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Research Article

Nash Equilibrium of Stochastic Partial Differential Game with Partial Information via Malliavin Calculus

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In this article, we consider the Nash equilibrium of stochastic differential game where the state process is governed by a controlled stochastic partial differential equation and the information available to the controllers is possibly less than the general information. All the system coefficients and the objective performance functionals are assumed to be random. We find an explicit strong solution of the linear stochastic partial differential equation with a generalized probabilistic representation for this solution with the benefit of Kunita's stochastic flow theory. We use Malliavin calculus to derive a stochastic maximum principle for the optimal control and obtain the Nash equilibrium of this type of stochastic differential game problem.

1. Introduction

Let $(\mathcal{O}, \mathcal{B}(\mathcal{O}), m)$ be a measure space with finite measure, here, $\mathcal{O}$ is a bounded, open subset of $\mathbb{R}^n$ with C^1 regular boundary $\partial\mathcal{O}$, and m is the Lebesgue measure. Suppose the dynamics of a state process $X_t(x) = X_t^{(u,v)}(\omega, x), t \in [0,T], \omega \in \Omega$ and $x \in \mathcal{O}$ is a controlled stochastic process in $\mathbb{R}$ of the form

$$X_t(x) = X_0(x) + \int_0^t \{\mathscr{L}_s X_s(x) + b(\omega, s, x, X_s(x), \nabla_x X_s(x), u_s(x), v_s(x))\}\mathrm{d}s + \int_0^t \sigma(\omega, s, x, X_s(x), \nabla_x X_s(x), u_s(x), v_s(x))\mathrm{d}B_t, \tag{1}$$

with boundary condition $X_t(x) = \zeta(t,x), (t,x) \in (0,T) \times \partial\mathcal{O}$, where the coefficients

$$b\left(\omega, t, x, y, y', u, v\right): \Omega \times [0,T] \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \times U \times U \longrightarrow \mathbb{R},$$
$$\sigma\left(\omega, t, x, y, y', u, v\right): \Omega \times [0,T] \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \times U \times U \longrightarrow \mathbb{R},$$

$$\begin{aligned} X_0(x)&: \overline{\mathcal{O}} \longrightarrow \mathbb{R},\\ \zeta(t,x)&: (0,T)\times\partial\mathcal{O} \longrightarrow \mathbb{R}. \end{aligned} \tag{2}$$

are Borel measurable functions, where $U \subset \mathbb{R}$ is a closed convex set, and $\mathcal{L}$ is a partial operator of order m and ∇_x is the gradient acting on the space variable $x \in \mathbb{R}^n$. Here, $B_t = B_t(\omega)$ is a one-dimensional Brownian motion on a given filtered probability measure space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. The stochastic processes $u: \Omega \times [0,T] \times \mathcal{O} \longrightarrow U$, $v: \Omega \times [0,T] \times \mathcal{O} \longrightarrow U$ are two control processes and have values in a given closed convex set $U \subset \mathbb{R}$ for all $t \in [0,T]$, for a given fixed $T > 0$. Also, u_t, v_t are adapted to a given filtration $\{\mathcal{E}_t\}_{t\geq 0}$, where $\mathcal{E}_t \subset \mathcal{F}_t$, for every $t \in [0,T]$. $\{\mathcal{E}_t\}_{t\geq 0}$ represents the information available to the controller at time t. For example, we could take

$$\mathcal{E}_t = \mathcal{F}_{(t-\Delta)^+}; \quad t \in [0,T], \Delta > 0 \text{ is a constant}, \tag{3}$$

meaning that the controller gets a delayed information compared to $\mathcal{F}_t$. We refer to [1, 2] for more details about optimal control under partial information or partial observation.

Let $l_i: \Omega \times [0,T] \times \mathcal{O} \times \mathbb{R} \times U \times U \longrightarrow \mathbb{R}$ and $h_i: \Omega \times \mathcal{O} \times \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1,2$ are given measurable functions, for every (ω, t, x, u, v), the functions $\gamma \longmapsto l_i(\omega, t, x, \gamma, u, v)$ and $\gamma \longmapsto l_i(\omega, x, \gamma), i = 1,2$ are bounded continuously differentiable functions. Suppose we are given two performance functionals of the following form, for $u, v \in \mathcal{E}_t \otimes \mathcal{B}(\mathbb{R})$,

$$\begin{aligned} J_1(u,v) &= E\left[\int_0^T \int_{\mathcal{O}} l_1(\omega,t,x,X_t(x),u_t(x),v_t(x))m(\mathrm{d}x)\mathrm{d}t\right] + E\left[\int_{\mathcal{O}} h_1(\omega,x,X_T(x))m(\mathrm{d}x)\right],\\ J_2(u,v) &= E\left[\int_0^T \int_{\mathcal{O}} l_2(\omega,t,x,X_t(x),u_t(x),v_t(x))m(\mathrm{d}x)\mathrm{d}t\right] + E\left[\int_{\mathcal{O}} h_2(\omega,x,X_T(x))m(\mathrm{d}x)\right], \end{aligned} \tag{4}$$

where m is a finite Lebesgue measure on the above given measurable space $(\mathcal{O}, \mathcal{B}(\mathcal{O}))$, $E = E_P$ denotes the expectation with respect to the probability measure P. Let $\mathcal{A}_u, \mathcal{A}_v$ denote the given family of controls u, v, which are contained in the set of $\mathcal{E}_t \otimes \mathcal{B}(\mathbb{R})$-adapted controls, such that (1) has a unique strong solution up to time T and for all $u \in \mathcal{A}_u, v \in \mathcal{A}_v$, $i = 1,2$

$$E\left[\int_0^T \int_{\mathcal{O}} \left|l_i(\omega,t,x,X_t(x),u_t(x),v_t(x))\right| m(\mathrm{d}x)\mathrm{d}t + \int_{\mathcal{O}} \left|h_i(\omega,x,X_T(x))\right| m(\mathrm{d}x)\right] < \infty. \tag{5}$$

The partial information nonzero-sum stochastic partial differential game problem under consideration is stated as follows:

Find $u^* \in \mathcal{A}_u$ and $v^* \in \mathcal{A}_v$ such that

$$\begin{aligned} J_1(u^*,v^*) &= \sup_{u\in\mathcal{A}_u} (u,v^*),\\ J_2(u^*,v^*) &= \sup_{v\in\mathcal{A}_v} J_2(u^*,v). \end{aligned} \tag{6}$$

Such a control (u^*, v^*) is called a Nash equilibrium. The intuitive idea is that there are two players, Player I and Player II. While Player I controls u, Player II controls v. Given that each player knows the equilibrium strategy chosen by the other player, none of the players has anything to gain by changing only his or her own strategy (i.e., by changing unilaterally). Note that since we allow b, σ, l_i, h_i to be stochastic processes and also because our controls are required to be $\mathcal{E}_t$-adapted, this problem is not of Markovian type and hence cannot be solved by dynamic programming. In this paper, we use Malliavin calculus techniques, see [3, 4] to obtain a maximum principle for this general non-Markovian stochastic partial differential game with partial information. Our approach still works when any finite number of players instead of two-player formulation.

The problem of finding sufficient conditions for optimality for a stochastic optimal control problem with infinite dimensional state equation, most along the lines of the Pontryagin maximum principle was already addressed in the early 1980s in the pioneering paper by [1]. The Pontryagin maximum principle for the dynamic systems modeled by stochastic partial differential equations (SPDEs) is a well-known result, and we refer to [1, 5–11], and therein, for more details about the maximum principle for SPDEs. Despite of the fact that the finite dimensional case has been completely solved by [12], the infinite dimensional case requires at least one of the following three assumptions, see [13, 14]:

(i) The control domain is convex;

(ii) The diffusion does not depend on the control;

(iii) The state equation and performance functional are both linear in the state variable.

So, the maximum principle for the infinite dimensional case still has important open issues both on the side of the generality of the abstract model and on the side of its applicability to systems modeled by SPDEs. In this paper, let us suppose that the diffusion is dependent on the control, the state equation and performance functional are both nonlinear in the state variable, but we will assume that the control domain U is convex. That is to say, we just assume that (i) holds, and we do not need (ii) and (iii) to hold.

But there are few references about the maximum principle for stochastic differential games of systems described by stochastic partial differential equations. In the present paper, we use Malliavin calculus techniques to obtain a maximum principle for this general non-Markovian stochastic differential game with partial information of systems described by stochastic partial differential equations, without the use of backward stochastic differential equations. To use Malliavin calculus, a strong solution of stochastic partial differential equations with a generalized probabilistic representation will be given with the benefit of Kunita's stochastic flow theory. This approach of stochastic flow has been used to derive optimal control of stochastic partial differential equations with jump in [15], and at the same time, the ideas of [15] give us great inspiration. Our paper is related to the recent paper [16], where a maximum principle for stochastic control problem (NOT for stochastic differential game problem) with partial information is dealt with. However, the approach in [16] needs the solution of the backward stochastic differential equation for the adjoint processes. This is often a difficult point, particularly in the partial information case.

We summarize the main contributions of this paper as follows: (i) we find a strong solution of a stochastic partial differential equation, which follows from the theory of stochastic flows for stochastic processes; (ii) all coefficients of the controlled stochastic partial differential equation we are studying in this paper are all random, and the coefficients of the objective performance functionals are also random; (iii) with the help of Malliavin calculus for Brownian motion, we get the Nash equilibrium for our stochastic partial differential game with partial information, as obtained by establishing the corresponding stochastic maximum principles for the stochastic optimal controls. It is worth noting that our diffusion term in the controlled stochastic partial differential equation can be dependent on two control variables from two players and the controlled stochastic partial differential equation or the objective performance functionals need not be linear in the state variable.

The article is organised in the following way: in Section 2, we present the explicit strong solution of a stochastic partial differential equation with the benefit of stochastic flow theory for stochastic processes. In Section 3, we provide some properties of Malliavin calculus for Brownian motion, especially the chain rule and duality formula of the Malliavin derivative. In Section 4, we give the Nash equilibrium for our stochastic partial differential game with partial information with the help of the explicit strong solution and Malliavin calculus via a stochastic maximum principle. Finally, in Section 5, an example is given to illustrate our main results, and the conclusion is given in the final section.

2. Strong Solution of Linear SPDE

In this section, we recall some definitions of stochastic flows and preliminary results, more details about stochastic flows see [17, 18]. Let $m \in \mathbb{N}, \delta \in (0,1]$. Denote by $C^{m,\delta}$ the space of all m-times continuously differentiable functions $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\|f\|_{m+\delta;K} = \|f\|_{m;K} + \sum_{|\alpha|=m} \sup_{x,y\in K, x\neq y} \frac{\left|D^\alpha f(x) - D^\alpha f(y)\right|}{\|x-y\|^\delta} < \infty, \tag{7}$$

where

$$\|f\|_{m;K} := \sup_{x\in K} \frac{|f(x)|}{1+\|x\|} + \sum_{1\le|\alpha|\le m} \sup_{x\in K}\left|D^\alpha f(x)\right|, \tag{8}$$

for all compact sets $K \subset \mathbb{R}^n$. For the multiindex of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_d)$, the operator D^α is defined as

$$D^\alpha = \frac{\partial|\alpha|}{\left(\partial x^1\right)^{\alpha_1}\cdots\left(\partial x^d\right)^{\alpha_d}}, \tag{9}$$

where $|\alpha| := \sum_{i=1}^d \alpha_i$. Further, introduce for sets $K \subset \mathbb{R}^n$, the norm

$$\|g\|^*_{m+\delta;K} := \|g\|^*_{m;K} + \sum_{|\alpha|=m} \left\|D_x^\alpha D_y^\alpha g\right\|^*_{\delta;K}, \tag{10}$$

where

$$\|g\|_{m;K}^{*} := \sup_{x,y\in K} \frac{|g(x,y)|}{(1+\|x\|)(1+\|y\|)} + \sum_{1\le|\alpha|\le m} \sup_{x,y\in K} \left|D_x^{\alpha} D_y^{\alpha} g(x,y)\right|,$$
$$\|g\|_{\delta;K}^{*} := \sup_{\substack{x,x',y,y'\in K\\ x\neq y, x'\neq y'}} \frac{\left|g(x,y) - g\left(x',y\right) - g\left(x,y'\right) + g\left(x',y'\right)\right|}{\left\|x-x'\right\|^{\delta}\left\|y-y'\right\|^{\delta}}. \tag{11}$$

We will simply write $\|g\|\|_{m+\delta}^{*}$ for $\|g\|\|_{m+\delta;\mathbb{R}^n}^{*}$.

Define

$$\begin{aligned}
\tilde{b}(t,x) &= \frac{\partial}{\partial\gamma} b(\omega,t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x)),\\
\tilde{\sigma}(t,x) &= \frac{\partial}{\partial\gamma} \sigma(\omega,t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x)),\\
b_i'(t,x) &= \frac{\partial}{\partial\gamma_i'} b(\omega,t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x)), \quad i=1,\ldots,n,\\
\sigma_i'(t,x) &= \frac{\partial}{\partial\gamma_i'} \sigma(\omega,t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x)), \quad i=1,\ldots,n,\\
b_u(t,x) &= \frac{\partial}{\partial u} b(\omega,t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x)),\\
\sigma_u(t,x) &= \frac{\partial}{\partial u} \sigma(\omega,t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x)).
\end{aligned} \tag{12}$$

Set

$$F_i(x,\mathrm{d}t) := b_i'(t,x)\mathrm{d}t + \sigma_i'(t,x)\mathrm{d}B(t), \quad i=1,\cdots n, \tag{13}$$

$$F_{n+1}(x,\mathrm{d}t) := \tilde{b}(t,x)\mathrm{d}t + \tilde{\sigma}(t,x)\mathrm{d}B(t). \tag{14}$$

Define the symmetric matrix function $A^{ij}(t,x,y)_{1\le i,j\le n+1}$ as

$$\begin{aligned}
A^{ij}(t,x,y) &= \sigma_i'(t,x)\sigma_j'(t,y), \quad i,j=1,\ldots,n,\\
A^{i,n+1}(t,x,y) &= \sigma_i'(t,x)\tilde{\sigma}(t,y), \quad i=1,\ldots,n,\\
A^{n+1,n+1}(t,x,y) &= \tilde{\sigma}(t,x)\tilde{\sigma}(t,y).
\end{aligned} \tag{15}$$

We assume that, for some $m\ge 3$ and $\delta>0$,

$$\begin{aligned}
&\sum_{i,j=1}^{n+1} \int_0^T \left\|A^{ij}(t,\cdot,\cdot)\right\|_{m+\delta}^{*} \mathrm{d}t < \infty,\\
&\int_0^T \left[\sum_{i=1}^{n} \left\|b_i'(t,\cdot)\right\|_{m+\delta} + \|\tilde{b}(t,\cdot)\|_{m+\delta}\right] \mathrm{d}s < \infty, a.e.
\end{aligned} \tag{16}$$

For all $u,v,\beta\in\mathscr{A}_u$, the stochastic process $Y(t,x) = Y^{\beta}(t,x) = \mathrm{d}/\mathrm{d}y X^{u+y\beta,v}(t,x)|_{y=0}$ exists and

$$\begin{aligned}
\mathscr{L}Y(t,x) &= \frac{\mathrm{d}}{\mathrm{d}y} \mathscr{L}X^{u+y\beta,v}(t,x)\Big|_{y=0},\\
\nabla_x Y(t,x) &= \frac{\mathrm{d}}{\mathrm{d}y} \nabla_x X^{u+y\beta,v}(t,x)\Big|_{y=0}.
\end{aligned} \tag{17}$$

Further, suppose that $Y(t,x)$ follows the SPDE.

$$\begin{aligned}Y(t,x) = &\int_0^t \left[\mathscr{L}_s Y(s,x) + Y(s,x)\tilde{b}(s,x) + \nabla_x Y(s,x) b'(s,x)\right] \mathrm{d}s \\ &+ \int_0^t \left[Y(s,x)\tilde{\sigma}(s,x) + \nabla_x Y(s,x)\sigma'(s,x)\right] \mathrm{d}B_s \\ &+ \int_0^t \beta(s,x) b_u(s,x) ds + \int_0^t \beta(s,x)\sigma_u(s,x) \mathrm{d}B_s,\end{aligned} \tag{18}$$

with obviously initial condition $Y(0,x) = 0, x \in \overline{\mathcal{O}}$, and boundary condition

$$Y(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\mathcal{O}, \tag{19}$$

where $(t,x) \in [0,T] \times \mathcal{O}$, and $\nabla_x = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$.

In the following, we assume that the differential operator $\mathscr{L}$ in the above SPDE (18) is of the form.

$$\mathscr{L}_t \Phi = \mathscr{L}_t^{(1)} \Phi + \mathscr{L}_t^{(s)} \Phi, \tag{20}$$

where

$$\begin{aligned}\mathscr{L}_t^{(1)} \Phi &:= \frac{1}{2}\sum_{i,j=1}^n G^{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} \Phi + \sum_{i=1}^n f^i(t,x) \frac{\partial}{\partial x_i} \Phi + d(t,x)\Phi, \\ \mathscr{L}_t^{(2)} \Phi &:= \frac{1}{2}\sum_{i,j=1}^n A^{ij}(t,x) \frac{\partial^2 \Phi}{\partial x_i \partial x_j} + \sum_{i=1}^n \left(A^{i,n+1}(t,x,x) + \frac{1}{2} C_i(t,x)\right) \frac{\partial \Phi}{\partial x_i} \\ &\quad + \frac{1}{2}\left(D(t,x) + A^{n+1,n+1}(t,x,x)\right)\Phi,\end{aligned} \tag{21}$$

where $d(t,x)$ is a continuous function in (t,x), belongs to $C^{m,\delta}$ for some $m \geq 3, \delta > 0$ and $d/(1+\|x\|)$ is bounded from the above. Here,

$$\begin{aligned}C_i(t,x) &:= \sum_{j=1}^n \frac{\partial A^{ij}}{\partial y_i}(t,x,y)\bigg|_{y=x}, \quad i = 1, \ldots, n, \\ D(t,x) &:= \sum_{j=1}^n \frac{\partial A^{i,n+1}}{\partial y_i}(t,x,y)\bigg|_{y=x}.\end{aligned} \tag{22}$$

Furthermore, we require the following condition,

(L-i) $L_t^{(1)}$ is an elliptic differential operator.

(L-ii) There exists a non-negative symmetric continuous matrix function $(G^{ij}(t,x,y))_{1 \leq i,j \leq n}$ such that $G^{ij}(t,x,y) = g^i(x,t) g^j(y,t)$, hence

$$\begin{aligned}G^{ij}(t,x,y) &= G^{ji}(t,x,y), \\ \sum_{i,j=1}^n \left\|G^{ij}(t,\cdot,\cdot)\right\|_{m+1+\delta} &\leq K,\end{aligned} \tag{23}$$

for all s, for a constant K and some $m \geq 3, \delta > 0$.

(L-iii) The functions $f_i(t,x), i = 1, \ldots, n$ are continuous in (t,x) and satisfy

$$\sum_{i=1}^n \left\|f_i(t,\cdot)\right\|_{m+\delta} \leq C, \quad \text{for all } s, \tag{24}$$

for a constant C and some $m \geq 3$ and $\delta > 0$.

(L-iv) The function $\tilde{b}, \tilde{\sigma}, G^{ij}$ and d are uniformly bounded.

Here, the operator $\mathscr{L}^{(1)}$ does not depend on controls u or v, that is, there are no controls in $G^{i,j}$ and f^i. In this section, aided by a stochastic flow theory, we will give a probabilistic representation of the explicit strong solution of the above linear SPDE (18).

Now, we derive the announced probabilistic representation of a solution $Y(t,x)$ of linear SPDE (18). Let $\Upsilon(x,t) = (\Upsilon_1(x,t), \ldots, \Upsilon_n(x,t))$ be a $C^{k,\gamma}$-valued Brownian motion, that is a continuous process $\Upsilon(t,\cdot) \in C^{k,\gamma}$ with independent increments on another probability space $(\widehat{\Omega}, \widehat{\mathscr{F}}, \widehat{P})$. Assume that this stochastic process has local characteristic $G^{ij}(x,y,t)$ and $m^i(x,t) = f^i(t,x) - c^i(t,x)$, where the correction term $c^i(t,x)$ is given by

$$c^i(t,x) = \frac{1}{2}\int_0^t \sum_{j=1}^n \frac{\partial G^{ij}}{\partial x_j}(s,x,y)\bigg|_{y=x} \mathrm{d}s, \quad i = 1, \ldots, n. \tag{25}$$

For instance, $\Upsilon(x,t)$ has a decomposition

$$\Upsilon(x,t) = M(x,t) + B(x,t), \tag{26}$$

where

$$\begin{aligned}\langle M^i(x,t), M^j(y,t)\rangle &= \int_0^t G^{ij}(x,y,s)\mathrm{d}s,\\ B^i(x,t) &= \int_0^t m^i(x,s)\mathrm{d}s,\\ M^i(x,t) &= \int_0^t g^i(x,s)\mathrm{d}W(s).\end{aligned} \tag{27}$$

Here, $W(s)$ is a Brownian motion defined on an auxiliary probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$.

Then, let us consider the SPDE on the product space $(\Omega \times \widehat{\Omega}, \widehat{\mathcal{F}} \times \mathcal{F}, P \times \widehat{P})$:

$$\begin{aligned}\Phi(x,t) = {}&\int_0^t \mathcal{L}_s\Phi(x,s)\mathrm{d}s + \sum_{i=1}^n \int_0^t \Upsilon_i^*(x,\mathrm{d}s)\frac{\partial}{\partial x_i}\Phi(x,s) + \sum_{i=1}^n \int_0^t F_i(x,\mathrm{d}s)\frac{\partial}{\partial x_i}\Phi(x,s)\\ &+ \int_0^t \Phi(x,s)F_{n+1}(x,ds) + F_{n+2}(x,t),\end{aligned} \tag{28}$$

where $\Upsilon^*(x,t) = (\Upsilon_1^*(x,t), \dots, \Upsilon_n^*(x,t))$ is the martingale part of $\Upsilon(x,t)$ and

$$\begin{aligned}F_{n+2}(x,t) := {}&\int_0^t \beta(s,x)b_u(s,x)\mathrm{d}s\\ &+ \int_0^t \beta(s,x)\sigma_u(s,x)\mathrm{d}B_s.\end{aligned} \tag{29}$$

So, taking the expectation $E_{\widehat{P}}$ to both sides of (28) gives the following representation for the solution to linear SPDE (18):

Theorem 1. *Under the above specified conditions, the following probabilistic representation of the solution to linear SPDE (18) holds:*

$$Y(t,x) = E_{\widehat{P}}[\Phi(x,t)]. \tag{30}$$

Proof. Taking the expectation $E_{\widehat{P}}$ to both sides of equation (28), we can obtain

$$\begin{aligned}E_{\widehat{P}}[\Phi(x,t)] = {}&E_{\widehat{P}}\left[\int_0^t \mathcal{L}_s\Phi(x,s)\mathrm{d}s\right] + \sum_{i=1}^n E_{\widehat{P}}\left[\int_0^t \Upsilon_i^*(x,\mathrm{d}s)\frac{\partial}{\partial x_i}\Phi(x,s)\right]\\ &+ \sum_{i=1}^n E_{\widehat{P}}\left[\int_0^t F_i(x,\mathrm{d}s)\frac{\partial}{\partial x_i}\Phi(x,s)\right] + E_{\widehat{P}}\left[\int_0^t \Phi(x,s)F_{n+1}(x,\mathrm{d}s)\right]\\ &+ E_{\widehat{P}}\left[\int_0^t \beta(s,x)b_u(s,x)\mathrm{d}s + \int_0^t \beta(s,x)\sigma_u(s,x)\mathrm{d}B_s\right].\end{aligned} \tag{31}$$

Since $\Upsilon^*(x,t)$ is the martingale part of $\Upsilon(x,t)$ in the probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$, the second term in the right side of (31) equals zero; hence, by Fubini's theorem, we arrive at

$$\begin{aligned}E_{\widehat{P}}[\Phi(x,t)] = {}&\int_0^t \mathcal{L}_s E_{\widehat{P}}[\Phi(x,s)]\mathrm{d}s\\ &+ \sum_{i=1}^n \int_0^t F_i(x,\mathrm{d}s)\frac{\partial}{\partial x_i}E_{\widehat{P}}[\Phi(x,s)] + \int_0^t E_{\widehat{P}}[\Phi(x,s)]F_{n+1}(x,\mathrm{d}s)\\ &+ \int_0^t \beta(s,x)b_u(s,x)\mathrm{d}s + \int_0^t \beta(s,x)\sigma_u(s,x)\mathrm{d}B_s.\end{aligned} \tag{32}$$

Hence, by using (49) and (60) in (32), we find

$$
\begin{aligned}
E_{\widehat{P}}[\Phi(x,t)] &= \int_0^t \mathscr{L}_s E_{\widehat{P}}[\Phi(x,s)]\mathrm{d}s \\
&\quad + \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i} E_{\widehat{P}}[\Phi(x,s)]\left(b_i'(s,x)ds + \sigma_i'(s,x)\mathrm{d}B_s\right) \\
&\quad + \int_0^t E_{\widehat{P}}[\Phi(x,s)]\left(\widetilde{b}(s,x)\mathrm{d}s + \widetilde{\sigma}(s,x)\mathrm{d}B_s\right) \\
&\quad + \int_0^t \beta(s,x)b_u(s,x)\mathrm{d}s + \int_0^t \beta(s,x)\sigma_u(s,x)\mathrm{d}B_s \\
&= \int_0^t \left[\mathscr{L}_s E_{\widehat{P}}[\Phi(x,s)] + E_{\widehat{P}}[\Phi(x,s)]\widetilde{b}(s,x) + \nabla_x E_{\widehat{P}}[\Phi(x,s)]b'(s,x)\right]\mathrm{d}s \\
&\quad + \int_0^t \left[E_{\widehat{P}}[\Phi(x,s)]\widetilde{\sigma}(s,x) + \nabla_x E_{\widehat{P}}[\Phi(x,s)]\sigma'(s,x)\right]\mathrm{d}B_s \\
&\quad + \int_0^t \beta(s,x)b_u(s,x)\mathrm{d}s + \int_0^t \beta(s,x)\sigma_u(s,x)\mathrm{d}B_s,
\end{aligned}
\tag{33}
$$

here, $b'(s,x) := (b_1'(s,x),\ldots,b_n'(s,x))$, $\sigma'(s,x) := (\sigma_1'(s,x),\ldots,\sigma_n'(s,x))$ and

$$
\nabla_x := \left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right). \tag{34}
$$

Therefore, let $Y(t,x) = E_{\widehat{P}}[\Phi(x,t)]$ in (33), we can see $Y(t,x)$ solve the linear SPDE (18). □

Remark 2.

(i) For the probabilistic representation of the solution to linear SPDE, we also refer to Theorem 6.2.5 in [18]. Different from Theorem 6.2.5 in [18], the linear SPDE (18) contains the derivative of the control term.

(ii) Using the definition of $\Upsilon(x,t)$ and noting that Υ^i (x,t) and $F^i(x,t)$ are independent, the above linear SPDE (28) can be recast as a first-order SPDE in the sense of the Stratonovich integral using the stochastic flows theory:

$$
\begin{aligned}
\Phi(x,t) &= \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i}\Phi(x,s)\left(\Upsilon_i(x,\circ\mathrm{d}s) + F_i(x,\circ\mathrm{d}s)\right) \\
&\quad + \int_0^t \Phi(x,s)\left(d(s,x)\mathrm{d}s + F_{n+1}(x,\circ\mathrm{d}s)\right) + F_{n+2}(x,t).
\end{aligned}
\tag{35}
$$

The connection between the Itô and Stratonovich integral of semimartingale f with respect to semimartingale g is given by

$$
\int_0^t f(s-)\circ\mathrm{d}g(s) = \int_0^t f(s-)\mathrm{d}g(s) + \frac{1}{2}[f,g]_t^c, \tag{36}
$$

the notation ∘ is called the Itô circle, ∘dt stands for nonlinear integration in the sense of the Stratonovich integral. For more details about Stratonovich integral, see [19].

In order to use this probabilistic representation (30) in the proof of our general stochastic maximum principle for stochastic partial differential games, we proceed to develop an expression for $\Phi(x,t)$ in Theorem 1. Let $Z_{s,t}$ be the solution of the Stratonovich SDE.

$$
Z_{s,t}^x = x - \int_s^t G\left(Z_{s,r}^x, \circ dr\right), \tag{37}
$$

where $G(x,t) := (\Upsilon_1(x,t) + F_1(x,t),\ldots,\Upsilon_n(x,t) + F_n(x,t))$ and ∘dt stands for nonlinear integration in the sense of the Stratonovich integral. Then, by the formula (86) of Section 6.1 in [18] (where $f = 0$ in (76) of Section 6.1 in [18]), we obtain the following representation of $\Phi(x,t)$:

$$\Phi(x,t) = \int_0^t \Gamma(s,t)\cdot\left[\beta(s,x)b_u(s,x)\mathrm{d}s + \beta(s,x)\sigma_u(s,x)\circ\widehat{\mathrm{d}}B_s\right], \tag{38}$$

where

$$\Gamma(s,t) = \exp\left\{\int_s^t \widetilde{b}\left(r, Z_r^{t,x}\right)\mathrm{d}r + \int_s^t \widetilde{\sigma}\left(r, Z_r^{t,x}\right)\widehat{\mathrm{d}}B_r + \int_s^t d\left(r, Z_r^{t,x}\right)\mathrm{d}r\right\}, \tag{39}$$

$\widehat{d}$ denotes backward integration and Z_s^t is the inverse flow of the stochastic flow $Z_{s,t}$.

For the general case, we consider the case with general initial condition $\zeta(x)$, that is,

$$\begin{aligned} Y(0,x) &= \zeta(x), \quad x \in \overline{\mathcal{O}}, \\ Y(t,x) &= 0, \quad (t,x) \in (0,T)\times\partial\mathcal{O}, \end{aligned} \tag{40}$$

holds, where $\zeta \in C^{m,\delta}$. Then, $\Phi(x,t)$ in the probabilistic representation, (30) is described by

$$\begin{aligned} \Phi(x,t) = \zeta(x) &+ \int_0^t \mathscr{L}_s\Phi(x,t)\mathrm{d}s + \sum_{i=1}^n \int_0^t \Upsilon_i^*(x,\mathrm{d}s)\frac{\partial}{\partial x_i}\Phi(x,s) \\ &+ \sum_{i=1}^n \int_0^t F_i(x,\mathrm{d}s)\frac{\partial}{\partial x_i}\Phi(x,s) + \int_0^t \Phi(x,s)F_{n+1}(x,\mathrm{d}s) \\ &+ F_{n+2}(x,t), \end{aligned} \tag{41}$$

and using the same reasoning as above we obtain:

$$\begin{aligned} \Phi(x,t) &= \Gamma(0,t)\zeta\left(Z_0^{t,x}\right) \\ &\quad + \int_0^t \Gamma(s,t)\cdot\left[\beta(s,x)b_u(s,x)ds + \beta(s,x)\sigma_u(s,x)\circ\widehat{d}B_s\right], \end{aligned} \tag{42}$$

where $\Gamma(s,t)$ is given by (39).

3. Malliavin Calculus for Brownian Motion

In this section, we recall the basic definition and properties of Malliavin calculus for Brownian motion related to this paper, for reader's convenience. A natural starting point is the Wiener–Itô chaos expansion theorem, which states that any $\xi \in L^2(\mathscr{F}, P)$ can be written as

$$\xi = \sum_{n=0}^{\infty} I_n(f_n), \tag{43}$$

for a unique sequence of symmetric deterministic functions $f_n \in L^2(\lambda^n)$, where λ is a Lebesgue measure on $[0,T]$ and

$$I_n(f_n) = n!\int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1,t_2,\ldots,t_n)dB(t_1)dB(t_2)\cdots dB(t_n), \tag{44}$$

(the n-times iterated integral of f_n with respect to $B(\cdot)$) for $n = 1, 2, \cdots$ and $I_0(f_0) = f_0$ when f_0 is a constant. Here, we use λ as the measure on time variable t, m as the measure on spatial variable x.

Moreover, we have the isometry

$$E\left[\xi^2\right] = \|\xi\|_{L^2(p)}^2 = \sum_{n=0}^{\infty} n!\left\|f_n\right\|_{L^2(\lambda^n)}^2. \tag{45}$$

We first present the Malliavin derivative $D_t\xi$ with respect to Brownian motion $B(\cdot)$ at t of a given Malliavin differentiable random variable $\xi(\omega)$; $\omega \in \Omega$, and then we present some basic properties about Malliavin derivative related to this paper.

Let $\mathbb{D}$ denote the set of all random variables which are Malliavin differentiable with respect to Brownian motion $B(\cdot)$, precisely, let $\mathbb{D}$ be the space of all $\xi \in L^2(\mathscr{F}, P)$ such that its chaos expansion satisfies

$$\|\xi\|_{\mathbb{D}}^2 = \sum_{n=1}^{\infty} n n! \|f_n\|_{L^2(\lambda^n)}^2 < \infty. \quad (46)$$

Definition 3. For any $\xi \in \mathbb{D}$, define the Malliavin derivative $D_t(\xi)$ of ξ at $t, t \in [0,T]$ with respect to Brownian motion $B(\cdot)$ as

$$D_t(\xi) = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), \quad (47)$$

where the notation $I_{n-1}(f_n(\cdot,t))$ means that we apply the $(n-1)$-times iterated integral to the first $n-1$ variables $t_1, t_2, \ldots, t_{n-1}$ of $f_n(t_1, t_2, \ldots, t_n)$ and keep the last variable $t_n = t$ as a parameter.

It is easy to check that

$$E\left[\int_0^T (D_t\xi)^2 dt\right] = \sum_{n=1}^{\infty} n n! \|f_n\|_{L^2(\lambda^n)}^2, \\ = \|\xi\|_{\mathbb{D}}^2, \quad (48)$$

so $(t,\omega) \longrightarrow D_t\xi(\omega)$ belongs to $L^2(\lambda \times P)$.

Some basic properties of the Malliavin derivative D_t are the following (a) chain rule and (b) duality formula.

(a) Suppose $\xi_1, \ldots, \xi_m \in \mathbb{D}$ and that $f\colon \mathbb{R}^m \longrightarrow \mathbb{R}$ is C^1 with bounded partial derivatives. Then, $f(\xi_1, \ldots, \xi_m) \in \mathbb{D}$ and

$$D_t f(\xi_1, \ldots, \xi_m) = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(\xi_1, \ldots, \xi_m) D_t(\xi_i). \quad (49)$$

(b) Suppose $\varphi(t)$ is $\mathcal{F}_t$-adapted with

$$E\left[\int_0^T \varphi^2(t) dt\right] < \infty, \quad (50)$$

and let $\xi \in \mathbb{D}$. Then,

$$E\left[\xi \int_0^T \varphi(t) dB(t)\right] = E\left[\int_0^T \varphi(t) D_t(\xi) dt\right]. \quad (51)$$

4. Nash Equilibrium of Nonzero-Sum SPD Games

In this section, we use Malliavin calculus to derive Nash equilibrium of a nonzero-sum stochastic partial differential game by establishing a stochastic maximum principle. After some assumptions and notations, we introduce the stochastic Hamiltonian function and then the maximum principle for nonzero-sum stochastic partial differential games with partial information is stated and proved.

4.1. *Assumptions and Stochastic Hamiltonian Function.*

We now return to the partial information nonzero-sum stochastic partial differential game problem given in the introduction. We make the following assumptions:

(A1) For all $s, r, t \in (0,T), t \le r$, and all bounded $\mathcal{E}_t \otimes \mathcal{B}(\mathbb{R})$-measurable random variables $\alpha = \alpha(\omega, x), \xi = \xi(\omega, x)$, the controls

$$\beta_\alpha^i(s,x) := \alpha^i(\omega,x) I_{[t,r]}(s), \\ \eta_\xi^i(s,x) := \xi^i(\omega,x) I_{[t,r]}(s);\ s \in [0,T], \quad (52)$$

Belong to $\mathcal{A}_u$ and $\mathcal{A}_v$, respectively, where $I_{[t,T]}$ denotes the indictor function on $[t,T]$.

(A2) For all $u, \beta \in \mathcal{A}_u$; $v, \eta \in \mathcal{A}_v$ with β and η are bounded, there exists $\delta > 0$ such that the controls $u(t,x) + y\beta(t,x)$ and $v(t,x) + z\eta(t,x), t \in [0,T]$, belong to $\mathcal{A}_u$ and $\mathcal{A}_v$, respectively, for all $y, z \in (-\delta, \delta)$, and such that the families

$$\left\{\frac{\partial l_1}{\partial \gamma}\left(t, x, X_t^{(u+y\beta,v)}(x), u(t,x) + y\beta(t,x), v(t,x)\right)\frac{d}{dy}X_t^{(u+y\beta,v)}(x) + \frac{\partial l_1}{\partial u}\left(t, x, X_t^{(u+y\beta,v)}(x), u(t,x) + y\beta(t,x), v(t,x)\right)\beta(t,x)\right\}_{y\in(-\delta,\delta)}, \\ \left\{\frac{\partial l_2}{\partial \gamma}\left(t, x, X_t^{(u,v+z\eta)}(x), u(t,x), v(t,x) + z\eta(t,x)\right)\frac{d}{dz}X_t^{(u,v+z\eta)}(x) + \frac{\partial l_2}{\partial v}\left(t, x, X_t^{(u,v+z\eta)}(x), u(t,x), v(t,x) + z\eta(t,x)\right)\eta(t,x)\right\}_{z\in(-\delta,\delta)}, \quad (53)$$

are $\lambda \times P \times m$-uniformly integrable and the families

$$\left\{\frac{\partial h_1}{\partial \gamma}\left(x, X_T^{(u+y\beta,v)}(x)\right)\frac{d}{dy}X_T^{(u+y\beta,v)}(x)\right\}_{y\in(-\delta,\delta)},$$
$$\left\{\frac{\partial h_2}{\partial \gamma}\left(x, X_T^{(u,v+z\eta)}(x)\right)\frac{d}{dz}X_T^{(u,v+z\eta)}(x)\right\}_{z\in(-\delta,\delta)}. \tag{54}$$

Are $P\times m$-uniformly integrable.

(A3) For all $u,\beta\in\mathscr{A}_u$; $v,\eta\in\mathscr{A}_v$ with β and η are bounded, the process

$$Y^\beta(t,x) = \frac{d}{dy}X^{(u+y\beta,v)}(t,x)\Big|_{y=0},$$
$$Y^\eta(t,x) = \frac{d}{dz}X^{(u,v+z\eta)}(t,x)\Big|_{z=0}. \tag{55}$$

Exist. Further, $Y^\beta(t,x)$ follows the SPDE, for $(t,x)\in[0,T]\times\mathcal{O}$

$$\begin{aligned}Y^\beta(t,x) = &\int_0^t\Big\{\mathscr{L}Y^\beta(s,x) + \frac{\partial b}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y^\beta(s,x)\\&+\nabla_x Y^\beta(s,x)\nabla_{\gamma'}b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\\&+\frac{\partial b}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\beta(s,x)\Big\}ds\\&+\int_0^t\Big\{\frac{\partial \sigma}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y^\beta(s,x)\\&+\nabla_x Y^\beta(s,x)\nabla_{\gamma'}\sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\\&+\frac{\partial \sigma}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\beta(s,x)\Big\}dB_s.\end{aligned} \tag{56}$$

And for all $x\in\overline{\mathcal{O}}$, $Y^\beta(0,x)=0$ and for all $(t,x)\in(0,T)\times\partial\mathcal{O}$, $Y^\beta(t,x)=0$; $Y^\eta(t,x)$ follows the SPDE, for $(t,x)\in[0,T]\times\mathcal{O}$

$$\begin{aligned}Y^\eta(t,x) = &\int_0^t\Big\{\mathscr{L}Y^\eta(s,x) + \frac{\partial b}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y^\eta(s,x)\\&+\nabla_x Y^\eta(s,x)\nabla_{\gamma'}b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\\&+\frac{\partial b}{\partial v}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\eta(s,x)\Big\}ds\\&+\int_0^t\Big\{\frac{\partial \sigma}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y^\eta(s,x)\\&+\nabla_x Y^\eta(s,x)\nabla_{\gamma'}\sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\\&+\frac{\partial \sigma}{\partial v}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\eta(s,x)\Big\}dB_s.\end{aligned} \tag{57}$$

And for all $x \in \overline{\mathcal{O}}$, $Y^{\eta}(0,x) = 0$ and for all $(t,x) \in (0,T)\times\partial\mathcal{O}$, $Y^{\eta}(t,x) = 0$.

(A4) For all $(u,v) \in \mathcal{A}_u \times \mathcal{A}_v$, the following processes, $i = 1,2$:

$$
\begin{aligned}
n_i(t,x) &:= \frac{\partial}{\partial\gamma} h_i(x, X(T,x)) + \int_t^T \frac{\partial}{\partial\gamma} l_i(s,x,X(s,x),u(s,x),v(s,x))ds,\\
\Psi_i^{\mathcal{L}}(s,x) &:= n_i(s,x)\Bigg(\mathcal{L} + \frac{\partial b}{\partial\gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x)),\\
&\quad + \nabla_x^* \nabla_{\gamma'} b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\Bigg),\\
&\quad + D_s(n_i(s,x))\left[\frac{\partial\sigma}{\partial\gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x)),\right.\\
&\quad \left. + \nabla_x^* \nabla_{\gamma'} \sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\right],\\
m_i(s,x) &:= \Psi_i^{\mathcal{L}}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\Gamma(t,s),
\end{aligned}
\tag{58}
$$

Are well defined and where $\Gamma(t,s)$, $Z_t^{s,x}$ are defined as in the proof, where the operator ∇_x^* stands for the adjoint of ∇_x.

We now define the Hamiltonians for this general stochastic partial differential game problem as follows:

Definition 4. The general stochastic Hamiltonians for the stochastic partial differential game are the functions

$$
H_i\left(t,x,\gamma,\gamma',u,v,\omega\right):\ [0,T]\times\mathcal{O}\times L(\mathbb{R};\mathbb{R})\times L\left(\mathbb{R};\mathbb{R}^n\right)\times U\times U\times\Omega \longrightarrow \mathbb{R},
\tag{59}
$$

defined by

$$
\begin{aligned}
&H_i\left(t,x,\gamma,\gamma',u(t,x),v(t,x),\omega\right)\\
&:= l_i(t,x,\gamma,u,v) + n_i(t,x)b\left(\omega,t,x,\gamma,\gamma',u,v\right) + D_t(n_i(t,x))\sigma\left(\omega,t,x,\gamma,\gamma',u,v\right)\\
&\quad + \int_t^T E_{\widehat{P}}\Big[m_i(s,x)b\left(\omega,t,Z_t^{s,x},\gamma(Z_t^{s,x}),\gamma'(Z_t^{s,x}),u,v\right)\\
&\quad + D_t(m_i(s,x))\sigma\left(\omega,t,Z_t^{s,x},\gamma(Z_t^{s,x}),\gamma'(Z_t^{s,x}),u,v\right)\Big]ds,\quad i = 1,2.
\end{aligned}
\tag{60}
$$

4.2. Stochastic Maximum Principle for Nonzero-Sum Games

Theorem 5

(i) Let $(u^*,v^*) \in \mathcal{A}_u \times \mathcal{A}_v$ be a Nash equilibrium with the corresponding state process $X^*(t,x) = X^{(u^*,v^*)}(t,x)$, that is,

$$
\begin{aligned}
&(a) J_1(u,v^*) \le J_1(u^*,v^*),\quad \text{for all } u \in \mathcal{A}_u,\\
&(b) J_2(u^*,v) \le J_2(u^*,v^*),\quad \text{for all } v \in \mathcal{A}_v.
\end{aligned}
\tag{61}
$$

Assume that for all random variables $F(\omega)$, $\omega\in\Omega$, its Malliavin derivative with respect to $B(\cdot)$ at t exists. Then,

$$E_P\left[\frac{\partial}{\partial u}H_1\big(t,x,X^{(u,v^*)}(t,x),\nabla_x X^{(u,v^*)}(t,x),u(t,x),v^*(t,x),\omega\big)\Big|_{u=u^*}\mathscr{E}_t\right]=0, \tag{62}$$

$$E_P\left[\frac{\partial}{\partial v}H_2\big(t,x,X^{(u^*,v)}(t,x),\nabla_x X^{(u^*,v)}(t,x),u^*(t,x),v(t,x),\omega\big)\Big|_{v=v^*}\mathscr{E}_t\right]=0. \tag{63}$$

For a.a. t,x,ω.

(ii) *Conversely, suppose that there exists* $(u^*,v^*)\in\mathscr{A}_u\times\mathscr{A}_v$ *such that equations (62) and (63) hold.*

Then,

$$\begin{aligned}\frac{\partial}{\partial y}J_1(u^*+y\beta,v^*)\Big|_{y=0}&=0,\quad \text{for all }\beta,\\ \frac{\partial}{\partial z}J_2(u^*,v^*+z\eta)\Big|_{z=0}&=0,\quad \text{for all }\eta.\end{aligned} \tag{64}$$

If $J_1(u,v^*)$ and $J_2(u^*,v)$ are concave with respect to u and v, *respectively, then* (u^*,v^*) is a Nash equilibrium.

Proof. (i) Suppose $(u^*,v^*)\in\mathscr{A}_u\times\mathscr{A}_v$ is a Nash equilibrium. Since (a) and (b) hold for all u and v, (u^*,v^*) is a directional critical point for $J_i(u,v), i=1,2$, in the sense that for all bounded $\beta\in\mathscr{A}_u$ and $\eta\in\mathscr{A}_v$, there exist $\delta>0$ such that $u^*+y\beta\in\mathscr{A}_u, v^*+z\eta\in\mathscr{A}_v$, for all $y,z\in(-\delta,\delta)$. For simplicity of notation, we write $u^*=u, v^*=v, X^*=X$ and $Y^*=Y$ in the following. For ease in writing, asterisks on optimal functions will sometimes be omitted where the meaning is clear from the context.

By the definition of $J_1(u,v)$, we have

$$\begin{aligned}&\frac{\partial}{\partial y}J_1(u+y\beta,v)|_{y=0}\\ &=E\left[\int_0^T\int_{\mathscr{O}}\left\{\frac{\partial l_1}{\partial\gamma}(t,x,X(t,x),u(t,x),v(t,x))Y(t,x)\right.\right.\\ &\quad\left.\left.+\frac{\partial l_1}{\partial u}(t,x,X(t,x),u(t,x),v(t,x))\beta_t\right\}m(dx)dt\right]\\ &\quad+E\left[\int_{\mathscr{O}}\frac{\partial h_1}{\partial\gamma}(x,X(T,x))Y(T,x)m(dx)\right],\end{aligned} \tag{65}$$

where

$$\begin{aligned}Y(t,x)&=Y^{(\beta)}(t,x)=\frac{d}{dy}X^{(u+y\beta)}(t,x)|_{y=0}\\ &=\int_0^t\left\{\mathscr{L}(s,x)Y(s,x)+\frac{\partial b}{\partial\gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y(s,x)\right.\\ &\quad+\nabla_x Y(s,x)\nabla_{\gamma'}b\big(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x)\big)\\ &\quad\left.+\frac{\partial b}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\beta_s\right\}ds\\ &\quad+\int_0^t\left\{\frac{\partial\sigma}{\partial\gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s(x)\right.\\ &\quad+\nabla_x Y(s,x)\nabla_{\gamma'}\sigma\big(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x)\big)\\ &\quad\left.+\frac{\partial\sigma}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\beta_s\right\}dB_s,\end{aligned} \tag{66}$$

with initial condition

$$Y_0(x)\equiv 0,\quad x\in\overline{\mathscr{O}}, \tag{67}$$

and boundary condition

$$Y(t,x)=0,\quad (t,x)\in(0,t)\times\partial\mathscr{O}. \tag{68}$$

By the duality formulae, we get

$$
\begin{aligned}
&E\left[\int_{\mathcal{O}} \frac{\partial h_1}{\partial \gamma}(x, X(T,x))Y(T,x)m(dx)\right] \\
&= E\left[\int_{\mathcal{O}} \frac{\partial h_1}{\partial \gamma}(x, X(T,x))\left(\int_0^T \{\mathscr{L}(s,x)Y(s,x)\right.\right. \\
&\quad + \frac{\partial b}{\partial \gamma}(s, x, X(s,x), \nabla_x X(s,x), u(s,x), v(s,x))Y(s,x) \\
&\quad + \nabla_x Y(s,x)\nabla_{\gamma'} b(s, x, X(s,x), \nabla_x X(s,x), u(s,x), v(s,x)) \\
&\quad + \frac{\partial b}{\partial u}(s, x, X(s,x), \nabla_x X(s,x), u(s,x), v(s,x))\beta_s\Big\} ds \\
&\quad + \int_0^T \left\{\frac{\partial \sigma}{\partial \gamma}(s, x, X(s,x), \nabla_x X(s,x), u(s,x), v(s,x))Y(s,x)\right. \\
&\quad + \nabla_x Y(s,x)\nabla_{\gamma'} \sigma(s, x, X(s,x), \nabla_x X(s,x), u(s,x), v(s,x)) \\
&\quad \left.\left.\left. + \frac{\partial \sigma}{\partial u}(s, x, X(s,x), \nabla_x X(s,x), u(s,x), v(s,x))\beta_s\right\} dB_s\right) m(dx)\right] \\
&= E\left[\int_0^T \int_{\mathcal{O}} \frac{\partial h_1}{\partial \gamma}(x, X(T,x))\{\mathscr{L}(t,x)Y(t,x)\right. \\
&\quad + \frac{\partial b}{\partial \gamma}(t, x, X(t,x), \nabla_x X(t,x), u(t,x), v(t,x))Y(t,x) \\
&\quad + \nabla_x Y(s,x)\nabla_{\gamma'} b(s, x, X(s,x), \nabla_x X(s,x), u(s,x), v(s,x)) \\
&\quad \left. + \frac{\partial b}{\partial u}(t, x, X(t,x), \nabla_x X(t,x), u(t,x), v(t,x))\beta_t\Big\} m(dx)dt\right] \\
&\quad + E\left[\int_0^T \int_{\mathcal{O}} D_t\left(\frac{\partial h_1}{\partial \gamma}(x, X(T,x))\right)\left\{\frac{\partial \sigma}{\partial \gamma}(t, x, X(t,x), \nabla_x X(t,x), u(t,x), v(t,x))Y(t,x)\right.\right. \\
&\quad + \nabla_x Y(s,x)\nabla_{\gamma'} \sigma(s, x, X(s,x), \nabla_x X(s,x), u(s,x), v(s,x)) \\
&\quad \left.\left. + \frac{\partial \sigma}{\partial u}(t, x, X(t,x), \nabla_x X(t,x), u(t,x), v(t,x))\beta_t\right\} m(dx)dt\right].
\end{aligned}
\tag{69}
$$

Similarly, we get

$$
\begin{aligned}
&E\left[\int_0^T\int_{\mathcal{O}}\left(\frac{\partial l_1}{\partial \gamma}\right)(t,x,X_t(x),u(t,x),v(t,x))Y_t(x)m(\mathrm{d}x)\mathrm{d}t\right]\\
&\quad= E\left[\int_0^T\int_{\mathcal{O}}\left(\frac{\partial l_1}{\partial \gamma}\right)(t,x,X_t(x),u(t,x),v(t,x))\left(\int_0^t\{\mathscr{L}(s,x)Y_s(x)\right.\right.\\
&\qquad+\left(\frac{\partial b}{\partial \gamma}\right)(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s(x)\\
&\qquad+\nabla_x Y(s,x)\nabla_{\gamma'} b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\\
&\qquad+\left(\frac{\partial b}{\partial u}\right)(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\beta_s\Big\}\mathrm{d}s\\
&\qquad+\int_0^t\Big\{\frac{\partial \sigma}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s(x)\\
&\qquad+\nabla_x Y(s,x)\nabla_{\gamma'}\sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\\
&\qquad\left.\left.+\left(\frac{\partial \sigma}{\partial u}\right)(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\beta_s\Big\}\mathrm{d}B_s\right)m(\mathrm{d}x)\mathrm{d}t\right]\\
&\quad= E\left[\int_0^T\int_{\mathcal{O}}\int_0^t\left(\frac{\partial l_1}{\partial \gamma}\right)(t,x,X_t(x),u(t,x),v(t,x))\{\mathscr{L}(s,x)Y_s(x)\right.\\
&\qquad+\left(\frac{\partial b}{\partial \gamma}\right)(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s(x)\\
&\qquad+\nabla_x Y(s,x)\nabla_{\gamma'} b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\\
&\qquad\left.+\left(\frac{\partial b}{\partial u}\right)(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\beta_s\Big\}\mathrm{d}s\,m(\mathrm{d}x)\mathrm{d}t\right]\\
&\qquad+E\left[\int_0^T\int_{\mathcal{O}}\int_0^t D_s\left(\left(\frac{\partial l_1}{\partial \gamma}\right)(t,x,X_t(x),u(t,x),v(t,x))\right)\right.\\
&\qquad\cdot\Big\{\frac{\partial \sigma}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s(x)\\
&\qquad+\nabla_x Y(s,x)\nabla_{\gamma'}\sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\\
&\qquad\left.+\left(\frac{\partial \sigma}{\partial u}\right)(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\beta_s\Big\}\mathrm{d}s\,m(\mathrm{d}x)\mathrm{d}t\right]\\
&\quad= E\left[\int_0^T\int_{\mathcal{O}}\left(\int_s^T\left(\frac{\partial l_1}{\partial \gamma}\right)(t,x,X_t(x),u(t,x),v(t,x))\mathrm{d}t\right)\{\mathscr{L}(s,x)Y_s(x)\right.\\
&\qquad+\left(\frac{\partial b}{\partial \gamma}\right)(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s(x)\\
&\qquad+\nabla_x Y(s,x)\nabla_{\gamma'} b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\\
&\qquad\left.+\left(\frac{\partial b}{\partial u}\right)(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\beta_s\Big\}m(\mathrm{d}x)\mathrm{d}s\right]\\
&\qquad+E\left[\int_0^T\int_{\mathcal{O}}\left\{\int_s^T D_s\left(\left(\frac{\partial l_1}{\partial \gamma}\right)(t,x,X_t(x),u(t,x),v(t,x))\right)\mathrm{d}t\right\}\right.\\
&\qquad\cdot\Big\{\frac{\partial \sigma}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s(x)\\
&\qquad+\nabla_x Y(s,x)\nabla_{\gamma'}\sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\\
&\qquad\left.+\left(\frac{\partial \sigma}{\partial u}\right)(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\beta_s\Big\}m(dx)ds\right]\\
&\quad= E\left[\int_0^T\int_{\mathcal{O}}\left(\int_t^T\left(\frac{\partial l_1}{\partial \gamma}\right)(s,x,X_s(x),u(s,x),v(s,x))ds\right)\{\mathscr{L}(t,x)Y_t(x)\right.\\
&\qquad+\left(\frac{\partial b}{\partial \gamma}\right)(t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x))Y_t(x)\\
&\qquad+\nabla_x Y(t,x)\nabla_{\gamma'} b(t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x))\\
&\qquad\left.+\left(\frac{\partial b}{\partial u}\right)(t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x))\beta_t\Big\}m(\mathrm{d}x)\mathrm{d}t\right]\\
&\qquad+E\left[\int_0^T\int_{\mathcal{O}}\left\{\int_t^T D_t\left(\left(\frac{\partial l_1}{\partial \gamma}\right)(s,x,X_s(x),u(s,x),v(s,x))\right)\mathrm{d}s\right\}\right.\\
&\qquad\cdot\Big\{\frac{\partial \sigma}{\partial \gamma}(t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x))Y_t(x)\\
&\qquad+\nabla_x Y(t,x)\nabla_{\gamma'}\sigma(t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x))\\
&\qquad\left.+\left(\frac{\partial \sigma}{\partial u}\right)(t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x))\beta_t\Big\}m(\mathrm{d}x)\mathrm{d}t\right].
\end{aligned}
\tag{70}
$$

Here, in the last equality, we changed the notation s to t.

Now, we define

$$n_1(t,x) := \int_t^T \frac{\partial l_1}{\partial \gamma}(s,x,X_s(x),u(s,x),v(s,x))ds + \frac{\partial h_1}{\partial \gamma}(x,X_T(x)). \tag{71}$$

Since

$$\frac{\partial}{\partial y} J_1(u+y\beta,v)\bigg|_{y=0} = 0, \tag{72}$$

we have, using (65), (69), and (70),

$$\begin{aligned}
&E\bigg[\int_0^T \int_{\mathcal{O}} n_1(t,x)\{\mathcal{L}(t,x)Y_t(x) \\
&+ \frac{\partial b}{\partial \gamma}(t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x))Y_t(x) \\
&+ \nabla_x Y(t,x)\nabla_{\gamma'} b(t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x)) \\
&+ \frac{\partial b}{\partial u}(t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x))\beta_t\bigg\}m(dx)dt\bigg] \\
&+ E\bigg[\int_0^T \int_{\mathcal{O}} \{D_t(n_1(t,x))\}\bigg\{\frac{\partial \sigma}{\partial \gamma}(t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x))Y_t(x) \\
&+ \nabla_x Y(t,x)\nabla_{\gamma'}\sigma(t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x)) \\
&+ \frac{\partial \sigma}{\partial u}(t,x,X(t,x),\nabla_x X(t,x),u(t,x),v(t,x))\beta_t\bigg\}m(dx)dt\bigg] \\
&+ E\bigg[\int_0^T \int_{\mathcal{O}} \frac{\partial l_1}{\partial u}(t,x,X_t(x),u(t,x),v(t,x))\beta_t m(dx)dt\bigg] = 0.
\end{aligned} \tag{73}$$

Next, we apply the above to $\beta = \beta^\alpha \in \mathcal{A}_u$ of the form

$$\beta^\alpha(s) = \alpha I_{[t,t+h]}(s), \tag{74}$$

for some $t,h \in (0,T), t+h \le T$, where $\alpha = \alpha(\omega,x)$ is bounded and $\mathcal{E}_t \otimes \mathcal{B}(\mathbb{R})$-measurable random variable. Then, we have

$$Y_s^{\beta^\alpha}(x) = 0, \quad \text{for all } s \in [0,t], \tag{75}$$

and hence (73) becomes

$$\begin{aligned}
0 &= E\left[\int_t^T \int_{\mathcal{O}} n_1(s,x)\left\{\mathcal{L}(s,x)Y_s^{\beta^\alpha}(x)\right.\right.\\
&+ \frac{\partial b}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s^{\beta^\alpha}(x)\\
&+ \nabla_x Y_s^{\beta^\alpha}(x)\nabla_{\gamma'} b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\\
&\left.\left.+ \frac{\partial b}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\beta_s^\alpha\right\} m(dx)ds\right]\\
&+ E\left[\int_t^T \int_{\mathcal{O}} \{D_s(n_1(s,x))\}\left\{\frac{\partial \sigma}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s^{\beta^\alpha}(x)\right.\right.\\
&+ \nabla_x Y_s^{\beta^\alpha}(x)\nabla_{\gamma'} \sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\\
&\left.\left.+ \frac{\partial \sigma}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\beta_s^\alpha\right\} m(dx)ds\right]\\
&+ E\left[\int_t^T \int_{\mathcal{O}} \frac{\partial l_1}{\partial u}(s,x,X_s(x),u(s,x),v(s,x))\beta_s^\alpha m(dx)ds\right]\\
&= E\left[\int_t^T \int_{\mathcal{O}} n_1(s,x)\left\{\left[\mathcal{L}(s,x) + \frac{\partial b}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\right]Y_s^{\beta^\alpha}(x)\right.\right.\\
&\left.+ \nabla_{\gamma'} b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\nabla_x Y_s^{\beta^\alpha}(x)\right\}\\
&+ D_s(n_1(s,x))\left[\frac{\partial \sigma}{\partial \gamma}(s,x,X_s(x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s^{\beta^\alpha}(x)\right.\\
&\left.\left.+ \nabla_{\gamma'} \sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\nabla_x Y_s^{\beta^\alpha}(x)\right] m(dx)ds\right]\\
&+ E\left[\int_t^{t+h} \int_{\mathcal{O}} \left\{n_1(s,x)\frac{\partial b}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\right.\right.\\
&+ D_s(n_1(s,x))\frac{\partial \sigma}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\\
&\left.\left.+ \frac{\partial l_1}{\partial u}(s,x,X_s(x),u(s,x),v(s,x))\right\}\alpha(x)m(dx)ds\right]\\
&=: A_1 + A_2.
\end{aligned} \tag{76}$$

Note that, by (66), with $Y_s(x) = Y_s^{\beta^\alpha}(x)$ and $s \geq t+h$, the process $Y_s(x)$ follows the following dynamics:

$$\begin{aligned}Y_s(x) = Y_{t+h}(x) &+ \int_{t+h}^{s} \left\{ \mathscr{L}(\tau, x) Y_\tau(x) + \frac{\partial b}{\partial \gamma}(\tau, x, X_\tau(x), \nabla_x X_\tau(x), u_\tau, v_\tau) Y_\tau(x) \right\} d\tau \\ &+ \int_{t+h}^{s} \nabla_x Y_\tau(x) \nabla_{\gamma'} b(\tau, x, X_\tau(x), \nabla_x X_\tau(x), u_\tau, v_\tau) d\tau \\ &+ \int_{t+h}^{s} \frac{\partial \sigma}{\partial \gamma}(\tau, x, X_\tau(x), \nabla_x X_\tau(x), u_\tau, v_\tau) Y_\tau(x) dB_\tau \\ &+ \int_{t+h}^{s} \nabla_x Y_\tau(x) \nabla_{\gamma'} \sigma(\tau, x, X_\tau(x), \nabla_x X_\tau(x), u_\tau, v_\tau) dB_\tau.\end{aligned} \tag{77}$$

By Theorem 1 and (42), we know that the previous dynamics has an explicit strong solution.

$$\begin{aligned}Y_s(x) = E_{\widehat{P}}\Bigg[Y_{t+h}(Z_s^{s,x}) \exp\Bigg\{ &\int_{t+h}^{s} \frac{\partial b}{\partial \gamma}(\tau, Z_{s-\tau}^{s,x}, X_\tau(Z_{s-\tau}^{s,x}), \nabla_x X_\tau(Z_{s-\tau}^{s,x}), u_\tau, v_\tau) d\tau \\ &+ \int_{t+h}^{s} \frac{\partial \sigma}{\partial \gamma}(\tau, Z_{s-\tau}^{s,x}, X_\tau(Z_{s-\tau}^{s,x}), \nabla_x X_\tau(Z_{s-\tau}^{s,x}), u_\tau, v_\tau) dB_\tau \\ &+ \frac{1}{2} \int_{t+h}^{s} \left| \frac{\partial \sigma}{\partial \gamma}(\tau, Z_{s-\tau}^{s,x}, X_\tau(Z_{s-\tau}^{s,x}), \nabla_x X_\tau(Z_{s-\tau}^{s,x}), u_\tau, v_\tau) \right|^2 d\tau \Bigg\} \Bigg],\end{aligned} \tag{78}$$

the process $\{Z_\tau^{s,x}\}_{\tau \geq 0}$ is the inverse flow of the stochastic flow $Z_{\tau,s}$. Here, $Z_{\tau,s}$ solves the following Stratonovich SDE:

$$Z_{\tau,s}^x = x + \int_\tau^s G\left(Z_{\tau,r}^x, \circ dr\right), \tag{79}$$

where $G(x,t) \coloneqq (\Upsilon_1(x,t) + F_1(x,t), \ldots, \Upsilon_n(x,t) + F_n(x,t))$, $\Upsilon_i(x,t), F_i(x,t)$ defined in Section 2. In fact, one could verify that $Z_{\tau,s}$ solves the following Itô SDE,

$$\begin{aligned}dZ_{\tau,s}^x = x &+ \sum_{i=1}^n \int_\tau^s f^i\left(Z_{\tau,r}^x, r\right) dr + \sum_{i=1}^n \int_\tau^s g^i\left(Z_{\tau,r}^x, r\right) dW_r \\ &+ \sum_{i=1}^n \int_\tau^s \frac{\partial b}{\partial \gamma_i'}\left(Z_{\tau,r}^x, r\right) dr + \sum_{i=1}^n \int_\tau^s \frac{\partial \sigma}{\partial \gamma_i'}\left(Z_{\tau,r}^x, r\right) dB_r,\end{aligned} \tag{80}$$

W_r is a Brownian motion defined on an auxiliary probability space $(\widehat{\Omega}, \widehat{\mathscr{F}}, \widehat{P})$.

We rewrite (78) as, for $s \geq t+h$,

$$Y_s(x) = E_{\widehat{P}}\left[Y_{t+h}(Z_s^{s,x}) \Gamma(t+h, s)\right], \tag{81}$$

here,

$$\begin{aligned}\Gamma(t,s) = \exp\Bigg\{ &\int_t^s \Bigg(\frac{\partial b}{\partial \gamma}(\tau, Z_{s-\tau}^{s,x}, X_\tau(Z_{s-\tau}^{s,x}), \nabla_x X_\tau(Z_{s-\tau}^{s,x}), u_\tau, v_\tau) \\ &+ \frac{1}{2} \left| \frac{\partial \sigma}{\partial \gamma}(\tau, Z_{s-\tau}^{s,x}, X_\tau(Z_{s-\tau}^{s,x}), \nabla_x X_\tau(Z_{s-\tau}^{s,x}), u_\tau, v_\tau) \right|^2 \Bigg) d\tau \\ &+ \int_t^s \frac{\partial \sigma}{\partial \gamma}(\tau, Z_{s-\tau}^{s,x}, X_\tau(Z_{s-\tau}^{s,x}), \nabla_x X_\tau(Z_{s-\tau}^{s,x}), u_\tau, v_\tau) dW_\tau \Bigg\}.\end{aligned} \tag{82}$$

We now deal with $(A1)$ in (76). Differentiating with respect to h at $h=0$, we get

$$\begin{aligned}
&\frac{d}{dh}A_1\Big|_{h=0}\\
&=\frac{d}{dh}E\bigg[\int_t^{t+h}\int_{\mathcal{O}}n_1(s,x)\bigg\{\bigg[\mathcal{L}(s,x)+\frac{\partial b}{\partial\gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\bigg]Y_s^{\beta^\alpha}(x)\\
&\quad+\nabla_{\gamma'}b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\nabla_x Y_s^{\beta^\alpha}(x)\bigg\}\\
&\quad+D_s(n_1(s,x))\bigg[\frac{\partial\sigma}{\partial\gamma}(s,x,X_s(x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s^{\beta^\alpha}(x)\\
&\quad+\nabla_{\gamma'}\sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\nabla_x Y_s^{\beta^\alpha}(x)\bigg]m(dx)ds\bigg]_{h=0}\\
&\quad+\frac{d}{dh}E\bigg[\int_{t+h}^{T}\int_{\mathcal{O}}n_1(s,x)\bigg\{\bigg[\mathcal{L}(s,x)+\frac{\partial b}{\partial\gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\bigg]Y_s^{\beta^\alpha}(x)\\
&\quad+\nabla_{\gamma'}b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\nabla_x Y_s^{\beta^\alpha}(x)\bigg\}\\
&\quad+D_s(n_1(s,x))\bigg[\frac{\partial\sigma}{\partial\gamma}(s,x,X_s(x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s^{\beta^\alpha}(x)\\
&\quad+\nabla_{\gamma'}\sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\nabla_x Y_s^{\beta^\alpha}(x)\bigg]m(dx)ds\bigg]_{h=0}.
\end{aligned}\tag{83}$$

For first term in (83), $s\in[t,t+h)$, since $Y_t^{\beta^\alpha}(x)=0$, we have

$$\begin{aligned}
&\frac{d}{dh}E\bigg[\int_t^{t+h}\int_{\mathcal{O}}n_1(s,x)\bigg\{\bigg[\mathcal{L}(s,x)+\frac{\partial b}{\partial\gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\bigg]Y_s^{\beta^\alpha}(x)\\
&\quad+\nabla_{\gamma'}b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\nabla_x Y_s^{\beta^\alpha}(x)\bigg\}\\
&\quad+D_s(n_1(s,x))\bigg[\frac{\partial\sigma}{\partial\gamma}(s,x,X_s(x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s^{\beta^\alpha}(x)\\
&\quad+\nabla_{\gamma'}\sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\nabla_x Y_s^{\beta^\alpha}(x)\bigg]m(dx)ds\bigg]_{h=0}\\
&=0.
\end{aligned}\tag{84}$$

For $s\geq t+h$, by (81) and (84), we have

$$\begin{aligned}
&\frac{d}{dh}A_1\Big|_{h=0} \\
&= \frac{d}{dh}E\Bigg[\int_{t+h}^{T}\int_{\mathcal{O}} n_1(s,x)\Big\{\Big[\mathscr{L}(s,x)+\frac{\partial b}{\partial\gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\Big]Y_s^{\beta^\alpha}(x) \\
&\quad+\nabla_{\gamma'}b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\nabla_x Y_s^{\beta^\alpha}(x)\Big\} \\
&\quad+D_s(n_1(s,x))\Big[\frac{\partial\sigma}{\partial\gamma}(s,x,X_s(x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s^{\beta^\alpha}(x) \\
&\quad+\nabla_{\gamma'}\sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\nabla_x Y_s^{\beta^\alpha}(x)\Big]m(dx)ds\Bigg]_{h=0} \\
&= \frac{d}{dh}E\Bigg[\int_{t+h}^{T}\int_{\mathcal{O}} n_1(s,x)\Big[\mathscr{L}(s,x)+\frac{\partial b}{\partial\gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x)) \\
&\quad+\nabla_x^*\nabla_{\gamma'}b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\Big]Y_s^{\beta^\alpha}(x) \\
&\quad+D_s(n_1(s,x))\Big[\frac{\partial\sigma}{\partial\gamma}(s,x,X_s(x),\nabla_x X(s,x),u(s,x),v(s,x)) \\
&\quad+\nabla_x^*\nabla_{\gamma'}\sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\Big]Y_s^{\beta^\alpha}(x)m(dx)ds\Bigg]_{h=0} \\
&= \frac{d}{dh}E\Bigg[\int_{t+h}^{T}\int_{\mathcal{O}}\Psi_1^{\mathscr{L}}(s,x)Y_s^{\beta^\alpha}(x)m(dx)ds\Bigg]_{h=0} \\
&= \int_t^T\int_{\mathcal{O}}\frac{d}{dh}E\Big[\Psi_1^{\mathscr{L}}(s,x)E_{\widehat{P}}\Big[Y_{t+h}^{\beta^\alpha}(Z_s^{s,x})\Gamma(t+h,s)\Big]\Big]_{h=0}m(dx)ds \\
&= \int_t^T\int_{\mathcal{O}}\frac{d}{dh}E\Big[\Psi_1^{\mathscr{L}}(s,x)E_{\widehat{P}}\Big[Y_{t+h}^{\beta^\alpha}(Z_s^{s,x})\Gamma(t,s)\Big]\Big]_{h=0}m(dx)ds,
\end{aligned} \tag{85}$$

where, the operator ∇_x^* stands for the adjoint of ∇_x, and we define

$$\begin{aligned}
\Psi_1^{\mathscr{L}}(s,x) &= \Psi_1^{\mathscr{L}}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x)) \\
&:= n_1(s,x)\Big[\mathscr{L}(s,x)+\frac{\partial b}{\partial\gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x)) \\
&\quad+\nabla_x^*\nabla_{\gamma'}b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\Big] \\
&\quad+D_s(n_1(s,x))\Big[\frac{\partial\sigma}{\partial\gamma}(s,x,X_s(x),\nabla_x X(s,x),u(s,x),v(s,x)) \\
&\quad+\nabla_x^*\nabla_{\gamma'}\sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\Big].
\end{aligned} \tag{86}$$

By (66) and $\beta(s)=\alpha I_{[t,t+h]}(s)$, we have

$$\begin{aligned}
Y_{t+h}(x) &= \int_t^{t+h}\bigg\{\mathscr{L}(s,x)Y_s(x) + \frac{\partial b}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s(x) \\
&\quad + \nabla_x Y_s(x)\nabla_{\gamma'} b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x)) \\
&\quad + \alpha\frac{\partial b}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\bigg\}ds \\
&\quad + \int_t^{t+h}\bigg\{\frac{\partial \sigma}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s(x) \\
&\quad + \nabla_x Y_s(x)\nabla_{\gamma'} \sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x)) \\
&\quad + \alpha\frac{\partial \sigma}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\bigg\}dB_s \\
&= \alpha\bigg\{\int_t^{t+h}\frac{\partial b}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))ds \\
&\quad + \frac{\partial \sigma}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))dB_s\bigg\} \\
&\quad + \int_t^{t+h}\bigg[\mathscr{L}(s,x) + \frac{\partial b}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x)) \\
&\quad + \nabla_x^*\nabla_{\gamma'} b(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\bigg]Y_s(x)ds \\
&\quad + \int_t^{t+h}\bigg[\frac{\partial \sigma}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x)) \\
&\quad + \nabla_x^*\nabla_{\gamma'} \sigma(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\bigg]Y_s(x)dB_s.
\end{aligned} \tag{87}$$

After putting (87) in (85), we get

$$\frac{d}{dh}A_1\bigg|_{h=0} = A_1' + A_1'', \tag{88}$$

where

$$\begin{aligned}
A_1' &= \int_t^T \int_{\mathcal{O}} \frac{d}{dh} E\bigg[\Psi_1^{\mathscr{L}}(s,x)\cdot \alpha E_{\widehat{P}}\bigg[\bigg\{\int_t^{t+h}\frac{\partial b}{\partial u}(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau)d\tau \\
&\quad + \frac{\partial \sigma}{\partial u}(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau)dB_\tau\bigg\}\Gamma(t,s)\bigg]\bigg]_{h=0} m(dx)ds,
\end{aligned} \tag{89}$$

and

$$
\begin{aligned}
A_1'' = \int_t^T \int_{\mathcal{O}} \frac{d}{dh} E\Bigg[\Psi_1^{\mathcal{L}}(s,x) E_{\widehat{P}}\Bigg[\Gamma(t,s)\Bigg(\int_t^{t+h} [\mathcal{L}(\tau,x) \\
&+ \frac{\partial b}{\partial \gamma}(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau) \\
&+ \nabla_x^* \nabla_{\gamma'} b(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau)] Y_\tau^{\beta^\alpha}(Z_\tau^{s,x}) d\tau \\
&+ \int_t^{t+h} \Bigg[\frac{\partial \sigma}{\partial \gamma}(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau\Bigg) \\
&+ \nabla_x^* \nabla_{\gamma'} \sigma(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau)] Y_\tau^{\beta^\alpha}(Z_\tau^{s,x}) dB_\tau\Bigg)\Bigg]\Bigg]_{h=0} m(dx) ds.
\end{aligned} \tag{90}
$$

For the latter term in (88), i.e., (90), since $Y_s^{\beta^\alpha}(x) \equiv 0$ for $0 \le s \le t$, we have by applying the mean theorem,

$$
\begin{aligned}
A_1'' &= \int_t^T \int_{\mathcal{O}} E\Bigg[\Psi_1^{\mathcal{L}}(s,x) \frac{d}{dh} E_{\widehat{P}}\Bigg[\Gamma(t,s)\Bigg(\int_t^{t+h} \{\mathcal{L}(\tau,x) \\
&+ \frac{\partial b}{\partial \gamma}(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau) \\
&+ \nabla_x^* \nabla_{\gamma'} b(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau)\Bigg] Y_\tau^{\beta^\alpha}(Z_\tau^{s,x}) d\tau \\
&+ \int_t^{t+h} \Bigg[\frac{\partial \sigma}{\partial \gamma}(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau\Bigg) \\
&+ \nabla_x^* \nabla_{\gamma'} \sigma(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau)] Y_\tau^{\beta^\alpha}(Z_\tau^{s,x}) dB_\tau\Bigg)\Bigg]_{h=0}\Bigg] m(dx) ds \\
&= 0.
\end{aligned} \tag{91}
$$

For (A_1'), by the duality formulae, we have

$$
\begin{aligned}
A_1' = {} & \int_t^T \int_{\mathcal{O}} \frac{d}{dh} E\Big[\Psi_1^{\mathscr{L}}(s, x, X_s(x), \nabla_x X_s(x), u(s,x), v(s,x)) \\
& \cdot \alpha E_{\widehat{P}}\Big[\Big\{ \int_t^{t+h} \frac{\partial b}{\partial u}(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau) d\tau \\
& + \frac{\partial \sigma}{\partial u}(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau) dB_\tau \Big\} \Gamma(t,s) \Big]\Big]_{h=0} m(dx) ds \\
= {} & \int_t^T \int_{\mathcal{O}} \frac{d}{dh} E\Big[\alpha \Psi_1^{\mathscr{L}}(s, x, X_s(x), \nabla_x X_s(x), u(s,x), v(s,x)) \\
& \cdot E_{\widehat{P}}\Big[\Gamma(t,s) \int_t^{t+h} \frac{\partial b}{\partial u}(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau) d\tau \Big]\Big]_{h=0} m(dx) ds \\
& + \int_t^T \int_{\mathcal{O}} \frac{d}{dh} E\Big[\alpha E_{\widehat{P}}\Big[\Psi_1^{\mathscr{L}}(s, x, X_s(x), \nabla_x X_s(x), u(s,x), v(s,x)) \\
& \cdot \Gamma(t,s) \int_t^{t+h} \frac{\partial \sigma}{\partial u}(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau) dB_\tau \Big]\Big]_{h=0} m(dx) ds \\
= {} & \int_t^T \int_{\mathcal{O}} \frac{d}{dh} E\Big[\alpha \Psi_1^{\mathscr{L}}(s, x, X_s(x), \nabla_x X_s(x), u(s,x), v(s,x)) \\
& \cdot E_{\widehat{P}}\Big[\Gamma(t,s) \int_t^{t+h} \frac{\partial b}{\partial u}(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau) d\tau \Big]\Big]_{h=0} m(dx) ds \\
& + \int_t^T \int_{\mathcal{O}} \frac{d}{dh} E\Big[\alpha E_{\widehat{P}}\Big[\int_t^{t+h} \frac{\partial \sigma}{\partial u}(\tau, Z_\tau^{s,x}, X_\tau(Z_\tau^{s,x}), \nabla_x X_\tau(Z_\tau^{s,x}), u_\tau, v_\tau) \\
& \cdot D_\tau\big(\Gamma(t,s) \Psi_1^{\mathscr{L}}(s, x, X_s(x), \nabla_x X_s(x), u(s,x), v(s,x))\big) d\tau \Big]\Big]_{h=0} m(dx) ds \\
= {} & \int_t^T \int_{\mathcal{O}} E\Big[\alpha \Psi_1^{\mathscr{L}}(s, x, X_s(x), \nabla_x X_s(x), u(s,x), v(s,x)) \\
& \cdot E_{\widehat{P}}\Big[\Gamma(t,s) \frac{\partial b}{\partial u}(t, Z_t^{s,x}, X_t(Z_t^{s,x}), \nabla_x X_t(Z_t^{s,x}), u(t,x), v(t,x)) \Big]\Big] m(dx) ds \\
& + \int_t^T \int_{\mathcal{O}} E\Big[\alpha E_{\widehat{P}}\Big[\frac{\partial \sigma}{\partial u}(t, Z_t^{s,x}, X_t(Z_t^{s,x}), \nabla_x X_t(Z_t^{s,x}), u(t,x), v(t,x)) \\
& \cdot D_t\big(\Gamma(t,s) \Psi_1^{\mathscr{L}}(s, x, X_s(x), \nabla_x X_s(x), u(s,x), v(s,x))\big) \Big]\Big] m(dx) ds.
\end{aligned} \tag{92}
$$

Combining (88)–(92), we obtain

$$\begin{aligned}
\frac{d}{dh}A_1\Big|_{h=0} &= \frac{d}{dh}E\Bigg[\int_{t+h}^{T}\int_{\mathcal{O}}\Big\{n_1(s,x)\Big(\mathcal{L}(s,x)+\frac{\partial b}{\partial \gamma}(s,x,X_s(x),\nabla_x X_s(x),u(s,x),v(s,x))\\
&\quad +\nabla_x^*\nabla_{\gamma'}b(s,x,X_s(x),\nabla_x X_s(x),u(s,x),v(s,x))\Big)\\
&\quad + D_s(n_1(s,x))\Big[\frac{\partial \sigma}{\partial \gamma}(s,x,X_s(x),\nabla_x X_s(x),u(s,x),v(s,x))\\
&\quad +\nabla_x^*\nabla_{\gamma'}\sigma(s,x,X_s(x),\nabla_x X_s(x),u(s,x),v(s,x))\Big]\Big\}Y_s(x)m(dx)ds\Bigg]_{h=0}\\
&= E\Bigg[\int_t^T\int_{\mathcal{O}}\alpha E_{\widehat{P}}\Big[\Psi_1^{\mathcal{L}}(s,x,X_s(x),\nabla_x X_s(x),u(s,x),v(s,x))\Gamma(t,s)\\
&\quad \cdot\frac{\partial b}{\partial u}(t,Z_t^{s,x},X_t(Z_t^{s,x}),\nabla_x X_t(Z_t^{s,x}),u(t,x),v(t,x))\Big]m(dx)ds\Bigg]\\
&\quad + E\Bigg[\int_t^T\int_{\mathcal{O}}\alpha E_{\widehat{P}}\Big[\frac{\partial \sigma}{\partial u}(t,Z_t^{s,x},X_t(Z_t^{s,x}),\nabla_x X_t(Z_t^{s,x}),u(t,x),v(t,x))\\
&\quad \cdot D_t\big(\Gamma(t,s)\Psi_1^{\mathcal{L}}(s,x,X_s(x),\nabla_x X_s(x),u(s,x),v(s,x))\big)\Big]m(dx)ds\Bigg].
\end{aligned} \tag{93}$$

For (A_2) in (76), we see directly that

$$\begin{aligned}
\frac{d}{dh}A_2\Big|_{h=0} &= \frac{d}{dh}E\Bigg[\int_t^T\int_{\mathcal{O}}\Big\{n_1(s,x)\frac{\partial b}{\partial u}(s,x,X_s(x),\nabla_x X_s(x),u(s,x),v(s,x))\\
&\quad + D_s(n_1(s,x))\frac{\partial \sigma}{\partial u}(s,x,X_s(x),\nabla_x X_s(x),u(s,x),v(s,x))\\
&\quad + \frac{\partial l_1}{\partial u}(s,x,X_s(x),u(s,x),v(s,x))\Big\}\beta_s m(dx)ds\Bigg]_{h=0}\\
&= \frac{d}{dh}E\Bigg[\int_t^{t+h}\int_{\mathcal{O}}\alpha\Big\{n_1(s,x)\frac{\partial b}{\partial u}(s,x,X_s(x),\nabla_x X_s(x),u(s,x),v(s,x))\\
&\quad + D_s(n_1(s,x))\frac{\partial \sigma}{\partial u}(s,x,X_s(x),\nabla_x X_s(x),u(s,x),v(s,x))\\
&\quad + \frac{\partial l_1}{\partial u}(s,x,X_s(x),u(s,x),v(s,x))\Big\}m(dx)ds\Bigg]_{h=0}\\
&= E\Bigg[\int_{\mathcal{O}}\alpha\Big\{n_1(t,x)\frac{\partial b}{\partial u}(t,x,X_t(x),\nabla_x X_t(x),u(t,x),v(t,x))\\
&\quad + D_t(n_1(t,x))\frac{\partial \sigma}{\partial u}(t,x,X_t(x),\nabla_x X_t(x),u(t,x),v(t,x))\\
&\quad + \frac{\partial l_1}{\partial u}(t,x,X_t(x),u(t,x),v(t,x))\Big\}m(dx)\Bigg].
\end{aligned} \tag{94}$$

Therefore, differentiating (76) with respect to h at $h=0$, we obtain the following equation from (93) and (94):

$$
\begin{aligned}
& E\left[\int_t^T \int_{\mathcal{O}} \alpha E_{\widehat{P}}\left[\Gamma(t,s)\Psi_1^{\mathscr{L}}(s,x,X_s(x),\nabla_x X_s(x),u(s,x),v(s,x))\right.\right. \\
& \left.\left.\cdot \frac{\partial b}{\partial u}(t,Z_t^{s,x},X_t(Z_t^{s,x}),\nabla_x X_t(Z_t^{s,x}),u(t,x),v(t,x))\right] m(dx)ds\right] \\
& + E\left[\int_t^T \int_{\mathcal{O}} \alpha E_{\widehat{P}}\left[\frac{\partial \sigma}{\partial u}(t,Z_t^{s,x},X_t(Z_t^{s,x}),\nabla_x X_t(Z_t^{s,x}),u(t,x),v(t,x))\right.\right. \\
& \left.\left.\cdot D_t\left(\Gamma(t,s)\Psi_1^{\mathscr{L}}(s,x,X_s(x),\nabla_x X_s(x),u(s,x),v(s,x))\right)\right] m(dx)ds\right] \\
& + E\left[\int_{\mathcal{O}} \alpha\left\{n_1(t,x)\frac{\partial b}{\partial u}(t,x,X_t(x),\nabla_x X_t(x),u(t,x),v(t,x))\right.\right. \\
& + D_t(n_1(t,x))\frac{\partial \sigma}{\partial u}(t,x,X_t(x),\nabla_x X_t(x),u(t,x),v(t,x)) \\
& \left.\left.+ \frac{\partial l_1}{\partial u}(t,x,X_t(x),u(t,x),v(t,x))\right\} m(dx)\right] \\
& = E\left[\left\{\int_t^T \int_{\mathcal{O}} \alpha E_{\widehat{P}}\left[\Gamma(t,s)\Psi_1^{\mathscr{L}}(s,x,X_s(x),\nabla_x X_s(x),u(s,x),v(s,x))\right.\right.\right. \\
& \cdot \frac{\partial b}{\partial u}(t,Z_t^{s,x},X_t(Z_t^{s,x}),\nabla_x X_t(Z_t^{s,x}),u(t,x),v(t,x)) \\
& + \frac{\partial \sigma}{\partial u}(t,Z_t^{s,x},X_t(Z_t^{s,x}),\nabla_x X_t(Z_t^{s,x}),u(t,x),v(t,x)) \\
& \left.\cdot D_t\left(\Gamma(t,s)\Psi_1^{\mathscr{L}}(s,x,X_s(x),\nabla_x X_s(x),u(s,x),v(s,x))\right)\right] m(dx)ds \\
& + \int_{\mathcal{O}} \alpha\left\{n_1(t,x)\frac{\partial b}{\partial u}(t,x,X_t(x),\nabla_x X_t(x),u(t,x),v(t,x))\right. \\
& + D_t(n_1(t,x))\frac{\partial \sigma}{\partial u}(t,x,X_t(x),\nabla_x X_t(x),u(t,x),v(t,x)) \\
& \left.\left.\left.+ \frac{\partial l_1}{\partial u}(t,x,X_t(x),u(t,x),v(t,x))\right\} m(dx)\right\}\right] \\
& = 0.
\end{aligned}
\tag{95}
$$

We denote

$$
\begin{aligned}
m_1(s,x) &:= \Psi_1^{\mathscr{L}}(s,x,X_s(x),\nabla_x X_s(x),u(s,x),v(s,x))\Gamma(t,s), \\
H_1(t,x,X,\nabla X u,v) &:= \int_t^T E_{\widehat{P}}\left[m_1(s,x)b(t,Z_t^{s,x},X(t,Z_t^{s,x}),\nabla_x X(t,Z_t^{s,x}),u,v)\right. \\
&\quad \left.+ D_t(m_1(s,x))\sigma(t,Z_t^{s,x},X(t,Z_t^{s,x}),\nabla_x X(t,Z_t^{s,x}),u,v)\right]ds \\
&\quad + \{n_1(t,x)b(t,x,X(t,x),\nabla_x X(t,x),u,v) \\
&\quad + D_t(n_1(s,x))\sigma(t,x,X_t(x),\nabla_x X(t,x),u,v) + l_1(t,x,X_t(x),u,v)\},
\end{aligned}
\tag{96}
$$

then, the above equation (95) can be written as follows:

$$E\left[\int_{\mathcal{O}}\alpha\frac{\partial}{\partial u}H_1\left(t,x,X(t,x),\nabla_x X(t,x),u,v(t,x)\right)\Big|_{u=u(t,x)}m(dx)\right]=0. \quad (97)$$

Since α be arbitrarily bounded $\mathscr{E}_t\otimes\mathscr{B}(\mathbb{R})$-measurable random variable, we conclude that, for all $(t,x)\in[0,T]\times\mathcal{O}$, a.s.,

$$E\left[\frac{\partial}{\partial u}H_1 t,x,X(t,x),\nabla_x X(t,x),u,v(t,x)\,|_{u=u(t,x)}\,|\,\mathscr{E}_t\right]=0. \quad (98)$$

Similarly, we have

$$\begin{aligned}0&=\frac{\partial}{\partial z}J_2(u,v+z\eta)\Big|_{z=0}\\&=E\left[\int_0^T\int_{\mathcal{O}}\left\{\frac{\partial l_1}{\partial\gamma}(t,x,X(t,x),u(t,x),v(t,x))Y(t,x)\right.\right.\\&\quad\left.\left.+\frac{\partial l_2}{\partial v}(t,x,X(t,x),u(t,x),v(t,x))\beta_t\right\}m(dx)dt\right]\\&\quad+E\left[\int_{\mathcal{O}}\frac{\partial h_2}{\partial\gamma}(x,X(T,x))Y(T,x)m(dx)\right].\end{aligned} \quad (99)$$

where

$$\begin{aligned}Y(t,x)&=Y^{(\eta)}(t,x)=\frac{d}{dy}X^{(u,v+z\eta)}(t,x)|_{z=0}\\&=\int_0^t\left\{\mathscr{L}(s,x)Y(s,x)+\frac{\partial b}{\partial\gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y(s,x)\right.\\&\quad+\frac{\partial b}{\partial\gamma'}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\nabla_x Y(s,x)\\&\quad\left.+\frac{\partial b}{\partial v}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\beta_s\right\}ds\\&\quad+\int_0^t\left\{\frac{\partial\sigma}{\partial\gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))Y_s(x)\right.\\&\quad+\frac{\partial\sigma}{\partial\gamma'}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\nabla_x Y(s,x)\\&\quad\left.+\frac{\partial\sigma}{\partial v}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\beta_s\right\}dB_s.\end{aligned} \quad (100)$$

and $Y_0(x) \equiv 0, x \in \overline{\mathcal{O}}$ and boundary condition $Y(t,x) = 0$, $(t,x) \in (0,t) \times \partial\mathcal{O}$.

By using similar arguments as J_1, we get

$$E\left[\frac{\partial}{\partial v}H_2(t,x,X(t,x),\nabla_x X(t,x),u(t,x),v)\,|_{v=v(t,x)}\,|\,\mathscr{E}_t\right] = 0. \tag{101}$$

This completes the proof of assertion (i).

(ii) Conversely, suppose that there exists $(u,v) \in \mathscr{A}_u \times \mathscr{A}_v$ such that (62) and (63) hold. In fact, the proof of the opposite direction is divided into two steps.

Firstly, consider $s \in [t, t+h)$. If (62) holds, then we obtain that (75) holds for all $\beta_s^\alpha = \alpha I_{(t,t+h]}(s)$, that is,

$$\begin{aligned}
0 = {} & E\bigg[\int_t^T \int_{\mathcal{O}} \bigg\{ n_1(s,x)\bigg(\mathscr{L}(s,x) + \frac{\partial b}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\bigg) \\
& + D_s(n_1(s,x))\frac{\partial \sigma}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\bigg\} Y_s^{\beta^\alpha}(x) m(dx) ds\bigg] \\
& + E\bigg[\int_t^T \int_{\mathcal{O}} \bigg\{ n_1(s,x)\frac{\partial b}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x)) \\
& + D_s(n_1(s,x))\frac{\partial \sigma}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x)) \\
& + \frac{\partial l_1}{\partial u}(s,x,X(s,x),u(s,x),v(s,x))\bigg\} \beta_s^\alpha m(dx) ds\bigg],
\end{aligned} \tag{102}$$

for all $t, h \in [0,T]$ with $t+h \le T$ and some bounded $\mathscr{E}_t \otimes \mathscr{B}(\mathbb{R})$-measurable random variable α.

Similarly, for all $\eta_s^\xi = \xi I_{(t,t+h]}(s)$, we have

$$\begin{aligned}
0 = {} & E\bigg[\int_t^T \int_{\mathcal{O}} \bigg\{ n_2(s,x)\bigg(\mathscr{L}(s,x) + \frac{\partial b}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\bigg) \\
& + D_s(n_2(s,x))\frac{\partial \sigma}{\partial \gamma}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x))\bigg\} Y_s^{\eta^\xi}(x) m(dx) ds\bigg] \\
& + E\bigg[\int_t^T \int_{\mathcal{O}} \bigg\{ n_2(s,x)\frac{\partial b}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x)) \\
& + D_s(n_2(s,x))\frac{\partial \sigma}{\partial u}(s,x,X(s,x),\nabla_x X(s,x),u(s,x),v(s,x)) \\
& + \frac{\partial l_2}{\partial u}(s,x,X(s,x),u(s,x),v(s,x))\bigg\} \eta_s^\xi m(dx) ds\bigg],
\end{aligned} \tag{103}$$

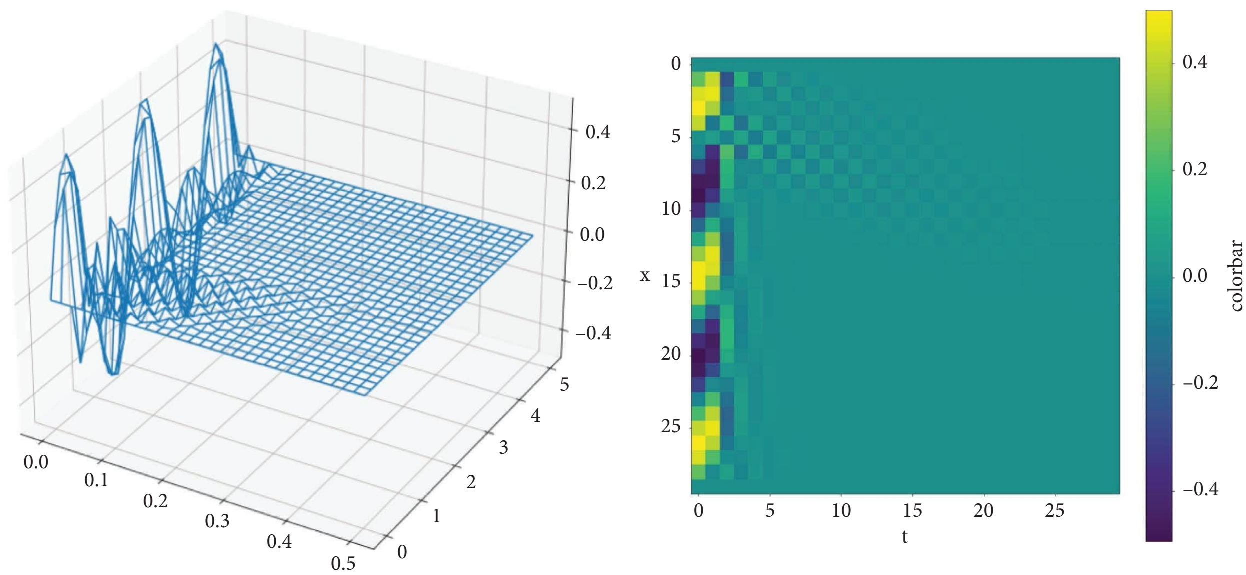
for all $t, h \in [0,T]$ with $t+h \le T$ and some bounded $\mathscr{E}_t \otimes \mathscr{B}(\mathbb{R})$-measurable random variable ξ.

Secondly, consider $s \in [t,T]$. These equalities above (102) and (103) hold for all linear combinations of such β^{α_i} and η^{ξ_i}. For any $\beta \in \mathscr{A}_u$ and $\eta \in \mathscr{A}_v$, since all bounded $\beta \in \mathscr{A}_u$ and $\eta \in \mathscr{A}_v$ can be approximated pointwise boundary in (t,x,ω) by such linear combinations, it follows that (102) and (103) hold for all bounded $\beta \in \mathscr{A}_u$ and $\eta \in \mathscr{A}_v$, that is, for any $\beta \in \mathscr{A}_u$, we can approximate β by

$$\beta_n = \sum_{i=1}^n \ell_i \beta_i^{\alpha_i}(x,\omega) I_{[t,t+ih)}(s), \quad s \in [t,T], \tag{104}$$

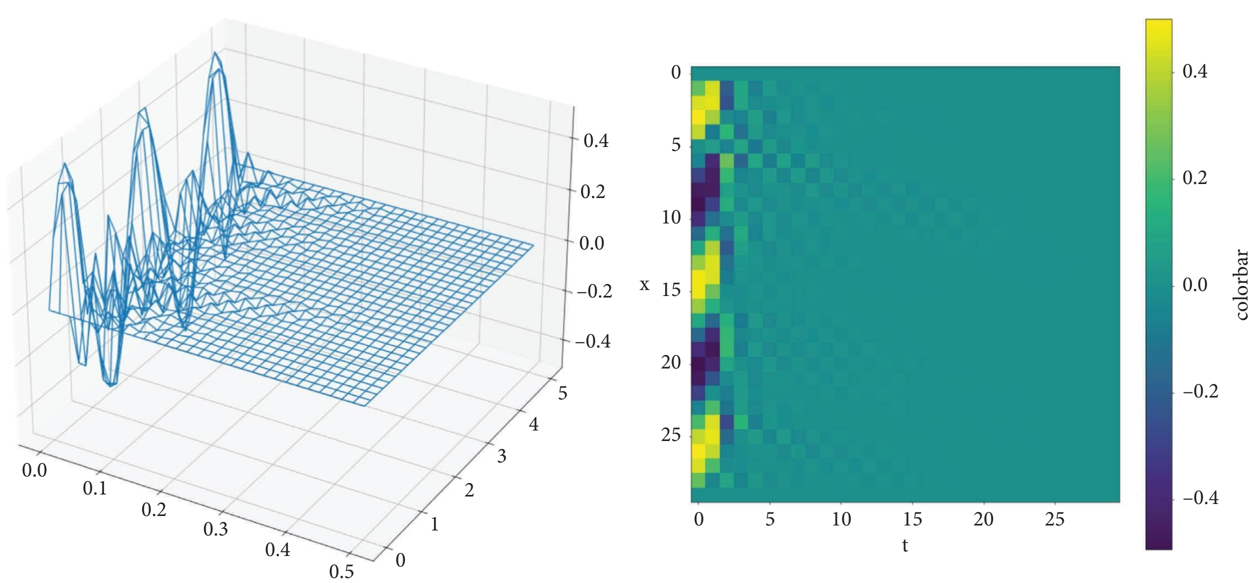
where ℓ_i is the coefficient, $\{t, t+h, \dots, t+nh = T\}$ is a partition of the interval $[t,T]$, α_i is a boundary random variable, and this approximation procedure is uniformly for (t,x,ω). Hence, we obtain (73) holds for any $\beta \in \mathscr{A}_u$, in the interval $[t,T]$.

Taking $t = 0$, we conclude that (73) holds for all bounded $\beta \in \mathscr{A}_u$, and this is equivalent to



(a) (b)

Figure 1: $Y(0,x) = 1/2\sin \pi x$ and $Y(t,0) = Y(t,5) = 0$.



(a) (b)

Figure 2: $Y(0,x) = 1/2\sin \pi x$ and $Y(t,0) = Y(t,5) = 0$.

$$\frac{\partial}{\partial y} J_1(u + y\beta, v)\bigg|_{y=0} = 0, \quad (105)$$

for all bounded $\beta \in \mathscr{A}_u$. Similarly, we get that

$$\frac{\partial}{\partial z} J_1(u, v + z\eta)\bigg|_{z=0} = 0, \quad (106)$$

for all bounded $\eta \in \mathscr{A}_v$. □

5. Numerical Simulations for the Linear SPDE

A strong solution of the linear stochastic partial differential equation with a generalized probabilistic representation has been given with the benefit of Kunita's stochastic flow theory. This section is concerned with the numerical simulations of solutions to the linear stochastic partial differential equations. First of all, we consider one space dimensional in the following linear stochastic partial differential equations:

$$\begin{aligned} Y_s(x) = Y_0(x) &+ \int_0^s \left\{ \mathscr{L}(\tau,x) Y_\tau(x) + \frac{\partial b}{\partial \gamma}\left(\tau, x, X_\tau(x), \nabla_x X_\tau(x), u_\tau, v_\tau\right) Y_\tau(x) \right\} d\tau \\ &+ \int_0^s \nabla_x Y_\tau(x) \nabla_{\gamma'} b\left(\tau, x, X_\tau(x), \nabla_x X_\tau(x), u_\tau, v_\tau\right) d\tau \\ &+ \int_0^s \frac{\partial \sigma}{\partial \gamma}\left(\tau, x, X_\tau(x), \nabla_x X_\tau(x), u_\tau, v_\tau\right) Y_\tau(x) dB_\tau \\ &+ \int_0^s \nabla_x Y_\tau(x) \nabla_{\gamma'} \sigma\left(\tau, x, X_\tau(x), \nabla_x X_\tau(x), u_\tau, v_\tau\right) dB_\tau, \end{aligned} \tag{107}$$

where $\mathscr{L}(\tau,x)$ has the form

$$\mathscr{L}(t,x)\Phi = \frac{1}{2} G(t,x) \frac{\partial^2}{\partial x^2}\Phi + f(t,x)\frac{\partial}{\partial x}\Phi + d(t,x)\Phi. \tag{108}$$

5.1. Example 1: Stochastic Equation with Volatility $Y(t,x)$. In this example, we solve the linear stochastic partial differential equation (107) on the domain $(t,x) \in [0,0.5] \times [0,5]$. The space and time steps are chosen as $\Delta x = 5/30$ and $\Delta t = 0.5/30$, respectively. The initial value $Y_0(x) = 1/2 \sin \pi x$ and boundary value $Y(t,0) = Y(t,5) = 0$, $\partial b/\partial \gamma = \nabla_{\gamma'} b = \nabla_{\gamma'}\sigma = 0, \partial\sigma/\partial\gamma = 1$, and the functions $G(t,x) = f(t,x) = 1$, $d(t,x) = 0$. The solutions of these linear stochastic partial differential equations are shown in Figure 1. In this case, the linear stochastic partial differential equation is

$$\begin{cases} Y(s,x) = \dfrac{1}{2}\sin \pi x + \displaystyle\int_0^s \left[\frac{1}{2}\frac{\partial^2}{\partial x^2} Y(\tau,x) + \frac{\partial}{\partial x} Y(\tau,x)\right] d\tau \\ + \displaystyle\int_0^s Y(\tau,x) dB_\tau, \quad (s,x) \in [0,0.5] \times [0,5], \\ Y(s,0) = Y(s,5) = 0, \quad s \in [0,0.5]. \end{cases} \tag{109}$$

5.2. Example 2: Stochastic Equation with Volatility $\nabla_x Y(t,x)$. In this example, we solve the linear stochastic partial differential equation (107) on the domain $(t,x) \in [0,0.5] \times [0,5]$. The space and time steps are chosen as $\Delta x = 5/30$ and $\Delta t = 0.5/30$, respectively. The initial value $Y_0(x) = 1/2 \sin \pi x$ and boundary value $Y(t,0) = Y(t,5) = 0$, $\partial b/\partial\gamma = \nabla_{\gamma'} b = \partial\sigma/\partial\gamma = 0, \nabla_{\gamma'}\sigma = 1$, and the functions $G(t,x) = f(t,x) = 1$, $d(t,x) = 0$. The solutions of these linear stochastic partial differential equations are shown in Figure 2. In this case, the linear stochastic partial differential equation is

$$\begin{cases} Y(s,x) = \dfrac{1}{2}\sin \pi x + \displaystyle\int_0^s \left[\frac{1}{2}\frac{\partial^2}{\partial x^2} Y(\tau,x) + \frac{\partial}{\partial x} Y(\tau,x)\right] \mathrm{d}\tau \\ + \displaystyle\int_0^s \nabla_x Y(\tau,x) \mathrm{d}B_\tau, \quad (s,x) \in [0,0.5] \times [0,5], \\ Y(s,0) = Y(s,5) = 0, \quad s \in [0,0.5]. \end{cases} \tag{110}$$

6. Conclusion

In this paper, we consider a Nash equilibrium of stochastic differential game where the state process is governed by a controlled stochastic partial differential equation. The problem of finding sufficient conditions for Nash equilibrium of stochastic differential game can be transformed into optimality conditions for a stochastic optimal control problem with infinite dimensional state equation. Applying Kunita's stochastic flow theory, we find an explicit strong solution of the linear stochastic partial differential equation, and this solution has a probabilistic representation. The probabilistic representation of solution and Malliavin calculus imply a stochastic maximum principle for the optimal control and obtain the Nash equilibrium of this type of stochastic differential game problem. We would like to point out that it is meaningful to consider a Nash equilibrium of stochastic differential game when the state process is governed by a controlled stochastic partial differential equation with jump-diffusion, which is a valuable future research direction.

Data Availability

All data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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