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Research Article

A Mathematical Model for the Dynamics of Income Distribution in the Presence of Production

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In this paper, a mathematical model is formulated, suitable to explain the evolution of income distribution over a population in the presence of production. The model is conceived from the perspective of complexity. Indeed, the income distribution emerges as the result of a myriad of economic exchanges taking place between individuals. In fact, the aim of the paper is to provide a framework and mathematical tools for the construction and the investigation of models having an exploratory character. The framework is expressed in the form of a system of nonlinear ordinary differential equations, as many as the income classes are, involving transition probabilities. Numerical solutions of these systems are constructed under different assumptions on the law of production and in the presence of different fiscal systems, which provides an example of the versatility of the method.

1. Introduction

We propose and investigate here a mathematical model for the formation of the income distribution of a population when some production takes place. The approach we suggest fits within a complex system perspective as the mentioned distribution arises as the result of a myriad of economic exchanges occurring between individuals of the population. The general framework for the formulation of the problem has been first discussed in [1] in connection with the construction of a model relative to a closed market society, namely, a case in which the total income of the population is conserved. The model explored in [1] was subsequently extended in various directions to address different issues. For example, in [2], it was generalized so as to incorporate a means-tested welfare program and to focus on economic inequality; in [3], it was employed to investigate the existence of a (negative) correlation between economic inequality and social mobility; in [4], a simplified version of it was considered that includes some noise terms.

In all these papers, the conservation of the total income was assumed to hold true. More precisely, in [4], also the possibility for the total income not to be conserved was admitted, which was, however, an effect of uncertainties related to the "external world" as for example import-export of goods, if any. In contrast, we deal here with cases characterised by growth in total income due to production, where the law according to which production is generated is assumed to be known.

Considering total income conservation corresponds to suppose that production and consumption in society are balanced. The dynamics in this case is only governed by the passing of money from one individual to another, this passing being motivated by payment of some service, job, or good, by payment of taxes, and benefit from the tax revenue. If society is divided into a finite number of income classes, every single payment results in a shift of "a small portion of an individual" from one of these classes to another. Of course, this is not but the microscopic description, and the process makes sense when viewed at an aggregate level at which the percentage of individuals in the various income classes changes over time.

The introduction of production can be modelled based on the observation that additional income—what we might colloquially and somewhat improperly call additional wealth—is typically generated by individuals belonging to certain income classes. A reasonable hypothesis in the case in point seems to be, for example, that it is the upper middle-income classes that produce. We adopt this hypothesis here. Accordingly, we integrate the framework first established in [1], expressed in the form of a system of nonlinear ordinary differential equations, with suitable terms which translate into formulas based on the observation just made. More generally, we consider various cases assuming that all classes but the poorest and the richest one contribute equally to production, or that only some of them do so, and unequally. We then study and explore the new model focusing on the scenarios emerging in various cases. Interest lies especially in finding the effects that different production laws have on different income classes. The model parameters and functions in the model, in particular the production laws we consider, may seem arbitrary and perhaps naive from an economic point of view. But here we emphasise that our aim is nothing more than to provide a mathematical tool for the formulation and analysis of exploratory models. In fact, this paper should be understood as providing a step forward (although still to be further improved) towards the paradigm for the study of economic issues which is advocated, for example, in works [5–8]. The contribution it makes so far is that the approach it proposes incorporates nonlinearity and evolution and, especially, links the observable "macroscopic" features and patterns at a multiplicity of interactions which take place at a "microscopic" level.

We are aware of only few works that formulate models and study the relationship between production and income distribution from a complex system perspective. One of these is [9], where an agent-based model incorporating wealth exchange, economic growth, and its distribution is proposed. The total wealth is assumed there to grow exponentially and the added wealth is assumed to be nonuniformly distributed to the agents according to a formula which involves a parameter λ. This parameter is found to play a role in the distinction of different possible "phases" of the system: for certain values of λ, a steady state is reached, whereas for others, the system is nonstationary and wealth condensation arises. Another paper which includes production among other features relative to economic exchange models is [10]. There, wealth is supposed to grow linearly, and a partial integrodifferential equation for the wealth distribution is derived. The conclusion reached after a rescaling of the variables in the equation is that the only effect of production is to change the yardstick by which wealth is measured and concentration of wealth takes place.

More generally, in recent decades, the distribution of income—or wealth—has been the subject of various works belonging to the field of econophysics (see, e.g., [11–15] and the references therein and see also [16]). In these works, income distribution has been obtained through methods inspired by kinetic theory and statistical physics as a limit of processes involving a large number of monetary exchanges between agents. We emphasise that an important and specific aspect of the model discussed here (and in [2–4]), which characterises and distinguishes the approach of the paper from that of the mentioned works in the econophysics literature, is the subdivision of the population into income classes, also accompanied by a differentiated taxation with redistribution.

Finally, we want to mention a few recent contributions on the relationship between economic growth and income distribution, belonging to economic literature [17–19]. In particular, the first of these three works provides a review with an extensive bibliography on the subject and shows that, at least in the economic literature, related research keeps being lively.

The article is organised as follows: In Section 2, the equations describing the model are derived, and in Section 3, some aspects of these equations are discussed, also in a comparison with the model in [1]. Section 4 reports results of numerical simulations obtained in correspondence of different production laws, namely, in correspondence of different prescribed functions which describe the evolution of the production. Cases are considered in the presence and the absence of a taxation and redistribution process. We anticipate here that in the cases in which a progressive taxation system is foreseen, the effect of production is the decrease in the quantity of individuals in the lower income classes and its simultaneous increase in the upper income classes. Furthermore, a decrease of the value of the Gini index can be seen taking place, which corresponds to a decrease of economic inequality. Without the corrective contribution of the taxes, the situation is somehow different, although similar in the end, as the changes in the middle classes exhibit less regular behavior. More detailed observations and quantitative data can be found below. Finally, Section 5 contains a short summary and a critical analysis of the paper.

2. The Model

We consider a population of individuals divided into a finite number n of classes characterised by their average incomes $r_1 \leq r_2 \leq \ldots \leq r_n$. These are defined as follows (we introduce here a small generalisation of the framework in [1] where all income intervals were assumed to have the same length): Let $0 = \rho_0 < \rho_1 < \ldots < \rho_n$ be $n+1$ nonnegative numbers with ρ_n representing an upper limit for the maximal conceivable income of each individual. Suppose in particular that $\rho_{i+1} - \rho_i = \alpha\,(\rho_i - \rho_{i-1})$ with $\alpha > 1$, for $i = 1, \ldots, n-1$. The n intervals $[\rho_{i-1}, \rho_i)$ have increasing length, which allows a more faithful representation of reality in comparison to [1] without altering the essence of things. Assuming for simplicity that the "density" of individuals with income in $[\rho_{i-1}, \rho_i)$ is the same at each point in this interval, we set $r_i = (\rho_i + \rho_{i-1})/2$ for $i = 1, \ldots, n$ and denote by $x_i(t)$ (with $x_i \colon \mathbf{R} \longrightarrow [0, +\infty)$ for $i = 1, \ldots, n$ and with the normalisation $\sum_{i=1}^{n} x_i = 1$) the fraction at time t of the population whose income is r_i. We call i-th class the corresponding class and call i-individuals the individuals of this class. We postulate the occurrence of economic exchanges taking place between individuals and the existence of a fiscal system with redistribution, according to which individuals have to pay a percentage tax on each earning, the tax rate for the i-th class being denoted by τ_i with $0 \leq \tau_i \leq 1$. The equations describing the evolution in time of each component $x_i(t)$ for the original model developed in [1] are of the form:

$$\frac{dx_i}{dt} = \sum_{h=1}^{n}\sum_{k=1}^{n}\left(C_{hk}^{i} + T_{[hk]}^{i}(x)\right)x_h x_k - x_i \sum_{k=1}^{n} x_k, \quad i = 1, 2, \ldots, n, \tag{1}$$

with

- the coefficients C_{hk}^{i} expressing transition probabilities associated with the direct pairwise money exchanges: $C_{hk}^{i} \in [0, +\infty)$ expresses the probability density that an h-individual will belong to the i-th class after a direct interaction with an k-individual. The identity $\sum_{i=1}^{n} C_{hk}^{i} = 1$ has to be satisfied for any fixed h and k;
- the functions $T_{[hk]}^{i}$ expressing density variations associated with the processes of taxation and redistribution related to each transaction: $T_{[hk]}^{i}: \mathbf{R}^n \longrightarrow \mathbf{R}$ expresses the variation density in the i-th class due to an interaction between an h-individual with a k-individual. These functions have to satisfy $\sum_{i=1}^{n} T_{[hk]}^{i}(x) = 0$ for any fixed h, k and $x \in \mathbf{R}^n$.

More details on the meaning of these coefficients and of these functions can be found in [1]. We merely recall their expression, for which we set out some preliminary notation: we denote by

- $p_{h,k}$ (for $h, k = 1, \ldots, n$) is the probability that in an encounter between an h-individual a k-individual, the one who pays is the former one (with the obvious requirement that $0 \le p_{h,k} \le 1$ and $p_{h,k} + p_{k,h} \le 1$);
- $S \ll \Delta r$is the amount of money paid in a transaction

It can be seen that the only possibly nonzero elements C_{hk}^{i} are the following ones (with the caveat that the expression for $C_{i+1,k}^{i}$ in (2) holds true for $i \le n-1$ and $k \le n-1$, the second addendum of the expression for $C_{i,k}^{i}$ is effectively present only provided $i \le n-1$ and $k \ge 2$, while its third addendum is present only provided $i \ge 2$ and $k \le n-1$, and the expression for $C_{i-1,k}^{i}$ holds true for $i \ge 2$ and $k \ge 2$):

$$\begin{aligned} C_{i+1,k}^{i} &= p_{i+1,k}\frac{S(1-\tau_k)}{r_{i+1}-r_i}, \\ C_{i,k}^{i} &= 1 - p_{k,i}\frac{S(1-\tau_i)}{r_{i+1}-r_i} - p_{i,k}\frac{S(1-\tau_k)}{r_i - r_{i-1}}, \\ C_{i-1,k}^{i} &= p_{k,i-1}\frac{S(1-\tau_{i-1})}{r_i - r_{i-1}}. \end{aligned} \tag{2}$$

As for $T_{[hk]}^{i}(x)$, they take the form $T_{[hk]}^{i}(x) = U_{[hk]}^{i}(x) + V_{[hk]}^{i}(x)$, where

$$U_{[hk]}^{i}(x) = \frac{p_{h,k} S\tau_k}{\sum_{j=1}^{n} x_j}\left(\frac{x_{i-1}}{r_i - r_{i-1}} - \frac{x_i}{r_{i+1}-r_i}\right), \tag{3}$$

and $V_{[hk]}^{i}(x)$ is different from zero only if $h = i+1$ or $h = i$. In particular,

$$\begin{aligned} V_{[hk]}^{i}(x) &= -p_{h,k} S\tau_k \frac{1}{r_i - r_{i-1}} \frac{\sum_{j=1}^{n-1} x_j}{\sum_{j=1}^{n} x_j} \quad \text{if } h = i, \\ V_{[hk]}^{i}(x) &= p_{h,k} S\tau_k \frac{1}{r_{i+1} - r_i} \frac{\sum_{j=1}^{n-1} x_j}{\sum_{j=1}^{n} x_j} \quad \text{if } h = i+1. \end{aligned} \tag{4}$$

To introduce now the contribution of production, we argue as follows: The variation in an interval of time Δt of the fraction $x_i(t)$ of individuals in the i-th class only due to production can be estimated observing that if the $(i-1)$-th class contributes to the production, there will be a quantity $\xi_{i-1}(t)\Delta t + o(\Delta t)$ of individuals leaving the $(i-1)$-th class and entering the i-th class, and analogously, if the i-th class contributes to the production, there will be a quantity $\xi_i(t)\Delta t + o(\Delta t)$ of individuals leaving the i-th class for the $(i+1)$-th one. In formulas,

$$x_i(t+\Delta t) - x_i(t) = \left(\xi_{i-1}(t) - \xi_i(t)\right)\Delta t + o(\Delta t). \tag{5}$$

In turn, the quantity $\xi_i(t)\Delta t$ is associated with the amount $\pi_i(t)\Delta t$ of money (income) corresponding to the production realised by i-individuals through the formula:

$$\xi_i(t) = \frac{\pi_i(t)}{r_{i+1} - r_i}. \tag{6}$$

Indeed, if $\pi_i(t)\Delta t$ is the amount produced by the class i in the time interval Δt and the difference of the average income of the classes i and $i+1$ is $r_{i+1} - r_i$, the "quantity" of individuals improving their economic status is given by $\pi_i(t)\Delta t/(r_{i+1} - r_i)$.

Focusing for the moment only on production-induced changes, we obtain in view of (6):

$$\frac{dx_i}{dt} = \frac{\pi_{i-1}(t)}{r_i - r_{i-1}} - \frac{\pi_i(t)}{r_{i+1} - r_i}. \tag{7}$$

At this point, a reasonable, however questionable, observation suggests that the production of the different classes is related, and possibly proportional, to their richness. In other words, one can suppose, and we do it that

$$\pi_i(t) = \frac{c_i r_i x_i}{\sum_j c_j r_j x_j}\pi_{\mathrm{tot}}(t), \tag{8}$$

where $\pi_{\mathrm{tot}}(t)$ denotes the total production and the nonnegative coefficients c_i express possibly different weights.

Pushing forward with the hypotheses, we assume here that the classes that contribute to production and wealth creation are the upper middle ones. In particular, the model we are going to discuss these classes includes neither the poorest nor the richest.

Putting together the terms which characterise the conservative model (i.e., those in the rhs. of (1)) and those which keep into account production (those in the rhs of (7)), we get the evolution equations:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{h=1}^{n}\sum_{k=1}^{n}\left(C_{\mathrm{hk}}^{i} + T_{[\mathrm{hk}]}^{i}(x)\right)x_h x_k - x_i \sum_{k=1}^{n} x_k + \left(\frac{c_{i-1}\, r_{i-1} x_{i-1}}{r_i - r_{i-1}} - \frac{c_i\, r_i x_i}{r_{i+1} - r_i}\right)\frac{\pi_{\mathrm{tot}}(t)}{\sum_j c_j\, r_j x_j}, \quad i = 1, 2, \ldots, n, \tag{9}$$

where the sum $\sum c_j r_j x_j$ in one denominator on the rhs must be understood as extending only to the indices j of the classes that are actually productive, say those from the j^*-th one to the $(j^* + K)$-th one for some $j^* > 1$ and $j^* + K < n$ for some positive K. Accordingly, $c_j \neq 0$ only for indices j with $j^* \leq j \leq j^* + K$.

3. Properties of the Model

Global existence and uniqueness of solutions of (9) satisfying suitable initial conditions hold true. Specifically, we have the following result.

Theorem 1. *In correspondence to any initial condition $x_0 = (x_{01}, \ldots, x_{0n})$ with $x_{0i} \geq 0$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^{n} x_{0i} = 1$, a unique solution $x(t) = (x_1(t), \ldots, x_n(t))$ of (6), satisfying $x(0) = x_0$, exists, defined for all $t \in [0, +\infty)$, and such that for all $t \geq 0$, both $x_i(t) \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} x_i(t) = 1$ hold true.*

Proof. The proof is essentially the same as that given for the conservative model in [1] to which we refer the reader. There are only two points where some additional reasoning is needed. As for the first, by taking the sum over $i = 1, \ldots, n$ of both terms on the lhs and on the rhs in (1), one finds (exactly as in [1]) $\sum_{i=1}^{n} x_i(t) = \sum_{i=1}^{n} x_{0i} = 1$. We explicitly point out that this conclusion remains true also when one takes the sum in (6), due to the fact that the sum $\sum_{i=1}^{n} (c_{i-1} r_{i-1} x_{i-1}/(r_i - r_{i-1}) - c_i r_i x_i/(r_{i+1} - r_i))\pi_{\mathrm{tot}}(t)/\sum_j c_j r_j x_j$ is a telescopic one and, due to what has been observed at the end of the previous section, only contains terms with index i ranging from j^* to $j^* + K + 1$. This guarantees that this sum is equal to zero. A second point deserving an additional check concerns the continuation of the nonnegativity of the components x_i: $x_i(t) \geq 0$. If there is a first time t^* at which the component x_i of the solution for some i vanishes, $x_i(t^*) = 0$, it is proved in [1] that $\mathrm{d}x_i/\mathrm{d}t(t^*) \geq 0$. In the model described by (6), the expression corresponding to this derivative also contains an additional term which reduces however to $(c_{i-1} r_{i-1}\ x_{i-1}/(r_i - r_{i-1}))\pi_{\mathrm{tot}}(t)/\sum_j c_j r_j x_j$ when $x_i = 0$. This term is plainly nonnegative, which proves that x_i cannot become negative. □

Of course, for the equation system (6) relative to the model with production, the scalar function $\mu(x) = \sum_{i=1}^{n} r_i x_i$, expressing the global income (total amount of money), is no more a first integral in general as was the case for the model in [1]. Also, the existence of asymptotic stationary solutions, one for each value of the global income, which all numerical simulations relative to the model in [1] suggested, cannot be expected in the presence of production.

4. Numerical Simulations

In this section, some results of numerical simulations are reported. Carrying out the simulations obviously requires that all parameters of the model are fixed. In this regard, various choices are made below. In particular, different expressions for the law governing production are considered, the interest being to analyse the effects on different income classes of different forms of production. We also point out that since an uninterrupted production over time would result in continuous growth of wealth, the simulations we develop refer to three subsequent time intervals:

First interval $[0, T_1]$ when there is no production. The equations whose solutions we look for are in this interval of equation (1), and T_1 is chosen large enough to ensure that "a stationary solution is reached" ($T_1 = 500000$ in the simulations). As briefly recalled in Section 3, for the conservative model, for any fixed value μ of the global income, a unique stationary solution exists to which all solutions evolving from initial conditions whose global income is μ tend in the long run.

Second interval $[T_1, T_2]$ during which the existence of some production is postulated, namely, an explicit expression for the function $\pi_{\mathrm{tot}}(t)$ is supposed to be given, together with a specific choice of the weights c_i appearing in (8). During this interval, we look for solutions of equation (9) ($T_2 - T_1 = 50000$ in the simulations).

Third interval $[T_2, T_3]$ during which again there is no production and the evolution of the system is again governed by equation (1). Again, the width $T_3 - T_2$ of the interval is chosen large, so as to ensure that solutions tend to an equilibrium ($T_3 - T_2 = 250000$ in the simulations).

We take some parameters to be the same in all simulations. Specifically, to fix ideas, we take $n = 9$, $\rho_1 = 15$, and $\alpha = 1.5$. Thus, rounding off the numbers gives

$$\begin{gathered}(\rho_0, \rho_1, \rho_2, \rho_3, \ldots, \rho_n) = (0.0,\ 15.0,\ 37.5,\ 71.25, \ldots, 1123.3),\\ (r_1, r_2, r_3, \ldots, r_n) = (7.5,\ 26.25,\ 54.375, \ldots, 931.084).\end{gathered} \tag{10}$$

This makes it possible to include incomes that differ by orders of magnitude, just as happens in real life. The tax rates are chosen as

$$\tau(i) = \tau_{\min} + (\tau_{\max} - \tau_{\min})\frac{(i-1)}{(n-1)} \quad \text{for } i = 1, \ldots, n, \tag{11}$$

with $\tau_{\min} = 0.15$ and $\tau_{\max} = 0.45$. $S = 0.1$, and the coefficients $p_{h,k}$ are chosen as

$$p_{h,k} = \frac{1}{3}\left(\frac{\min\{h,k\}}{n}\right)^2, \quad \text{if } h = 2,\ldots,n$$
$$k = 1,\ldots,n-1, \tag{12}$$
$$p_{1,k} = 0 \quad \text{if } k = 1,\ldots,n,$$
$$p_{h,n} = 0 \quad \text{if } h = 1,\ldots,n.$$

Remark 2. The choice of the parameters $p_{h,k}$ is highly arbitrary. The rationale of the one made here is to reproduce the fact that poor individuals usually earn and pay less than rich ones. Of course, other choices could be—and in fact have been—considered, e.g., in [1–3].

As for the law governing production, valid in the interval of time $[T_1, T_2]$, we assume

$$\pi_{\text{tot}}(t) = c, \tag{13}$$

with a positive c, in such a way that, denoting $\mathscr{R}(t)$ the "richness" at time t, one has $\mathscr{R}(t+T_1) = \mathscr{R}(T_1) + ct$. The constant c in (13) will be taken in different simulations as $c = 0.001$ (similar values provide similar results), and both for the case in which taxation and redistribution are present (the different tax rates being as in (11) with $\tau_{\min} = 0.15$ and $\tau_{\max} = 0.45$) as well as the case in which there are no taxation and redistribution (for which it suffices to take $\tau_{\min} = 0 = \tau_{\max}$ in (11)), we consider two possible choices of the weights c_i appearing in (8), namely,

$$\text{(I) } c_i = 1 \text{ for } i = 2,\ldots,n-1, c_1 = 0, c_n = 0,$$
$$\text{(II) } c_i = (3n-2i)\, i \text{ for } i = 2,\ldots,n-1, c_1 = 0, c_n = 0. \tag{14}$$

Each of these choices entails that neither the poorest nor the richest class contributes to production. Apart from that, according to the choice (I), all other classes contribute equally; the choice (II) is designed to express the assumption that the classes contribute differently, and the most productive class is in the upper middle-income segment. Indeed, it implies that productivity increases with average income, but at some point, it starts to decrease.

By summarising, four versions of the model are considered:

$$\begin{aligned}&\text{(I) with taxes and redistribution, (I) without taxes and redistribution,}\\&\text{(II) with taxes and redistribution, (II) without taxes and redistribution.}\end{aligned} \tag{15}$$

For each of them, we report the results collected in one of Figures 1–4. Each figure contains the following:

(i) Three panels displaying the income distributions of the model version at hand, at time T_1 (i.e., after a period without production), T_2 (after a period with production), and T_3 (again, after a period without production).

(ii) Three panels with histograms displaying the fraction of population in the nine income classes at different times. In the caption of the figure, also the values of these fractions are given.

(iii) Three panels with histograms displaying the variation of the fraction of population in the nine income classes at different times. Going from the left to the right, the panels refer to the difference in each class, respectively, between the "number of individuals":

$$\begin{aligned}&\text{at } t = T_2 \text{ and at } t = T_1,\\&\text{at } t = T_3 \text{ and at } t = T_2,\\&\text{at } t = T_3 \text{ and at } t = T_1.\end{aligned} \tag{16}$$

(iv) Two more panels: That one which shows the solutions of the model during the second and the third period, namely, during an interval of time when there is some production, followed by an interval of time when no production is present. It is evident from the figures that there is a singularity of the solutions after a time period of length $50000 = T_2 - T_1$ from the initial instant, i.e., at the change from the production phase to the nonproduction phase. The last panel shows the behaviour of the Gini index (a well-known tool, usually employed to measure economic inequality, [20]), and also in this panel (at least for some model versions), a singular point can be seen in correspondence of $t = 50000$.

What can be immediately observed is that for all model versions, the effect of production (to be visualised in each figure in panel (g)) is a decrease in the number of individuals in lower income classes together with an increase in the upper income classes. What happens to the middle classes depends on the models. From a qualitative point of view, at least for the model versions including taxation and redistribution, production seems to prove capable of yielding in each class an improvement in the economic condition. Quantitative measures of population fraction increases and decreases in the various classes obviously depend on the specific growth laws.

One more point is that the Gini index is found to be decreasing or at least nonincreasing in all model versions.

Finally, notice that the components of the solutions at time T_3 of the model versions (I), (II) with taxes (respectively, without taxes) are the same. This is due to the fact that the global income at time T_2 is the same in the three cases, the model is conservative in $[T_2, T_3]$, and hence, uniqueness of the "asymptotic equilibrium" holds true. Analogous observation holds true for the components of the solutions at time T_1.

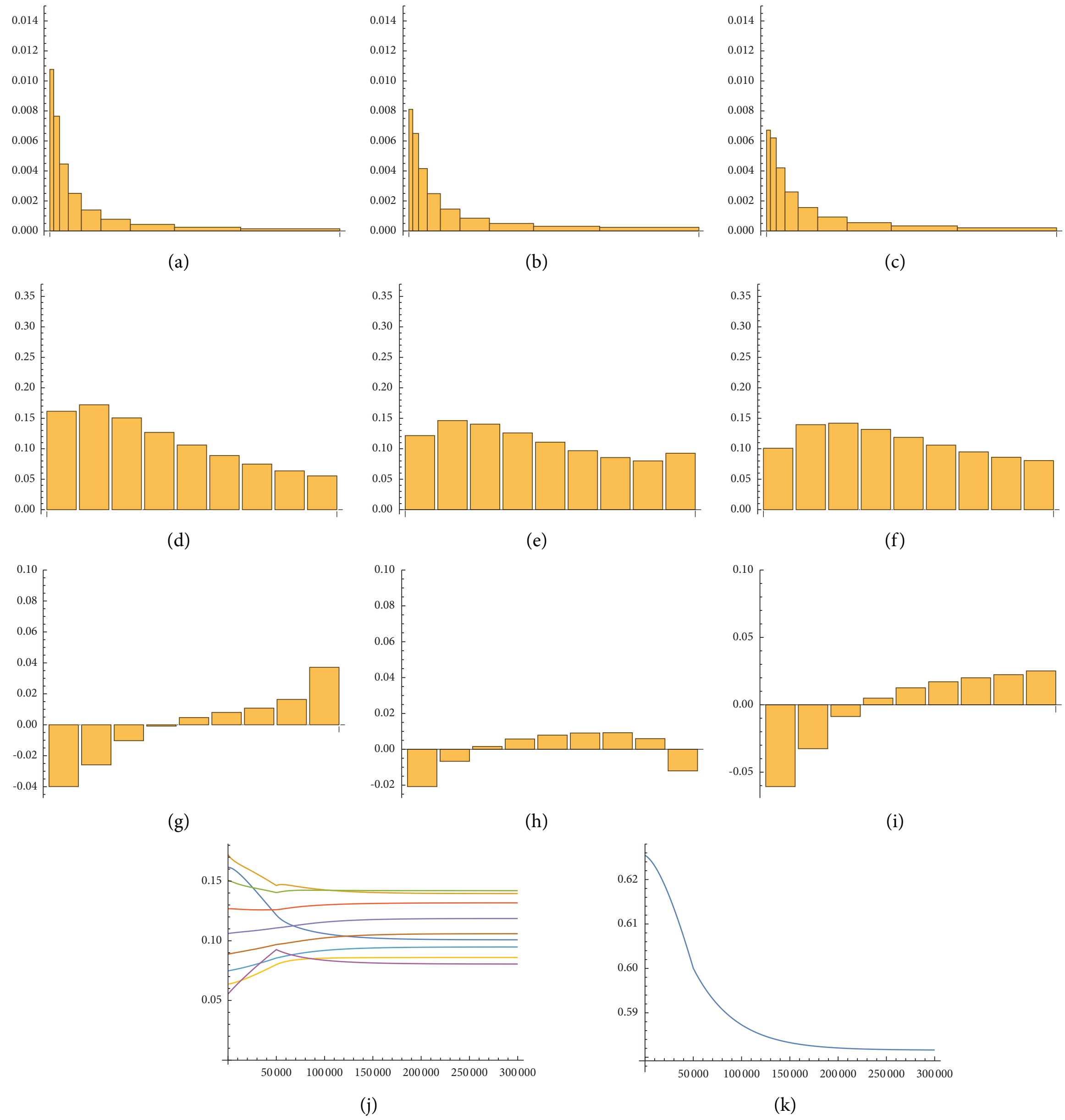

Figure 1: The figure refers to the model version (I) when a taxation process is in place. The panels (a–c) show the income distributions at time T_1, T_2, and T_3. The panels (d–f) show the fraction of population in each income class at time T_1, T_2, and T_3. The rounded components of these fractions are, respectively, (0.162, 0.172, 0.151, 0.127, 0.106, 0.089, 0.075, 0.064, 0.056), (0.122, 0.146, 0.140, 0.126, 0.111, 0.097, 0.086, 0.080, 0.093), and (0.101, 0.140, 0.142, 0.132, 0.119, 0.106, 0.095, 0.086, 0.081). The panels (g–i) represent the variation of the fraction of population in the nine income classes at the different times specified in the captions. Panel (j) displays the solutions in $[T_1, T_3]$; panel (k) displays the behaviour of the Gini index in the same interval of time. (a) At time T_1. (b) At time T_2. (c) At time T_3. (d) $x(T_1)$. (e) $x(T_2)$. (f) $x(T_3)$. (g) $x(T_2) - x(T_1)$. (h) $x(T_3) - x(T_2)$. (i) $x(T_3) - x(T_1)$. (j) The solutions in $[T_1, T_3]$. (k) The Gini index in $[T_1, T_3]$.

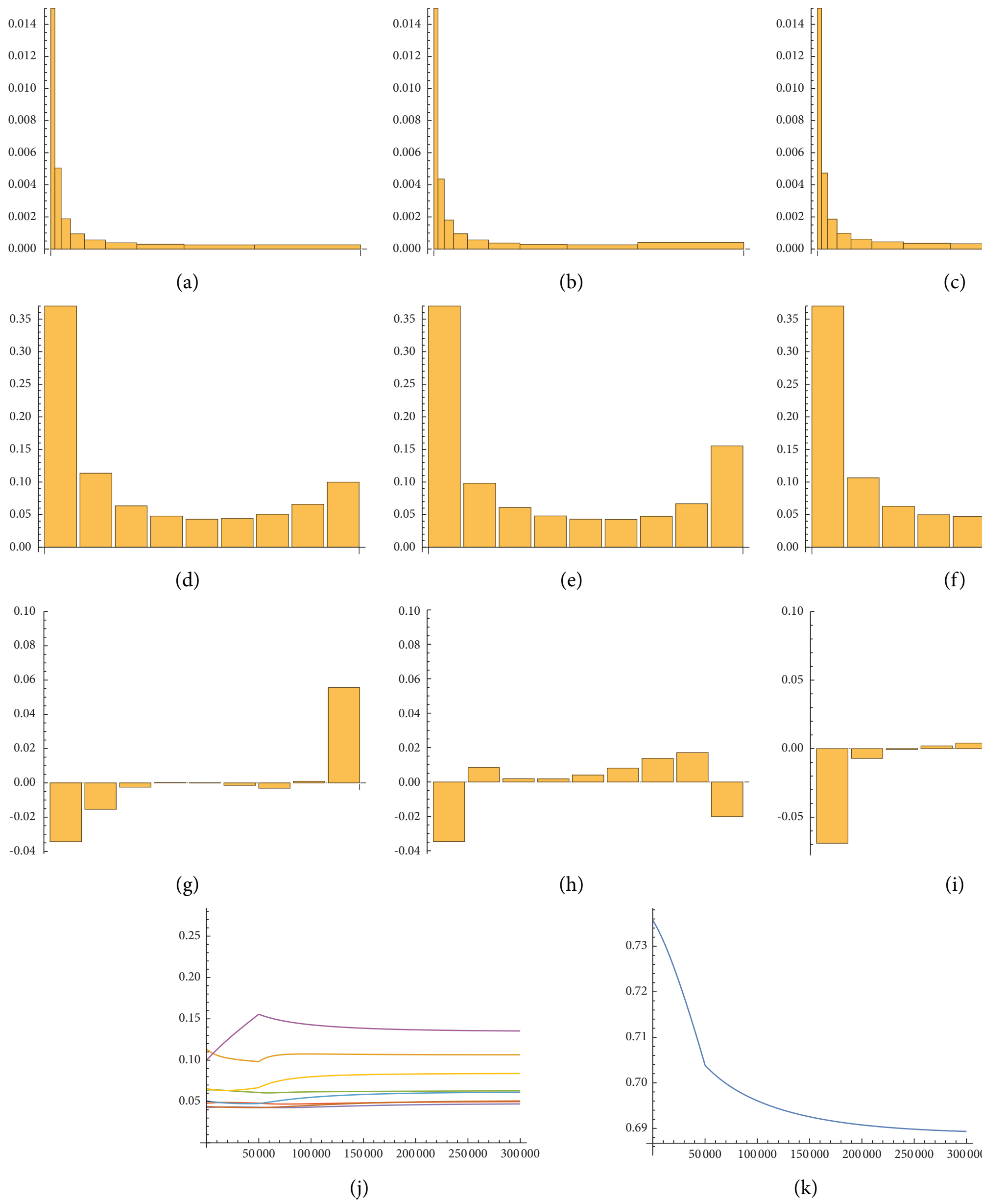

Figure 2: The figure refers to the model version (I) with no taxes. The panels (a–c) show the income distributions at time T_1, T_2, and T_3. The panels (d–f) show the fraction of population in each income class at time T_1, T_2, and T_3. The rounded components of these fractions are, respectively, (0.472, 0.114, 0.064, 0.048, 0.043, 0.044, 0.051, 0.066, 0.100), (0.437, 0.098, 0.061, 0.048, 0.043, 0.043, 0.048, 0.067, 0.156), and (0.403, 0.107, 0.063, 0.050, 0.047, 0.051, 0.061, 0.084, 0.135). The panels (g–i) represent the variation of the fraction of population in the nine income classes at the different times specified in the captions. Panel (j) displays the solutions in $[T_1, T_3]$; panel (k) displays the behaviour of the Gini index in the same interval of time. (a) At time T_1. (b) At time T_2. (c) At time T_3. (d) $x(T_1)$. (e) $x(T_2)$. (f) $x(T_3)$. (g) $x(T_2) - x(T_1)$. (h) $x(T_3) - x(T_2)$. (i) $x(T_3) - x(T_1)$. (j) The solutions in $[T_1, T_3]$. (k) The Gini index in $[T_1, T_3]$.

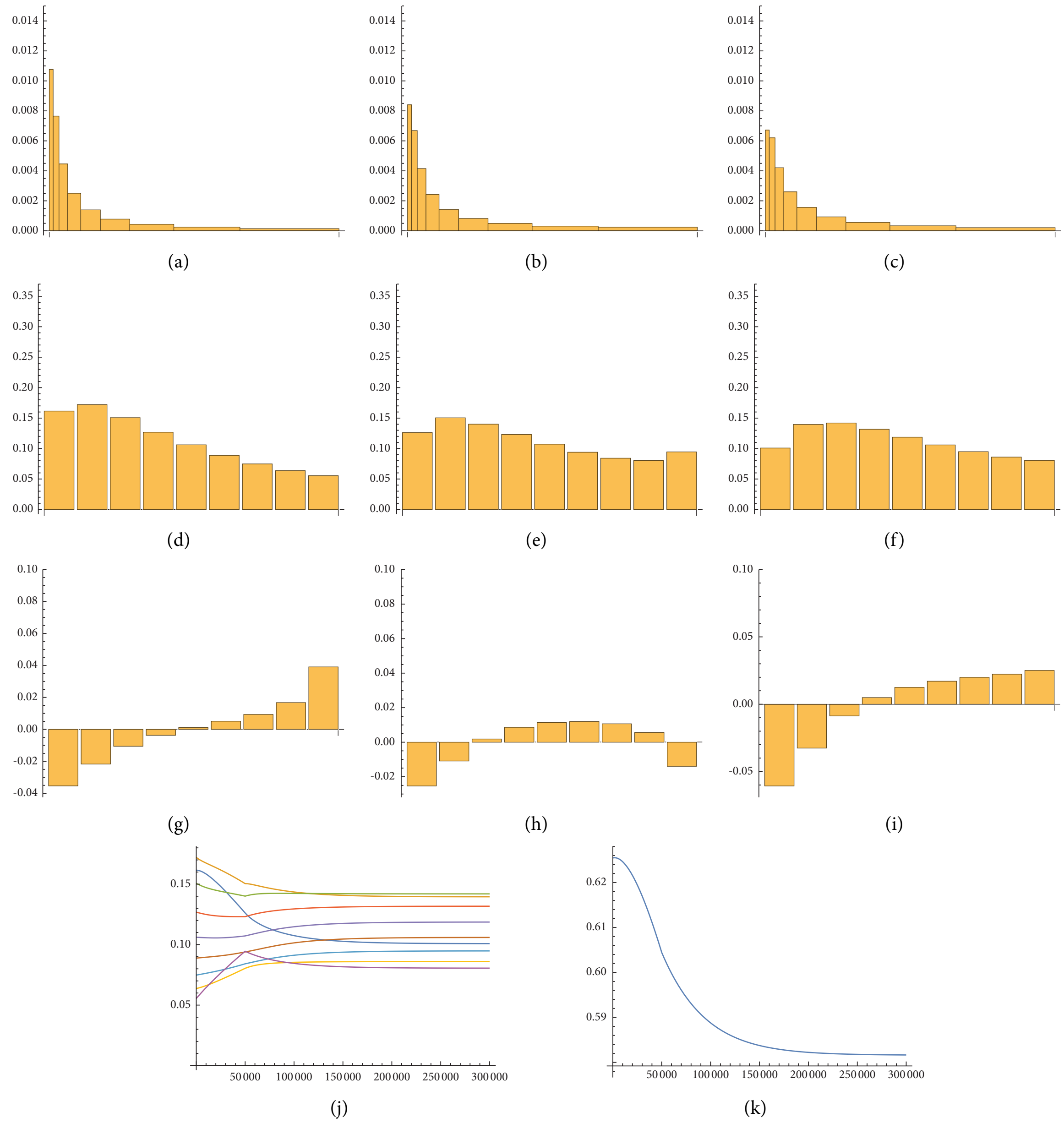

Figure 3: The figure refers to the model version (II) when a taxation process is in place. The panels (a–c) show the income distributions at time T_1, T_2, and T_3. The panels (d–f) show the fraction of population in each income class at time T_1, T_2, and T_3. The rounded components of these fractions are, respectively, (0.162, 0.172, 0.151, 0.127, 0.106, 0.089, 0.075, 0.064, 0.056), (0.126, 0.150, 0.140, 0.123, 0.107, 0.094, 0.084, 0.080, 0.095), and (0.101, 0.140, 0.142, 0.132, 0.119, 0.106, 0.095, 0.086, 0.081). The panels (g–i) represent the variation of the fraction of population in the nine income classes at the different times specified in the captions. Panel (j) displays the solutions in $[T_1, T_3]$; panel (k) displays the behaviour of the Gini index in the same interval of time. (a) At time T_1. (b) At time T_2. (c) At time T_3. (d) $x(T_1)$. (e) $x(T_2)$. (f) $x(T_3)$. (g) $x(T_2) - x(T_1)$. (h) $x(T_3) - x(T_2)$. (i) $x(T_3) - x(T_1)$. (j) The solutions in $[T_1, T_3]$. (k) The Gini index in $[T_1, T_3]$.

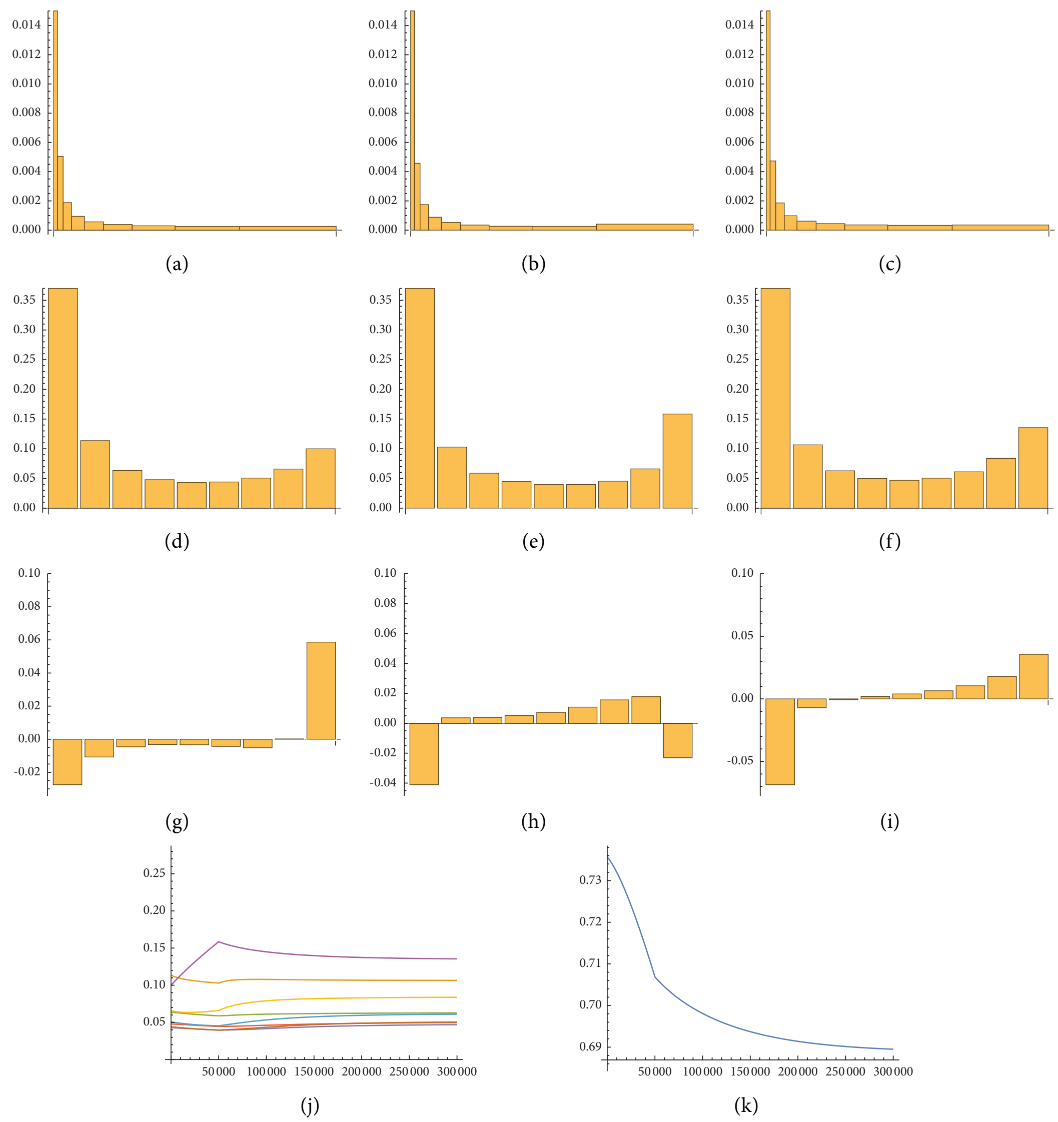

Figure 4: The figure refers to the model version (II) with no taxes. The panels (a–c) show the income distributions at time T_1, T_2, and T_3. The panels (d–f) show the fraction of population in each income class at time T_1, T_2, and T_3. The rounded components of these fractions are, respectively, (0.472, 0.114, 0.064, 0.048, 0.043, 0.044, 0.051, 0.066, 0.100), (0.444, 0.103, 0.059, 0.045, 0.040, 0.040, 0.046, 0.066, 0.159), and (0.403, 0.107, 0.063, 0.050, 0.047, 0.051, 0.061, 0.084, 0.136). The panels g–i represent the variation of the fraction of population in the nine income classes at the different times specified in the captions. Panel (j) displays the solutions in $[T_1, T_3]$; panel (k) displays the behaviour of the Gini index in the same interval of time. (a) At time T_1. (b) At time T_2. (c) At time T_3. (d) $x(T_1)$. (e) $x(T_2)$. (f) $x(T_3)$. (g) $x(T_2) - x(T_1)$. (h) $x(T_3) - x(T_2)$. (i) $x(T_3) - x(T_1)$. (j) The solutions in $[T_1, T_3]$. (k) The Gini index in $[T_1, T_3]$.

5. Concluding Remarks

In this paper, a mathematical model for the evolution in time of the income distribution of a population in the presence of production is formulated and investigated. The model is expressed by a system of nonlinear ordinary differential equations (containing probability transitions), as many as the classes, distinguished by the average income, in which the population is divided. Individuals exchange money through a myriad of interactions which represent payments due to the provision of goods and services, payment of taxes, and benefits from the redistribution of the tax revenue (the process of taxation and redistribution being described by an algorithm which bypasses the necessity of introducing the state tax agency, see [1]) and gains deriving from the production (which increases the total amount of circulating money). These economic exchanges result in the displacement of individuals, more precisely “little portions of individuals” from one class to another, a phenomenon that of course only makes sense on a collective level.

Assuming that production obeys one or other of two slightly different laws, i.e., the amount of generated “richness” in time can be described by two different prescribed functions, and letting the ODE system evolve, we find in particular that the effect of production is a decrease in the number of individuals in lower (in some model version also in lower-middle) income classes together with an increase in the upper-income classes. Hence, at least when parameters are as those chosen in Section 4, and the production laws are of the form here postulated, we find a confirmation of (or at least compatibility with) the famous aphorism "A rising tide lifts all boats". How this occurs depends on the model details.

In conclusion, we again emphasise that our objective here is simply methodological: it is to provide a mathematical framework and tools with which we possibly carry out experiments and simulations, aimed at predicting emergent scenarios in correspondence to different parameters and production laws. We believe that explorative models such as this one could help, if supplemented with parameters estimated from real-world data, in the adoption of appropriate policies.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author is a member of G.N.F.M. (Gruppo Nazionale per la Fisica Matematica) of INdAM (Istituto Nazionale di Alta Matematica).

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