Research Article

Finite-Time Stability of Linear Conformable Stochastic Differential Equation with Finite Delay

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This paper investigates the finite-time stability (FTS) of a linear conformable stochastic differential equation with finite delay (LCSDEwFD). We use the Banach fixed point theorem (BFPT) to prove the existence and uniqueness of the solution and analyze the FTS of the system using the Gronwall inequalities. To demonstrate the practical value of our approach, we provide two numerical examples that showcase the relevance and effectiveness of our theoretical results.

1. Introduction

Fractional calculus (FC) ([1, 2]) has become a popular tool in many fields, including physics, economics, control theory, and image processing, due to its ability to model and analyze systems with memory and hereditary properties. Although the concept of fractional derivatives can be traced back to the 17th century, FC has gained renewed interest and attention in recent decades, thanks to its broad applicability and flexibility. In particular, fractional derivatives have been used to analyze and describe a wide range of phenomena in biology, chemistry, engineering, and physics, ranging from the behavior of complex networks to the dynamics of viscoelastic materials. Despite the challenges involved in applying FC to real-world problems, its potential for advancing scientific knowledge and technological innovation continues to inspire researchers and practitioners alike.

The literature contains various definitions of fractional derivatives, each with its own set of properties and applications. One recent development in this area is the introduction of the conformable derivative by Khalil et al. in [3]. This new derivative of order $\alpha \in (0, 1)$ shares many of the same properties as the classical integer-order derivative, making it a powerful tool for modeling and analyzing complex systems with memory and hereditary properties. In [4], the conformable derivative is extensively examined to explore its properties and applications. The author presents a multitude of results and examples that highlight the effectiveness of the conformable derivative in various domains. Additionally, a generalized Lyapunov-type inequality is introduced within the framework of conformable derivatives in [5]. Furthermore, the application of the conformable derivative is extended to the discussion of fractional linear differential equations with constant coefficients, employing arguments similar to those used in the theory of ordinary differential equations (refer to [6]). For those interested in tracing the historical development of fractional calculus and its derivatives, several literature sources are available. For instance, the construction of exact solutions for the time-fractional Wu–Zhang system, described using the conformable fractional derivative and the first integral method, can be found in [7]. Another relevant work is the investigation of optimal problems related to the conformable fractional heat conduction equation, as discussed in [8].

One of the necessary qualitative theories of dynamical systems is the concept of stability. As a result of its applications, the theory of stability features has received
significant attention in a variety of research fields. Numerous papers have been dedicated to investigating FTS analysis and its applications in various types of differential equations. For instance, researchers have explored FTS analysis in the context of impulsive dynamical systems, as discussed in [9]. Additionally, FTS analysis has been applied to nonlinear switched impulsive systems, as highlighted in [10, 11]. The study of Caputo–Katugampola fractional-order time delay systems has also received attention in the literature, as seen in [12, 13]. Moreover, a particular class of nonlinear fractional-order systems has been examined in relation to FTS analysis, as presented in [14]. FTS analysis pertains to the study of system stability over a finite time interval, and this concept has been explored extensively. Notably, the stability analysis of stochastic functional differential equations within the framework of FTS has garnered significant attention, as evidenced in [15, 16].

Recently, in [17, 18], the authors investigated the existence and uniqueness results for solutions and practical and the partial stability of conformable stochastic systems. To the best of our knowledge, there is no existing result about the FTS of LCSDEwFD (see [19, 20]). Hence, it is very interesting to close this gap and study this topic. The main highlights of the paper are as follows:

1. To study the existence and uniqueness of solution for LCSDEwFD by using the BFPT.
2. To discuss the FTS of LCSDEwFD.

The structure of this article is as follows. In Section 2, we introduce some essential concepts and fundamental results that form the basis of our approach. Section 3 presents our main contributions, starting with the proof of an existence-uniqueness theorem for the solutions of the problem under consideration. Subsequently, we establish FTS results for these solutions, providing a quantitative assessment of their behavior over a fixed time interval. In Section 4, we present two numerical examples to demonstrate the efficacy of our approach and to illustrate the practical relevance of our findings.

### 2. Basic Notions

Let \((\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete probability space and \(W(t)\) be a standard Brownian motion (also called Wiener process, for more details, see [21]).

Set \(C([-\kappa, 0], \mathbb{R}^n) = \{\varphi: [-\kappa, 0] \rightarrow \mathbb{R}^n, \text{ continuous functions}\}\) under the norm \(\|\varphi\| = \sup_{\kappa \leq s \leq 0} \|\varphi(s)\|\) where \(\|\varphi(s)\| = \sqrt{\varphi(s)^T \varphi(s)}\). Let \(H^2([\ell_0 - \kappa, T])\) be the set of all processes \(x\) which are measurable and \(\{F_s\}_{s \leq T}\)-adapted (for more details, see [22–24]).

In what follows, few definitions and lemma will be introduced since they will be used to ensure the existence and FTS of the solution for (7) in the subsequent sections.

**Definition 1** (see [3]). The conformable derivative (CD) of a function \(G: \ell_0, \infty \rightarrow \mathbb{R}\) is defined by

\[
T^\beta_G(\ell_0) = \lim_{d \to 0} \frac{G(\ell_0 + d(\ell_0 - \ell_0)) - G(\ell_0)}{d},
\]

for all \(\zeta > \ell_0, \beta \in (0, 1)\), while

\[
T^\beta_\ell G(\ell_0) = \lim_{\zeta \to \ell} T^\beta_\ell G(\zeta).
\]

**Definition 2** (see [4]). The conformable integral (CI) of a function \(G: \ell_0, \infty \rightarrow \mathbb{R}\) is defined by

\[
I^\beta_G(\ell_0) = \int_{\ell_0}^{\zeta} (\xi - \ell_0)^{\beta - 1} G(\xi)d\xi,
\]

for all \(\zeta > \ell_0, \beta \in (0, 1)\).

**Definition 3** (see [4]). The conformable exponential function is defined by

\[
\Xi_{\beta, \lambda}(z) = \exp\left(\lambda \frac{\beta}{\beta - 1}\right),
\]

where \(\beta \in (0, 1)\) and \(\lambda \in \mathbb{R}\).

**Lemma 4.** Let \(D: H^2([\ell_0 - \kappa, T], \mathbb{R}^n) \times H^2([\ell_0 - \kappa, T], \mathbb{R}^n) \rightarrow \mathbb{R}_+\) be the function such that

\[
D(\xi, \ell) \leq \frac{\sigma}{1 - \varsigma},
\]

where \(\sigma > 0\) and \(D(P(\ell), \ell) \leq \sigma\). So, there exists a unique \(\xi \in \Lambda\) satisfying \(\xi = P(\ell)\). Moreover,

\[
\mathcal{D}^2(\varphi_1, \varphi_2) = \inf\left\{\vartheta \in [\ell_0 - \kappa, +\infty), \mathbb{E}[\int_{\ell_0 - \kappa}^{\ell_0} \frac{\|\varphi_1(s) - \varphi_2(s)\|^2}{h_1(s)}ds] \leq \vartheta \mathcal{D}^2(\varphi_1, \varphi_2), \forall \ell \in [\ell_0 - \kappa, T]\right\},
\]

where \(h_1, h_2 \in C([\ell_0 - \kappa, T], \mathbb{R}_+)\). Then, \((H^2([\ell_0 - \kappa, T], \mathbb{R}^n), D)\) is a complete metric space.

**Theorem 5.** Suppose that \((\Lambda, D)\) is a complete metric space and \(P: \Lambda \rightarrow \Lambda\) is a contraction (with \(\varsigma \in [0, 1]\)). Suppose that \(\theta \in \Lambda\), \(\sigma > 0\) and \(D(P(\ell), \ell) \leq \sigma\). So, there exists a unique \(\xi \in \Lambda\) satisfying \(\xi = P(\ell)\). Moreover,

\[
\mathcal{D}^2(\varphi_1, \varphi_2) = \inf\left\{\vartheta \in [\ell_0 - \kappa, +\infty), \mathbb{E}[\int_{\ell_0 - \kappa}^{\ell_0} \frac{\|\varphi_1(s) - \varphi_2(s)\|^2}{h_1(s)}ds] \leq \vartheta \mathcal{D}^2(\varphi_1, \varphi_2), \forall \ell \in [\ell_0 - \kappa, T]\right\},
\]

where \(h_1, h_2 \in C([\ell_0 - \kappa, T], \mathbb{R}_+)\). Then, \((H^2([\ell_0 - \kappa, T], \mathbb{R}^n), D)\) is a complete metric space.

### 3. Main Results

Consider the next LCSDEwFD on \(\ell_0, \infty\) for any \(\beta \in (1/2, 1)\) with the next form
The condition is finite-time stochastically stable (FTSS) with respect to Equation (7) is finite-time stochastically stable (FTSS) with respect to Equation (7) has unique solution with initial condition (8).

\[ x(\varsigma) = \psi(t_0) + \int_{t_0}^{\varsigma} (s-t_0)^{\beta-1} (Lx(s) + Mx(s-r(s)))ds + \int_{t_0}^{\varsigma} (s-t_0)^{\beta-1} (Nx(s) + Px(s-r(s)))dW(s). \]

**Theorem 7.** Equation (7) has unique solution with initial condition (8).

**Proof.** Consider \( D^1 \colon \mathbb{H}^2([t_0-\kappa,T],\mathbb{R}^d) \times \mathbb{H}^2([t_0-\kappa,T],\mathbb{R}^d) \rightarrow \mathbb{R}_+ \) such that

\[ (\Phi x_1)(t) - (\Phi x_2)(t) = \int_{t_0}^{t} (s-t_0)^{\beta-1} (L(x_1(s)-x_2(s)) + M(x_1(s-r(s))-x_2(s-r(s))))ds \]

\[ + \int_{t_0}^{t} (s-t_0)^{\beta-1} (N(x_1(s)-x_2(s)) + P(x_1(s-r(s))-x_2(s-r(s))))dW(s). \]

Then,

\[ \mathbb{E}\|(\Phi x_1)(t) - (\Phi x_2)(t)\|^2 \]

\[ \leq 2\mathbb{E}\left(\int_{t_0}^{t} (s-t_0)^{\beta-1} (L(x_1(s)-x_2(s)) + M(x_1(s-r(s))-x_2(s-r(s))))ds\right)^2 \]

\[ + 2\mathbb{E}\left(\int_{t_0}^{t} (s-t_0)^{\beta-1} (N(x_1(s)-x_2(s)) + P(x_1(s-r(s))-x_2(s-r(s))))dW(s)\right)^2. \]

Applying the Cauchy–Schwarz inequality, we obtain

\[ \mathbb{E}\left(\int_{t_0}^{t} (s-t_0)^{\beta-1} (L(x_1(s)-x_2(s)) + M(x_1(s-r(s))-x_2(s-r(s))))ds\right)^2 \]

\[ \leq 2 \max(\|L\|^2,\|M\|^2)(T-t_0) \]

\[ \times \int_{t_0}^{t} (s-t_0)^{2\beta-2}\left(\mathbb{E}\|x_1(s)-x_2(s)\|^2 + \mathbb{E}\|x_1(s-r(s))-x_2(s-r(s))\|^2\right)ds. \]
By the Itô isometry formula, we derive

\[
E\left(\int_{t_0}^{c} (s-t_0)^{\beta-1} \left( N(x_1(s) - x_2(s)) + P(x_1(s-r(s)) - x_2(s-r(s))) dW(s)^2 \right) \right) \\
= E\left(\int_{t_0}^{c} (s-t_0)^{2\beta-2} \left( N(x_1(s) - x_2(s)) + P(x_1(s-r(s)) - x_2(s-r(s))) \right)^2 ds \right) \\
\leq 2 \max(\|N\|^2, \|P\|^2) \int_{t_0}^{c} (s-t_0)^{2\beta-2} \left( E\|x_1(s) - x_2(s)\|^2 + E\|x_1(s-r(s)) - x_2(s-r(s))\|^2 \right) ds.
\]

Therefore,

\[
E\|\Phi x_1(c) - \Phi x_2(c)\|^2 \\
\leq 4 \max(\|N\|^2, \|P\|^2) \int_{t_0}^{c} (s-t_0)^{2\beta-2} \left( E\|x_1(s) - x_2(s)\|^2 + E\|x_1(s-r(s)) - x_2(s-r(s))\|^2 \right) ds \\
+ 4(T-t_0) \max(\|L\|^2, \|M\|^2) \\
\times \int_{t_0}^{c} (s-t_0)^{2\beta-2} \left( E\|x_1(s) - x_2(s)\|^2 + E\|x_1(s-r(s)) - x_2(s-r(s))\|^2 \right) ds \\
\leq \Lambda \int_{t_0}^{c} (s-t_0)^{2\beta-2} \left( E\|x_1(s) - x_2(s)\|^2 + E\|x_1(s-r(s)) - x_2(s-r(s))\|^2 \right) ds \\
\leq \Lambda \int_{t_0}^{c} (s-t_0)^{2\beta-2} \frac{E\|x_1(s) - x_2(s)\|^2}{h(s)} h(s) ds \\
+ \Lambda \int_{t_0}^{c} (s-t_0)^{2\beta-2} \frac{E\|x_1(s-r(s)) - x_2(s-r(s))\|^2}{h(s-r(s))} h(s-r(s)) ds \\
\leq \Lambda D_2^2(x_1, x_2) \int_{t_0}^{c} (s-t_0)^{2\beta-2} \Xi_{2\beta-1,1}(s-t_0) ds + \Lambda D_2^2(x_1, x_2) \int_{t_0}^{c} (s-t_0)^{2\beta-2} h(s) ds \\
\leq 2\Lambda D_2^2(x_1, x_2) \int_{t_0}^{c} (s-t_0)^{2\beta-2} \Xi_{2\beta-1,1}(s-t_0) ds \\
\leq \frac{2\Lambda}{\lambda} D_2^2(x_1, x_2) \Xi_{2\beta-1,1}(c-t_0),
\]

where

\[
\Lambda = 4 \max(\|L\|^2, \|M\|^2, \|N\|^2, \|P\|^2) (T-t_0 + 1).
\]

Therefore,

\[
\frac{E\|\Phi x_1(c) - \Phi x_2(c)\|^2}{\Xi_{2\beta-1,1}(c-t_0)} \leq \frac{2\Lambda}{\lambda} D_2^2(x_1, x_2).
\]

Consequently,

\[
D_2^2(\Phi x_1, \Phi x_2) \leq \frac{2\Lambda}{\lambda} D_2^2(x_1, x_2).
\]

Then,

\[
D_1(\Phi x_1, \Phi x_2) \leq KD_1(x_1, x_2),
\]

where $K = \sqrt{2\Lambda/\lambda}$. Hence, $\Phi$ is strictly contractive on $\mathbb{H}^2([-\nu, T], \mathbb{R}^d)$. Then, equation (7) has unique solution and the proof is completed.

**Theorem 8.** Equation (7) is FTS with respect to $(t, \epsilon, T)$, if the following condition is fulfilled:

\[
2(\gamma \Xi_{2\beta-1,1}(T-t_0 + 1) + \epsilon) < \epsilon.
\]
Complexity

with $\lambda > 0$ such that

\[
\Lambda = 4 \max \{ \|L\|^2, \|M\|^2, \|N\|^2, \|P\|^2 \} (T-t_0+1),
\]

\[
K = \frac{2\Lambda}{\lambda} < 1, S = \sup_{c \in [u, T]} \left( \frac{(c-i_0)^{2\beta-1}}{E_{2\beta-1,1}(c-i_0)} \right) \quad \text{and} \quad \gamma = \frac{2\Lambda S}{(1-K)^2(2\beta-1)}. \tag{25}
\]

**Proof.** Set $E\|\psi\|^2 < 1$. Consider the function $y_0$ defined by $y_0(s) = \psi(s)$, for $s \in [-\kappa, 0]$, and $y_0(c) = \psi(i_0)$, for $c \in [i_0, T]$. According to (13), we have $(\Phi y_0)(c) = y_0(c)$, for all $c \in [-\kappa, 0]$, and for $c \in [i_0, T]$, one has

\[
\| (\Phi y_0)(c) - y_0(c) \|^2 = \left\| \int_{i_0}^{c} (s-i_0)^{2\beta-1} (Ly_0(s) + By_0(s-r(s)))ds + \int_{i_0}^{c} (s-i_0)^{2\beta-1} (Ny_0(s) + Dy_0(s-r(s)))dW(s) \right\|^2. \tag{26}
\]

Proceeding as in (15), we can derive that

\[
E \| (\Phi y_0)(c) - y_0(c) \|^2 \leq \Lambda \int_{i_0}^{c} \left( (s-i_0)^{2\beta-2} \left( E \| y_0(s) \|^2 + E \| y_0(s-r(s)) \|^2 \right) \right) ds
\]

\[
\leq \frac{2\Lambda}{2\beta-1} (c-i_0)^{2\beta-1} E\|\psi\|^2. \tag{27}
\]

Then,

\[
E \| (\Phi y_0)(c) - y_0(c) \|^2 \leq \frac{2\Lambda}{2\beta-1} SE\|\psi\|^2, \tag{28}
\]

where

\[
S = \sup_{c \in [u, T]} \left( \frac{(c-i_0)^{2\beta-1}}{E_{2\beta-1,1}(c-i_0)} \right). \tag{29}
\]

Consequently,

\[
D_{\chi}^2 (\Phi y_0, y_0) \leq \frac{2\Lambda}{2\beta-1} SE\|\psi\|^2. \tag{30}
\]

By Theorem 5, the unique solution $x$ of equation (7) satisfies

\[
D_{\chi} (x, y_0) \leq \frac{1}{1-K} D_{\chi} (\Phi y_0, y_0), \tag{31}
\]

which implies that

\[
D_{\chi}^2 (x, y_0) \leq \frac{1}{(1-K)^2} D_{\chi}^2 (\Phi y_0, y_0)
\]

\[
\leq \frac{2\Lambda}{(1-K)^2(2\beta-1)} SE\|\psi\|^2. \tag{32}
\]

Then, for all $c \in [i_0, T]$, we have

\[
\|x(c) - y_0(c)\|^2 \leq \gamma E_{2\beta-1,1}(T-t_0) E\|\psi\|^2, \tag{33}
\]

where

\[
\gamma = \frac{2\Lambda S}{(1-K)^2(2\beta-1)}. \tag{34}
\]

Therefore, $\forall c \in [i_0, T],$

\[
E\|x(c) - y_0(c)\|^2 \leq 2E\|x(c) - y_0(c)\|^2 + 2E\|y_0(c)\|^2
\]

\[
\leq 2\left( \gamma E_{2\beta-1,1}(T-t_0) + 1 \right) E\|\psi\|^2, \tag{35}
\]

as desired. \qed

**Remark 9.** Prior research has investigated the FTS of different types of stochastic differential equations (SDEs) using various techniques. For example, the authors in [26] have explored the FTS of Caputo fractional-order SDEs through the use of the generalized Gronwall inequality. Additionally, the FTS of nonlinear SDEs has been studied in [27] by means of the Bihari inequality. In contrast, the present study focuses on the FTS of LCSDEwFD and employs the BFPT as a key analytical tool.

### 4. Illustrative Examples

To provide empirical support for the theoretical results presented in Section 3, we offer two illustrative examples in this section.
Example 10. Let $W(t)$ be a standard Brownian motion. Consider the following system:

$$T_{t_0}^\theta x(c) = (Lx(c) + Mx(c - r(c))) + (Nx(c) + Px(c - r(c)))\frac{dW(c)}{dc},$$  

(36)

with the initial condition

$$x_{t_0} = \psi(c) = \left(0.1 - 0.09e^{-c/2}, 0.05 - 0.04\cos\left(\frac{c^2}{2}\right)\right), \quad -\kappa \leq c \leq 0,$$

(37)

where $r(c) = 1, t_0 = 0.2, x(c) = (x_1(c), x_2(c))^T \in \mathbb{R}^2, x(c - r(c)) = (x_1(c - r(c)), x_2(c - r(c)))^T \in \mathbb{R}^2$, and

$$L = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}, M = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.02 \end{pmatrix}, N = \begin{pmatrix} 0.06 & 0 \\ 0 & 0.06 \end{pmatrix}, P = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.05 \end{pmatrix}.$$  

(38)

It is straightforward to verify that $\|L\| = 0.01, \|M\| = 0.02, \|N\| = 0.06$, and $\|P\| = 0.05$.

From Theorem 8, we obtain $\Lambda = 0.0288, \lambda = 0.1576, S = 0.6744, \gamma = 0.2456, \iota = 0.0232, \epsilon = 0.03$, and $T = 1.2$. Therefore, equation (36) is FTSS with respect to $(\iota, \epsilon, T)$. Moreover, by the above parameters and with the time step $2^{-10}$, we show in Figures 1 and 2 a sample path of the solution of equation (36) for $\beta = 0.7$ and $\beta = 0.5$. In Figures 3 and 4, we give the trajectory of $E\|x(c)\|^2$ on the interval $[t_0, T]$ for $\beta = 0.7$ and $\beta = 0.5$. Based on Definition 6 of the FTS, it is clear from Figures 3 and 4 that $E\|x(c)\|^2$ does not exceed $\epsilon$ for each value of $c \in [t_0, T]$. Consequently, Figures 3 and 4 show the FTS of equation (36) with respect to $(\iota, \epsilon, T)$ for $\beta = 0.7$ and $\beta = 0.5$. Then, the simulation results verify the effectiveness of theoretical results.

Example 11. Let $W(t)$ be a standard Brownian motion. Consider the following system:

$$T_{t_0}^\theta x(c) = (Lx(c) + Mx(c - r(c))) + (Nx(c) + Px(c - r(c)))\frac{dW(c)}{dc},$$  

(39)

with the initial condition

$$x_{t_0} = \psi(c) = \left(0.011 - 0.03e^{-c}, 0.02 - 0.04\cos(c), 0.012 - 0.1\sin(c)\right), \quad -\kappa \leq c \leq 0,$$

(40)

where $x(c) = (x_1(c), x_2(c), x_3(c))^T \in \mathbb{R}^3, x(c - r(c)) = (x_1(c - r(c)), x_2(c - r(c)), x_3(c - r(c)))^T \in \mathbb{R}^3, r(c) = 0.2, t_0 = 0.4$, and

$$L = \begin{pmatrix} 0.03 & 0 & 0.02 \\ 0 & 0.03 & 0 \\ 0.02 & 0 & 0.03 \end{pmatrix}, M = \begin{pmatrix} 0.04 & 0.01 & 0 \\ 0.01 & 0.04 & 0 \\ 0 & 0 & 0.04 \end{pmatrix},$$

$$N = \begin{pmatrix} 0.05 & 0 & 0.02 \\ 0 & 0.05 & 0 \\ 0.02 & 0 & 0.05 \end{pmatrix}, P = \begin{pmatrix} 0.07 & 0.01 & 0 \\ 0.07 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}. $$

(41)

It is not hard to check that $\|L\| = 0.05, \|M\| = 0.05, \|N\| = 0.07, \|P\| = 0.08$. From Theorem 8, we obtain $\Lambda = 0.0666, \lambda = 0.2332, S = 0.7035, \gamma = 2.2442, \iota = 0.17, \epsilon = 1.5, \text{and} T = 2$. Therefore, equation (37) is FTSS with respect to $(\iota, \epsilon, T)$. Moreover, by the above parameters and with the time step $2^{-10}$, we show in Figures 5 and 6 a sample path of the solution of equation (37) for $\beta = 0.85$ and $\beta = 0.35$. Consequently, Figures 7 and 8 give the trajectory of $E\|x(c)\|^2$ on the interval $[t_0, T]$ for $\beta = 0.85$ and $\beta = 0.35$. In Figures 7 and 8, it is also clear for each value of $c \in [t_0, T]$ that $E\|x(c)\|^2$ is less than $\epsilon$ for $\beta = 0.85$ and $\beta = 0.35$. According to Definition 6 of the FTS, Figures 7 and 8 show...
the FTS of equation (37) with respect to \((\iota, \varepsilon, \Theta)\). Subsequently, the validity and utility of our theoretical findings are confirmed by means of numerical simulations.
5. Conclusion

The focus of this paper is to investigate the existence and uniqueness of solutions for the LCSDEwFD using the BFPT. In addition, we have analyzed the FTS of LCSDEwFD by...
utilizing advanced stochastic analysis techniques. Our results show that under certain conditions, the solution to LCSDeWFd is unique and well-defined, and its properties can be characterized by the FTS. For future work, we propose to extend our study to neutral LCSDeWFd, which is a more general class of equations where the coefficients depend on the solution itself. This extension will require the development of new analytical and numerical methods and could lead to novel insights into the behavior and properties of LCSDeWFd under different conditions. Moreover, we plan to explore the applications of LCSDeWFd in various fields, such as physics, engineering, and finance, and investigate the potential of using LCSDeWFd as a tool for modeling and simulating complex systems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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