)$, as well as $\gamma_1 = \gamma_2 = \cdots = \gamma_g = \gamma$. There are only two unknown parameters π and γ. In this case, the corresponding log-likelihood function can be expressed as

$$l_0^b(\pi,\gamma \mid \mathbf{m}) = \sum_{i=2}^{g}\left[m_{0i}\log(1-2g_i(\pi)+g_i(\pi)\gamma)+m_{1i}\log(2g_i(\pi)(1-\gamma))+m_{2i}\log(g_i(\pi)\gamma)\right]+\log C, \tag{10}$$

where $C = \prod_{i=1}^{g}(m_i!/m_{0i}!m_{1i}!m_{2i}!)$ is a constant. To obtain the MLEs of parameters, we take the first partial derivative of each parameter equal to zero, i.e.,

$$\frac{\partial l_0^b}{\partial \pi} = 0, \qquad \frac{\partial l_0^b}{\partial \gamma} = 0, \tag{11}$$

where

$$\frac{\partial l_0^b}{\partial \pi} = \sum_{i=1}^{g}\frac{\partial g_i(\pi)}{\partial \pi}\left[\frac{m_{0i}(\gamma-2)}{1-2g_i(\pi)+g_i(\pi)\gamma}+\frac{m_{1i}+m_{2i}}{g_i(\pi)}\right],$$
$$\frac{\partial l_0^b}{\partial \gamma} = \sum_{i=1}^{g}\frac{m_{0i}g_i(\pi)}{1-2g_i(\pi)+g_i(\pi)\gamma}-\frac{m_{1i}}{1-\gamma}+\frac{m_{2i}}{\gamma}. \tag{12}$$

When $g_i(\pi) = \varrho_i\pi_i\,(\varrho_1 = \varrho_2 = \cdots = \varrho_g = \varrho)$, we have

$$\tilde{\pi}^b = \frac{S_1+S_2}{2\varrho N}, \qquad \tilde{\gamma}^b = \frac{2S_2}{S_1+2S_2}. \tag{13}$$

When the function is complex, the system of equations may have no explicit solution. In this case, the Fisher-score algorithm is used to solve the equations because of its strong convergence. The $(t+1)$ update of π, γ is given by the Fisher-score algorithm as follows:

$$\begin{bmatrix}\pi^{(t+1)}\\ \gamma^{(t+1)}\end{bmatrix} = \begin{bmatrix}\pi^{(t)}\\ \gamma^{(t)}\end{bmatrix} + [I]^{-\pi}(\pi,\gamma)\times\left.\begin{bmatrix}\dfrac{\partial l_0^b}{\partial \pi}\\ \dfrac{\partial l_0^b}{\partial \gamma}\end{bmatrix}\right|_{\pi=\pi^{(t)},\gamma=\gamma^{(t)}}, \tag{14}$$

where I^{-1} is the inverse matrix of the Fisher information matrix I satisfying

$$I = -\begin{bmatrix} E\left(\dfrac{\partial^2 l_0^b}{\partial \pi^2}\right) & E\left(\dfrac{\partial^2 l_0^b}{\partial \pi\partial\gamma}\right)\\ E\left(\dfrac{\partial^2 l_0^b}{\partial \pi\partial\gamma}\right) & E\left(\dfrac{\partial^2 l_0^b}{\partial \gamma^2}\right)\end{bmatrix}, \tag{15}$$

where

$$E\left(\frac{\partial^2 l_b^0}{\partial \pi^2}\right) = -\sum_{i=1}^{g}\left(\frac{\partial g_i(\pi)}{\partial \pi}\right)^2\left[\frac{m_i(\gamma-2)^2}{1-2g_i(\pi)+g_i(\pi)\gamma} - \frac{m_i(\gamma-2)}{g_i(\pi)}\right],$$

$$E\left(\frac{\partial^2 l_b^0}{\partial \pi \partial \gamma}\right) = \sum_{i=1}^{g}\frac{\partial g_i(\pi)}{\partial \pi}\,\frac{m_i}{1-2g_i(\pi)+g_i(\pi)\gamma}, \tag{16}$$

$$E\left(\frac{\partial^2 l_b^0}{\partial \gamma^2}\right) = -\sum_{i=1}^{g}\left[\frac{m_i g_i^2(\pi)}{1-2g_i(\pi)+g_i(\pi)\gamma} + \frac{2m_i\pi_i}{1-\gamma} + \frac{m_i\pi_i}{\gamma}\right].$$

3.2. MLEs under H_{1a} and H_{1b}. Under the alternative hypothesis H_{1a}, let the global MLEs of $\pi_i\,(i=1,\ldots,g)$ and $\gamma_i\,(i=1,\ldots,g)$ be $\widehat{\pi}_i^a$ and $\widehat{\gamma}_i^a$, respectively. Then,

$$\widehat{\pi}_i^a = \frac{m_{1i}+2m_{2i}}{2m_i}, \qquad \widehat{\gamma}_i^a = \frac{2m_i}{m_{1i}+2m_{2i}}. \tag{17}$$

For the alternative hypothesis H_{1b}, we suppose that $\widehat{\pi}_i^b$ and $\widehat{\gamma}^b$ are the global MLEs of $\pi_i\,(i=1,\ldots,g)$ and γ, respectively. Then,

$$\widehat{\pi}_i^b = \frac{m_{1i}+2m_{2i}(S_1+2S_2)}{2m_i}, \qquad \widehat{\gamma}_i^b = \frac{2S_2}{S_1+2S_2}. \tag{18}$$

4. The Proposed Methods

4.1. Likelihood Ratio Tests. To test the hypothesis H_{01}, the likelihood ratio test statistic is given by

$$T_L^a = 2\left[l\left(\widehat{\boldsymbol{\pi}}^a, \widehat{\boldsymbol{\gamma}}^a \mid \mathbf{m}\right) - l_0^a\left(\widetilde{\boldsymbol{\pi}}^a, \widetilde{\boldsymbol{\gamma}}^a \mid \mathbf{m}\right)\right], \tag{19}$$

where $\widehat{\pi}^a = (\widehat{\pi}_1^a,\ldots,\widehat{\pi}_g^a)$, $\widehat{\gamma}^a = (\widehat{\gamma}_1^a,\ldots,\widehat{\gamma}_g^a)$, $\widetilde{\pi}^a = (g_1(\widetilde{\pi}^a), g_2(\widetilde{\pi}^a),\ldots,g_g(\widetilde{\pi}^a))$, and $\widetilde{\gamma}^a = (\widetilde{\gamma}_1^a,\ldots,\widetilde{\gamma}_g^a)$. It can be simplified as

$$T_L^a = 2\sum_{i=1}^{g}\left[m_{0i}\log\left(\frac{1-2\widehat{\pi}_i^a+\widehat{\pi}_i^a\widehat{\gamma}_i^a}{1-2g_i(\widetilde{\pi}^a)+g_i(\widetilde{\pi}^a)\widetilde{\gamma}_i^a}\right) + m_{1i}\log\left(\frac{\widehat{\pi}_i^a(1-\widehat{\gamma}_i^a)}{g_i(\widetilde{\pi}^a)(1-\widetilde{\gamma}_i^a)}\right) + m_{2i}\log\left(\frac{\widehat{\pi}_i^a\widehat{\gamma}_i^a}{g_i(\widetilde{\pi}^a)\widetilde{\gamma}_i^a}\right)\right]. \tag{20}$$

For convenience, we next provide three special expressions of the likelihood ratio test under the risk difference, relative risk ratio, and odds ratio.

(i) RD: $\pi_i = g_i(\pi) = \pi + \varrho_i\,(i=1,\ldots,g)$. From (20), it is easy to get

$$T_L^a = 2\sum_{i=1}^{g}\left[m_{0i}\log\left(\frac{1-2\widehat{\pi}_i^a+\widehat{\pi}_i^a\widehat{\gamma}_i^a}{1-(\widetilde{\pi}^a+\varrho_i)(2+\widetilde{\gamma}_i^a)}\right) + m_{1i}\log\left(\frac{\widehat{\pi}_i^a(1-\widehat{\gamma}_i^a)}{(\widetilde{\pi}^a+\varrho_i)(1-\widetilde{\gamma}_i^a)}\right) + m_{2i}\log\left(\frac{\widehat{\pi}_i^a\widehat{\gamma}_i^a}{(\widetilde{\pi}^a+\varrho_i)\widetilde{\gamma}_i^a}\right)\right]. \tag{21}$$

(ii) RR: $\pi_i = g_i(\pi) = \varrho_i\pi\,(i=1,\ldots,g)$. It follows that

$$T_L^a = 2\sum_{i=1}^{g}\left[m_{0i}\log\left(\frac{1-2\widehat{\pi}_i^a+\widehat{\pi}_i^a\widehat{\gamma}_i^a}{1-2\varrho_i\widetilde{\pi}^a+\varrho_i\widetilde{\pi}^a\widetilde{\gamma}_i^a}\right) + m_{1i}\log\left(\frac{\widehat{\pi}_i^a(1-\widehat{\gamma}_i^a)}{\varrho_i\widetilde{\pi}^a(1-\widetilde{\gamma}_i^a)}\right) + m_{2i}\log\left(\frac{\widehat{\pi}_i^a\widehat{\gamma}_i^a}{\varrho_i\widetilde{\pi}^a\widetilde{\gamma}_i^a}\right)\right]. \tag{22}$$

(iii) OR: $\pi_i = g_i(\pi) = (\varrho_i\pi)/(1-\pi+\varrho_i\pi)\,(i=1,\ldots,g)$. Through (20), we have

$$T_L^a = 2\sum_{i=1}^{g}\left[m_{0i}\log\left(\frac{(1-\tilde{\pi}^a-\varrho_i\tilde{\pi}^a)(1-2\hat{\pi}_i^a+\hat{\pi}_i^a\hat{\gamma}_i^a)}{1-\tilde{\pi}^a-\varrho_i\tilde{\pi}^a-2\varrho_i\tilde{\pi}^a+\varrho_i\tilde{\pi}^a\tilde{\gamma}_i^a}\right)+m_{1i}\log\left(\frac{\hat{\pi}_i^a(1-\tilde{\pi}^a-\varrho_i\tilde{\pi}^a)(1-\hat{\gamma}_i^a)}{\varrho_i\tilde{\pi}^a(1-\tilde{\gamma}_i^a)}\right)\right.$$
$$\left.+m_{2i}\log\left(\frac{(1-\tilde{\pi}^a-\varrho_i\tilde{\pi}^a)(\hat{\pi}_i^a\hat{\gamma}_i^a)}{\varrho_i\tilde{\pi}^a\tilde{\gamma}_i^a}\right)\right]. \tag{23}$$

For testing the hypothesis H_{02}, the likelihood ratio test statistic is given by

$$T_L^b = 2\left[l\left(\hat{\boldsymbol{\pi}}^b,\hat{\gamma}^b\mid\mathbf{m}\right)-l_0^b\left(\tilde{\boldsymbol{\pi}}^b,\tilde{\gamma}^b\mid\mathbf{m}\right)\right], \tag{24}$$

where $\hat{\boldsymbol{\pi}}^b=(\hat{\pi}_1^b,\ldots,\hat{\pi}_g^b)$ and $\tilde{\boldsymbol{\pi}}^b=(g_1(\tilde{\pi}^b),g_2(\tilde{\pi}^b),\ldots,g_g(\tilde{\pi}^b))$. It can be simplified as

$$T_L^b = 2\sum_{i=1}^{g}\left[m_{0i}\log\left(\frac{1-2\hat{\pi}_i^b+\hat{\pi}_i^b\hat{\gamma}_i^b}{1-2g_i(\tilde{\pi}^b)+g_i(\tilde{\pi}^b)\tilde{\gamma}_i^b}\right)+m_{1i}\log\left(\frac{\hat{\pi}_i^b(1-\hat{\gamma}_i^b)}{g_i(\tilde{\pi}^b)(1-\tilde{\gamma}_i^b)}\right)+m_{2i}\log\left(\frac{\hat{\pi}_i^b\hat{\gamma}_i^b}{g_i(\tilde{\pi}^b)\tilde{\gamma}_i^b}\right)\right]. \tag{25}$$

Similar to T_L^a, $g_i(\pi)$ can be given different forms to obtain T_L^b corresponding to three shapes of the risk difference, risk ratio, and odds ratio. Under H_{01} and H_{02}, T_L^a and T_L^b are, respectively, asymptotically distributed as the chi-squared distribution with $g-1$ degrees of freedom [11].

4.2. Wald-Type Tests. The null hypothesis H_{01}: $f_1(\pi_1)=f_2(\pi_2)=\cdots=f_g(\pi_g)$ is equivalent to H_{01}: $f_1(\pi_1)-f_2(\pi_2)=f_2(\pi_2)-f_3(\pi_3)=\cdots=f_{g-1}(\pi_{g-1})-f_g(\pi_g)=0$. For $f_i(\pi_i)-f_{i+1}(\pi_{i+1})=0$, that is, $\pi_i-f_i^{-1}f_{i+1}(\pi_{i+1})=0$. Then, the null hypothesis H_{01} can be written as $\mathbf{C}_1\boldsymbol{\beta}_1^T=0$ in the matrix form, where $\boldsymbol{\beta}_1=(\pi_1,\ldots,\pi_g,\gamma_1,\ldots,\gamma_g)$ and

$$\mathbf{C}_1=\begin{pmatrix} 1 & -\frac{f_1^{-1}f_2(\pi_2)}{\pi_2} & 0 & \cdots\cdots & 0 & 0 & \cdots\cdots\cdots\cdots & 0\\ 0 & 1 & -\frac{f_2^{-1}f_3(\pi_3)}{\pi_3} & \cdots\cdots & 0 & 0 & \cdots\cdots\cdots\cdots & 0\\ \vdots & \ddots & \ddots & \ddots\ \ddots & \vdots & \vdots & & \vdots\\ \vdots & \ddots & \ddots & \ddots\ \ddots & \vdots & \vdots & & \vdots\\ 0 & \cdots & \cdots & 0\quad 1 & -\frac{f_{g-1}^{-1}f_g(\pi_g)}{\pi_g} & 0 & \cdots\cdots\cdots\cdots & 0 \end{pmatrix}_{(g-1)\times(2g)}. \tag{26}$$

Let $\hat{\boldsymbol{\beta}}_1=(\hat{\pi}_1^a,\ldots,\hat{\pi}_g^a,\hat{\gamma}_1^a,\ldots,\hat{\gamma}_g^a)$. Following Agresti [14], the Wald-type test statistic is given by

$$T_W^a=\left(\boldsymbol{\beta}_1\mathbf{C}_1^T\right)\left(\mathbf{C}_1\mathbf{I}_1^{-1}\mathbf{C}_1^{\mathrm{T}}\right)^{-1}\left(\mathbf{C}_1\boldsymbol{\beta}_1^{\mathrm{T}}\right)\Big|_{\boldsymbol{\beta}_1=\hat{\boldsymbol{\beta}}_1}, \tag{27}$$

where $\mathbf{I_1}$ is the information matrix in the Appendix A. By calculation, we have

$$\mathbf{I_1^{-1}}=\begin{pmatrix} a_1 & \cdots & 0 & b_1 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & a_g & 0 & \cdots & b_g\\ b_1 & \cdots & 0 & c_1 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & b_g & 0 & \cdots & c_g \end{pmatrix}_{(2g)\times(2g)}, \tag{28}$$

where

$$a_i = \frac{\widehat{\pi}_i^a\left(1+\widehat{\gamma}_i^a-2\widehat{\pi}_i^a\right)}{2m_i}, b_i = \frac{-\widehat{\gamma}_i^a\left(\widehat{\gamma}_i^a-1\right)}{2m_i}, c_i = \frac{\widehat{\gamma}_i^a\left(\widehat{\gamma}_i^a-1\right)\left(\widehat{\gamma}_i^a-2\right)}{2m_i\widehat{\pi}_i^a}. \quad (29)$$

We denote $\mathbf{A}_1 \triangleq \mathbf{C}_1\mathbf{I}_1^{-1}\mathbf{C}_1^T$. Then,

$$\mathbf{A}_1 = \begin{pmatrix} \widehat{\mu}_1+\widehat{e}_2^2\widehat{\mu}_2 & -\widehat{e}_2\widehat{\mu}_2 & 0 & 0 & \cdots & 0 & 0 \\ -\widehat{e}_2\widehat{\mu}_2 & \widehat{\mu}_2+\widehat{e}_3^2\widehat{\mu}_3 & -\widehat{e}_3\widehat{\mu}_3 & 0 & \cdots & 0 & 0 \\ 0 & -\widehat{e}_3\widehat{\mu}_3 & \widehat{\mu}_3+\widehat{e}_4^2\widehat{\mu}_4 & -\widehat{e}_4\widehat{\mu}_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & -\widehat{e}_{g-1}\widehat{\mu}_{g-1} & \widehat{\mu}_{g-1}+\widehat{e}_g^2\widehat{\mu}_g \end{pmatrix}_{(g-1)\times(g-1)}, \quad (30)$$

where

$$\widehat{e}_i = \frac{f_{i-1}^{-1}f_i\left(\widehat{\pi}_i^a\right)}{\widehat{\pi}_i^a}, \widehat{\mu}_i = \frac{\widehat{\pi}_i^a\left(1-\widehat{\gamma}_i^a-2\widehat{\pi}_i^a\right)}{2m_i} = \frac{m_{1i}\left(m_{2i}+m_i\right)+4m_im_{2i}}{4m_i^3}. \quad (31)$$

Note that $\mathbf{A}_1$ is a symmetric tridiagonal matrix. The elements of $\mathbf{A}_1^{-1}$ can be derived as follows:

$$\left(\mathbf{A}_1^{-1}\right)_{i,j} = \begin{cases} \dfrac{\left(\widehat{\mu}_{i+1}\cdots\widehat{\mu}_j\right)\left(d_{j+1}\cdots d_g\right)}{\delta_i\cdots\delta_{g-1}}, & i \neq j, i, j = 1,\ldots,g-1, \\ \dfrac{d_{i+1}\cdots d_{g-1}}{\delta_i\cdots\delta_{g-1}}, & i = j, i = 1,\ldots,g-1, \end{cases} \quad (32)$$

in which

$$\begin{aligned} \delta_1 &= \widehat{\mu}_1+\widehat{e}_2^2\widehat{\mu}_2, \delta_i = \widehat{\mu}_i+\widehat{e}_{i+1}^2\widehat{\mu}_{i+1}-\frac{\widehat{e}_i^2\widehat{\mu}_i^2}{\delta_{i-1}}, \quad (i = 2,\ldots,g-2), \\ d_{g-1} &= \widehat{\mu}_{g-1}+\widehat{\mu}_g, d_j = \widehat{\mu}_j+\widehat{e}_{j+1}^2\widehat{\mu}_{j+1}-\frac{\widehat{e}_{j+1}^2}{d_{j+1}}, \quad (j = g-2,\ldots,1). \end{aligned} \quad (33)$$

By the abovementioned calculation, the Wald-type statistic T_W^a can be simplified to

$$T_W^a = \sum_{i=1}^{g-1}\sum_{j=1}^{g-1}\left[\widehat{\pi}_i^a - f_i^{-1}f_{i+1}\left(\widehat{\pi}_{i+1}^a\right)\right]\left[\widehat{\pi}_j^a - f_j^{-1}f_{j+1}\left(\widehat{\pi}_{j+1}^a\right)\right]\left(\mathbf{A}_1^{-1}\right)_{ij}. \quad (34)$$

To investigate specific problems, we only need to give the corresponding function forms of $f_i(\pi_i)$ or $g_i(\pi)$.

(i) RD: $\pi_i = g_i(\pi) = \pi + \varrho_i\,(i = 1, \ldots, g)$. In this case, the Wald-type statistic T_W^a becomes

$$T_W^a = \sum_{i=1}^{g-1}\sum_{j=1}^{g-1}\left[\left(\widehat{\pi}_i^a - \varrho_i\right) - \left(\widehat{\pi}_{i+1}^a - \varrho_{i+1}\right)\right]\left[\left(\widehat{\pi}_j^a - \varrho_j\right) - \left(\widehat{\pi}_{j+1}^a - \varrho_{j+1}\right)\right]\left(\mathbf{A}_{11}^{-1}\right)_{ij}, \quad (35)$$

where

$$\mathbf{A}_{11} = \begin{pmatrix} \widehat{\mu}_1 + \widehat{\phi}_2^2\widehat{\mu}_2 & -\widehat{\phi}_2\widehat{\mu}_2 & 0 & 0 & \cdots & 0 & 0 \\ -\widehat{\phi}_2\widehat{\mu}_2 & \widehat{\mu}_2 + \widehat{\phi}_3^2\widehat{\mu}_3 & -\widehat{\phi}_3\widehat{\mu}_3 & 0 & \cdots & 0 & 0 \\ 0 & -\widehat{\phi}_3\widehat{\mu}_3 & \widehat{\mu}_3 + \widehat{\phi}_4^2\widehat{\mu}_4 & -\widehat{\phi}_4\widehat{\mu}_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & -\widehat{\phi}_{g-1}\widehat{\mu}_{g-1} & \widehat{\mu}_{g-1} + \widehat{\phi}_g^2\widehat{\mu}_g \end{pmatrix}_{(g-1)\times(g-1)}, \quad (36)$$

and $\widehat{\phi}_i = \widehat{\pi}_i + \alpha_{i-1} - \alpha_i/\widehat{\pi}_i$.

(ii) RR: $\pi_i = g_i(\pi) = \varrho_i\pi\,(i = 1, \ldots, g)$. From (34), we have

$$T_W^a = \sum_{i=1}^{g-1}\sum_{j=1}^{g-1}\left(\widehat{\pi}_i^a - \frac{\varrho_i}{\varrho_{i+1}}\widehat{\pi}_{i+1}^a\right)\left(\widehat{\pi}_j^a - \frac{\varrho_j}{\varrho_{j+1}}\widehat{\pi}_{j+1}^a\right)\left(\mathbf{A}_{12}^{-1}\right)_{ij}, \quad (37)$$

where

$$\mathbf{A}_{12} = \begin{pmatrix} \widehat{\mu}_1 + \left(\frac{\varrho_1}{\varrho_2}\right)^2\widehat{\mu}_2 & -\frac{\varrho_1}{\varrho_2}\widehat{\mu}_2 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{\varrho_1}{\varrho_2}\widehat{\mu}_2 & \widehat{\mu}_2 + \left(\frac{\varrho_2}{\varrho_3}\right)^2\widehat{\mu}_3 & -\frac{\varrho_2}{\varrho_3}\widehat{\mu}_3 & 0 & \cdots & 0 & 0 \\ 0 & -\frac{\varrho_2}{\varrho_3}\widehat{\mu}_3 & \widehat{\mu}_3 + \left(\frac{\varrho_3}{\varrho_4}\right)^2\widehat{\mu}_4 & -\frac{\varrho_3}{\varrho_4}\widehat{\mu}_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & -\frac{\varrho_{g-2}}{\varrho_{g-1}}\widehat{\mu}_{g-1} & \widehat{\mu}_{g-1} + \left(\frac{\varrho_{g-1}}{\varrho_g}\right)^2\widehat{\mu}_g \end{pmatrix}_{(g-1)\times(g-1)}. \quad (38)$$

(iii) OR: $\pi_i = g_i(\pi) = (\varrho_i\pi)/(1 - \pi + \varrho_i\pi)\,(i = 1, \ldots, g)$. Hence,

$$T_W^a = \sum_{i=1}^{g-1}\sum_{j=1}^{g-1}\left[\hat{\pi}_i^a - \frac{\varrho_i \hat{\pi}_{i+1}^a}{\varrho_{i+1} + \left(\varrho_i - \varrho_{i+1}\right)\hat{\pi}_{i+1}^a}\right]\left[\hat{\pi}_j^a - \frac{\varrho_j \hat{\pi}_{j+1}^a}{\varrho_{j+1} + \left(\varrho_j - \varrho_{j+1}\right)\hat{\pi}_{j+1}^a}\right]\left(\mathbf{A}_{13}^{-1}\right)_{ij}, \tag{39}$$

where

$$\mathbf{A}_{13} = \begin{pmatrix} \hat{\mu}_1 + \hat{\varphi}_2^2\hat{\mu}_2 & -\hat{\varphi}_2\hat{\mu}_2 & 0 & 0 & \cdots & 0 & 0 \\ -\hat{\varphi}_2\hat{\mu}_2 & \hat{\mu}_2 + \hat{\varphi}_3^2\hat{\mu}_3 & -\hat{\varphi}_3\hat{\mu}_3 & 0 & \cdots & 0 & 0 \\ 0 & -\hat{\varphi}_3\hat{\mu}_3 & \hat{\mu}_3 + \hat{\varphi}_4^2\hat{\mu}_4 & -\hat{\varphi}_4\hat{\mu}_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & -\hat{\varphi}_{g-1}\hat{\mu}_{g-1} & \hat{\mu}_{g-1} + \hat{\varphi}_g^2\hat{\mu}_g \end{pmatrix}_{(g-1)\times(g-1)}, \tag{40}$$

and $\hat{\varphi}_i = \alpha_{i-1}/\alpha_i(1-\hat{\pi}_i) + \alpha_{i-1}\hat{\pi}_i$.

The hypothesis H_{02} can be written as $\mathbf{C}_2\beta_2^T = 0$ in the matrix form, where $\beta_2 = (\pi_1, \ldots, \pi_g, \gamma)$ and

$$\mathbf{C}_2 = \begin{pmatrix} 1 & -\frac{f_1^{-1}f_2(\pi_2)}{\pi_2} & 0 & \cdots\cdots & 0 & 0 \\ 0 & 1 & \frac{f_2^{-1}f_3(\pi_3)}{\pi_3} & & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots\ \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 \quad 1 & -\frac{f_{g-1}^{-1}f_g(\pi_g)}{\pi_g} & 0 \end{pmatrix}_{(g-1)\times g}. \tag{41}$$

Let $\hat{\beta}_2 = (\hat{\pi}_1^b, \ldots, \hat{\pi}_g^b, \hat{\gamma}^b)$. Then, another Wald-type test statistic is given by

$$T_W^b = \left(\boldsymbol{\beta}_2\mathbf{C}_2^T\right)\left(\mathbf{C}_2\mathbf{I}_2^{-1}\mathbf{C}_2^T\right)^{-1}\left(\mathbf{C}_2\boldsymbol{\beta}_2^T\right)\Big|_{\boldsymbol{\beta}_2=\hat{\boldsymbol{\beta}}_2}, \tag{42}$$

where $\mathbf{I_2}$ is the information matrix in the Appendix B. We denote $\mathbf{A}_2 \triangleq \mathbf{C}_2\mathbf{I}_2^{-1}\mathbf{C}_2^T$. Then, T_W^b can be simplified to

$$T_W^b = \sum_{i=1}^{g-1}\sum_{j=1}^{g-1}\left[\hat{\pi}_i^b - f_i^{-1}f_{i+1}\left(\hat{\pi}_{i+1}^b\right)\right]\left[\hat{\pi}_j^b - f_j^{-1}f_{j+1}\left(\hat{\pi}_{j+1}^b\right)\right]\left(\mathbf{A}_2^{-1}\right)_{ij}. \tag{43}$$

See the Appendix C for the derivation process of $\mathbf{A}_2^{-1}$.

Identically, T_W^b can be obtained by giving the forms of $f_i(\pi_i)$ corresponding to the risk difference, relative risk ratio, and odds ratio. Under the hypotheses H_{01} and H_{02}, T_W^a and T_W^b asymptotically obey the chi-square distribution with the degree of freedom $g-1$.

4.3. Score Tests. For Case (i), we note that $\gamma_i \neq \gamma_j$ for some $i \neq j \in \{1, \ldots, g\}$. Let $\mathbf{U}_1 = ((\partial l/\partial\pi_1), (\partial l/\partial\pi_2), \ldots, (\partial l/\partial\pi_g), (\partial l/\partial\ \gamma_1), (\partial l/\partial\gamma_2), \ldots, (\partial l/\partial\gamma_g))$, $\tilde{\gamma}^a = (\tilde{\gamma}_1^a, \tilde{\gamma}_2^a, \ldots, \tilde{\gamma}_g^a)$. Under the null hypothesis H_{01}: $f_1(\pi_1) = f_2(\pi_2) = \cdots = f_g(\pi_g)$, according to [12, 13], the score test statistic can be defined as follows:

Table 2: Parameter settings for computing empirical type I error rates.

Parameters	Cases	Number of groups		
		$g = 2$	$g = 3$	$g = 4$
	EQ	(π, π)	(π, π, π)	(π, π, π, π)
π	DR	$(\pi, \pi + \eta)$	$(\pi, \pi + \eta, \pi + \xi)$	$(\pi, \pi + \eta, \pi + \xi, \pi + \zeta)$
$(\pi_1, \pi_2, \cdots, \pi_g)$	RR	$(\pi, \lambda\pi)$	$(\pi, \lambda\pi, \mu\pi)$	$(\pi, \lambda\pi, \mu\pi, \delta\pi)$
	OR	$(\pi, (\lambda\pi)/(1 - \pi + \lambda\pi))$	$(\pi, (\lambda\pi)/(1 - \pi + \lambda\pi), (\mu\pi)/(1 - \pi + \mu\pi))$	$(\pi, (\lambda\pi)/(1 - \pi + \lambda\pi), (\mu\pi)/(1 - \pi + \mu\pi), (\delta\pi)/(1 - \pi + \delta\pi))$
	a	(0.4, 0.5)	(0.4, 0.5, 0.4)	(0.4, 0.5, 0.4, 0.5)
	b	(0.5, 0.6)	(0.5, 0.6, 0.5)	(0.5, 0.6, 0.5, 0.6)
γ	c	(0.6, 0.7)	(0.6, 0.7, 0.6)	(0.6, 0.7, 0.6, 0.7)
$(\gamma_1, \gamma_2, \cdots, \gamma_g)$	d	(0.4, 0.4)	(0.4, 0.4, 0.4)	(0.4, 0.4, 0.4, 0.4)
	e	(0.5, 0.5)	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5, 0.5)
	f	(0.6, 0.6)	(0.6, 0.6, 0.6)	(0.6, 0.6, 0.6, 0.6)

[1]Note: EQ, DR, RR, and OR represent the equality ratio, risk difference, relative risk ratio, and odds ratio, respectively.

Table 3: Partial power efficiency setting.

Cases	Number of groups		
	$g = 2$	$g = 3$	$g = 4$
EQ	$(\pi, \pi + 0.15)$	$(\pi, \pi + 0.15, \pi)$	$(\pi, \pi + 0.15, \pi, \pi + 0.15)$
RR	$(\pi, \lambda\pi + 0.15)$	$(\pi, \lambda\pi + 0.15, \mu\pi)$	$(\pi, \lambda\pi + 0.15, \mu\pi, \delta\pi + 0.15)$

Table 4: The empirical type I error rates under H_{01}.

Cases	g	γ	π	$m = 30$				$m = 50$				$m = 100$			
				T_L^a	T_{SC}^a	T_W^a	T_R^a	T_L^a	T_{SC}^a	T_W^a	T_R^a	T_L^a	T_{SC}^a	T_W^a	T_R^a
			0.30	0.0500	0.0470	0.0550	0.0470	0.0550	0.0540	0.0580	0.0540	0.0500	0.0490	0.0520	0.0490
		a	0.40	0.0500	0.0480	0.0550	0.0480	0.0500	0.0490	0.0530	0.0490	0.0510	0.0500	0.0520	0.0500
			0.50	0.0560	0.0550	0.0590	0.0550	0.0510	0.0490	0.0530	0.0490	0.0550	0.0540	0.0570	0.0540
			0.30	0.0570	0.0540	**0.0630**	0.0540	0.0510	0.0500	0.0550	0.0500	0.0520	0.0510	0.0530	0.0510
	2	b	0.40	0.0520	0.0500	0.0570	0.0500	0.0550	0.0530	0.0590	0.0530	0.0510	0.0510	0.0530	0.0510
			0.50	0.0530	0.0520	0.0590	0.0520	0.0510	0.0490	0.0530	0.0490	0.0510	0.0500	0.0530	0.0500
			0.30	0.0520	0.0500	0.0570	0.0500	0.0510	0.0490	0.0530	0.0490	0.0490	0.0490	0.0510	0.0490
		c	0.40	0.0530	0.0520	0.0580	0.0520	0.0490	0.0480	0.0520	0.0480	0.0520	0.0520	0.0540	0.0520
			0.50	0.0540	0.0510	0.0580	0.0510	0.0520	0.0510	0.0550	0.0510	0.0510	0.0510	0.0520	0.0510
			0.30	0.0470	0.0410	0.0570	0.0410	0.0540	0.0500	**0.0610**	0.0500	0.0530	0.0510	0.0570	0.0510
		a	0.40	0.0550	0.0490	**0.0660**	0.0490	0.0570	0.0530	**0.0620**	0.0530	0.0500	0.0480	0.0530	0.0480
			0.50	0.0520	0.0460	**0.0630**	0.0460	0.0560	0.0520	**0.0610**	0.0520	0.0530	0.0510	0.0550	0.0510
			0.30	0.0560	0.0490	**0.0690**	0.0490	0.0540	0.0510	**0.0620**	0.0510	0.0510	0.0490	0.0550	0.0490
EQ	3	b	0.40	0.0540	0.0480	**0.0670**	0.0480	0.0570	0.0540	**0.0630**	0.0540	0.0560	0.0550	**0.0600**	0.0550
			0.50	0.0550	0.0500	**0.0650**	0.0500	0.0540	0.0500	**0.0610**	0.0500	0.0520	0.0510	0.0550	0.0510
			0.30	0.0510	0.0450	**0.0620**	0.0450	0.0530	0.0510	**0.0610**	0.0510	0.0560	0.0540	0.0590	0.0540
		c	0.40	0.0560	0.0530	**0.0670**	0.0530	0.0500	0.0480	0.0570	0.0480	0.0500	0.0490	0.0550	0.0490
			0.50	0.0520	0.0470	**0.0660**	0.0470	0.0540	0.0510	**0.0600**	0.0510	0.0540	0.0520	0.0560	0.0520
			0.30	0.0480	0.0410	**0.0670**	0.0410	0.0520	0.0470	**0.0620**	0.0470	0.0520	0.0500	0.0570	0.0500
		a	0.40	0.0520	0.0450	**0.0670**	0.0450	0.0530	0.0490	**0.0630**	0.0490	0.0510	0.0490	0.0550	0.0490
			0.50	**0.0600**	0.0510	**0.0720**	0.0510	0.0570	0.0520	**0.0650**	0.0520	0.0530	0.0500	0.0580	0.0500
			0.30	0.0590	0.0500	**0.0770**	0.0500	0.0530	0.0490	**0.0620**	0.0490	0.0530	0.0510	**0.0600**	0.0510
	4	b	0.40	0.0540	0.0470	**0.0700**	0.0470	0.0530	0.0490	**0.0630**	0.0490	0.0490	0.0470	0.0540	0.0470
			0.50	0.0570	0.0510	**0.0720**	0.0510	0.0520	0.0480	**0.0620**	0.0480	0.0520	0.0500	0.0570	0.0500
			0.30	0.0540	0.0470	**0.0750**	0.0470	0.0590	0.0540	**0.0670**	0.0540	0.0510	0.0490	0.0550	0.0490
		c	0.40	0.0550	0.0490	**0.0700**	0.0490	0.0550	0.0520	**0.0650**	0.0520	0.0540	0.0520	**0.0600**	0.0520
			0.50	0.0530	0.0460	**0.0720**	0.0460	0.0540	0.0510	**0.0650**	0.0510	0.0520	0.0500	0.0560	0.0500

Note: The empirical TIEs of the liberal region (more than 0.06) are shown in bold.

$$T_{SC}^a = \mathbf{U}_1 \mathbf{I}_1^{-1} \mathbf{U}_1^T \Big|_{\pi_i = g_i(\tilde{\pi}^a), \gamma = \tilde{\gamma}^a}, \tag{44}$$

where $\mathbf{I_1}$ is the Fisher information matrix. Hence, T_{SC}^a can be simplified as

$$T_{SC}^a = \sum_{i=1}^{g} \frac{M_i^2 g_i(\tilde{\pi}^a)[H_i \tilde{\gamma}_i^a - 2g_i(\tilde{\pi}^a)]}{2m_i} - \frac{H_i M_i \tilde{\gamma}_i^a (\tilde{\gamma}_i^a - 1)}{m_i} + \frac{H_i^2 \tilde{\gamma}_i^a (\tilde{\gamma}_i^a - 1)(\tilde{\gamma}_i^a - 2)}{2\tilde{\pi}^a m_i}, \tag{45}$$

where

$$M_i = \frac{m_{1i} + m_{2i}}{g_i(\tilde{\pi}^a)} + \frac{m_{0i}(\tilde{\gamma}_i^a - 2)}{g_i(\tilde{\pi}^a)\tilde{\gamma}_i^a - 2g_i(\tilde{\pi}^a) + 1},$$

$$H_i = \frac{m_{2i}}{\tilde{\gamma}_i^a} + \frac{m_{1i}}{\tilde{\gamma}_i^a - 1} + \frac{m_{0i} g_i(\tilde{\pi}^a)}{g_i(\tilde{\pi}^a)\tilde{\gamma}_i^a - 2g_i(\tilde{\pi}^a) + 1}, \quad i = 1, 2, \ldots, g. \tag{46}$$

For Case (ii) $\gamma_1 = \gamma_2 = \cdots = \gamma_g = \gamma$, let $\mathbf{U}_2 = ((\partial l / \partial \pi_1), (\partial l / \partial \pi_2), \cdots, (\partial l / \partial \pi_g), 0)$. Under the null hypothesis H_{02}: $f_1(\pi_1) = f_2(\pi_2) = \cdots = f_g(\pi_g)$, the score test statistic can be given by

$$T_{SC}^b = \mathbf{U}_2 \mathbf{I}_2^{-1} \mathbf{U}_2^T \Big|_{\pi_i = g_i(\tilde{\pi}^b), \gamma = \tilde{\gamma}^b}, \tag{47}$$

where $\mathbf{I_2}$ is the Fisher information matrix. T_{SC}^b can be simplified as

Table 5: The empirical type I error rates under H_{02}.

Cases	g	γ	π	$m = 30$				$m = 50$				$m = 100$			
				T_L^b	T_{SC}^b	T_W^b	T_R^b	T_L^b	T_{SC}^b	T_W^b	T_R^b	T_L^b	T_{SC}^b	T_W^b	T_R^b
			0.30	0.0540	0.0530	0.0530	0.0530	0.0570	0.0570	0.0570	0.0570	0.0510	0.0510	0.0500	0.0510
		a	0.40	0.0530	0.0460	0.0530	0.0460	0.0550	0.0550	0.0540	0.0550	0.0530	0.0530	0.0550	0.0530
			0.50	**0.0650**	0.0530	**0.0600**	0.0530	0.0500	0.0490	0.0550	0.0490	0.0510	0.0510	0.0500	0.0510
			0.30	0.0500	0.0500	0.0500	0.0500	0.0570	0.0570	0.0570	0.0570	0.0530	0.0530	0.0530	0.0530
	2	b	0.40	0.0550	0.0520	0.0550	0.0520	0.0540	0.0540	0.0550	0.0540	0.0490	0.0490	0.0500	0.0490
			0.50	0.0560	0.0450	0.0560	0.0450	0.0500	0.0500	0.0550	0.0500	0.0490	0.0470	0.0490	0.0470
			0.30	0.0510	0.0500	0.0500	0.0500	0.0510	0.0510	0.0510	0.0510	0.0580	0.0580	0.0580	0.0580
		c	0.40	0.0560	0.0550	0.0560	0.0550	0.0590	0.0590	0.0590	0.0590	0.0510	0.0510	0.0510	0.0510
			0.50	0.0560	0.0470	0.0560	0.0470	0.0510	0.0510	0.0570	0.0510	0.0510	0.0510	0.0540	0.0510
			0.30	0.0520	0.0500	**0.0700**	0.0500	0.0510	0.0470	**0.0600**	0.0470	0.0490	0.0490	0.0530	0.0490
		a	0.40	0.0560	0.0490	**0.0760**	0.0490	0.0560	0.0550	**0.0610**	0.0550	0.0500	0.0490	0.0520	0.0490
			0.50	0.0550	0.0500	**0.0840**	0.0500	0.0560	0.0490	**0.0630**	0.0490	0.0510	0.0500	0.0550	0.0500
			0.30	0.0570	0.0530	**0.0750**	0.0530	0.0520	0.0500	**0.0600**	0.0500	0.0530	0.0520	0.0590	0.0520
EQ	3	b	0.40	0.0530	0.0480	**0.0700**	0.0480	0.0530	0.0520	**0.0600**	0.0520	0.0490	0.0480	0.0510	0.0480
			0.50	0.0530	0.0490	**0.0740**	0.0490	0.0540	0.0500	**0.0600**	0.0500	0.0520	0.0500	0.0530	0.0500
			0.30	0.0540	0.0500	**0.0710**	0.0500	0.0570	0.0560	**0.0650**	0.0560	0.0480	0.0470	0.0530	0.0470
		c	0.40	0.0520	0.0490	**0.0700**	0.0490	0.0540	0.0510	**0.0610**	0.0510	0.0500	0.0490	0.0540	0.0490
			0.50	0.0530	0.0480	**0.0740**	0.0480	0.0520	0.0480	0.0580	0.0480	0.0510	0.0500	0.0550	0.0500
			0.30	0.0550	0.0510	**0.0720**	0.0510	0.0500	0.0480	0.0570	0.0480	0.0490	0.0480	0.0520	0.0480
		a	0.40	0.0580	0.0540	**0.0680**	0.0540	0.0560	0.0530	**0.0660**	0.0530	0.0530	0.0510	0.0570	0.0510
			0.50	0.0570	0.0500	**0.0710**	0.0500	0.0590	0.0530	**0.0710**	0.0530	0.0570	0.0540	**0.0640**	0.0540
			0.30	0.0530	0.0490	**0.0670**	0.0490	0.0490	0.0470	0.0590	0.0470	0.0520	0.0510	0.0560	0.0510
	4	b	0.40	0.0510	0.0480	**0.0640**	0.0480	0.0540	0.0520	**0.0640**	0.0520	0.0530	0.0510	0.0590	0.0510
			0.50	0.0580	0.0500	**0.0680**	0.0500	0.0550	0.0490	**0.0690**	0.0490	0.0530	0.0510	0.0580	0.0510
			0.30	0.0560	0.0530	**0.0690**	0.0530	0.0510	0.0490	0.0610	0.0490	0.0550	0.0540	0.0580	0.0540
		c	0.40	0.0520	0.0480	**0.0660**	0.0480	0.0470	0.0450	0.0570	0.0450	0.0510	0.0500	0.0550	0.0500
			0.50	0.0550	0.0520	**0.0640**	0.0520	0.0560	0.0530	**0.0670**	0.0530	0.0510	0.0490	0.0570	0.0490

Note: The empirical TIEs of the liberal region (more than 0.06) are shown in bold.

Table 6: The empirical type I error rates under H_{01}.

Cases	g	γ	π	$m = 30$				$m = 50$				$m = 70$			
				T_L^a	T_{SC}^a	T_W^a	T_R^a	T_L^a	T_{SC}^a	T_W^a	T_R^a	T_L^a	T_{SC}^a	T_W^a	T_R^a
			0.30	0.0520	0.0500	**0.0630**	0.0500	0.0540	0.0520	**0.0680**	0.0520	0.0460	0.0450	0.0560	0.0450
		a	0.40	0.0530	0.0500	**0.0640**	0.0500	0.0540	0.0520	**0.0620**	0.0520	0.0540	0.0530	**0.0620**	0.0530
			0.50	0.0540	0.0520	**0.0640**	0.0520	0.0530	0.0510	**0.0620**	0.0510	0.0530	0.0520	0.0590	0.0520
			0.30	0.0530	0.0500	**0.0670**	0.0500	0.0540	0.0530	**0.0670**	0.0530	0.0530	0.0520	**0.0600**	0.0520
	2	b	0.40	0.0540	0.0520	**0.0650**	0.0520	0.0510	0.0490	**0.0620**	0.0490	0.0490	0.0490	0.0570	0.0490
			0.50	0.0510	0.0490	**0.0610**	0.0490	0.0570	0.0550	**0.0620**	0.0550	0.0510	0.0500	0.0570	0.0500
			0.30	0.0560	0.0530	**0.0700**	0.0530	0.0500	0.0490	**0.0620**	0.0490	0.0570	0.0560	**0.0660**	0.0560
		c	0.40	0.0560	0.0520	**0.0680**	0.0520	0.0580	0.0570	**0.0660**	0.0570	0.0500	0.0500	0.0570	0.0500
			0.50	0.0510	0.0480	**0.0610**	0.0480	0.0510	0.0510	**0.0610**	0.0510	0.0500	0.0500	0.0570	0.0500
			0.30	0.0560	0.0490	**0.0780**	0.0490	0.0490	0.0460	**0.0640**	0.0460	0.0540	0.0520	**0.0680**	0.0520
		a	0.40	0.0530	0.0470	**0.0710**	0.0470	0.0510	0.0480	**0.0640**	0.0480	0.0510	0.0500	**0.0620**	0.0500
			0.50	0.0540	0.0480	**0.0720**	0.0480	0.0500	0.0470	**0.0600**	0.0470	0.0490	0.0470	0.0580	0.0470
			0.30	0.0530	0.0490	**0.0760**	0.0490	0.0530	0.0510	**0.0720**	0.0510	0.0540	0.0530	**0.0670**	0.0530
DR	3	b	0.40	0.0540	0.0490	**0.0760**	0.0490	0.0550	0.0510	**0.0680**	0.0510	0.0510	0.0490	**0.0620**	0.0490
			0.50	0.0550	0.0490	**0.0730**	0.0490	0.0500	0.0470	**0.0630**	0.0470	0.0500	0.0490	**0.0610**	0.0490
			0.30	0.0540	0.0480	**0.0750**	0.0480	0.0500	0.0480	**0.0710**	0.0480	0.0480	0.0460	**0.0630**	0.0460
		c	0.40	0.0580	0.0530	**0.0780**	0.0530	0.0550	0.0530	**0.0720**	0.0530	0.0490	0.0470	**0.0600**	0.0470
			0.50	**0.0610**	0.0560	**0.0790**	0.0560	0.0540	0.0500	**0.0660**	0.0500	0.0490	0.0490	0.0580	0.0490
			0.30	0.0500	0.0450	**0.0620**	0.0450	0.0550	0.0490	**0.0610**	0.0490	0.0540	0.0520	0.0540	0.0520
		a	0.40	0.0540	0.0460	**0.0640**	0.0460	0.0580	0.0520	**0.0610**	0.0520	0.0540	0.0510	0.0520	0.0500
			0.50	**0.0600**	0.0470	**0.0660**	0.0460	0.0550	0.0490	0.0570	0.0470	0.0540	0.0490	0.0520	0.0470
			0.30	0.0530	0.0440	**0.0660**	0.0440	0.0550	0.0510	**0.0620**	0.0500	0.0520	0.0520	0.0520	0.0510
	4	b	0.40	0.0540	0.0480	**0.0670**	0.0480	0.0520	0.0480	0.0550	0.0470	0.0570	0.0540	0.0580	0.0540
			0.50	**0.0620**	0.0540	**0.0720**	0.0540	0.0550	0.0500	**0.0620**	0.0490	0.0550	0.0530	0.0540	0.0510
			0.30	0.0540	0.0490	**0.0690**	0.0490	0.0550	0.0500	**0.0630**	0.0500	0.0510	0.0500	0.0510	0.0500
		c	0.40	**0.0610**	0.0540	**0.0740**	0.0540	0.0510	0.0470	0.0570	0.0470	0.0530	0.0520	0.0540	0.0520
			0.50	0.0580	0.0510	**0.0710**	0.0500	0.0510	0.0470	0.0570	0.0470	0.0530	0.0510	0.0530	0.0500

Table 6: Continued.

Cases	g	γ	π	$m = 30$				$m = 50$				$m = 70$			
				T_L^a	T_{SC}^a	T_W^a	T_R^a	T_L^a	T_{SC}^a	T_W^a	T_R^a	T_L^a	T_{SC}^a	T_W^a	T_R^a
			0.30	0.0510	0.0480	0.0560	0.0480	0.0530	0.0520	0.0570	0.0520	0.0550	0.0540	0.0560	0.0540
		a	0.40	0.0500	0.0480	0.0550	0.0480	0.0540	0.0520	0.0570	0.0520	0.0510	0.0500	0.0530	0.0500
			0.50	0.0540	0.0520	0.0580	0.0520	0.0520	0.0500	0.0550	0.0500	0.0510	0.0500	0.0530	0.0500
			0.30	0.0520	0.0480	0.0590	0.0480	0.0520	0.0500	0.0550	0.0500	0.0560	0.0550	0.0570	0.0550
	2	b	0.40	0.0520	0.0490	0.0570	0.0490	0.0540	0.0530	0.0560	0.0530	0.0520	0.0520	0.0540	0.0520
			0.50	0.0550	0.0520	**0.0600**	0.0520	0.0530	0.0510	0.0570	0.0510	0.0500	0.0500	0.0520	0.0500
			0.30	0.0560	0.0520	**0.0600**	0.0520	0.0510	0.0490	0.0530	0.0490	0.0530	0.0520	0.0540	0.0520
		c	0.40	0.0530	0.0520	0.0580	0.0520	0.0560	0.0550	0.0590	0.0550	0.0520	0.0520	0.0550	0.0520
			0.50	0.0540	0.0510	0.0590	0.0510	0.0500	0.0490	0.0520	0.0490	0.0510	0.0500	0.0530	0.0500
			0.30	0.0520	0.0460	**0.0610**	0.0460	0.0530	0.0510	**0.0600**	0.0510	0.0500	0.0490	0.0520	0.0490
		a	0.40	0.0580	0.0520	**0.0690**	0.0520	0.0530	0.0500	0.0590	0.0500	0.0530	0.0510	0.0570	0.0510
			0.50	0.0550	0.0470	**0.0650**	0.0470	0.0510	0.0460	0.0570	0.0460	0.0540	0.0520	0.0570	0.0520
			0.30	0.0550	0.0510	**0.0680**	0.0510	0.0540	0.0520	**0.0620**	0.0520	0.0530	0.0510	0.0560	0.0510
RR	3	b	0.40	0.0570	0.0510	**0.0690**	0.0510	0.0560	0.0520	**0.0650**	0.0520	0.0490	0.0480	0.0520	0.0480
			0.50	0.0540	0.0470	**0.0630**	0.0470	0.0510	0.0470	0.0560	0.0470	0.0500	0.0480	0.0540	0.0480
			0.30	0.0560	0.0510	**0.0700**	0.0510	0.0540	0.0510	**0.0620**	0.0510	0.0520	0.0510	0.0560	0.0510
		c	0.40	0.0520	0.0480	**0.0640**	0.0480	0.0490	0.0470	0.0560	0.0470	0.0490	0.0470	0.0520	0.0470
			0.50	0.0570	0.0520	**0.0690**	0.0520	0.0510	0.0480	0.0580	0.0480	0.0560	0.0550	0.0590	0.0550
			0.30	0.0490	0.0420	**0.0680**	0.0420	0.0530	0.0470	**0.0630**	0.0470	0.0520	0.0500	0.0570	0.0500
		a	0.40	0.0520	0.0440	**0.0670**	0.0440	0.0560	0.0500	**0.0640**	0.0500	0.0530	0.0500	0.0570	0.0500
			0.50	0.0560	0.0450	**0.0700**	0.0450	0.0550	0.0500	**0.0630**	0.0500	0.0540	0.0510	0.0580	0.0510
			0.30	0.0580	0.0500	**0.0790**	0.0500	0.0520	0.0480	**0.0620**	0.0480	0.0520	0.0500	0.0580	0.0500
	4	b	0.40	0.0510	0.0450	**0.0670**	0.0450	0.0550	0.0510	**0.0650**	0.0510	0.0520	0.0500	0.0550	0.0500
			0.50	0.0580	0.0490	**0.0730**	0.0490	0.0540	0.0490	**0.0630**	0.0490	0.0470	0.0460	0.0520	0.0460
			0.30	0.0550	0.0470	**0.0740**	0.0470	0.0550	0.0530	**0.0660**	0.0530	0.0520	0.0500	0.0580	0.0500
		c	0.40	0.0560	0.0510	**0.0720**	0.0510	0.0530	0.0500	**0.0640**	0.0500	0.0510	0.0490	0.0560	0.0490
			0.50	0.0540	0.0480	**0.0730**	0.0480	0.0520	0.0490	**0.0630**	0.0490	0.0510	0.0490	0.0550	0.0490
			0.30	0.0560	0.0520	0.0580	0.0520	0.0490	0.0470	0.0480	0.0470	0.0520	0.0520	0.0510	0.0520
		a	0.40	0.0540	0.0520	0.0550	0.0520	0.0520	0.0510	0.0510	0.0510	0.0550	0.0540	0.0530	0.0540
			0.50	0.0520	0.0480	0.0500	0.0480	0.0470	0.0460	0.0460	0.0460	0.0500	0.0490	0.0460	0.0490
			0.30	0.0590	0.0550	**0.0600**	0.0550	0.0500	0.0490	0.0500	0.0490	0.0500	0.0490	0.0480	0.0490
	2	b	0.40	0.0530	0.0510	0.0530	0.0510	0.0550	0.0540	0.0540	0.0540	0.0510	0.0500	0.0480	0.0500
			0.50	0.0560	0.0530	0.0560	0.0530	0.0520	0.0500	0.0490	0.0500	0.0500	0.0500	0.0480	0.0500
			0.30	0.0540	0.0510	0.0570	0.0510	0.0550	0.0530	0.0550	0.0530	0.0520	0.0500	0.0490	0.0500
		c	0.40	0.0530	0.0510	0.0540	0.0510	0.0510	0.0500	0.0500	0.0500	0.0490	0.0480	0.0460	0.0480
			0.50	0.0550	0.0530	0.0550	0.0530	0.0540	0.0530	0.0520	0.0530	0.0490	0.0490	0.0460	0.0490
			0.30	0.0520	0.0470	0.0580	0.0470	0.0560	0.0540	**0.0600**	0.0540	0.0530	0.0510	0.0520	0.0510
		a	0.40	0.0540	0.0500	0.0570	0.0500	0.0530	0.0490	0.0530	0.0490	0.0530	0.0510	0.0490	0.0510
			0.50	0.0560	0.0480	0.0580	0.0480	0.0540	0.0500	0.0530	0.0500	0.0530	0.0510	0.0490	0.0510
			0.30	0.0520	0.0470	0.0590	0.0470	0.0540	0.0510	0.0580	0.0510	0.0500	0.0480	0.0490	0.0480
OR	3	b	0.40	0.0570	0.0530	**0.0630**	0.0530	0.0560	0.0540	0.0580	0.0540	0.0520	0.0490	0.0490	0.0490
			0.50	0.0560	0.0490	0.0580	0.0490	0.0510	0.0480	0.0500	0.0480	0.0530	0.0520	0.0490	0.0520
			0.30	0.0560	0.0500	**0.0640**	0.0500	0.0530	0.0510	0.0580	0.0510	0.0540	0.0530	0.0540	0.0530
		c	0.40	0.0570	0.0530	**0.0640**	0.0530	0.0500	0.0480	0.0510	0.0480	0.0530	0.0520	0.0520	0.0520
			0.50	0.0550	0.0500	0.0590	0.0500	0.0480	0.0460	0.0480	0.0460	0.0530	0.0520	0.0500	0.0520
			0.30	0.0500	0.0420	**0.0630**	0.0420	0.0540	0.0500	**0.0620**	0.0500	0.0500	0.0490	0.0530	0.0490
		a	0.40	0.0550	0.0470	**0.0650**	0.0470	0.0520	0.0470	0.0580	0.0470	0.0520	0.0490	0.0520	0.0490
			0.50	0.0560	0.0450	**0.0630**	0.0450	0.0550	0.0500	0.0570	0.0500	0.0510	0.0470	0.0510	0.0470
			0.30	0.0550	0.0480	**0.0700**	0.0480	0.0540	0.0490	**0.0610**	0.0490	0.0530	0.0510	0.0540	0.0510
	4	b	0.40	0.0580	0.0510	**0.0700**	0.0510	0.0500	0.0450	0.0550	0.0450	0.0500	0.0490	0.0510	0.0490
			0.50	0.0580	0.0500	**0.0680**	0.0500	0.0560	0.0510	0.0590	0.0510	0.0520	0.0500	0.0530	0.0500
			0.30	0.0580	0.0490	**0.0730**	0.0490	0.0520	0.0480	**0.0610**	0.0480	0.0540	0.0520	0.0570	0.0520
		c	0.40	0.0540	0.0490	**0.0680**	0.0490	0.0490	0.0460	0.0550	0.0460	0.0530	0.0520	0.0560	0.0520
			0.50	0.0570	0.0510	**0.0700**	0.0510	0.0530	0.0500	0.0570	0.0500	0.0540	0.0530	0.0540	0.0530

Note: The empirical TIEs of the liberal region (more than 0.06) are shown in bold.

Table 7: The empirical type I error rates under H_{02}.

Cases	g	γ	π	$m = 30$				$m = 50$				$m = 70$			
				T_L^b	T_{SC}^b	T_W^b	T_R^b	T_L^b	T_{SC}^b	T_W^b	T_R^b	T_L^b	T_{SC}^b	T_W^b	T_R^b
DR	2	a	0.30	0.0540	0.0540	0.0580	0.0540	0.0470	0.0470	**0.0710**	0.0470	0.0470	0.0460	0.0530	0.0460
			0.40	0.0500	0.0480	**0.0650**	0.0480	0.0540	0.0530	**0.0610**	0.0530	0.0520	0.0510	0.0590	0.0510
			0.50	0.0540	0.0510	**0.0620**	0.0510	0.0530	0.0500	0.0580	0.0500	0.0520	0.0500	0.0570	0.0500
		b	0.30	0.0510	0.0510	0.0580	0.0510	0.0500	0.0490	**0.0720**	0.0490	0.0540	0.0530	0.0550	0.0530
			0.40	0.0570	0.0550	**0.0680**	0.0550	0.0550	0.0530	**0.0670**	0.0530	0.0510	0.0510	0.0570	0.0510
			0.50	0.0500	0.0460	0.0550	0.0460	0.0540	0.0510	**0.0600**	0.0510	0.0510	0.0490	0.0560	0.0500
		c	0.30	0.0510	0.0500	**0.0620**	0.0500	0.0450	0.0440	**0.0630**	0.0440	0.0560	0.0560	**0.0640**	0.0560
			0.40	0.0560	0.0560	**0.0680**	0.0560	0.0530	0.0510	**0.0690**	0.0510	0.0490	0.0490	0.0540	0.0490
			0.50	0.0520	0.0490	**0.0610**	0.0490	0.0530	0.0520	**0.0600**	0.0520	0.0480	0.0470	0.0540	0.0480
	3	a	0.30	0.0500	0.0470	**0.0760**	0.0470	0.0550	0.0540	0.0730	0.0540	0.0520	0.0510	**0.0650**	0.0510
			0.40	0.0550	0.0520	**0.0730**	0.0520	0.0520	0.0490	**0.0680**	0.0490	0.0490	0.0480	0.0590	0.0480
			0.50	0.0540	0.0450	**0.0730**	0.0450	0.0530	0.0480	**0.0690**	0.0480	0.0520	0.0490	**0.0620**	0.0490
		b	0.30	0.0480	0.0450	**0.0690**	0.0450	0.0540	0.0530	**0.0710**	0.0530	0.0460	0.0460	**0.0600**	0.0460
			0.40	0.0570	0.0530	**0.0750**	0.0530	0.0520	0.0510	**0.0710**	0.0510	0.0510	0.0500	**0.0600**	0.0500
			0.50	0.0560	0.0490	**0.0750**	0.0500	0.0550	0.0510	**0.0670**	0.0510	0.0550	0.0530	**0.0640**	0.0530
		c	0.30	0.0520	0.0490	**0.0740**	0.0490	0.0510	0.0500	**0.0680**	0.0500	0.0510	0.0500	0.0650	0.0500
			0.40	0.0520	0.0490	**0.0700**	0.0490	0.0480	0.0460	**0.0650**	0.0460	0.0570	0.0560	**0.0670**	0.0560
			0.50	0.0520	0.0480	**0.0710**	0.0480	0.0550	0.0510	**0.0670**	0.0510	0.0500	0.0480	0.0590	0.0480
	4	a	0.30	0.0570	0.0530	**0.0690**	0.0530	0.0530	0.0520	0.0590	0.0520	0.0550	0.0540	0.0540	0.0540
			0.40	0.0550	0.0510	**0.0690**	0.0510	0.0540	0.0500	**0.0600**	0.0500	0.0490	0.0480	0.0510	0.0480
			0.50	0.0550	0.0450	**0.0790**	0.0450	0.0570	0.0500	**0.0690**	0.0500	0.0520	0.0490	0.0570	0.0490
		b	0.30	0.0530	0.0500	**0.0670**	0.0500	0.0480	0.0460	0.0530	0.0460	0.0470	0.0470	0.0490	0.0470
			0.40	0.0530	0.0490	**0.0670**	0.0490	0.0530	0.0510	**0.0600**	0.0510	0.0520	0.0510	0.0530	0.0510
			0.50	0.0560	0.0470	**0.0760**	0.0470	0.0520	0.0510	**0.0630**	0.0510	0.0540	0.0510	0.0560	0.0510
		c	0.30	0.0540	0.0500	**0.0650**	0.0500	0.0510	0.0490	0.0540	0.0490	0.0480	0.0470	0.0500	0.0470
			0.40	0.0570	0.0520	**0.0690**	0.0520	0.0510	0.0490	0.0540	0.0490	0.0500	0.0490	0.0510	0.0490
			0.50	0.0550	0.0490	**0.0730**	0.0490	0.0500	0.0460	0.0590	0.0460	0.0550	0.0540	0.0580	0.0550
RR	2	a	0.30	0.0510	0.0490	0.0580	0.0490	0.0460	0.0450	0.0490	0.0450	0.0550	0.0550	0.0560	0.0550
			0.40	0.0580	0.0540	**0.0650**	0.0540	0.0510	0.0490	0.0510	0.0490	0.0510	0.0500	0.0510	0.0500
			0.50	0.0510	0.0500	**0.0660**	0.0500	0.0540	0.0510	**0.0590**	0.0510	0.0500	0.0500	0.0530	0.0500
		b	0.30	0.0500	0.0470	0.0560	0.0470	0.0550	0.0550	0.0590	0.0550	0.0520	0.0520	0.0530	0.0520
			0.40	0.0590	0.0550	**0.0730**	0.0550	0.0470	0.0460	0.0470	0.0460	0.0520	0.0510	0.0530	0.0510
			0.50	0.0520	0.0520	0.0580	0.0520	0.0560	0.0530	**0.0610**	0.0530	0.0520	0.0520	0.0530	0.0520
		c	0.30	0.0490	0.0480	0.0530	0.0480	0.0540	0.0530	0.0580	0.0530	0.0500	0.0500	0.0530	0.0500
			0.40	0.0570	0.0520	**0.0690**	0.0520	0.0460	0.0440	0.0460	0.0440	0.0520	0.0510	0.0540	0.0510
			0.50	0.0550	0.0530	0.0590	0.0530	0.0550	0.0490	0.0560	0.0490	0.0500	0.0500	0.0510	0.0500
	3	a	0.30	0.0550	0.0520	**0.0650**	0.0520	0.0500	0.0480	0.0550	0.0480	0.0540	0.0530	0.0560	0.0530
			0.40	0.0550	0.0510	**0.0660**	0.0510	0.0470	0.0440	0.0530	0.0440	0.0540	0.0520	0.0580	0.0520
			0.50	0.0420	0.0300	**0.0760**	0.0300	0.0500	0.0390	**0.0670**	0.0390	0.0510	0.0440	0.0560	0.0440
		b	0.30	0.0580	0.0560	**0.0720**	0.0560	0.0510	0.0490	0.0580	0.0490	0.0520	0.0510	0.0560	0.0510
			0.40	0.0540	0.0510	**0.0660**	0.0510	0.0510	0.0490	0.0580	0.0490	0.0560	0.0550	0.0590	0.0550
			0.50	0.0450	0.0360	**0.0650**	0.0360	0.0530	0.0460	**0.0610**	0.0460	0.0530	0.0520	0.0570	0.0520
		c	0.30	0.0530	0.0510	**0.0640**	0.0510	0.0530	0.0510	**0.0600**	0.0510	0.0530	0.0520	0.0560	0.0520
			0.40	0.0580	0.0540	**0.0700**	0.0540	0.0530	0.0510	0.0580	0.0510	0.0510	0.0500	0.0550	0.0500
			0.50	0.0540	0.0480	**0.0680**	0.0480	0.0510	0.0480	0.0580	0.0480	0.0550	0.0530	**0.0600**	0.0530
	4	a	0.30	0.0540	0.0500	**0.0720**	0.0500	0.0520	0.0500	**0.0600**	0.0500	0.0510	0.0500	0.0560	0.0500
			0.40	0.0520	0.0470	**0.0720**	0.0470	0.0540	0.0520	**0.0650**	0.0520	0.0530	0.0510	0.0590	0.0510
			0.50	0.0410	0.0320	**0.0890**	0.0320	0.0500	0.0430	**0.0750**	0.0430	0.0540	0.0490	**0.0680**	0.0490
		b	0.30	0.0540	0.0500	**0.0720**	0.0500	0.0520	0.0500	**0.0600**	0.0500	0.0510	0.0490	0.0560	0.0490
			0.40	0.0560	0.0480	**0.0740**	0.0480	0.0510	0.0500	**0.0630**	0.0500	0.0490	0.0470	0.0550	0.0470
			0.50	0.0530	0.0470	**0.0790**	0.0470	0.0560	0.0500	**0.0700**	0.0500	0.0520	0.0480	0.0580	0.0480
		c	0.30	0.0530	0.0490	**0.0710**	0.0490	0.0530	0.0510	**0.0620**	0.0510	0.0500	0.0500	0.0560	0.0500
			0.40	0.0540	0.0500	**0.0710**	0.0500	0.0530	0.0500	**0.0620**	0.0500	0.0500	0.0480	0.0550	0.0480
			0.50	0.0530	0.0450	**0.0720**	0.0450	0.0520	0.0470	**0.0660**	0.0470	0.0490	0.0480	0.0550	0.0480

Table 7: Continued.

Cases	g	γ	π	$m = 30$ T_L^b	$m = 30$ T_{SC}^b	$m = 30$ T_W^b	$m = 30$ T_R^b	$m = 50$ T_L^b	$m = 50$ T_{SC}^b	$m = 50$ T_W^b	$m = 50$ T_R^b	$m = 70$ T_L^b	$m = 70$ T_{SC}^b	$m = 70$ T_W^b	$m = 70$ T_R^b
			0.30	0.0570	0.0570	0.0580	0.0570	0.0490	0.0480	0.0480	0.0480	0.0480	0.0470	0.0470	0.0470
		a	0.40	0.0490	0.0460	0.0470	0.0460	0.0520	0.0510	0.0500	0.0510	0.0510	0.0490	0.0470	0.0490
			0.50	0.0580	0.0490	0.0500	0.0490	0.0530	0.0490	0.0470	0.0500	0.0510	0.0490	0.0460	0.0490
			0.30	0.0510	0.0500	0.0510	0.0500	0.0460	0.0450	0.0460	0.0450	0.0550	0.0550	0.0530	0.0550
	2	b	0.40	0.0460	0.0450	0.0450	0.0450	0.0480	0.0470	0.0450	0.0470	0.0470	0.0460	0.0430	0.0460
			0.50	0.0520	0.0490	0.0490	0.0490	0.0490	0.0460	0.0440	0.0460	0.0470	0.0450	0.0410	0.0450
			0.30	0.0530	0.0520	0.0540	0.0520	0.0440	0.0430	0.0450	0.0430	0.0520	0.0520	0.0470	0.0520
		c	0.40	0.0480	0.0480	0.0480	0.0480	0.0470	0.0460	0.0440	0.0460	0.0480	0.0470	0.0420	0.0470
			0.50	0.0560	0.0540	0.0540	0.0540	0.0520	0.0490	0.0480	0.0490	0.0510	0.0490	0.0450	0.0490
			0.30	0.0510	0.0480	0.0580	0.0480	0.0510	0.0490	0.0520	0.0490	0.0510	0.0490	0.0520	0.0490
		a	0.40	0.0550	0.0520	**0.0600**	0.0520	0.0520	0.0500	0.0520	0.0500	0.0520	0.0500	0.0520	0.0500
			0.50	0.0530	0.0430	**0.0610**	0.0430	0.0570	0.0510	0.0580	0.0510	0.0570	0.0510	0.0580	0.0510
			0.30	0.0540	0.0510	**0.0620**	0.0510	0.0510	0.0500	0.0520	0.0500	0.0510	0.0500	0.0520	0.0500
OR	3	b	0.40	0.0510	0.0480	0.0550	0.0490	0.0530	0.0520	0.0540	0.0520	0.0530	0.0520	0.0540	0.0520
			0.50	0.0560	0.0480	**0.0610**	0.0480	0.0570	0.0540	0.0560	0.0550	0.0570	0.0540	0.0560	0.0550
			0.30	0.0500	0.0470	0.0560	0.0470	0.0560	0.0530	0.0570	0.0530	0.0560	0.0530	0.0570	0.0530
		c	0.40	0.0550	0.0500	**0.0610**	0.0500	0.0510	0.0490	0.0490	0.0490	0.0510	0.0490	0.0490	0.0490
			0.50	0.0580	0.0540	**0.0610**	0.0540	0.0550	0.0520	0.0540	0.0520	0.0550	0.0520	0.0540	0.0520
			0.30	0.0540	0.0510	**0.0690**	0.0510	0.0540	0.0510	**0.0610**	0.0510	0.0500	0.0490	0.0520	0.0490
		a	0.40	0.0520	0.0470	**0.0680**	0.0470	0.0520	0.0500	0.0590	0.0500	0.0520	0.0510	0.0520	0.0510
			0.50	0.0550	0.0450	**0.0770**	0.0450	0.0570	0.0510	**0.0690**	0.0510	0.0510	0.0490	0.0520	0.0490
			0.30	0.0510	0.0480	**0.0640**	0.0480	0.0530	0.0500	**0.0590**	0.0500	0.0480	0.0470	0.0500	0.0470
	4	b	0.40	0.0510	0.0460	**0.0640**	0.0460	0.0520	0.0500	0.0580	0.0500	0.0480	0.0470	0.0500	0.0470
			0.50	**0.0600**	0.0520	**0.0780**	0.0520	0.0560	0.0510	**0.0660**	0.0510	0.0520	0.0490	0.0540	0.0490
			0.30	0.0560	0.0530	**0.0700**	0.0530	0.0550	0.0520	**0.0610**	0.0520	0.0530	0.0520	0.0550	0.0520
		c	0.40	0.0550	0.0500	**0.0670**	0.0500	0.0530	0.0500	0.0590	0.0500	0.0560	0.0550	0.0560	0.0550
			0.50	0.0540	0.0470	**0.0730**	0.0470	0.0540	0.0510	**0.0610**	0.0510	0.0510	0.0490	0.0540	0.0490

Note: The empirical TIEs of the liberal region (more than 0.06) are shown in bold.

$$T_{SC}^b = -\sum_{i=1}^{g} Y_i \left[\frac{g_i(\tilde{\pi}^b) Y_i (2G + D_i)}{2m_i G(\tilde{\gamma}^b - 2)} - \sum_{j=1, j\neq i}^{g} \frac{g_i(\tilde{\pi}^b) g_j(\tilde{\pi}^b) \tilde{\gamma}^b (\tilde{\gamma}^b - 1) Y_j}{2G(\tilde{\gamma}^b - 2)} \right], \tag{48}$$

where

$$Y_i = \frac{m_{1i} + m_{2i}}{g_i(\tilde{\pi}^b)} + \frac{m_{0i}(\tilde{\gamma}^b - 2)}{g_i(\tilde{\pi}^b)\tilde{\gamma}^b - 2g_i(\tilde{\pi}^b) + 1}, G = \sum_{i=1}^{g} m_i g_i(\tilde{\pi}^b),$$

$$D_i = m_i g_i(\tilde{\pi}^b)\tilde{\gamma}^b(1 - \tilde{\pi}^b) + 2g_i(\tilde{\pi}^b)G(\tilde{\gamma}^b - 2), \quad i = 1, \ldots, g. \tag{49}$$

Under H_{01} and H_{02}, T_{SC}^a and T_{SC}^b asymptotically follow the chi-square distribution with $g-1$ degree of freedom [11], respectively.

4.4. Rosner-Type Tests. Let $\hat{\pi}_i$ and $\tilde{\pi}$ be the MLEs under H_{1a} and H_{01}, respectively. To test the null hypothesis H_{01}, the Rosner-type test statistic is given by

$$T_R^a = \sum_{i=1}^{g} \left[\frac{(\hat{\pi}_i^a - g_i(\tilde{\pi}^a))^2}{\widehat{\mathrm{Var}}(\hat{\pi}_i^a)} \right], \tag{50}$$

where

$$\widehat{\mathrm{Var}}(\hat{\pi}_i^a) = Var(\hat{\pi}_i^a)\big|_{\pi_i = g_i(\tilde{\pi}^a), \gamma_i = \tilde{\gamma}^a} = \frac{\pi_i(1 - 2\pi_i + \gamma_i)}{2m_i}\bigg|_{\pi_i = g_i(\tilde{\pi}^a), \gamma_i = \tilde{\gamma}^a}. \tag{51}$$

Appendix D shows the specific calculation process. Then, one has

$$T_R^a = \sum_{i=1}^{g} \frac{[m_{1i} + 2m_{2i} - 2m_i g_i(\tilde{\pi}^a)]^2}{2m_i g_i(\tilde{\pi}^a)(1 - 2g_i(\tilde{\pi}^a) + \tilde{\gamma}_i^a)}. \tag{52}$$

To test H_{02}, the test statistic T_R^b is given by the following expression:

$$T_R^b = \sum_{i=1}^{g} \left[\frac{(\hat{\pi}_i^b - g_i(\tilde{\pi}^b))^2}{\widehat{\mathrm{Var}}(\hat{\pi}_i^b)} \right], \tag{53}$$

where $\widehat{\mathrm{Var}}(\hat{\pi}_i^b)$ is given in Appendix E. Then, one has

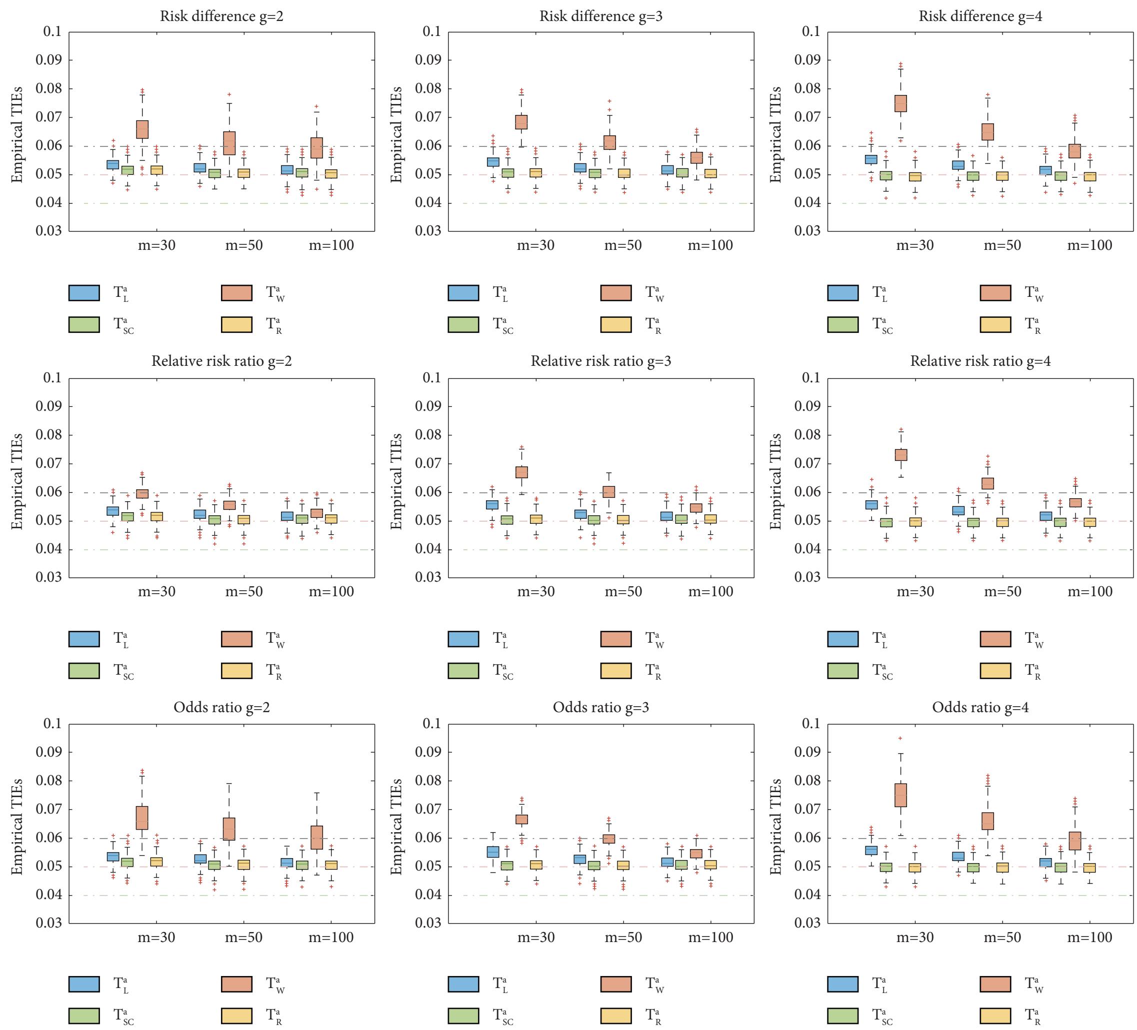

Figure 1: Box plots of empirical TIEs with 1000 parameter configurations under H_{01}.

$$T_R^b = \sum_{i=1}^{g} \frac{4m_i\left[\tilde{\pi} - g_i\left(\tilde{\pi}^b\right)\right]^2 (S_1 + S_2)^2}{g_i\left(\tilde{\pi}^b\right)\left[2 - \tilde{\gamma}^b - 4g_i\left(\tilde{\pi}^b\right) + 4g_i\left(\tilde{\pi}^b\right)\tilde{\gamma}^b - g_i\left(\tilde{\pi}^b\right)\left(\tilde{\gamma}^b\right)^2\right](S_1 + 2S_2)^2}. \tag{54}$$

Here, $\tilde{\pi}_i^b$ and $\tilde{\pi}^b$ are the MLEs under H_{02} and H_{1b}, respectively. Under H_{01} and H_{02}, T_R^a and T_R^b asymptotically submit to the chi-square distribution with $g-1$ degree of freedom.

5. Comparison with Test Methods

This section will evaluate the eight test statistics with different forms of Section 4 from the empirical TIEs and power through simulation experiments. We randomly generate 10000 replications from null hypotheses H_{01} and H_{02}. The empirical TIEs are calculated as (the number of rejections)/10000 at the significance level $\alpha = 0.05$. According to [15], a test is defined as liberal (or conservative) if its TIE is greater than 0.06 (or less than 0.04). Otherwise, it is robust.

Recently, there have been some exciting research results under Dallal's model. Our proposed method extends the existing studies to a more general situation. For this, we make $g_i(\pi)$ take the different forms, and its three conditions are as follows: (i) risk difference: $\pi_i = g_i(\pi) = \pi + \varrho_i$, (ii) relative risk ratio: $\pi_i = g_i(\pi) = \varrho_i\pi$, and (iii) odds ratio: $\pi_i = g_i(\pi) = (\varrho_i\pi)/(1 - \pi + \varrho_i\pi)$.

The specific value of $\boldsymbol{\pi}, \boldsymbol{\gamma}$ is shown in Table 2, where $\pi = 0.2, 0.25, 0.3, \eta = 0.02, \xi = 0.04, \zeta = 0, \lambda = 1.1, \mu = 1.2, \delta = 1$. Table 3 shows the parameter settings of empirical power.

Tables 4 and 5 present only the empirical TIEs of the partial cases under H_{01} and H_{02}, and the rest of the cases are shown in the Appendix F (i.e., Tables 6 and 7). It can be easily seen that the TIEs of T_W^a and T_W^b are the most inflated, followed by $T_L^a, T_L^b, T_{SC}^a, T_{SC}^b, T_R^a$, and T_R^b. Moreover, the TIEs of $T_{SC}^a, T_{SC}^b, T_R^a$, and T_R^b are close to 0.05. Thus, $T_{SC}^a, T_{SC}^b, T_R^a$, and T_R^b are robust, while T_W^a, T_W^b, T_L^a, and T_L^b are liberal.

Further, we consider the general result that ϱ_i in cases DR, RR, and OR are unequal. Firstly, 1000 parameters (π, γ_i)

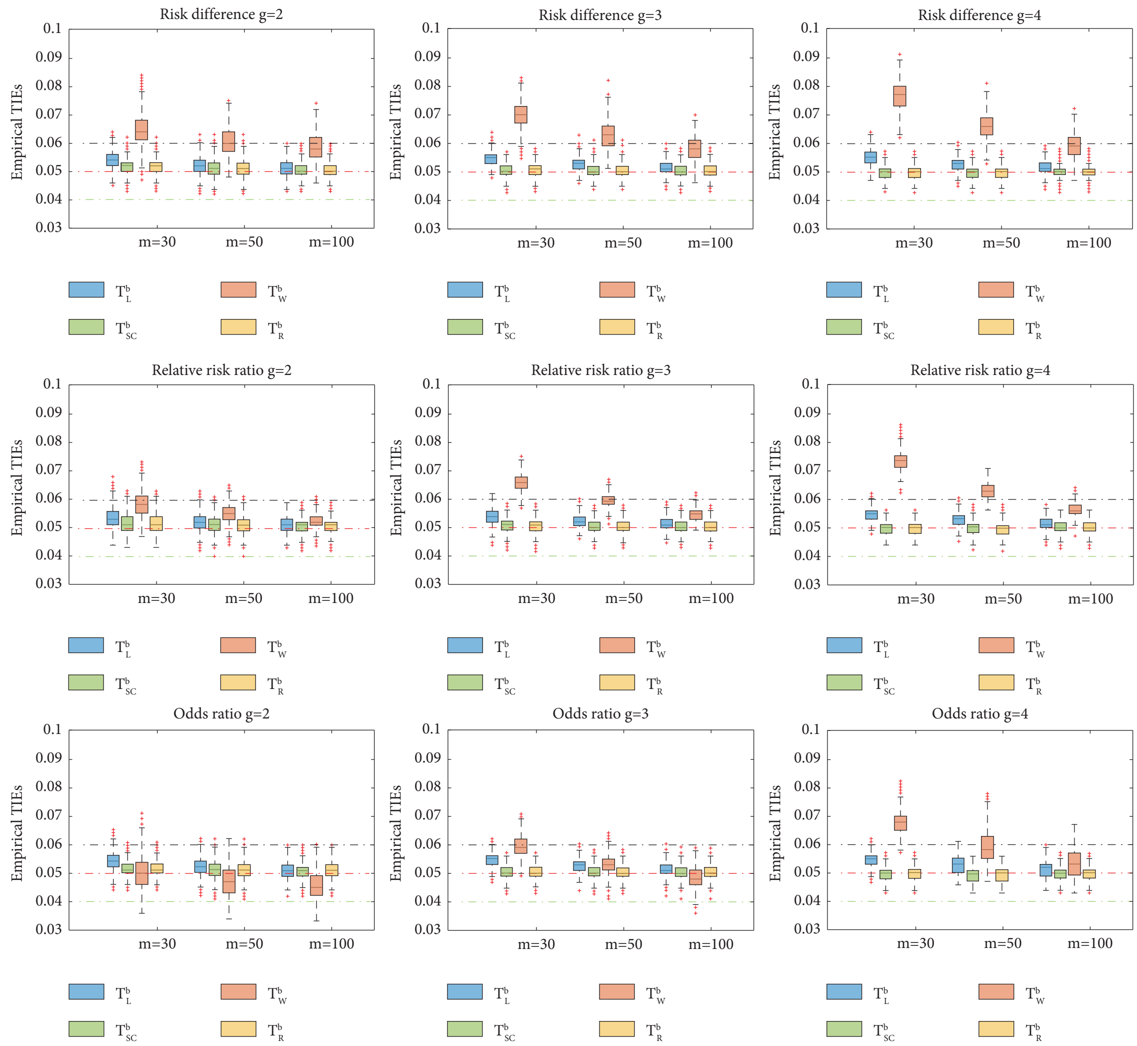

Figure 2: Box plots of empirical TIEs with 1000 parameter configurations under H_{02}.

and (π, γ) are randomly generated under H_{01} and H_{02}, respectively. In each parameter setting, $\zeta = 0$, $\delta = 1$, and λ, μ, η, and ξ are randomly generated results, which are drawn in Figures 1 and 2.

Whether to study the risk difference, relative risk ratio, or odds ratio, the empirical TIEs of the eight test statistics under H_{01} and H_{02} will approach 0.05 as the sample size increases. However, when the sample size increases in $g = 3, 4$, the change of TIEs of $T^a_{SC}, T^b_{SC}, T^a_R, T^b_R$ is not significant. The empirical TIEs of $T^a_W, T^b_W, T^a_L, T^b_L$ increase; thereinto, T^a_W andT^b_W are the most significant. Since $T^a_{SC}, T^b_{SC}, T^a_R$, and T^b_R change around 0.05, they are all robust, while T^a_W and T^b_W always have the unstable TIEs, followed by T^a_L and T^b_L.

Next, we compare the power of the eight test statistics of $m = 30, 50, 100$ and set the parameters under H_{01} and H_{1a}, respectively. Without loss of generality, we only list the parameter settings for cases EQ and RR in Table 3 and the rest of the cases are similar.

Figure 3 shows that the power of each test statistic increases as m increases, and when m is large, their power is close to each other.

As can be seen from the box plots (Figures 1 and 2), in the same number of groups and sample size, the type I error rate of T^a_W and T^b_W will be obviously different under different response rate function forms; however, the other seven test statistics have no obvious changes.

6. Two Real Examples

In this section, two real examples are given to investigate the performance of the proposed methods at the nominal level $\alpha = 0.05$.

The first example is a retinitis pigmentosa (RP) clinical trial in [1]. In this trial, 216 patients aged 20–39 with PR from different families are divided into four genetic groups, that is, autosomal dominant RP (DOM), autosomal recessive RP

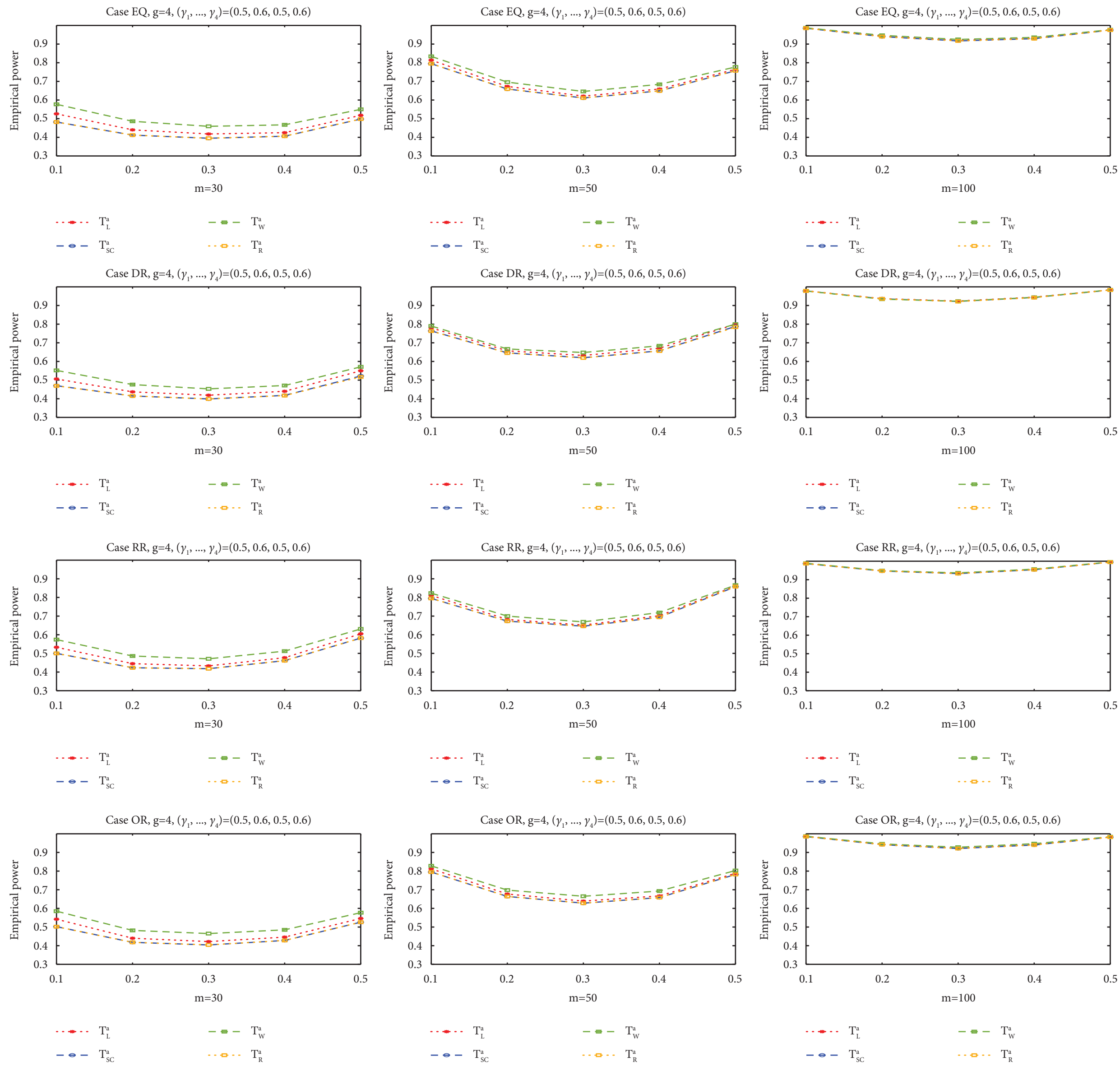

FIGURE 3: Schematic representation of the power of the different statistics under H_{01}.

TABLE 8: The number of affected eyes for patients in genetic-type groups.

Response (l)	Genetic type			
	DOM	AR	SL	ISO
0	15	7	3	67
1	6	5	2	24
2	7	9	14	57

(AR), sex-linked RP (SL), and isolate RP (ISO). Table 8 shows the response conduction of patients' eyes.

Li et al. [16] derived $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4$ by using the likelihood ratio, Score, and Wald-type statistics. Under Dallal's model, it is interesting to test whether the response rate of four groups is a functional relationship, i.e., H_{02}: $f_1(\pi_1) = f_2(\pi_2) = \cdots = f_g(\pi_g)$. The test statistics $T_l^b\,(l = L, W, SC, R)$ and their p values are listed in Table 9.

TABLE 9: Statistics and p values of the first example.

Cases	Value	T_L^b	T_{SC}^b	T_w^b	T_R^b
EQ	Statistic	11.5577	10.8830	15.6875	10.8830
	p values	0.0091	0.0124	0.0013	0.0124
DR	Statistic	9.3743	8.7395	12.7249	8.7394
	p values	0.0247	0.0330	0.0053	0.0330
RR	Statistic	6.8231	6.3412	8.7169	6.3412
	p values	0.0778	0.0961	0.0333	0.0961
OR	Statistic	9.0831	8.4657	11.1568	8.4657
	p values	0.0282	0.0373	0.0109	0.0373

The p value of $T_l^b\,(l = L, W, SC, R)$ is less than 0.05 in cases of EQ, DR, and OR, but the p value of $T_l^b\,(l = L, SC, R)$ is more significant than 0.05 in cases RR. The function setting of the risk ratio is more suitable for our data in cases RR.

Table 10: Number of affected forearms of patients taking different drugs.

Response (l)	Placebo	Collagen
0	55	36
1	3	4
2	3	6

Table 11: Statistic values and p values.

Cases	Values	T_L^a	T_{SC}^a	T_W^a	T_R^a	T_L^b	T_{SC}^b	T_W^b	T_R^b
EQ	Statistic	3.0108	3.0150	2.8097	3.0150	2.8973	2.9216	2.7030	2.9216
	p values	0.0827	0.0825	0.0937	0.0825	0.0887	0.0874	0.1002	0.0874
DR	Statistic	1.9217	1.9460	2.1448	1.9460	1.7739	1.7973	1.9895	1.7990
	p values	0.1657	0.1630	0.1431	0.1630	0.1829	0.1800	0.1584	0.1798
RR	Statistic	2.3739	2.3538	2.2874	2.3538	2.2389	2.2343	2.1550	2.2343
	p values	0.1234	0.1250	0.1304	0.1250	0.1346	0.1350	0.1421	0.1350
OR	Statistic	2.4452	2.4318	2.3268	2.4318	2.3120	2.3135	2.1979	2.3135
	p values	0.1179	0.1189	0.1272	0.1189	0.1284	0.1283	0.1382	0.1283

Another example is the two-arm multi-center phase II double placebo control clinical trial given by Pei et al. [17]. One hundred seventy patients with diffuse scleroderma were randomly assigned to receive 500 g/day of oral natural collagen or a similar placebo. The total duration of the treatment phase was 12 months, and the safety follow-up was conducted in the 15th month (3 months after drug withdrawal). The MRSS measured disease improvement within 170 patients is shown in Table 10.

In the hypothesis test, we take the function $\pi_i = g_i(\pi) = \varrho_i\pi$. Table 11 describes the eight test statistics, $T_l^a, T_l^b\,(l = L, W, SC, R)$, and its p values. It is easy to see that the p values of eight statistics are all greater than 0.05. Among them, the p value of case DR is more significant than that of other cases, which shows that the function setting of the risk difference (case DR) is most suitable for our data.

7. Conclusions

In this paper, a novel general hypothesis test was proposed to test the homogeneity of response rate function values of each group. Our proposed method can effectively test the consistency of paired data with a general function form, which is a generalization of the existing parametric hypothesis test method. Furthermore, it is of great significance to make the function $f_i(x)$ of null hypotheses take the corresponding form for different research problems. The eight test statistics T_l^a and $T_l^b\,(l = L, W, SC, R)$ were proposed to test null hypotheses.

Simulation studies are given to explore test statistics' performance in power and TIEs under general hypothesis tests. When parameter ϱ_i is unknown or not all equal in the risk difference, relative risk ratio, and odds ratio analysis, T_W and T_L have satisfactory power. However, they produce inflated TIEs, especially for T_W. It is worth noting that T_{SC}^a, T_{SC}^b, T_R^a, and T_R^b always have robust TIEs and higher power. Therefore, $T_{SC}^a, T_{SC}^b, T_R^a$, and T_R^b are recommended under the abovementioned circumstances.

We notice little research to investigate the general hypothesis test for small samples. Moreover, when there is a random zero structure in the contingency table, this may lead to the failure of test statistics. It is meaningful to study the above problems, which will be considered in our future work.

Appendix

A. Derivation of the Fisher Information Matrix I_1

If $\gamma_i \neq \gamma_j$ for some $i \neq j \in \{1, \ldots, g\}$, by taking the second partial derivative of l with respect to π_i and γ_i, we get

$$\begin{aligned}
\frac{\partial^2 l}{\partial \pi_i^2} &= -\frac{m_{0i}(\gamma_i - 2)^2}{(\pi_i\gamma_i - 2\pi_i + 1)^2} - \frac{m_{1i} + m_{2i}}{\pi_i^2},\\
\frac{\partial^2 l}{\partial \gamma_i^2} &= -\frac{m_{0i}\pi_i^2}{(\pi_i\gamma_i - 2\pi_i + 1)^2} - \frac{m_{1i}}{(\gamma_i - 1)^2} - \frac{m_{2i}}{\gamma_i^2},\\
\frac{\partial^2 l}{\partial \pi_i \partial \gamma_i} &= \frac{m_{0i}}{(\pi_i\gamma_i - 2\pi_i + 1)^2},\\
\frac{\partial^2 l}{\partial \pi_i \partial \pi_j} &= \frac{\partial^2 l}{\partial \gamma_i \partial \gamma_j} = \frac{\partial^2 l}{\partial \pi_i \partial \gamma_j} = 0, \quad i \neq j.
\end{aligned} \tag{A.1}$$

It follows that the information matrix I_1 is given by

$$I_1 = -E\begin{pmatrix} \frac{\partial^2 l}{\partial \pi_1^2} & \cdots & 0 & \frac{\partial^2 l}{\partial \pi_1 \partial \gamma_1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\partial^2 l}{\partial \pi_g^2} & 0 & \cdots & \frac{\partial^2 l}{\partial \pi_g \partial \gamma_g} \\ \frac{\partial^2 l}{\partial \pi_1 \partial \gamma_1} & \cdots & 0 & \frac{\partial^2 l}{\partial \gamma_1^2} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\partial^2 l}{\partial \pi_g \partial \gamma_g} & 0 & \cdots & \frac{\partial^2 l}{\partial \gamma_g^2} \end{pmatrix}, \quad \text{(A.2)}$$

where

$$\begin{aligned} E\left(-\frac{\partial^2 l}{\partial \pi_i^2}\right) &= m_{+i}\left[\frac{(\gamma_i - 2)^2}{1 - 2\pi_i + \pi_i\gamma_i} + \frac{2-\gamma_i}{\pi_i}\right], \\ E\left(-\frac{\partial^2 l}{\partial \gamma_i}\right) &= m_{+i}\left[\frac{\pi_i^2}{1 - 2\pi_i + \pi_i\gamma_i} + \frac{2\pi_i}{1-\gamma_i} + \frac{\pi_i}{\gamma_i}\right], \quad \text{(A.3)} \\ E\left(-\frac{\partial^2 l}{\partial \pi_i \partial \gamma_i}\right) &= \frac{m_{+i}}{1 - 2\pi_i + \pi_i\gamma_i}, \quad i = 1, \ldots, g. \end{aligned}$$

B. Derivation of the Information Matrix I_2

If $\gamma_1 = \cdots = \gamma_g \triangleq \gamma$, by taking the second partial derivative of l_1 with respect to $\pi_i \, (i = 1, \ldots, g)$ and γ, we get

$$\begin{aligned} \frac{\partial^2 l_1}{\partial \pi_i^2} &= -\frac{m_{0i}(\gamma - 2)^2}{(\pi_i\gamma - 2\pi_i + 1)^2} - \frac{m_{1i} + m_{2i}}{\pi_i^2}, \\ \frac{\partial^2 l_1}{\partial \gamma^2} &= \sum_{i=1}^{g}\left[-\frac{m_{0i}\pi_i^2}{(\pi_i\gamma - 2\pi_i + 1)^2} - \frac{m_{1i}}{(\gamma - 1)^2} - \frac{m_{2i}}{\gamma^2}\right], \quad \text{(B.1)} \\ \frac{\partial^2 l_1}{\partial \pi_i \partial \gamma} &= \frac{m_{0i}}{(\pi_i\gamma - 2\pi_i + 1)^2}, \end{aligned}$$

and $(\partial^2 l_1/\partial \pi_i \partial \pi_j) = 0, i \neq j$. So, the information matrix I_2 is given by

$$I_2 = -E\begin{pmatrix} \frac{\partial^2 l_1}{\partial \pi_1^2} & \cdots & 0 & \frac{\partial^2 l_1}{\partial \pi_1 \partial \gamma} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \frac{\partial^2 l_1}{\partial \pi_g^2} & \frac{\partial^2 l_1}{\partial \pi_g \partial \gamma} \\ \frac{\partial^2 l_1}{\partial \pi_1 \partial \gamma} & \cdots & \frac{\partial^2 l_1}{\partial \pi_g \partial \gamma} & \frac{\partial^2 l_1}{\partial \gamma^2} \end{pmatrix}, \quad \text{(B.2)}$$

where

$$\begin{aligned} E\left(-\frac{\partial^2 l_1}{\partial \pi_i^2}\right) &= m_{+i}\left[\frac{(\gamma - 2)^2}{1 - 2\pi_i + \pi_i\gamma} + \frac{2-\gamma}{\pi_i}\right], \quad i = 1, \ldots, g, \\ E\left(-\frac{\partial^2 l_1}{\partial \pi_i \partial \gamma}\right) &= \frac{m_{+i}}{1 - 2\pi_i + \pi_i\gamma}, \quad i = 1, \ldots, g, \\ E\left(-\frac{\partial^2 l_1}{\partial \gamma^2}\right) &= \sum_{i=1}^{g}\left[m_{+i}\left(\frac{\pi_i^2}{1 - 2\pi_i + \pi_i\gamma} + \frac{2\pi_i}{1-\gamma} + \frac{\pi_i}{\gamma}\right)\right]. \end{aligned} \quad \text{(B.3)}$$

Furthermore, we have

$$I_2^{-1} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1g} & d_1 \\ c_{21} & c_{22} & \cdots & c_{2g} & d_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{g1} & c_{g2} & \cdots & c_{gg} & d_g \\ d_1 & d_2 & \cdots & d_g & f \end{pmatrix}, \quad \text{(B.4)}$$

where

$$\begin{aligned} f &= \left(I_{g+1,g+1} - \sum_{k-1}^{g}\frac{I_{k,g+1}^2}{I_{kk}}\right)^{-1}, c_{ii} = \frac{1}{I_{ii}} + \frac{I_{i,g+1}^2 f}{I_{ii}^2}, \quad i = 1, \ldots, g, \\ c_{ij} &= \frac{I_{i,g+1}I_{j,g+1}f}{I_{ii}I_{jj}}, \quad i \neq j, d_i = -\frac{I_{i,g+1}f}{I_{ii}}, \quad i = 1, \ldots, g. \end{aligned} \quad \text{(B.5)}$$

C. Derivation of the Inverse Matrix A_2

For $g = 2$, it is easy to get $A_2 = (c_{11} - c_{21}b_{12} - c_{12}b_{12} + c_{22}b_{12}^2)$. If $g = 3$, then

$$A_2 = \begin{pmatrix} c_{11} - c_{21}b_{12} - c_{12}b_{12} + c_{22}b_{12}^2 & c_{12} - c_{22}b_{12} - c_{13}b_{23} + c_{23}b_{12}b_{23}, \\ c_{21} - c_{31}b_{23} - c_{22}b_{12} + c_{32}b_{12}b_{23} & c_{22} - c_{32}b_{23} - c_{32}b_{23} + c_{22}b_{12}^2. \end{pmatrix}. \quad \text{(C.1)}$$

When $g = 4$, we have

$$A_2 = \begin{pmatrix} c_{11} - c_{21}b_{12} - c_{12}b_{12} + c_{22}b_{12}^2 & c_{12} - c_{22}b_{12} - c_{13}b_{23} + c_{23}b_{12}b_{23} & c_{13} - c_{23}b_{12} - c_{14}b_{34} + c_{24}b_{12}b_{34}, \\ c_{21} - c_{31}b_{23} - c_{22}b_{12} + c_{32}b_{12}b_{23} & c_{22} - c_{32}b_{23} - c_{32}b_{23} + c_{22}b_{12}^2 & c_{23} - c_{33}b_{23} - c_{24}b_{34} + c_{34}b_{23}b_{34}, \\ c_{31} - c_{41}b_{34} - c_{32}b_{12} + c_{12}b_{12}b_{34} & c_{32} - c_{42}b_{34} - c_{33}b_{23} + c_{43}b_{23}b_{34} & c_{33} - c_{43}b_{34} - c_{34}b_{34} + c_{44}b_{34}^2. \end{pmatrix}, \quad \text{(C.2)}$$

where $b_{i,i+1} = f_i^{-1} f_{i+1}(\widehat{\pi}^b)/\widehat{\pi}^b$ $i = 1, 2, 3$. Thus, $A_2^{-1} = 1/|A_2| A_2^*$ and A_2^* is the adjoint matrix of A_2.

D. Derivation of $\mathrm{Var}(\widehat{\pi}_i^a)$ for Rosner-Type Statistic

Due to $\widehat{\pi}_i^a = (m_{1i} + 2m_{2i})/(2m_i)$, $m_{1i} \sim B(m_i, 2\pi_i(1-\gamma_i))$, and $m_{2i} \sim B(m_i, \pi_i\gamma_i)$, the expectation and variance of m_{1i}, m_{2i} are, respectively, $E(m_{0i}) = m_{+i}(1 - 2\pi_i + \pi_i\gamma_i)$, $E(m_{1i}) = 2m_i\pi_i(1-\gamma_i)$, $E(m_{2i}) = m_i\pi_i\gamma_i$, $\mathrm{Var}(m_{1i}) = 2m_i \pi_i(1-\gamma_i)(1 - 2\pi_i(1-\gamma_i))$, and $\mathrm{Var}(m_{2i}) = m_i\pi_i\gamma_i(1 - \pi_i\gamma_i)$. Based on the abovementioned conditions, the covariance of m_{1i} and m_{2i} is calculated as

$$\begin{aligned} cov(m_{1i}, m_{2i}) &= E(m_{1i}m_{2i}) - E(m_{1i})E(m_{2i}) \\ &= 2m_i(m_i - 1)\pi_i^2\gamma_i(1-\gamma_i) - 2m_i^2\pi_i^2\gamma_i(1-\gamma_i) \\ &= 2m_i\pi_i^2\gamma_i(\gamma_i - 1). \end{aligned} \quad \text{(D.1)}$$

Then, we have

$$\begin{aligned} \mathrm{Var}(m_{1i} + 2m_{2i}) &= \mathrm{Var}(m_{1i}) + 4Var(m_{2i}) + 4cov(m_{1i}, m_{2i}) \\ &= 2m_i\pi_i(1-\gamma_i)(1 - 2\pi_i(1-\gamma_i)) + 4m_i\pi_i\gamma_i(1 - \pi_i\gamma_i) + 8m_i\pi_i^2\gamma_i(\gamma_i - 1) \\ &= 2m_i\pi_i(1 - 2\pi_i + \gamma_i). \end{aligned} \quad \text{(D.2)}$$

Furthermore, the expression of $\mathrm{Var}(\widehat{\pi}_i^a)$ is

$$\mathrm{Var}(\widehat{\pi}_i^a) = \mathrm{Var}\left(\frac{m_{1i} + 2m_{2i}}{2m_i}\right) = \frac{\mathrm{Var}(m_{1i} + 2m_{2i})}{4m_i^2} = \frac{\pi_i(1 - 2\pi_i + \gamma_i)}{2m_i}. \quad \text{(D.3)}$$

E. Derivation of the Rosner-Type Statistic $\mathrm{Var}(\widehat{\pi}_i^b)$

For case (ii), since $m_{1i} \sim B(m_i, 2\pi_i(1-\gamma))$ and $m_{2i} \sim B(m_i, \pi_i\gamma)$, we can get

$$\begin{aligned} \mathrm{Var}(m_{1i} + m_{2i}) &= \mathrm{Var}(m_{1i}) + \mathrm{Var}(m_{2i}) + 2cov(m_{1i}, m_{2i}) \\ &= 2m_i\pi_i(1-\gamma)(1 - 2\pi_i(1-\gamma)) + m_i\pi_i\gamma(1 - \pi_i\gamma) + 8m_i\pi_i^2\gamma(\gamma - 1) \\ &= m_i\pi_i\left(2 - \gamma - 4\pi + 4\pi\gamma_i - \pi_i\gamma^2\right). \end{aligned} \quad \text{(E.1)}$$

The expression of $\mathrm{Var}(\widehat{\pi}_i^b)$ is

$$\begin{aligned} \mathrm{Var}\left(\widehat{\pi}_i^b\right) &= \mathrm{Var}\left(\frac{(m_{1i} + m_{2i})(S_1 + 2S_2)}{2m_i(S_1 + S_2)}\right) = \frac{(S_1 + 2S_2)^2}{(2m_i(S_1 + S_2))^2}\mathrm{Var}(m_{1i} + m_{2i}) \\ &= \frac{m_i\pi_i\left(2 - \gamma - 4\pi + 4\pi\gamma_i - \pi_i\gamma^2\right)(S_1 + 2S_2)^2}{(2m_i(S_1 + S_2))^2}. \end{aligned} \quad \text{(E.2)}$$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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