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Research Article

Global Stability with Lyapunov Function and Dynamics of SEIR-Modified Lassa Fever Model in Sight Power Law Kernel

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Lassa fever is an acute viral hemorrhagic disease that affects humans and is endemic in various West African nations. In this study, a fractional-order model is constructed using the Caputo operator for SEIR-type Lassa fever transmission, including the control strategy. The proposed model examines the dynamics of Lassa fever transmission from rodents to humans and from person to person and in territories with infection in society. The model is analyzed both qualitatively and quantitatively. We examine the positively invariant area and demonstrate positive, bounded solutions to the model. We also show the equilibrium states for the occurrence and extinction of infection. The proposed nonlinear system is verified to be present, and a unique solution is shown to exist using fixed point theorems. Using the Volterra-type Lyapunov function, we investigate the global stability of the suggested system with a fractional Caputo derivative. To study the impact of the fractional operator through computational simulations, results are generated employing a two-step Lagrange polynomial in the generalized version of the power law kernel. A graphical evaluation is provided to show the simplicity and dependability of the model, and all rodents that could be source viruses are important in ecological research. The findings with a value equal to 1 are stronger, according to the comparison of outcomes with different fractional orders. The adverse effect of Lassa fever increases when all modes of transmission are taken into account, according to the study, with fractional-order findings indicating less detrimental effects on specific transmission routes.

1. Introduction

Lassa fever (LF), also known as Lassa hemorrhagic fever (LHF), is a severe viral infection that poses a serious threat to the public's health in sub-Saharan African nations. The Lassa virus (LASV), which belongs to the Arenaviradae family, is the virus that causes LF. The LASV is a bisegmented, linear, ambisense, and single-stranded RNA virus, and it is transmitted by the multimammate mouse (Mastomys natalensis), which is located in sub-Saharan Africa [1]. All age groups and both sexes are susceptible to Lassa fever. People who live in remote locations with low sanitary conditions are more vulnerable. The Lassa virus can be spread from person to person by coming into contact with its excretions, blood, tissues, or secretions. LASV does not spread through common contact. However, when the wrong personal protective equipment (PPE) is not employed, human-to-human transfer and nosocomial infections are frequent in healthcare environments. Additionally, compromised medical supplies like reused needles might spread LASV [2]. A normal LF epidemic lasts seven months, from November to May, with the majority of cases occurring in the first three months and sporadic instances on certain occasions throughout the year [2, 3]. In endemic African regions, LF is a substantial cause of morbidity and mortality, with 80% mild cases and 20% severe cases, as well as sporadic epidemics with a 50% case-fatality rate. Fever, discomfort, and neurological problems such as encephalitis, hearing loss, and tremors are among the symptoms of LF. Due to the overall nature of these symptoms, which often appear

1–3 weeks after contact, making a clinical diagnosis is challenging [4]. According to recent estimates, LF kills 5000 to 10,000 people annually, mostly in West Africa, and causes two million infections [5]. Several epidemiological models have been developed and utilized to better understand the LF transmission; for example, see [1, 6, 7]. The dynamics of LF transmission were shown using a nonautonomous system of nonlinear ordinary differential equations that took seasonal fluctuation in the occurrence of Mastomys into account [8, 9], as well as other mathematical studies connected to the Lassa virus [10–12]. Pregnant women who have LHF disease suffer harmful effects that first surfaced in Africa [13, 14].

In many scientific fields, particularly engineering and physics, fractional calculus is employed extensively [15]. Because fractional-order models distinguish between genetic and memory features of mathematical models, they are more factual and empirical than conventional integer-order models [16–19]. With the use of the Caputo fractional-order derivative, the authors of [20] created a new integer-order ordinary differential equation Lassa fever model, from which the FODE that it corresponds to was created. On the rodent population, a culling approach was used to reduce disease. Even though this method lessens the number of infected rats, it does not totally cure the illness in people. A fractional-order model of Lassa illness is presented in a different study [21]. Utilizing the Laplace Adomian decomposition method, the answer was found. According to the sensitivity analysis, the contact rate with exposed, infected, and isolated people needs to be under control. The authors of [22] present a fractional-order model for the kinetics of Lassa fever transmission that takes into account person-to-person, mastomy, rat-to-human, and polluted surroundings. Numerical solutions to the problem were found using the Adams–Bashforth–Moulton method. It was discovered that self-protection strategies advised for those with prior diseases identical to the one in question can contain a potentially explosive outbreak. With a fatality rate of roughly 80%, this lethal illness kills pregnant women more frequently than the Ebola hemorrhagic fever. For patients who were pregnant, a novel analysis using the time-fractional Lassa hemorrhagic fever model that has been suggested was conducted in [23]. The fractional variation iteration approach was used to find the numerical solution to this model. Researchers changed the derivative to a time-fractional derivative with nonsingular and nonlocal kernels in order to expand the model representing Lassa hemorrhagic fever [24]. Using the Banach fixed point theorem, a thorough examination of the existence and uniqueness of the exact solution was presented. Finally, it was demonstrated how several numerical simulations supported the efficiency of the employed derivative. Atangana [25] just created the novel concept of fractal-fractional derivative. In many circumstances, this novel concept is highly useful for solving some challenging issues. The operator has two orders: the fractional order, which is the first, and the fractal dimension, which is the second. The fractal-fractional derivative is a new concept that is superior to fractional derivatives and conventional ones. This is because working with fractal-fractional derivatives allows us to simultaneously examine the fractional operator and fractal dimension. Using a fractal-fractional derivative, Farman et al. [26] studied a sustainable method to observe the dynamics of infection in the plant. Solutions were produced using a two-step Lagrange polynomial in the generalized form of the Mittag–Leffler kernel to represent a time-fractional-order plant virus model with disease effects. Researchers employed fractal-fractional Atangana–Baleanu derivatives and integrals in the sense of Caputo to study the dynamics of Q fever transmission in livestock and ticks as well as the bacterial load in the environment [27]. The newly created Newton polynomial was used in a numerical technique that was presented. Baleanu et al. [28] recently developed constant-proportional Caputo (CPC), a hybrid fractional operator that is more flexible than Caputo's fractional derivative operator. Using the CPC operator and modified parameters, the new stochastic fractional coronavirus model was successfully constructed in [29]. They developed a novel method known as the CPC-Milstein approach to solve the hybrid stochastic fractional-order system. The constant-proportional Caputo (CPC) operator was used to analytically assess a nonlinear fractional-order smoking problem, which is a significant challenge in applied sciences, in [30]. Using the Atangana–Baleanu technique, the dynamical behavior of the fractal-fractional HBV model with modified vaccination effects was investigated in [31]. They discovered that the fractal-fractional operators produce more flexible results and gave numerous graphical examples. In [32], a fractal-fractional model for the syphilis disease was created using the Mittag–Leffler kernel. The fractional-order system was analyzed both qualitatively and quantitatively. In order to satisfy the requirements for the existence and uniqueness of the exact solution, fixed point theory and the Lipschitz condition were also applied. Numerical simulations demonstrating the impact of fractional-order derivatives on the fluctuations of syphilis transmission within the human population provided reinforcement for the analytical solution. In a different work, the mathematical model of the varicella-zoster virus was investigated using the Mittag–Leffler fractional operator [33]. To test the well-posedness of the proposed fractional-order model, the existence requirement, positive solution, Hyers–Ulam stability, and boundedness of outcomes were obtained. In order to demonstrate the validity of the discovered results, a few numerical illustrations for the suggested model of different fractional orders were provided using the generalized Adams–Bashforth–Moulton technique.

As a result of the preceding discussion, this article proposes a fractional-order model for modeling Lassa fever transmission using the Caputo fractional operator. In comparison to standard integer-order derivatives, the Caputo fractional derivative concept performs better. This makes our model and procedure different from the Lassa fever model that has been published in the past up to this point. Section "2" offers a generalized version of the model and an analysis of the description of the suggested model. Also, we describe the fundamentals of the proposed fractional operator. Section "3" investigates the well-posedness of the proposed model and qualitative aspects such as equilibrium states, reproductive number, existence and originality, and global stability. Section "4" contains the

numerical solution to the fractional Lassa fever model with a power law kernel. The numerical simulations, findings, and conclusions will be covered in Sections "5" and "6".

2. Lassa Fever (LF) Model

We take into account the model created in [2] that describes the epidemiological dynamics of LF transmission by using a traditional SEIR-typed model to analyze the transmission dynamics and control strategies of the LF outbreak in Nigeria, considering moderate and serious instances plus ecological transmission. The entire human population at time t, $\mathbf{N}_h(t)$, is separated into subpopulations of susceptible $\mathbf{S}_h(t)$, exposed $\mathbf{E}(t)$, symptomatically moderate infected persons $\mathbf{I}_m(t)$, and symptomatically severe infected persons $\mathbf{I}_s(t)$, and persons who were hospitalized $\mathbf{H}(t)$ and then recovered $\mathbf{R}_h(t)$, so that

$$\mathbf{N}_h(t) = \mathbf{S}_h(t) + \mathbf{E}(t) + \mathbf{I}_m(t) + \mathbf{I}_s(t) + \mathbf{H}(t) + \mathbf{R}_h(t). \quad (1)$$

The two subpopulations of susceptible and infectious rodents make up the entire rodent (reservoir) population at time t, represented by $\mathbf{N}_r(t)$. Thus, we have

$$\mathbf{N}_r(t) = \mathbf{S}_r(t) + \mathbf{I}_r(t). \quad (2)$$

Additionally, let $\mathbf{V}(t)$ stands for the concentration of LASV in the environment at time t, such that both human beings and rodents are vulnerable to LF when coming into touch with a polluted environment. The frequency of the disease, reservoir population, human behaviors, and seasonality all play a significant role in the spread of LF [2]. We explain the model using the successive systems of ordinary differential equations that are nonlinear and provided by

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{S}_h(t)}{\mathrm{d}t} &= \Phi_h + \varphi\mathbf{R} - \left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{vh}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_h - \eta_h\mathbf{S}_h,\\
\frac{\mathrm{d}\mathbf{E}(t)}{\mathrm{d}t} &= \left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{vh}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_h - (\lambda + \eta_h)\mathbf{E},\\
\frac{\mathrm{d}\mathbf{I}_m(t)}{\mathrm{d}t} &= \lambda\mathbf{E} - \left(\chi + \kappa_{\mathbf{I}_m} + \beta_{\mathbf{I}_m} + \eta_h\right)\mathbf{I}_m,\\
\frac{\mathrm{d}\mathbf{I}_s(t)}{\mathrm{d}t} &= \chi\mathbf{I}_m - \left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \mu_{\mathbf{I}_s} + \eta_h\right)\mathbf{I}_s,\\
\frac{\mathrm{d}\mathbf{H}(t)}{\mathrm{d}t} &= \kappa_{\mathbf{I}_m}\mathbf{I}_m + \kappa_{\mathbf{I}_s}\mathbf{I}_s - (\beta_h + \mu_h + \eta_h)\mathbf{H},\\
\frac{\mathrm{d}\mathbf{R}(t)}{\mathrm{d}t} &= \beta_{\mathbf{I}_m}\mathbf{I}_m + \beta_{\mathbf{I}_s}\mathbf{I}_s + \beta_h\mathbf{H} - (\varphi + \eta_h)\mathbf{R},\\
\frac{\mathrm{d}\mathbf{V}(t)}{\mathrm{d}t} &= \alpha_h(\mathbf{I}_m + \mathbf{I}_s) + \alpha_r\mathbf{I}_r - \vartheta\mathbf{V},\\
\frac{\mathrm{d}\mathbf{S}_r(t)}{\mathrm{d}t} &= \Phi_r - \left(\frac{\xi_r\mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{vr}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_r - \eta_r\mathbf{S}_r,\\
\frac{\mathrm{d}\mathbf{I}_r(t)}{\mathrm{d}t} &= \left(\frac{\xi_r\mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{vr}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_r - \eta_r\mathbf{I}_r.
\end{aligned} \quad (3)$$

The human recruitment rate is shown by Φ_h, and the environment's LASV pathogens' decay rate is indicated by Φ_r. The pace of reversion from $\mathbf{R}$ to $\mathbf{S}_h$ is represented by φ. $\xi_{\mathbf{I}_m}$, $\xi_{\mathbf{I}_s}$, ξ_{vh}, $\xi_{\mathbf{I}_r}$, and ξ_{vr} represent transmission rates of human-to-human, Mastomys rat-to-human, human-to-infected surfaces or environments, Mastomys rat-to-Mastomys rat, and Mastomys rat-to-infected surfaces or environments, respectively. In the contaminated environment, the concentration of LASV pathogens is defined by c. λ indicates the progression of an exposed person who develops symptoms of infection. χ is the LF progression rate from $\mathbf{I}_m$ to $\mathbf{I}_s$. Hospitalization rate from $\mathbf{I}_m$ is provided by $\kappa_{\mathbf{I}_m}$, whereas hospitalization rate from $\mathbf{I}_s$ is provided by $\kappa_{\mathbf{I}_s}$. The recovery rates for patients who are asymptomatic, symptomatic, and hospitalized are represented by the parameters $\beta_{\mathbf{I}_m}$, $\beta_{\mathbf{I}_s}$, and β_h, respectively. The death rates caused by LF are $\mu_{\mathbf{I}_s}$ and μ_h. The natural death rates for the populations of humans and rodents are η_h and η_r, respectively. α_h represents the rate at which an infected human sheds the virus into the environment, while α_r represents the rate at which an infected Mastomys rat sheds the virus into the environment. The rodents' maximum growth rate is shown by the symbol ϑ.

A close connection exists between fractional calculus and the dynamics of intricate real-world phenomena. We will review some recent and practical results in fractional calculus here.

Definition 1 (see [34]). The Caputo fractional derivative of a differentiable function $\Psi(t)$ to order $\rho \in (0,1)$ with starting point $t = 0$ is defined as follows:

$${}_0^C D_t^\rho \Psi(t) = \frac{1}{\Gamma(1-\rho)}\int_0^t \Psi'(\zeta)(t-\zeta)^{-\rho}\mathrm{d}\zeta. \quad (4)$$

Definition 2. If $\Psi(t)$ is an integrable function with $0 < \rho < 1$, the fractional integral is specified as follows [35]:

$${}_0^C I_t^\rho \Psi(t) = \frac{1}{\Gamma(\rho)}\int_0^t (t-\zeta)^{\rho-1}\Psi(\zeta)\mathrm{d}\zeta. \quad (5)$$

Remark 3. A fixed point $\omega^\star$ is perceived to be the equilibrium point of the Caputo system

$${}_0^C D_t^\rho \Psi(t) = \Psi(t, \omega(t)), \quad \rho \in (0,1), \quad (6)$$

if and only if $\Psi(t, \omega^\star) = 0$.

Lemma 4 (see [36]). *Consider that the function $\sigma(t) \in \mathbb{R}^+$ is differentiable. Then, for $\rho \in (0,1)$,*

$${}_0^C D_t^\rho\left(\sigma(t) - \sigma^* - \sigma^* \ln\frac{\sigma(t)}{\sigma^*}\right) \le \left[1 - \frac{\sigma^*}{\sigma(t)}\right]{}_0^C D_t^\rho \sigma(t), \quad \forall t \ge 0. \quad (7)$$

The order of fractional derivative regulates memory strength, which is important in the gradual development of human epidemiological functions. Greater comprehension of the conduct of models is made possible by the incorporation of memory effects into epidemiological

investigations of real dynamical procedures, which improves our knowledge of how diseases spread. To improve our understanding of the dynamics of the models, we expand the model (3) to the Caputo fractional operator.

$$
\begin{aligned}
{}_0^C D_t^\rho \mathbf{S}_h(t) &= \Phi_h + \varphi \mathbf{R} - \left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_h - \eta_h \mathbf{S}_h,\\
{}_0^C D_t^\rho \mathbf{E}(t) &= \left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_h - (\lambda + \eta_h)\mathbf{E},\\
{}_0^C D_t^\rho \mathbf{I}_m(t) &= \lambda \mathbf{E} - \left(\chi + \kappa_{\mathbf{I}_m} + \beta_{\mathbf{I}_m} + \eta_h\right)\mathbf{I}_m,\\
{}_0^C D_t^\rho \mathbf{I}_s(t) &= \chi \mathbf{I}_m - \left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \mu_{\mathbf{I}_s} + \eta_h\right)\mathbf{I}_s,\\
{}_0^C D_t^\rho \mathbf{H}(t) &= \kappa_{\mathbf{I}_m}\mathbf{I}_m + \kappa_{\mathbf{I}_s}\mathbf{I}_s - (\beta_h + \mu_h + \eta_h)\mathbf{H},\\
{}_0^C D_t^\rho \mathbf{R}(t) &= \beta_{\mathbf{I}_m}\mathbf{I}_m + \beta_{\mathbf{I}_s}\mathbf{I}_s + \beta_h \mathbf{H} - (\varphi + \eta_h)\mathbf{R},\\
{}_0^C D_t^\rho \mathbf{V}(t) &= \alpha_h(\mathbf{I}_m + \mathbf{I}_s) + \alpha_r \mathbf{I}_r - \vartheta \mathbf{V},\\
{}_0^C D_t^\rho \mathbf{S}_r(t) &= \Phi_r - \left(\frac{\xi_r \mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_r - \eta_r \mathbf{S}_r,\\
{}_0^C D_t^\rho \mathbf{I}_r(t) &= \left(\frac{\xi_r \mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_r - \eta_r \mathbf{I}_r,
\end{aligned}
\tag{8}
$$

where ${}^C D^\rho$ represents the Caputo fractional derivative of order $0 < \rho \le 1$, and the corresponding nonnegative initial conditions are such that

$$
\begin{aligned}
&\mathbf{S}_h(0) \ge 0, \mathbf{E}(0) \ge 0, \mathbf{I}_m(0) \ge 0, (0) \ge 0, \mathbf{H}(0) \ge 0,\\
&\mathbf{R}(0) \ge 0, \mathbf{V}(0) \ge 0, \mathbf{S}_r(0) \ge 0, \mathbf{I}_r(0) \ge 0.
\end{aligned}
\tag{9}
$$

3. Analysis of the Proposed Model

3.1. Well-Posedness and Positively Invariant Region. We look at the requirements essential to guarantee the favourable outcomes of the system under examination, assuming that they represent settings with realistic values in the real world.

In the case of integer-order derivative, we have for all $t \ge 0$:

$$
\begin{aligned}
&\mathbf{E}(t) \ge \mathbf{E}(0)e^{-(\lambda+\eta_h)t};\ \mathbf{I}_m(t) \ge \mathbf{I}_m(0)e^{-\left(\chi+\kappa_{\mathbf{I}_m}+\beta_{\mathbf{I}_m}+\eta_h\right)t};\\
&\mathbf{H}(t) \ge \mathbf{H}(0)e^{-(\beta_h+\mu_h+\eta_h)t};\ \mathbf{I}_s(t) \ge \mathbf{I}_s(0)e^{-\left(\kappa_{\mathbf{I}_s}+\beta_{\mathbf{I}_s}+\mu_{\mathbf{I}_s}+\eta_h\right)t};\\
&\mathbf{R}(t) \ge \mathbf{R}(0)e^{-(\varphi+\eta_h)t};\ \mathbf{V}(t) \ge \mathbf{V}(0)e^{-(\vartheta)t};\\
&\mathbf{I}_r(t) \ge \mathbf{I}_r(0)e^{-(\eta_r)t}.
\end{aligned}
\tag{10}
$$

Consider the norm

$$
\|\mathbf{f}\|_\infty = \sup_{t \in D_{\mathbf{f}}} |\mathbf{f}(t)|,
\tag{11}
$$

where $D_{\mathbf{f}}$ is the domain of $\mathbf{f}$.

We have for the function $\mathbf{S}_h(t)$

$$
\begin{aligned}
{}_0^C D_t^\rho \mathbf{S}_h(t) &= \Phi_h + \varphi \mathbf{R} - \left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_h - \eta_h \mathbf{S}_h\\
&\ge -\left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_h - \eta_h \mathbf{S}_h \ge -\left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}}\mathbf{V}}{c+\mathbf{V}} + \eta_h\right)\mathbf{S}_h\\
&\ge -\left(\frac{\xi_{\mathbf{I}_m}\sup_{t\in D_{\mathbf{I}_m}}|\mathbf{I}_m| + \xi_{\mathbf{I}_s}\sup_{t\in D_{\mathbf{I}_s}}|\mathbf{I}_s}{\sup_{t\in D_{\mathbf{N}_h}}|\mathbf{N}_h|} + \frac{\xi_{\mathrm{vh}}\sup_{t\in D_{\mathrm{V}}}|\mathbf{V}|}{c+\sup_{t\in D_{\mathrm{V}}|\mathbf{V}|}} + \eta_h\right)\mathbf{S}_h\\
&\ge -\left(\frac{\xi_{\mathbf{I}_m}\|\mathbf{I}_m\|_\infty + \xi_{\mathbf{I}_s}\|\mathbf{I}_s\|_\infty}{\|\mathbf{N}_h\|_\infty} + \frac{\xi_{vh}\|\mathbf{V}\|_\infty}{c+\|\mathbf{V}\|_\infty} + \eta_h\right)\mathbf{S}_h\\
&\Longrightarrow \mathbf{S}_h(t)\\
&\ge \mathbf{S}_h(0)e^{-\left(\xi_{\mathbf{I}_m}\|\mathbf{I}_m\|_\infty + \xi_{\mathbf{I}_s}\|\mathbf{I}_s\|_\infty/\|\mathbf{N}_h\|_\infty + \xi_{vh}\|\mathbf{V}\|_\infty/c+\|\mathbf{V}\|_\infty + \eta_h\right)t}, \quad \forall t \ge 0.
\end{aligned}
\tag{12}
$$

Also, we have the function $\mathbf{S}_r(t)$

$$
\begin{aligned}
{}_0^C D_t^\rho \mathbf{S}_r(t) &= \Phi_r - \left(\frac{\xi_r \mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}} \mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_r - \eta_r \mathbf{S}_r \\
&\geq -\left(\frac{\xi_r \mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}} \mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_r - \eta_r \mathbf{S}_r \geq -\left(\frac{\xi_r \mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}} \mathbf{V}}{c+\mathbf{V}} + \eta_r\right)\mathbf{S}_r \\
&\geq -\left(\frac{\xi_r \sup_{t\in D_{\mathrm{I}_r}} |\mathbf{I}_r|}{\sup_{t\in D_{\mathrm{N}_r}} |\mathbf{N}_r|} + \frac{\xi_{\mathrm{vr}} \sup_{t\in D_{\mathrm{V}}} |\mathbf{V}|}{c + \sup_{t\in D_{\mathrm{V}}|\mathbf{V}|}} + \eta_r\right)\mathbf{S}_r \\
&\geq -\left(\frac{\xi_r \|\mathbf{I}_r\|_\infty}{\|\mathbf{N}_r\|_\infty} + \frac{\xi_{\mathrm{vr}} \|\mathbf{V}\|_\infty}{c+\|\mathbf{V}\|_\infty} + \eta_r\right)\mathbf{S}_r \\
&\Longrightarrow \mathbf{S}_r(t) \geq \mathbf{S}_r(0) e^{-\left(\xi_r \|\mathbf{I}_r\|_\infty / \|\mathbf{N}_r\|_\infty + \xi_{\mathrm{vr}} \|\mathbf{V}\|_\infty / c + \|\mathbf{V}\|_\infty + \eta_r\right)t}, \quad \forall t \geq 0.
\end{aligned}
\tag{13}
$$

The positive solutions under fractional Caputo derivative are

$$
\begin{aligned}
&\mathbf{S}_h(t) \geq \mathbf{S}_h(0)\, E_\rho\left[-\left(\frac{\xi_{\mathbf{I}_m}\|\mathbf{I}_m\|_\infty + \xi_{\mathbf{I}_s}\|\mathbf{I}_s\|_\infty}{\|\mathbf{N}_h\|_\infty} + \frac{\xi_{\mathrm{vh}}\|\mathbf{V}\|_\infty}{c+\|\mathbf{V}\|_\infty} + \eta_h\right)t^\rho\right]; \\
&\mathbf{E}(t) \geq \mathbf{E}(0)\, E_\rho\left[-(\lambda + \eta_h)t^\rho\right]; \\
&\mathbf{I}_m(t) \geq \mathbf{I}_m(0)\, E_\rho\left[-\left(\chi + \kappa_{\mathbf{I}_m} + \beta_{\mathbf{I}_m} + \eta_h\right)t^\rho\right]; \\
&\mathbf{H}(t) \geq \mathbf{H}(0)\, E_\rho\left[-(\beta_h + \mu_h + \eta_h)t^\rho\right]; \\
&\mathbf{I}_s(t) \geq \mathbf{I}_s(0)\, E_\rho\left[-\left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \mu_{\mathbf{I}_s} + \eta_h\right)t^\rho\right]; \\
&\mathbf{R}(t) \geq \mathbf{R}(0)\, E_\rho\left[-(\varphi + \eta_h)t^\rho\right]; \\
&\mathbf{V}(t) \geq \mathbf{V}(0)\, E_\rho\left[-(\vartheta)t^\rho\right]; \\
&\mathbf{I}_r(t) \geq \mathbf{I}_r(0)\, E_\rho\left[-(\eta_r)t^\rho\right]; \\
&\mathbf{S}_r(t) \geq \mathbf{S}_r(0)\, E_\rho\left[-\left(\frac{\xi_r\|\mathbf{I}_r\|_\infty}{\|\mathbf{N}_r\|_\infty} + \frac{\xi_{vr}\|\mathbf{V}\|_\infty}{c+\|\mathbf{V}\|_\infty} + \eta_r\right)t^\rho\right]; \quad \forall t \geq 0,
\end{aligned}
\tag{14}
$$

where E_ρ represents Mittag–Leffler function.

We shall now show that system (8)'s feasibility region is positively invariant.

Lemma 5. *The feasible region* $\mho \in \mathbb{R}_+^9$

$$
\mho = \left\{\left(\mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r\right) \in \mathbb{R}_+^9 : 0 \leq \mathbf{N}_h(t) \leq \frac{\Phi_h}{\eta_h}, \mathbf{V}(t) \leq \frac{\alpha_h(\Phi_h/\eta_h) + \alpha_r(\Phi_r/\eta_h)}{\vartheta}, 0 \leq \mathbf{N}_r(t) \leq \frac{\Phi_r}{\eta_r}\right\}, \tag{15}
$$

attracts all solutions of system (8) and is positively invariant subjected to nonnegative initial constraints for the proposed system in $\mathbb{R}_+^9$.

Proof. We shall demonstrate the system's (8) positive solution, and the outcomes are given as follows:

$$\begin{aligned}
\left[{}_0^C D_t^\rho \mathbf{S}_h(t)\right]_{\mathbf{S}_h=0} &= \Phi_h + \varphi \mathbf{R}0,\\
\left[{}_0^C D_t^\rho \mathbf{E}(t)\right]_{\mathbf{E}=0} &= \left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_h \geq 0,\\
\left[{}_0^C D_t^\rho \mathbf{I}_m(t)\right]_{\mathbf{I}_m=0} &= \lambda \mathbf{E} \geq 0,\\
\left[{}_0^C D_t^\rho \mathbf{I}_s(t)\right]_{\mathbf{I}_s=0} &= \chi \mathbf{I}_m \geq 0,\\
\left[{}_0^C D_t^\rho \mathbf{H}(t)\right]_{\mathbf{H}=0} &= \kappa_{\mathbf{I}_m}\mathbf{I}_m + \kappa_{\mathbf{I}_s}\mathbf{I}_s \geq 0,\\
\left[{}_0^C D_t^\rho \mathbf{R}(t)\right]_{\mathbf{R}=0} &= \beta_{\mathbf{I}_m}\mathbf{I}_m + \beta_{\mathbf{I}_s}\mathbf{I}_s + \beta_h \mathbf{H} \geq 0,\\
\left[{}_0^C D_t^\rho \mathbf{V}(t)\right]_{\mathbf{V}=0} &= \alpha_h\left(\mathbf{I}_m + \mathbf{I}_s\right) + \alpha_r \mathbf{I}_r \geq 0,\\
\left[{}_0^C D_t^\rho \mathbf{S}_r(t)\right]_{\mathbf{S}_r=0} &= \Phi_r \geq 0,\\
\left[{}_0^C D_t^\rho \mathbf{I}_r(t)\right]_{\mathbf{I}_r=0} &= \left(\frac{\xi_r \mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_r \geq 0.
\end{aligned} \tag{16}$$

The system (16) indicates that the vector field lies in the region $\mathbb{R}_+^9$ on each hyperplane enclosing the nonnegative orthant with $t \geq 0$.

We now take into account the rate of change in the total populations of humans and rodents. We begin with the human population as

$$\begin{cases}
{}_0^C D_t^\rho \mathbf{N}_h(t) = {}_0^C D_t^\rho \mathbf{S}_h(t) + {}_0^C D_t^\rho \mathbf{E}(t) + {}_0^C D_t^\rho \mathbf{I}_m(t) + {}_0^C D_t^\rho \mathbf{I}_s(t) + {}_0^C D_t^\rho \mathbf{H}(t) + {}_0^C D_t^\rho \mathbf{R}(t),\\
{}_0^C D_t^\rho \mathbf{N}_h(t) = \Phi_h - \mu_{\mathbf{I}_s}\mathbf{I}_s - \mu_h \mathbf{H} - \eta_h\left(\mathbf{S}_h + \mathbf{E} + \mathbf{I}_m + \mathbf{I}_s + \mathbf{H} + \mathbf{R}\right),\\
{}_0^C D_t^\rho \mathbf{N}_h(t) \geq \Phi_h - \eta_h \mathbf{N}_h.
\end{cases} \tag{17}$$

Furthermore, this follows $\lim_{t\longrightarrow\infty}\sup\ [\mathbf{N}_h(t)] \leq \Phi_h/\eta_h$. Hence,

$$0 \leq \mathbf{N}_h(t) \leq \frac{\Phi_h}{\eta_h} \text{ for } 0 \leq \mathbf{N}_h(0) \leq \frac{\Phi_h}{\eta_h}, \quad \forall t \geq 0. \tag{18}$$

Therefore, the region $\mho_h$,

$$\mho_h = \left\{\left(\mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}\right) \in \mathbb{R}_+^6 \colon 0 \leq \mathbf{N}_h(t) \leq \frac{\Phi_h}{\eta_h}\right\}, \tag{19}$$

is a positively invariant region.

Now, we add the rodent population as

$$\begin{cases}
{}_0^C D_t^\rho \mathbf{N}_r(t) = {}_0^C D_t^\rho \mathbf{S}_r + {}_0^C D_t^\rho \mathbf{I}_r,\\
{}_0^C D_t^\rho \mathbf{N}_r(t) = \Phi_r - \eta_r\left(\mathbf{S}_r + \mathbf{I}_r\right) = \Phi_r - \eta_r \mathbf{N}_r.
\end{cases} \tag{20}$$

We have $\lim_{t\longrightarrow\infty}\sup\ [\mathbf{N}_r(t)] \leq \Phi_r/\eta_r$. Therefore,

$$0 \leq \mathbf{N}_h(t) \leq \frac{\Phi_r}{\eta_r} \text{ for } 0 \leq \mathbf{N}_r(0) \leq \frac{\Phi_r}{\eta_r}, \quad \forall t \geq 0. \tag{21}$$

Therefore, we can say that the region $\mho_r$,

$$\mho_r = \left\{\left(\mathbf{S}_r, \mathbf{I}_r\right) \in \mathbb{R}_+^2 \colon 0 \leq \mathbf{N}_r(t) \leq \frac{\Phi_r}{\eta_r}\right\}, \tag{22}$$

is a positively invariant region.

For the concentration of virus in the environment, we have

$${}_0^C D_t^\rho \mathbf{V}(t) = \alpha_h\left(\mathbf{I}_m + \mathbf{I}_s\right) + \alpha_r \mathbf{I}_r - \vartheta \mathbf{V}. \tag{23}$$

From equations (18) and (21), we can write

$${}_0^C D_t^\rho \mathbf{V}(t) \leq \alpha_h\left(\frac{\Phi_h}{\eta_h}\right) + \alpha_r\left(\frac{\Phi_r}{\eta_r}\right) - \vartheta \mathbf{V}. \tag{24}$$

This implies that $\lim_{t\longrightarrow\infty}\sup\ [\mathbf{V}(t)] \leq \alpha_h(\Phi_h/\eta_h) + \alpha_r(\Phi_r/\eta_r)/\vartheta$. Hence, the region $\mho_v$,

$$\mho_v = \left\{(\mathbf{V}) \in \mathbb{R}_+ \colon 0 \leq \mathbf{V}(t) \leq \frac{\alpha_h(\Phi_h/\eta_h) + \alpha_r(\Phi_r/\eta_r)}{\vartheta}\right\}, \tag{25}$$

is a positively invariant region.

Consequently, system (8)'s biologically feasible region is given by

$$\mho = \mho_h \times \mho_r \times \mho_v \in \mathbb{R}_+^6 \times \mathbb{R}_+^2 \times \mathbb{R}_+. \tag{26}$$

Hence, for every $t > 0$, every solution of the fractional model (8) with initial constraints in $\mho$ continues to exist in $\mho$. Therefore, we can investigate our model (8) in the feasible region $\mho$. □

3.2. Equilibrium Points Analysis. The two different kinds of equilibrium points are disease-present equilibrium points and complaint-free equilibrium points. To locate them, the right-hand side of the system is set to zero.

(1) We have disease-free equilibrium points $\mathscr{P}^0$:

$$\begin{aligned}\mathscr{P}^0 &= \left\{\mathbf{S}_h^0, \mathbf{E}^0, \mathbf{I}_m^0, \mathbf{I}_s^0, \mathbf{H}^0, \mathbf{R}^0, \mathbf{V}^0, \mathbf{S}_r^0, \mathbf{I}_r^0\right\} \\ &= \left\{\frac{\Phi_h}{\eta_h}, 0, 0, 0, 0, 0, 0, \frac{\Phi_r}{\eta_r}, 0\right\}.\end{aligned} \tag{27}$$

(2) We obtain an endemic equilibrium state after performing certain algebraic calculations by setting the vector field of system (8) to zero. The endemic equilibrium point is

$$\mathscr{P}^* = \left\{\mathbf{S}_h^*, \mathbf{E}^*, \mathbf{I}_m^*, \mathbf{I}_s^*, \mathbf{H}^*, \mathbf{R}^*, \mathbf{V}^*, \mathbf{S}_r^*, \mathbf{I}_r^*\right\}. \tag{28}$$

The endemic equilibrium points $\mathscr{P}^*$ in terms of $\mathbf{E}^*$, $\mathfrak{M}_h^*$, and $\mathfrak{M}_r^*$, as stated in [2], are

$$\begin{aligned}
\mathbf{S}_h^* &= \frac{\left[\left(\mathbf{E}^*\varphi\lambda\beta_m + \Phi_h\mathfrak{I}_1\mathfrak{I}_4\right)\mathfrak{I}_3 + \mathbf{E}^*\kappa_m\beta_h\varphi\lambda\right]\mathfrak{I}_2 + \mathbf{E}^*\varphi\chi\lambda\left(\beta_h\kappa_s + \beta_s\mathfrak{I}_3\right)}{\mathfrak{I}_1\mathfrak{I}_2\mathfrak{I}_3\mathfrak{I}_4\left(\mathfrak{M}_h^* + \eta_h\right)}, \\
\mathbf{I}_m^* &= \frac{\lambda\mathbf{E}^*}{\mathfrak{I}_1}, \mathbf{I}_s^* = \frac{\lambda\chi\mathbf{E}^*}{\mathfrak{I}_1\mathfrak{I}_2}, \mathbf{H}^* = \left(\kappa_{\mathbf{I}_m} + \frac{\kappa_{\mathbf{I}_m}\chi}{\mathfrak{I}_1}\right)\frac{\lambda\mathbf{E}^*}{\mathfrak{I}_1\mathfrak{I}_3}, \\
\mathbf{R}^* &= \frac{\lambda\mathbf{E}^*\chi\left(\beta_h\kappa_s + \beta_s\mathfrak{I}_3\right) + \lambda\mathbf{E}^*\mathfrak{I}_2\left(\beta_h\kappa_m + \beta_m\mathfrak{I}_3\right)}{\mathfrak{I}_1\mathfrak{I}_2\mathfrak{I}_3\mathfrak{I}_4}, \\
\mathbf{V}^* &= \frac{\lambda\mathbf{E}^*\alpha_h\left(\chi + \mathfrak{I}_2\right)\eta_r\left(\eta_r + \mathfrak{M}_r^*\right) + \alpha_r\Phi_r\mathfrak{M}_r^*\mathfrak{I}_1\mathfrak{I}_2}{\mathfrak{I}_2\mathfrak{I}_3\eta_r\left(\mathfrak{M}_r^* + \eta_r\right)}, \\
\mathbf{S}_r^* &= \frac{\Phi_r}{\mathfrak{M}_r^* + \eta_r}, \mathbf{I}_r^* = \frac{\Phi_r\mathfrak{M}_r^*}{\eta_r\left(\mathfrak{M}_r^* + \eta_r\right)},
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
&\mathfrak{M}_h^* = \frac{\xi_{\mathbf{I}_m}\mathbf{I}_m^* + \xi_{\mathbf{I}_s}\mathbf{I}_s^*}{\mathbf{N}_h^*} + \frac{\xi_{\mathrm{vh}}\mathbf{V}^*}{c + \mathbf{V}^*}, \mathfrak{M}_r^* = \frac{\xi_r\mathbf{I}_r^*}{\mathbf{N}_r^*} + \frac{\xi_{vr}\mathbf{V}^*}{c + \mathbf{V}^*}, \\
&\mathfrak{I}_1 = \chi + \kappa_{\mathbf{I}_m} + \beta_{\mathbf{I}_m} + \eta_h, \mathfrak{I}_2 = \kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \eta_h, \mathfrak{I}_3 = \beta_h + \mu_h + \eta_h, \mathfrak{I}_4 = \varphi + \eta_h.
\end{aligned} \tag{30}$$

3.3. Reproductive Number. The total number of subsequent cases that a typical initial case generates within a susceptible population throughout the infectious phase is known as the reproductive number, and we calculate it by using the next-generation matrix technique. We take into account the following equations to determine the reproduction number:

$$
\begin{aligned}
{}_0^C D_t^\rho \mathbf{E}(t) &= \left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_h - (\lambda + \eta_h)\mathbf{E},\\
{}_0^C D_t^\rho \mathbf{I}_m(t) &= \lambda\mathbf{E} - \left(\chi + \kappa_{\mathbf{I}_m} + \beta_{\mathbf{I}_m} + \eta_h\right)\mathbf{I}_m,\\
{}_0^C D_t^\rho \mathbf{I}_s(t) &= \chi\mathbf{I}_m - \left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \mu_{\mathbf{I}_s} + \eta_h\right)\mathbf{I}_s,\\
{}_0^C D_t^\rho \mathbf{H}(t) &= \kappa_{\mathbf{I}_m}\mathbf{I}_m + \kappa_{\mathbf{I}_s}\mathbf{I}_s - (\beta_h + \mu_h + \eta_h)\mathbf{H},\\
{}_0^C D_t^\rho \mathbf{V}(t) &= \alpha_h(\mathbf{I}_m + \mathbf{I}_s) + \alpha_r\mathbf{I}_r - \vartheta\mathbf{V},\\
{}_0^C D_t^\rho \mathbf{I}_r(t) &= \left(\frac{\xi_r\mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_r - \eta_r\mathbf{I}_r.
\end{aligned}
\tag{31}
$$

Matrix F represents the terms that are new to infection, and matrix V provides the other terms that are transferred.

$$
F = \begin{pmatrix}
0 & \frac{\xi_{\mathbf{I}_m}\eta_h}{\Phi_h} & \frac{\xi_{\mathbf{I}_s}\eta_h}{\Phi_h} & 0 & \frac{\xi_{vh}}{c} & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{\xi_{vr}}{c} & \frac{\xi_r\eta_r}{\Phi_r}
\end{pmatrix},
$$

$$
V = \begin{pmatrix}
\lambda + \eta_h & 0 & 0 & 0 & 0 & 0\\
-\lambda & \chi + \kappa_{\mathbf{I}_m} + \beta_{\mathbf{I}_m} + \eta_h & 0 & 0 & 0 & 0\\
0 & -\chi & \kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \mu_{\mathbf{I}_s} + \eta_h & 0 & 0 & 0\\
0 & -\kappa_{\mathbf{I}_m} & -\kappa_{\mathbf{I}_s} & \beta_h + \mu_h + \eta_h & 0 & 0\\
0 & -\alpha_h & -\alpha_h & 0 & \vartheta & -\alpha_r,\\
0 & 0 & 0 & 0 & 0 & \eta_r
\end{pmatrix}.
\tag{32}
$$

The reproductive number ($\mathscr{R}_0$) of our suggested model is obtained from $\det|F(\mathbf{E}^0)V^{-1} - \lambda I| = 0$.

$$\mathscr{R}_0 = \mathscr{R}_a + \mathscr{R}_b + \mathscr{R}_c, \tag{33}$$

where

$$\begin{cases} \mathscr{R}_a = \dfrac{1}{2\eta_r \vartheta \Theta}, \\ \mathscr{R}_b = \Theta\left(\dfrac{\vartheta \xi_r \eta_r}{\Phi_r} + \dfrac{\alpha_r \xi_{\mathrm{vr}}}{c}\right) + \eta_r \lambda\left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \eta_h\right)\left(\dfrac{\vartheta \xi_m \eta_h}{\Phi_h} + \dfrac{\alpha_h \xi_{\mathrm{vh}}}{c}\right) + \chi \eta_r \lambda\left(\dfrac{\vartheta \xi_s \eta_h}{\Phi_h} + \dfrac{\alpha_h \xi_{\mathrm{vh}}}{c}\right), \\ \mathscr{R}_c = \sqrt{\mathscr{R}_{b_1} + \mathscr{R}_{b_2} + \mathscr{R}_{b_3}}, \\ \Theta = (\lambda + \eta_h)\left(\chi + \kappa_{\mathbf{I}_m} + \beta_{\mathbf{I}_m} + \eta_h\right)\left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \eta_h\right), \\ \mathscr{R}_{c_1} = \left[\dfrac{\eta_h \eta_r \lambda}{\Phi_h}\left(\chi \xi_s + \xi_m\left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \eta_h\right)\right) - \Theta \dfrac{\xi_r \eta_r}{\Phi_r}\right]^2 \vartheta^2, \\ \mathscr{R}_{c_2} = 2\vartheta\left[\dfrac{\eta_r \lambda \xi_{\mathrm{vh}}}{c}\left(\chi + \xi_m\left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \eta_h\right)\right) - \Theta \dfrac{\xi_{\mathrm{vr}} \alpha_r}{c}\right]\left(\dfrac{\eta_h \eta_r \lambda}{\Phi_h}\left(\chi \xi_s + \xi_m\left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \eta_h\right)\right) - \Theta \dfrac{\xi_r \eta_r}{\Phi_r}\right), \\ \mathscr{R}_{c_3} = \left[\dfrac{\eta_r \lambda \xi_{\mathrm{vh}}}{c}\left(\chi + \left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \eta_h\right)\right) + \Theta \dfrac{\alpha_r \xi_{\mathrm{vr}}}{c}\right]^2. \end{cases} \tag{34}$$

3.4. Existence and Uniqueness of Solutions. We show that system (8) under examination has solutions by executing fixed-point results. Think about the function:

$$\begin{aligned} \mathscr{G}_1(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \Phi_h + \varphi \mathbf{R} - \left(\frac{\xi_{\mathbf{I}_m} \mathbf{I}_m + \xi_{\mathbf{I}_s} \mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}} \mathbf{V}}{c + \mathbf{V}}\right)\mathbf{S}_h - \eta_h \mathbf{S}_h; \\ \mathscr{G}_2(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \left(\frac{\xi_{\mathbf{I}_m} \mathbf{I}_m + \xi_{\mathbf{I}_s} \mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}} \mathbf{V}}{c + \mathbf{V}}\right)\mathbf{S}_h - (\lambda + \eta_h)\mathbf{E}; \\ \mathscr{G}_3(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \lambda \mathbf{E} - \left(\chi + \kappa_{\mathbf{I}_m} + \beta_{\mathbf{I}_m} + \eta_h\right)\mathbf{I}_m; \\ \mathscr{G}_4(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \chi \mathbf{I}_m - \left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \mu_{\mathbf{I}_s} + \eta_h\right)\mathbf{I}_s; \\ \mathscr{G}_5(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \kappa_{\mathbf{I}_m} \mathbf{I}_m + \kappa_{\mathbf{I}_s} \mathbf{I}_s - (\beta_h + \mu_h + \eta_h)\mathbf{H}; \\ \mathscr{G}_6(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \beta_{\mathbf{I}_m} \mathbf{I}_m + \beta_{\mathbf{I}_s} \mathbf{I}_s + \beta_h \mathbf{H} - (\varphi + \eta_h)\mathbf{R}; \\ \mathscr{G}_7(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \alpha_h(\mathbf{I}_m + \mathbf{I}_s) + \alpha_r \mathbf{I}_r - \vartheta \mathbf{V}; \\ \mathscr{G}_8(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \Phi_r - \left(\frac{\xi_r \mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}} \mathbf{V}}{c + \mathbf{V}}\right)\mathbf{S}_r - \eta_r \mathbf{S}_r; \\ \mathscr{G}_9(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \left(\frac{\xi_r \mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}} \mathbf{V}}{c + \mathbf{V}}\right)\mathbf{S}_r - \eta_r \mathbf{I}_r. \end{aligned} \tag{35}$$

Establish a Banach space $A[0,\mathbb{T}] = \mathscr{B}$ under the norm

$$\|\mathscr{G}\| = \operatorname*{Sup}_{t\in[0,\mathbb{T}]}\left[\left|\mathbf{S}_h(t)\right| + \left|\mathbf{E}(t)\right| + \left|\mathbf{I}_m(t)\right| + \left|\mathbf{I}_s(t)\right| + \left|\mathbf{H}(t)\right| + \left|\mathbf{R}(t)\right| + \left|\mathbf{V}(t)\right| + \left|\mathbf{S}_r(t)\right| + \left|\mathbf{I}_r(t)\right|\right], \tag{36}$$

where

$$\varpi(t) = \begin{cases} \mathbf{S}_h(t), \\ \mathbf{E}(t), \\ \mathbf{I}_m(t), \\ \mathbf{I}_s(t), \\ \mathbf{H}(t), \\ \mathbf{R}(t), \\ \mathbf{V}(t), \\ \mathbf{S}_r(t), \\ \mathbf{I}_r(t). \end{cases} \varpi_0(t) = \begin{cases} \mathbf{S}_h(0), \\ \mathbf{E}(0), \\ \mathbf{I}_m(0), \\ \mathbf{I}_s(0), \\ \mathbf{H}(0), \\ \mathbf{R}(0), \\ \mathbf{V}(0), \\ \mathbf{S}_r(0), \\ \mathbf{I}_r(0). \end{cases} = \begin{cases} \mathscr{G}_1(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r), \\ \mathscr{G}_2(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r), \\ \mathscr{G}_3(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r), \\ \mathscr{G}_4(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r), \\ \mathscr{G}_5(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r), \\ \mathscr{G}_6(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r), \\ \mathscr{G}_7(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r), \\ \mathscr{G}_8(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r), \\ \mathscr{G}_9(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r). \end{cases} \tag{37}$$

From the aforementioned collection, we can denote the system (8) as

$$\begin{cases} {}_0^C D_t^\rho \Psi(t) = \varpi(t,\Psi(t)), \quad t \in [0,\mathbb{T}], \\ \Psi(0) = \Psi_0. \end{cases} \tag{38}$$

Equation (38) can be expressed as follows:

$$\Psi(t) = \Psi_0 + \frac{1}{\Gamma(\rho)}\int_0^t (t-\zeta)^{\rho-1}\varpi(\zeta,\Psi(\zeta))\mathrm{d}\zeta, \quad t \in [0,\mathbb{T}]. \tag{39}$$

The following presumptions are made in order to demonstrate that the solution exists:

$$\{\mathbf{y}_1\}: \exists \text{ constants } \mathscr{X}_K, \mathscr{X}_M > 0$$
$$|\varpi(t,\Psi(t))| \le \mathscr{X}_K|\Psi(t)|^w + \mathscr{X}_M,$$
$$\{\mathbf{y}_2\}: \text{for each } \Psi, \overline{\Psi} \ \exists \text{ constant } \mathscr{X}_L > 0$$
$$|\varpi(t,\Psi(t))| - |\varpi(t,\overline{\Psi}(t))| \le \mathscr{X}_L\|\Psi(t) - \overline{\Psi}(t)\|. \tag{40}$$

Establish an operator $\wp$: $\mathscr{B} \longrightarrow \mathscr{B}$ as

$$\wp\Psi(t) = \Psi_0 + \frac{1}{\Gamma(\rho)}\int_0^t (t-\zeta)^{\rho-1}\varpi(\zeta,\Psi(\zeta))\mathrm{d}\zeta. \tag{41}$$

Theorem 6. *When both of the presumptions $\{\mathbf{y}_1\}$ and $\{\mathbf{y}_2\}$ are correct, it is confirmed that the problem (33) contains a minimum of one fixed point, which suggests that the system under consideration also possesses a minimum of one solution.*

Proof. We proceed as follows:

(i) At first, pretend that $\wp$ is continuous. Since ϖ is continuous, hence, assume that $\varpi(\zeta,\Psi(\zeta))$ is also continuous. Furthermore, if for $\Psi, \Psi_k \in \mathbb{G}$ $\exists$ $\Psi_k \longrightarrow \Psi$, then we find $\wp\Psi_k \longrightarrow \wp\Psi$. For this, we consider

$$\begin{aligned} \|\wp\Psi_k \longrightarrow \wp\Psi\| &= \max_{t\in[0,\mathbb{T}]}\left|\frac{1}{\Gamma(\rho)}\int_0^t (t-\zeta)^{\rho-1}\varpi_k(\zeta,\Psi_k(\zeta))\mathrm{d}\zeta - \frac{1}{\Gamma(\rho)}\int_0^t (t-\zeta)^{\rho-1}\varpi(\zeta,\Psi(\zeta))\mathrm{d}\zeta\right| \\ &= \max_{t\in[0,\mathbb{T}]}\frac{1}{\Gamma(\rho)}\int_0^t \left|(t-\zeta)^{\rho-1}\right|\left|\varpi_k(\zeta,\Psi_k(\zeta))\mathrm{d}\zeta - \varpi(\zeta,\Psi(\zeta))\right|\mathrm{d}\zeta \\ &\le \frac{T^\rho}{\Gamma(\rho+1)}\|\varpi_k - \varpi\| \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{42}$$

As ϖ is continuous, hence, $\wp\Psi_k \longrightarrow \wp\Psi$ implies that $\wp$ is also continuous.

(ii) Here, we demonstrate that $\wp$ is bounded for any $\Psi \in \mathbb{G}$. For this purpose, assume that $\wp$ satisfies the growth requirement.

$$\begin{aligned}\|\wp\Psi\| &= \max_{t\in[0,\mathbb{T}]}\left|\Psi_0 + \frac{1}{\Gamma(\rho)}\int_0^t (t-\zeta)^{\rho-1}\varpi(\zeta,\Psi(\zeta))\mathrm{d}\zeta\right| \\ &\le |\Psi_0| + \max_{t\in[0,\mathbb{T}]}\frac{1}{\Gamma(\rho)}\int_0^t \left|(t-\zeta)^{\rho-1}\right| |\varpi(\zeta,\Psi(\zeta))|\mathrm{d}\zeta \le |\Psi_0| + \frac{T^\rho}{\Gamma(\rho+1)}\left(\mathcal{X}_K\|\Psi_0\|^w + \mathcal{X}_M\right).\end{aligned} \tag{43}$$

Here, we suppose that J is a bounded subset of $\mathbb{G}$, and we must demonstrate that $\wp$J has the same property. In order to get where we are going, we suppose that for any $\Psi \in$ J, J is bounded such that there exists a constant $\mathcal{X}_w \ge 0$, where

$$\|\Psi\| \le \mathcal{X}_w, \quad \forall\, \Psi \in \mathrm{J}. \tag{44}$$

Additionally, considering the growth condition, we have

$$\begin{aligned}\|\wp\Psi\| &\le |\Psi_0| + \frac{T^\rho}{\Gamma(\rho+1)}\left(\mathcal{X}_K\|\Psi_0\|^w + \mathcal{X}_M\right) \\ &\le |\Psi_0| + \frac{T^\rho}{\Gamma(\rho+1)}\left(\mathcal{X}_K\mathcal{X}_w + \mathcal{X}_M\right).\end{aligned} \tag{45}$$

Therefore, $\wp$ is bounded.

(iii) Now, we try to prove that $\wp$ is equicontinuous. In this regard, consider $t_{\varepsilon_2} \le t_{\varepsilon_1} = [0, \mathbf{T}]$, then

$$\begin{aligned}\left|\wp\Psi(t_{\varepsilon_1}) - \wp\Psi(t_{\varepsilon_2})\right| &= \left|\frac{1}{\Gamma(\rho)}\int_0^t (t_{\varepsilon_1}-\zeta)^{\rho-1}\varpi(\zeta,\Psi(\zeta))\mathrm{d}\zeta - \frac{1}{\Gamma(\rho)}\int_0^t (t_{\varepsilon_2}-\zeta)^{\rho-1}\varpi(\zeta,\Psi(\zeta))\mathrm{d}\zeta\right| \\ &\le \left|\frac{1}{\Gamma(\rho)}\int_0^t (t_{\varepsilon_1}-\zeta)^{\rho-1}\Psi^{\eta-1} - \frac{1}{\Gamma(\rho)}\int_0^t (t_{\varepsilon_2}-\zeta)^{\rho-1}\right| |\varpi(\zeta,\Psi(\zeta))|\mathrm{d}\zeta \\ &\le \frac{T^\rho}{\Gamma(\rho+1)}\left(\mathcal{X}_K\|\Psi\|^w + \mathcal{X}_M\right)\left(t_{\varepsilon_1} - t_{\varepsilon_2}\right).\end{aligned} \tag{46}$$

As a result, $\wp$ is compact according to the Arzela–Ascoli theorem.

(iv) We must demonstrate the boundedness of the set specified below in this phase.

$$\mathfrak{E} = \{\Psi \in \mathbb{G}: \Psi = d\wp\Psi\}, \quad d \in (0,1). \tag{47}$$

Regarding this, assume that $\forall\, t \in [0, \mathbb{T}]$, we have $\Psi \in \mathfrak{E}$, then

$$\|\Psi\| = d\|\wp\Psi\| \le d\left[|\Psi_0| + \frac{T^\rho}{\Gamma(\rho+1)}\left(\mathcal{X}_K\|\Psi\|^w + \mathcal{X}_M\right)\right]. \tag{48}$$

We can then assert that the set specified above is bounded. The model we explored in this work has at least one solution because, according to Schaefer's fixed point theorem, the operator we defined, $\wp$, has at least one fixed point. □

Theorem 7. *The problem (33) has a unique solution if*

$$\frac{T^\rho}{\Gamma(\rho+1)}\mathcal{X}_K < 1. \tag{49}$$

Proof. Suppose that $\Psi, \widetilde{\Psi} \in \mathbb{G}$, then we find

$$\|\wp\Psi - \wp\widetilde{\Psi}\| \le \max_{t\in[0,\mathbb{T}]}\int_0^t \left|\frac{(t-\zeta)^{\rho-1}}{\Gamma(\rho)}\right| \left|\varpi(\zeta,\Psi_k(\zeta)) - \varpi(\zeta,\widetilde{\Psi}(\zeta))\right|\mathrm{d}\zeta \le \frac{T^\rho\mathcal{X}_\varpi}{\Gamma(\rho+1)}\|\Psi - \widetilde{\Psi}\|. \tag{50}$$

We may, therefore, claim that the fixed point is unique and that our solution is unique as a result. □

3.5. Stability Analysis. We now present some findings pertaining to the global stability.

Theorem 8. *In the case where $t > 0$ and $0 < \rho \leq 1$, the model (8) is said to be globally asymptotically stable at the disease-free equilibrium $\mathscr{P}^0 = \{\mathbf{S}_h^0, \mathbf{E}^0, \mathbf{I}_m^0, \mathbf{I}_s^0, \mathbf{H}^0, \mathbf{R}^0, \mathbf{V}^0, \mathbf{S}_r^0, \mathbf{I}_r^0\} = \{\Phi_h/\eta_h, 0, 0, 0, 0, 0, 0, \Phi_r/\eta_r, 0\}$, which is contained in the region $\mho$ if $\mathscr{R}_0 < 1$ and unstable otherwise.*

Proof. Define a Volterra-kind Lyapunov function:

$$\mathfrak{Q} = \left(\mathbf{S}_h - \mathbf{S}_h^0 - \mathbf{S}_h^0 \ln \frac{\mathbf{S}_h}{\mathbf{S}_h^0}\right) + \mathbf{E} + \mathbf{I}_m + \mathbf{I}_s + \mathbf{H} + \mathbf{R} + \mathbf{V} + \left(\mathbf{S}_r - \mathbf{S}_r^0 - \mathbf{S}_r^0 \ln \frac{\mathbf{S}_r}{\mathbf{S}_r^0}\right) + \mathbf{I}_r. \tag{51}$$

Implementing Lemma 4, we get

$$ {}_0^C D_t^\rho \mathfrak{Q} \leq \left(1 - \frac{\mathbf{S}_h^0}{\mathbf{S}_h}\right) {}_0^C D_t^\rho \mathbf{S}_h + {}_0^C D_t^\rho \mathbf{E} + {}_0^C D_t^\rho \mathbf{I}_m + {}_0^C D_t^\rho \mathbf{I}_s + {}_0^C D_t^\rho \mathbf{H} + {}_0^C D_t^\rho \mathbf{R} + {}_0^C D_t^\rho \mathbf{V} + \left(1 - \frac{\mathbf{S}_r^0}{\mathbf{S}_r}\right) {}_0^C D_t^\rho \mathbf{S}_r + {}_0^C D_t^\rho \mathbf{I}_r. \tag{52}$$

Putting the values of ${}_0^C D_t^\rho \mathbf{S}_h$, ${}_0^C D_t^\rho \mathbf{E}$, ${}_0^C D_t^\rho \mathbf{I}_m$, ${}_0^C D_t^\rho \mathbf{I}_s$, ${}_0^C D_t^\rho \mathbf{H}$, ${}_0^C D_t^\rho \mathbf{R}$, ${}_0^C D_t^\rho \mathbf{V}$, ${}_0^C D_t^\rho \mathbf{S}_r$, and ${}_0^C D_t^\rho \mathbf{I}_r$ from (8), we have

$$\begin{aligned} {}_0^C D_t^\rho \mathfrak{Q} \leq{}& \left(1 - \frac{\mathbf{S}_h^0}{\mathbf{S}_h}\right)\left[\Phi_h + \varphi \mathbf{R} - \left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}}\mathbf{V}}{c + \mathbf{V}}\right)\mathbf{S}_h - \eta_h \mathbf{S}_h\right] + \left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}}\mathbf{V}}{c + \mathbf{V}}\right)\mathbf{S}_h \\ &- (\lambda + \eta_h)\mathbf{E} + \lambda \mathbf{E} - \left(\chi + \kappa_{\mathbf{I}_m} + \beta_{\mathbf{I}_m} + \eta_h\right)\mathbf{I}_m + \chi \mathbf{I}_m - \left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \mu_{\mathbf{I}_s} + \eta_h\right)\mathbf{I}_s + \kappa_{\mathbf{I}_m}\mathbf{I}_m + \kappa_{\mathbf{I}_s}\mathbf{I}_s \\ &- (\beta_h + \mu_h + \eta_h)\mathbf{H} + \beta_{\mathbf{I}_m}\mathbf{I}_m + \beta_{\mathbf{I}_s}\mathbf{I}_s + \beta_h \mathbf{H} - (\varphi + \eta_h)\mathbf{R} + \alpha_h(\mathbf{I}_m + \mathbf{I}_s) + \alpha_r \mathbf{I}_r - \vartheta \mathbf{V} \\ &+ \left(1 - \frac{\mathbf{S}_r^0}{\mathbf{S}_r}\right)\left[\Phi_r - \left(\frac{\xi_r \mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}}\mathbf{V}}{c + \mathbf{V}}\right)\mathbf{S}_r - \eta_r \mathbf{S}_r\right] + \left(\frac{\xi_r \mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}}\mathbf{V}}{c + \mathbf{V}}\right)\mathbf{S}_r - \eta_r \mathbf{I}_r, \end{aligned} \tag{53}$$

where ${}_0^C D_t^\rho \mathfrak{Q} \leq 0$ for $\mathscr{R}_0 < 1$ and ${}_0^C D_t^\rho \mathfrak{Q} = 0$ only when $\mathbf{S}_h = \mathbf{S}_h^0$, $\mathbf{E} = \mathbf{E}^0$, $\mathbf{I}_m = \mathbf{I}_m^0$, $\mathbf{I}_s = \mathbf{I}_s^0$, $\mathbf{H} = \mathbf{H}^0$, $\mathbf{R} = \mathbf{R}^0$, $\mathbf{V} = \mathbf{V}^0$, $\mathbf{S}_r = \mathbf{S}_r^0$, and $\mathbf{I}_r = \mathbf{I}_r^0$. Hence, we conclude that a disease-free equilibrium state $\mathscr{P}^0$ is globally asymptotically stable.

For the endemic Lyapunov function, we set all independent variables in the suggested model; in our case, $\{\mathbf{S}_h, \mathrm{E}_h, \mathrm{I}_h, \mathbf{I}_s, \mathrm{S}_v, \mathrm{I}_v, \mathbf{V}, \mathbf{B}_b\}$, $\mathfrak{P} < 0$ is the endemic equilibrium ($\mathscr{P}^*$). □

Theorem 9. *In the case where $t > 0$ and $0 < \rho \leq 1$, the model (8) is said to be globally asymptotically stable at the endemic equilibrium $\mathscr{P}^* = \{\mathbf{S}_h^*, \mathbf{E}^*, \mathbf{I}_m^*, \mathbf{I}_s^*, \mathbf{H}^*, \mathbf{R}^*, \mathbf{V}^*, \mathbf{S}_r^*, \mathbf{I}_r^*\}$, which is contained in region $\mho$ if $\mathscr{R}_0 > 1$.*

Proof. We can express Volterra-kind Lyapunov function as follows:

$$\begin{aligned} \mathfrak{P}(t) ={}& \mathfrak{U}_1\left\{\mathbf{S}_h - \mathbf{S}_h^* - \mathbf{S}_h^* \ln \frac{\mathbf{S}_h}{\mathbf{S}_h^*}\right\} + \mathfrak{U}_2\left\{\mathbf{E} - \mathbf{E}^* - \mathbf{E}^* \ln \frac{\mathbf{E}}{\mathbf{E}^*}\right\} + \mathfrak{U}_3\left\{\mathbf{I}_m - \mathbf{I}_m^* - \mathbf{I}_m^* \ln \frac{\mathbf{I}_m}{\mathbf{I}_m^*}\right\} \\ &+ \mathfrak{U}_4\left\{\mathbf{I}_s - \mathbf{I}_s^* - \mathbf{I}_s^* \ln \frac{\mathbf{I}_s(t)}{\mathbf{I}_s^*}\right\} + \mathfrak{U}_5\left\{\mathbf{H} - \mathbf{H}^* - \mathbf{H}^* \ln \frac{\mathbf{H}}{\mathbf{H}^*}\right\} + \mathfrak{U}_6\left\{\mathbf{R} - \mathbf{R}^* - \mathbf{R}^* \ln \frac{\mathbf{R}}{\mathbf{R}^*}\right\} \\ &+ \mathfrak{U}_7\left\{\mathbf{V} - \mathbf{V}^* - \mathbf{V}^* \ln \frac{\mathbf{V}}{\mathbf{V}^*}\right\} + \mathfrak{U}_8\left\{\mathbf{S}_r - \mathbf{S}_r^* - \mathbf{S}_r^* \ln \frac{\mathbf{S}_r}{\mathbf{S}_r^*} + \mathfrak{U}_9\left\{\mathbf{I}_r - \mathbf{I}_r^* - \mathbf{I}_r^* \ln \frac{\mathbf{I}_r}{\mathbf{I}_r^*}\right\}, \end{aligned} \tag{54}$$

where $\mathfrak{U}_i, i = 1, 2, \ldots 9$, are positive constants that we can choose later. Substituting equation (54) into system (8) and utilizing Lemma 4, we find

$$
\begin{aligned}
{}_0^C D_t^\rho \mathfrak{P}(t) \le{}& \mathfrak{U}_1\left(\frac{\mathbf{S}_h-\mathbf{S}_h^*}{\mathbf{S}_h}\right){}_0^C D_t^\rho \mathbf{S}_h + \mathfrak{U}_2\left(\frac{\mathbf{E}-\mathbf{E}^*}{\mathbf{E}}\right){}_0^C D_t^\rho \mathbf{E} + \mathfrak{U}_3\left(\frac{\mathbf{I}_m-\mathbf{I}_m^*}{\mathbf{I}_m}\right){}_0^C D_t^\rho \mathbf{I}_m \\
&+ \mathfrak{U}_4\left(\frac{\mathbf{I}_s-\mathbf{I}_s^*}{\mathbf{I}_s}\right){}_0^C D_t^\rho \mathbf{I}_s + \mathfrak{U}_5\left(\frac{\mathbf{H}-\mathbf{H}^*}{\mathbf{H}}\right){}_0^C D_t^\rho \mathbf{H} + \mathfrak{U}_6\left(\frac{\mathbf{R}-\mathbf{R}^*}{\mathbf{R}}\right){}_0^C D_t^\rho \mathbf{R} \\
&+ \mathfrak{U}_7\left(\frac{\mathbf{V}-\mathbf{V}^*}{\mathbf{V}}\right){}_0^C D_t^\rho \mathbf{V} + \mathfrak{U}_8\left(\frac{\mathbf{S}_r-\mathbf{S}_r^*}{\mathbf{S}_r}\right){}_0^C D_t^\rho \mathbf{S}_r + \mathfrak{U}_9\left(\frac{\mathbf{I}_r-\mathbf{I}_r^*}{\mathbf{I}_r}\right){}_0^C D_t^\rho \mathbf{I}_r,
\end{aligned}
\tag{55}
$$

writing their expressions for derivatives (8) as follows:

$$
\begin{aligned}
{}_0^C D_t^\rho \mathfrak{P}(t) \le{}& \mathfrak{U}_1\left(\frac{\mathbf{S}_h-\mathbf{S}_h^*}{\mathbf{S}_h}\right)\left[\Phi_h + \varphi\mathbf{R} - \left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_h - \eta_h\mathbf{S}_h\right] \\
&+ \mathfrak{U}_2\left(\frac{\mathbf{E}-\mathbf{E}^*}{\mathbf{E}}\right)\left[\left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_h - (\lambda+\eta_h)\mathbf{E}\right] \\
&+ \mathfrak{U}_3\left(\frac{\mathbf{I}_m-\mathbf{I}_m^*}{\mathbf{I}_m}\right)\left[\lambda\mathbf{E} - \left(\chi + \kappa_{\mathbf{I}_m} + \beta_{\mathbf{I}_m} + \eta_h\right)\mathbf{I}_m\right] \\
&+ \mathfrak{U}_4\left(\frac{\mathbf{I}_s-\mathbf{I}_s^*}{\mathbf{I}_s}\right)\left[\chi\mathbf{I}_m - \left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \mu_{\mathbf{I}_s} + \eta_h\right)\mathbf{I}_s\right] \\
&+ \mathfrak{U}_5\left(\frac{\mathbf{H}-\mathbf{H}^*}{\mathbf{H}}\right)\left[\kappa_{\mathbf{I}_m}\mathbf{I}_m + \kappa_{\mathbf{I}_s}\mathbf{I}_s - (\beta_h + \mu_h + \eta_h)\mathbf{H}\right] \\
&+ \mathfrak{U}_6\left(\frac{\mathbf{R}-\mathbf{R}^*}{\mathbf{R}}\right)\left[\beta_{\mathbf{I}_m}\mathbf{I}_m + \beta_{\mathbf{I}_s}\mathbf{I}_s + \beta_h\mathbf{H} - (\varphi+\eta_h)\mathbf{R}\right] \\
&+ \mathfrak{U}_7\left(\frac{\mathbf{V}-\mathbf{V}^*}{\mathbf{V}}\right)\left[\alpha_h\left(\mathbf{I}_m + \mathbf{I}_s\right) + \alpha_r\mathbf{I}_r - \vartheta\mathbf{V}\right] \\
&+ \mathfrak{U}_8\left(\frac{\mathbf{S}_r-\mathbf{S}_r^*}{\mathbf{S}_r}\right)\left[\Phi_r - \left(\frac{\xi_r\mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_r - \eta_r\mathbf{S}_r\right] \\
&+ \mathfrak{U}_9\left(\frac{\mathbf{I}_r-\mathbf{I}_r^*}{\mathbf{I}_r}\right)\left[\left(\frac{\xi_r\mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_r - \eta_r\mathbf{I}_r\right].
\end{aligned}
\tag{56}
$$

Putting

$$
\begin{aligned}
&\mathbf{S}_h = \mathbf{S}_h - \mathbf{S}_h^*, \mathbf{E} = \mathbf{E} - \mathbf{E}^*, \mathbf{I}_m = \mathbf{I}_m - \mathbf{I}_m^*, \mathbf{I}_s = \mathbf{I}_s - \mathbf{I}_s^*, \mathbf{H} = \mathbf{H} - \mathbf{H}^*, \\
&\mathbf{R} = \mathbf{R} - \mathbf{R}^*, \mathbf{V} = \mathbf{V} - \mathbf{V}^*, \mathbf{S}_r = \mathbf{S}_r - \mathbf{S}_r^*, \mathbf{I}_r = \mathbf{I}_r - \mathbf{I}_r^*,
\end{aligned}
\tag{57}
$$

we have

$$
\begin{aligned}
{}_0^C D_t^{\rho}\mathfrak{P}(t) \le{}& \mathfrak{U}_1\left(\frac{\mathbf{S}_h-\mathbf{S}_h^*}{\mathbf{S}_h}\right)\left[\Phi_h+\varphi(\mathbf{R}-\mathbf{R}^*)-\left(\frac{\xi_{\mathbf{I}_m}(\mathbf{I}_m-\mathbf{I}_m^*)+\xi_{\mathbf{I}_s}(\mathbf{I}_s-\mathbf{I}_s^*)}{\mathbf{N}_h}+\frac{\xi_{\mathrm{vh}}(\mathbf{V}-\mathbf{V}^*)}{c+(\mathbf{V}-\mathbf{V}^*)}-\eta_h\right)(\mathbf{S}_h-\mathbf{S}_h^*)\right]\\
&+\mathfrak{U}_2\left(\frac{\mathbf{E}-\mathbf{E}^*}{\mathbf{E}}\right)\left[\left(\frac{\xi_{\mathbf{I}_m}(\mathbf{I}_m-\mathbf{I}_m^*)+\xi_{\mathbf{I}_s}(\mathbf{I}_s-\mathbf{I}_s^*)}{\mathbf{N}_h}+\frac{\xi_{\mathrm{vh}}(\mathbf{V}-\mathbf{V}^*)}{c+(\mathbf{V}-\mathbf{V}^*)}\right)(\mathbf{S}_h-\mathbf{S}_h^*)-(\lambda+\eta_h)(\mathbf{E}-\mathbf{E}^*)\right]\\
&+\mathfrak{U}_3\left(\frac{\mathbf{I}_m-\mathbf{I}_m^*}{\mathbf{I}_m}\right)\left[\lambda(\mathbf{E}-\mathbf{E}^*)-\left(\chi+\kappa_{\mathbf{I}_m}+\beta_{\mathbf{I}_m}+\eta_h\right)(\mathbf{I}_m-\mathbf{I}_m^*)\right]\\
&+\mathfrak{U}_4\left(\frac{\mathbf{I}_s-\mathbf{I}_s^*}{\mathbf{I}_s}\right)\left[\chi(\mathbf{I}_m-\mathbf{I}_m^*)-\left(\kappa_{\mathbf{I}_s}+\beta_{\mathbf{I}_s}+\mu_{\mathbf{I}_s}+\eta_h\right)(\mathbf{I}_s-\mathbf{I}_s^*)\right]\\
&+\mathfrak{U}_5\left(\frac{\mathbf{H}-\mathbf{H}^*}{\mathbf{H}}\right)\left[\kappa_{\mathbf{I}_m}(\mathbf{I}_m-\mathbf{I}_m^*)+\kappa_{\mathbf{I}_s}(\mathbf{I}_s-\mathbf{I}_s^*)-(\beta_h+\mu_h+\eta_h)(\mathbf{H}-\mathbf{H}^*)\right]\\
&+\mathfrak{U}_6\left(\frac{\mathbf{R}-\mathbf{R}^*}{\mathbf{R}}\right)\left[\beta_{\mathbf{I}_m}(\mathbf{I}_m-\mathbf{I}_m^*)+\beta_{\mathbf{I}_s}(\mathbf{I}_s-\mathbf{I}_s^*)+\beta_h(\mathbf{H}-\mathbf{H}^*)-(\varphi+\eta_h)(\mathbf{R}-\mathbf{R}^*)\right]\\
&+\mathfrak{U}_7\left(\frac{\mathbf{V}-\mathbf{V}^*}{\mathbf{V}}\right)\left[\alpha_h((\mathbf{I}_m-\mathbf{I}_m^*)+(\mathbf{I}_s-\mathbf{I}_s^*))+\alpha_r(\mathbf{I}_r-\mathbf{I}_r^*)-\vartheta(\mathbf{V}-\mathbf{V}^*)\right]\\
&+\mathfrak{U}_8\left(\frac{\mathbf{S}_r-\mathbf{S}_r^*}{\mathbf{S}_r}\right)\left[\Phi_r-\left(\frac{\xi_r(\mathbf{I}_r-\mathbf{I}_r^*)}{\mathbf{N}_r}+\frac{\xi_{\mathrm{vr}}(\mathbf{V}-\mathbf{V}^*)}{c+(\mathbf{V}-\mathbf{V}^*)}-\eta_r\right)(\mathbf{S}_r-\mathbf{S}_r^*)\right]\\
&+\mathfrak{U}_9\left(\frac{\mathbf{I}_r-\mathbf{I}_r^*}{\mathbf{I}_r}\right)\left[\left(\frac{\xi_r(\mathbf{I}_r-\mathbf{I}_r^*)}{\mathbf{N}_r}+\frac{\xi_{\mathrm{vr}}(\mathbf{V}-\mathbf{V}^*)}{c+(\mathbf{V}-\mathbf{V}^*)}\right)(\mathbf{S}_r-\mathbf{S}_r^*)-\eta_r(\mathbf{I}_r-\mathbf{I}_r^*)\right],
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
{}_0^C D_t^{\rho}\mathfrak{P}(t) \le{}& \mathfrak{U}_1\Phi_h-\mathfrak{U}_1\frac{\mathbf{S}_h^*}{\mathbf{S}_h}\Phi_h+\mathfrak{U}_1\varphi(\mathbf{R}-\mathbf{R}^*)-\mathfrak{U}_1\varphi\frac{\mathbf{S}_h^*}{\mathbf{S}_h}(\mathbf{R}-\mathbf{R}^*)-\mathfrak{U}_1\left[\frac{\xi_{\mathbf{I}_m}(\mathbf{I}_m-\mathbf{I}_m^*)+\xi_{\mathbf{I}_s}(\mathbf{I}_s-\mathbf{I}_s^*)}{\mathbf{N}_h}\right.\\
&\left.+\frac{\xi_{vh}(\mathbf{V}-\mathbf{V}^*)}{c+(\mathbf{V}-\mathbf{V}^*)}-\eta_h\right]\frac{(\mathbf{S}_h-\mathbf{S}_h^*)^2}{\mathbf{S}_h}+\mathfrak{U}_2\left[\frac{\xi_{\mathbf{I}_m}(\mathbf{I}_m-\mathbf{I}_m^*)+\xi_{\mathbf{I}_s}(\mathbf{I}_s-\mathbf{I}_s^*)}{\mathbf{N}_h}+\frac{\xi_{\mathrm{vh}}(\mathbf{V}-\mathbf{V}^*)}{c+(\mathbf{V}-\mathbf{V}^*)}\right](\mathbf{S}_h-\mathbf{S}_h^*)\\
&-\mathfrak{U}_2\left[\frac{\xi_{\mathbf{I}_m}(\mathbf{I}_m-\mathbf{I}_m^*)+\xi_{\mathbf{I}_s}(\mathbf{I}_s-\mathbf{I}_s^*)}{\mathbf{N}_h}+\frac{\xi_{\mathrm{vh}}(\mathbf{V}-\mathbf{V}^*)}{c+(\mathbf{V}-\mathbf{V}^*)}\right]\frac{\mathbf{E}^*}{\mathbf{E}}(\mathbf{S}_h-\mathbf{S}_h^*)-\mathfrak{U}_2(\lambda+\eta_h)\frac{(\mathbf{E}-\mathbf{E}^*)^2}{\mathbf{E}}\\
&+\mathfrak{U}_3\lambda(\mathbf{E}-\mathbf{E}^*)-\mathfrak{U}_3\frac{\mathbf{I}_m^*}{\mathbf{I}_m}\lambda(\mathbf{E}-\mathbf{E}^*)-\mathfrak{U}_3\left(\chi+\kappa_{\mathbf{I}_m}+\beta_{\mathbf{I}_m}+\eta_h\right)\frac{(\mathbf{I}_m-\mathbf{I}_m^*)^2}{\mathbf{I}_m}+\mathfrak{U}_4\chi(\mathbf{I}_m-\mathbf{I}_m^*)\\
&-\mathfrak{U}_4\frac{\mathbf{I}_m^*}{\mathbf{I}_s}\chi(\mathbf{I}_m-\mathbf{I}_m^*)-\mathfrak{U}_4\left(\kappa_{\mathbf{I}_s}+\beta_{\mathbf{I}_s}+\mu_{\mathbf{I}_s}+\eta_h\right)\frac{(\mathbf{I}_s-\mathbf{I}_s^*)^2}{\mathbf{I}_s}+\mathfrak{U}_5\kappa_{\mathbf{I}_m}(\mathbf{I}_m-\mathbf{I}_m^*)-\mathfrak{U}_5\kappa_{\mathbf{I}_m}\frac{\mathbf{H}^*}{\mathbf{H}}(\mathbf{I}_m-\mathbf{I}_m^*)\\
&+\mathfrak{U}_5\kappa_{\mathbf{I}_s}(\mathbf{I}_s-\mathbf{I}_s^*)-\mathfrak{U}_5\kappa_{\mathbf{I}_s}\frac{\mathbf{H}^*}{\mathbf{H}}(\mathbf{I}_s-\mathbf{I}_s^*)-\mathfrak{U}_5(\beta_h+\mu_h+\eta_h)\frac{(\mathbf{H}-\mathbf{H}^*)^2}{\mathbf{H}}+\mathfrak{U}_6\beta_{\mathbf{I}_m}(\mathbf{I}_m-\mathbf{I}_m^*)\\
&-\mathfrak{U}_6\beta_{\mathbf{I}_m}\frac{\mathbf{R}^*}{\mathbf{R}}(\mathbf{I}_m-\mathbf{I}_m^*)+\mathfrak{U}_6\beta_{\mathbf{I}_s}(\mathbf{I}_s-\mathbf{I}_s^*)-\mathfrak{U}_6\beta_{\mathbf{I}_s}\frac{\mathbf{R}^*}{\mathbf{R}}(\mathbf{I}_s-\mathbf{I}_s^*)+\mathfrak{U}_6\beta_h(\mathbf{H}-\mathbf{H}^*)-\mathfrak{U}_6\beta_h\frac{\mathbf{R}^*}{\mathbf{R}}(\mathbf{H}-\mathbf{H}^*)\\
&-\mathfrak{U}_6(\beta_h+\mu_h+\eta_h)\frac{(\mathbf{H}-\mathbf{H}^*)^2}{\mathbf{H}}+\mathfrak{U}_7\alpha_h(\mathbf{I}_m-\mathbf{I}_m^*)-\mathfrak{U}_7\alpha_h\frac{\mathbf{V}^*}{\mathbf{V}}(\mathbf{I}_m-\mathbf{I}_m^*)+\mathfrak{U}_7\alpha_h(\mathbf{I}_s-\mathbf{I}_s^*)\\
&-\mathfrak{U}_7\alpha_h\frac{\mathbf{V}^*}{\mathbf{V}}(\mathbf{I}_s-\mathbf{I}_s^*)+\mathfrak{U}_7\alpha_r(\mathbf{I}_r-\mathbf{I}_r^*)-\mathfrak{U}_7\alpha_r\frac{\mathbf{V}^*}{\mathbf{V}}(\mathbf{I}_r-\mathbf{I}_r^*)-\mathfrak{U}_7\vartheta\frac{(\mathbf{V}-\mathbf{V}^*)^2}{\mathbf{V}}+\mathfrak{U}_8\Phi_r-\mathfrak{U}_8\Phi_r\frac{\mathbf{S}_r^*}{\mathbf{S}_r}\\
&-\mathfrak{U}_8\left[\frac{\xi_r(\mathbf{I}_r-\mathbf{I}_r^*)}{\mathbf{N}_r}+\frac{\xi_{\mathrm{vr}}(\mathbf{V}-\mathbf{V}^*)}{c+(\mathbf{V}-\mathbf{V}^*)}-\eta_r\right]\frac{(\mathbf{S}_r-\mathbf{S}_r^*)^2}{\mathbf{S}_r}+\mathfrak{U}_9\left[\frac{\xi_r(\mathbf{I}_r-\mathbf{I}_r^*)}{\mathbf{N}_r}+\frac{\xi_{\mathrm{vr}}(\mathbf{V}-\mathbf{V}^*)}{c+(\mathbf{V}-\mathbf{V}^*)}\right](\mathbf{S}_r-\mathbf{S}_r^*)\\
&-\mathfrak{U}_9\left[\frac{\xi_r(\mathbf{I}_r-\mathbf{I}_r^*)}{\mathbf{N}_r}+\frac{\xi_{\mathrm{vr}}(\mathbf{V}-\mathbf{V}^*)}{c+(\mathbf{V}-\mathbf{V}^*)}\right]\frac{\mathbf{I}_r^*}{\mathbf{I}_r}(\mathbf{S}_r-\mathbf{S}_r^*)-\mathfrak{U}_9\eta_r\frac{(\mathbf{I}_r-\mathbf{I}_r^*)^2}{\mathbf{I}_r}.
\end{aligned}
\tag{59}
$$

Let $\mathfrak{U}_1 = \mathfrak{U}_2 = \mathfrak{U}_3 = \mathfrak{U}_4 = \mathfrak{U}_5 = \mathfrak{U}_6 = \mathfrak{U}_7 = \mathfrak{U}_8 = \mathfrak{U}_9 = 1$ and after simplifying (59), we can write

$$ {}_0^C D_t^\rho \mathfrak{P}(t) \le \Xi_x - \Xi_y, \tag{60} $$

where

$$ \begin{cases} \Xi_x = \Phi_h + \varphi(\mathbf{R} - \mathbf{R}^*) + \left[\dfrac{\xi_{\mathbf{I}_m}(\mathbf{I}_m - \mathbf{I}_m^*) + \xi_{\mathbf{I}_s}(\mathbf{I}_s - \mathbf{I}_s^*)}{\mathbf{N}_h} + \dfrac{\xi_{\mathrm{vh}}(\mathbf{V} - \mathbf{V}^*)}{c + (\mathbf{V} - \mathbf{V}^*)}\right](\mathbf{S}_h - \mathbf{S}_h^*) + \lambda(\mathbf{E} - \mathbf{E}^*) + \chi(\mathbf{I}_m - \mathbf{I}_m^*) \\ + \kappa_{\mathbf{I}_m}(\mathbf{I}_m - \mathbf{I}_m^*) + \kappa_{\mathbf{I}_s}(\mathbf{I}_s - \mathbf{I}_s^*) + \beta_{\mathbf{I}_m}(\mathbf{I}_m - \mathbf{I}_m^*) + \beta_{\mathbf{I}_s}(\mathbf{I}_s - \mathbf{I}_s^*) + \beta_h(\mathbf{H} - \mathbf{H}^*) + \alpha_h(\mathbf{I}_m - \mathbf{I}_m^*) \\ + \alpha_h(\mathbf{I}_s - \mathbf{I}_s^*) + \alpha_r(\mathbf{I}_r - \mathbf{I}_r^*) + \Phi_r + \left[\dfrac{\xi_r(\mathbf{I}_r - \mathbf{I}_r^*)}{\mathbf{N}_r} + \dfrac{\xi_{\mathrm{vr}}(\mathbf{V} - \mathbf{V}^*)}{c + (\mathbf{V} - \mathbf{V}^*)}\right](\mathbf{S}_r - \mathbf{S}_r^*), \end{cases} $$

$$ \begin{cases} \Xi_y = \dfrac{\mathbf{S}_h^*}{\mathbf{S}_h}\Phi_h + \varphi\dfrac{\mathbf{S}_h^*}{\mathbf{S}_h}(\mathbf{R} - \mathbf{R}^*) + \left[\dfrac{\xi_{\mathbf{I}_m}(\mathbf{I}_m - \mathbf{I}_m^*) + \xi_{\mathbf{I}_s}(\mathbf{I}_s - \mathbf{I}_s^*)}{\mathbf{N}_h} + \dfrac{\xi_{\mathrm{vh}}(\mathbf{V} - \mathbf{V}^*)}{c + (\mathbf{V} - \mathbf{V}^*)} - \eta_h\right]\dfrac{(\mathbf{S}_h - \mathbf{S}_h^*)^2}{\mathbf{S}_h} \\ + \left[\dfrac{\xi_{\mathbf{I}_m}(\mathbf{I}_m - \mathbf{I}_m^*) + \xi_{\mathbf{I}_s}(\mathbf{I}_s - \mathbf{I}_s^*)}{\mathbf{N}_h} + \dfrac{\xi_{\mathrm{vh}}(\mathbf{V} - \mathbf{V}^*)}{c + (\mathbf{V} - \mathbf{V}^*)}\right]\dfrac{\mathbf{E}^*}{\mathbf{E}}(\mathbf{S}_h - \mathbf{S}_h^*) + (\lambda + \eta_h)\dfrac{(\mathbf{E} - \mathbf{E}^*)^2}{\mathbf{E}} + \dfrac{\mathbf{I}_m^*}{\mathbf{I}_m}\lambda(\mathbf{E} - \mathbf{E}^*) \\ + \left(\chi + \kappa_{\mathbf{I}_m} + \beta_{\mathbf{I}_m} + \eta_h\right)\dfrac{(\mathbf{I}_m - \mathbf{I}_m^*)^2}{\mathbf{I}_m} + \dfrac{\mathbf{I}_m^*}{\mathbf{I}_s}\chi(\mathbf{I}_m - \mathbf{I}_m^*) + \left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \mu_{\mathbf{I}_s} + \eta_h\right)\dfrac{(\mathbf{I}_s - \mathbf{I}_s^*)^2}{\mathbf{I}_s} \\ + \kappa_{\mathbf{I}_m}\dfrac{\mathbf{H}^*}{\mathbf{H}}(\mathbf{I}_m - \mathbf{I}_m^*) + \kappa_{\mathbf{I}_s}\dfrac{\mathbf{H}^*}{\mathbf{H}}(\mathbf{I}_s - \mathbf{I}_s^*) + (\beta_h + \mu_h + \eta_h)\dfrac{(\mathbf{H} - \mathbf{H}^*)^2}{\mathbf{H}} + \beta_{\mathbf{I}_m}\dfrac{\mathbf{R}^*}{\mathbf{R}}(\mathbf{I}_m - \mathbf{I}_m^*) \\ + \beta_{\mathbf{I}_s}\dfrac{\mathbf{R}^*}{\mathbf{R}}(\mathbf{I}_s - \mathbf{I}_s^*) + \beta_h\dfrac{\mathbf{R}^*}{\mathbf{R}}(\mathbf{H} - \mathbf{H}^*) + (\beta_h + \mu_h + \eta_h)\dfrac{(\mathbf{H} - \mathbf{H}^*)^2}{\mathbf{H}} + \alpha_h\dfrac{\mathbf{V}^*}{\mathbf{V}}(\mathbf{I}_m - \mathbf{I}_m^*) + \alpha_h\dfrac{\mathbf{V}^*}{\mathbf{V}}(\mathbf{I}_s - \mathbf{I}_s^*) \\ + \alpha_r\dfrac{\mathbf{V}^*}{\mathbf{V}}(\mathbf{I}_r - \mathbf{I}_r^*) + \vartheta\dfrac{(\mathbf{V} - \mathbf{V}^*)^2}{\mathbf{V}} + \Phi_r\dfrac{\mathbf{S}_r^*}{\mathbf{S}_r} + \left[\dfrac{\xi_r(\mathbf{I}_r - \mathbf{I}_r^*)}{\mathbf{N}_r} + \dfrac{\xi_{\mathrm{vr}}(\mathbf{V} - \mathbf{V}^*)}{c + (\mathbf{V} - \mathbf{V}^*)} - \eta_r\right]\dfrac{(\mathbf{S}_r - \mathbf{S}_r^*)^2}{\mathbf{S}_r} \\ + \left[\dfrac{\xi_r(\mathbf{I}_r - \mathbf{I}_r^*)}{\mathbf{N}_r} + \dfrac{\xi_{\mathrm{vr}}(\mathbf{V} - \mathbf{V}^*)}{c + (\mathbf{V} - \mathbf{V}^*)}\right]\dfrac{\mathbf{I}_r^*}{\mathbf{I}_r}(\mathbf{S}_r - \mathbf{S}_r^*) + \eta_r\dfrac{(\mathbf{I}_r - \mathbf{I}_r^*)^2}{\mathbf{I}_r}. \end{cases} \tag{61} $$

We observe that

(i) if $\Xi_x < \Xi_y \Longrightarrow {}_0^C D_t^\rho \mathfrak{P}(t) \le 0$

(ii) However, when $\mathbf{S}_h(t) = \mathbf{S}_h^*$, $\mathbf{E}(t) = \mathbf{E}^*$, $\mathbf{I}_m(t) = \mathbf{I}_m^*$, $\mathbf{I}_s(t) = \mathbf{I}_s^*$, $\mathbf{H}(t) = \mathbf{H}^*$, $\mathbf{R}(t) = \mathbf{R}^*$, $\mathbf{V}(t) = \mathbf{V}^*$, $\mathbf{S}_r(t) = \mathbf{S}_r^*$, and $\mathbf{I}_r(t) = \mathbf{I}_r^*$, then

$$ \Xi_x - \Xi_y = 0 \Longrightarrow {}_0^C D_t^\rho \mathfrak{P}(t) = 0. \tag{62} $$

We find the proposed model's largest compact invariant set in

$$ \left\{(\mathbf{S}_h^*, \mathbf{E}^*, \mathbf{I}_m^*, \mathbf{I}_s^*, \mathbf{H}^*, \mathbf{R}^*, \mathbf{V}^*, \mathbf{S}_r^*, \mathbf{I}_r^*) \in \mho : {}_0^C D_t^\rho \mathfrak{P}(t) = 0\right\}, \tag{63} $$

is the point $\mathscr{P}^*$, the endemic equilibrium of the proposed model. From Lasalle's invariance concept, therefore, we can conclude that $\mathscr{P}^*$ is globally asymptotically stable in $\mho$ if $\Xi_x < \Xi_y$. □

4. Numerical Scheme

The literature has suggested that the Caputo derivative is the best model to simulate power-law processes in practical

problems. Our method of solving system (8) numerically is based on Newton's polynomial.

$$
\begin{aligned}
{}_0^C D_t^\rho \mathbf{S}_h(t) &= \Phi_h + \varphi \mathbf{R} - \left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_h - \eta_h \mathbf{S}_h,\\
{}_0^C D_t^\rho \mathbf{E}(t) &= \left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_h - (\lambda + \eta_h)\mathbf{E},\\
{}_0^C D_t^\rho \mathbf{I}_m(t) &= \lambda \mathbf{E} - \left(\chi + \kappa_{\mathbf{I}_m} + \beta_{\mathbf{I}_m} + \eta_h\right)\mathbf{I}_m,\\
{}_0^C D_t^\rho \mathbf{I}_s(t) &= \chi \mathbf{I}_m - \left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \mu_{\mathbf{I}_s} + \eta_h\right)\mathbf{I}_s,\\
{}_0^C D_t^\rho \mathbf{H}(t) &= \kappa_{\mathbf{I}_m}\mathbf{I}_m + \kappa_{\mathbf{I}_s}\mathbf{I}_s - (\beta_h + \mu_h + \eta_h)\mathbf{H},\\
{}_0^C D_t^\rho \mathbf{R}(t) &= \beta_{\mathbf{I}_m}\mathbf{I}_m + \beta_{\mathbf{I}_s}\mathbf{I}_s + \beta_h \mathbf{H} - (\varphi + \eta_h)\mathbf{R},\\
{}_0^C D_t^\rho \mathbf{V}(t) &= \alpha_h(\mathbf{I}_m + \mathbf{I}_s) + \alpha_r \mathbf{I}_r - \vartheta \mathbf{V},\\
{}_0^C D_t^\rho \mathbf{S}_r(t) &= \Phi_r - \left(\frac{\xi_r \mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_r - \eta_r \mathbf{S}_r,\\
{}_0^C D_t^\rho \mathbf{I}_r(t) &= \left(\frac{\xi_r \mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_r - \eta_r \mathbf{I}_r.
\end{aligned}
\tag{64}
$$

For ease of use, we will write the aforementioned system as follows:

$$
\begin{aligned}
\mathscr{K}_1(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \Phi_h + \varphi \mathbf{R} - \left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_h - \eta_h \mathbf{S}_h,\\
\mathscr{K}_2(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \left(\frac{\xi_{\mathbf{I}_m}\mathbf{I}_m + \xi_{\mathbf{I}_s}\mathbf{I}_s}{\mathbf{N}_h} + \frac{\xi_{\mathrm{vh}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_h - (\lambda + \eta_h)\mathbf{E},\\
\mathscr{K}_3(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \lambda \mathbf{E} - \left(\chi + \kappa_{\mathbf{I}_m} + \beta_{\mathbf{I}_m} + \eta_h\right)\mathbf{I}_m,\\
\mathscr{K}_4(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \chi \mathbf{I}_m - \left(\kappa_{\mathbf{I}_s} + \beta_{\mathbf{I}_s} + \mu_{\mathbf{I}_s} + \eta_h\right)\mathbf{I}_s,\\
\mathscr{K}_5(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \kappa_{\mathbf{I}_m}\mathbf{I}_m + \kappa_{\mathbf{I}_s}\mathbf{I}_s - (\beta_h + \mu_h + \eta_h)\mathbf{H},\\
\mathscr{K}_6(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \beta_{\mathbf{I}_m}\mathbf{I}_m + \beta_{\mathbf{I}_s}\mathbf{I}_s + \beta_h \mathbf{H} - (\varphi + \eta_h)\mathbf{R},\\
\mathscr{K}_7(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \alpha_h(\mathbf{I}_m + \mathbf{I}_s) + \alpha_r \mathbf{I}_r - \vartheta \mathbf{V},\\
\mathscr{K}_8(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \Phi_r - \left(\frac{\xi_r \mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_r - \eta_r \mathbf{S}_r,\\
\mathscr{K}_9(t, \mathbf{S}_h, \mathbf{E}, \mathbf{I}_m, \mathbf{I}_s, \mathbf{H}, \mathbf{R}, \mathbf{V}, \mathbf{S}_r, \mathbf{I}_r) &= \left(\frac{\xi_r \mathbf{I}_r}{\mathbf{N}_r} + \frac{\xi_{\mathrm{vr}}\mathbf{V}}{c+\mathbf{V}}\right)\mathbf{S}_r - \eta_r \mathbf{I}_r.
\end{aligned}
\tag{65}
$$

After applying fractional integral, we have the following:

$$\mathbf{S}_h(t_k+1)=\mathbf{S}_h(0)+\frac{1}{\Gamma(\rho)}\sum_{i=2}^{k}\int_{t_i}^{t_{i+1}}\mathscr{K}_1(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r)(t_{k+1}-\zeta)^{\rho-1}d\zeta, \tag{66}$$

$$\mathbf{E}(t_k+1)=\mathbf{E}(0)+\frac{1}{\Gamma(\rho)}\sum_{i=2}^{k}\int_{t_i}^{t_{i+1}}\mathscr{K}_2(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r)(t_{k+1}-\zeta)^{\rho-1}d\zeta, \tag{67}$$

$$\mathbf{I}_m(t_k+1)=\mathbf{I}_m(0)+\frac{1}{\Gamma(\rho)}\sum_{i=2}^{k}\int_{t_i}^{t_{i+1}}\mathscr{K}_3(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r)(t_{k+1}-\zeta)^{\rho-1}d\zeta, \tag{68}$$

$$\mathbf{I}_s(t_k+1)=\mathbf{I}_s(0)+\frac{1}{\Gamma(\rho)}\sum_{i=2}^{k}\int_{t_i}^{t_{i+1}}\mathscr{K}_4(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r)(t_{k+1}-\zeta)^{\rho-1}d\zeta, \tag{69}$$

$$\mathbf{H}(t_k+1)=\mathbf{H}(0)+\frac{1}{\Gamma(\rho)}\sum_{i=2}^{k}\int_{t_i}^{t_{i+1}}\mathscr{K}_5(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r)(t_{k+1}-\zeta)^{\rho-1}d\zeta, \tag{70}$$

$$\mathbf{R}(t_k+1)=\mathbf{R}(0)+\frac{1}{\Gamma(\rho)}\sum_{i=2}^{k}\int_{t_i}^{t_{i+1}}\mathscr{K}_6(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r)(t_{k+1}-\zeta)^{\rho-1}d\zeta, \tag{71}$$

$$\mathbf{V}(t_k+1)=\mathbf{V}(0)+\frac{1}{\Gamma(\rho)}\sum_{i=2}^{k}\int_{t_i}^{t_{i+1}}\mathscr{K}_7(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r)(t_{k+1}-\zeta)^{\rho-1}d\zeta, \tag{72}$$

$$\mathbf{S}_r(t_k+1)=\mathbf{S}_r(0)+\frac{1}{\Gamma(\rho)}\sum_{i=2}^{k}\int_{t_i}^{t_{i+1}}\mathscr{K}_8(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r)(t_{k+1}-\zeta)^{\rho-1}d\zeta, \tag{73}$$

$$\mathbf{I}_r(t_k+1)=\mathbf{I}_r(0)+\frac{1}{\Gamma(\rho)}\sum_{i=2}^{k}\int_{t_i}^{t_{i+1}}\mathscr{K}_9(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r,\mathbf{I}_r)(t_{k+1}-\zeta)^{\rho-1}d\zeta. \tag{74}$$

Here, we recall the Newton polynomial as follows:

$$\begin{aligned}
&\mathbb{P}(t,\mathbf{S}_h,\mathbf{E},\mathbf{I}_m,\mathbf{I}_s,\mathbf{H},\mathbf{R},\mathbf{V},\mathbf{S}_r\mathbf{I}_r)\\
&\simeq\mathbb{P}\left(t_{k-2},\mathbf{S}_h^{k-2},\mathbf{E}^{k-2},\mathbf{I}_m^{k-2},\mathbf{I}_s^{k-2},\mathbf{H}^{k-2},\mathbf{R}^{k-2},\mathbf{V}^{k-2},\mathbf{S}_r^{k-2},\mathbf{I}_r^{k-2}\right)\\
&\quad+\frac{1}{\Delta t}\left\{\mathbb{P}\left(t_{k-1},\mathbf{S}_h^{k-1},\mathbf{E}^{k-1},\mathbf{I}_m^{k-1},\mathbf{I}_s^{k-1},\mathbf{H}^{k-1},\mathbf{R}^{k-1},\mathbf{V}^{k-1},\mathbf{S}_r^{k-1},\mathbf{I}_r^{k-1}\right)\right.\\
&\quad\left.-\mathbb{P}\left(t_{k-2},\mathbf{S}_h^{k-2},\mathbf{E}^{k-2},\mathbf{I}_m^{k-2},\mathbf{I}_s^{k-2},\mathbf{H}^{k-2},\mathbf{R}^{k-2},\mathbf{V}^{k-2},\mathbf{S}_r^{k-2},\mathbf{I}_r^{k-2}\right)\right\}\\
&\quad\times(\zeta-t_{k-2})+\frac{1}{2\Delta t^2}\left\{\mathbb{P}\left(t_k,\mathbf{S}_h^{k},\mathbf{E}^{k},\mathbf{I}_m^{k},\mathbf{I}_s^{k},\mathbf{H}^{k},\mathbf{R}^{k},\mathbf{V}^{k},\mathbf{S}_r^{k},\mathbf{I}_r^{k}\right)\right.\\
&\quad-2\mathbb{P}\left(t_{k-2},\mathbf{S}_h^{k-1},\mathbf{E}^{k-1},\mathbf{I}_m^{k-1},\mathbf{I}_s^{k-1},\mathbf{H}^{k-1},\mathbf{R}^{k-1},\mathbf{V}^{k-1},\mathbf{S}_r^{k-1},\mathbf{I}_r^{k-1}\right)\\
&\quad\left.+\mathbb{P}\left(t_{k-2},\mathbf{S}_h^{k-2},\mathbf{E}^{k-2},\mathbf{I}_m^{k-2},\mathbf{I}_s^{k-2},\mathbf{H}^{k-2},\mathbf{R}^{k-2},\mathbf{V}^{k-2},\mathbf{S}_r^{k-2},\mathbf{I}_r^{k-2}\right)\right\}\\
&\quad\times(\zeta-t_{k-2})(\zeta-t_{k-1}).
\end{aligned} \tag{75}$$

Replacing the Newton polynomial (75) into equations (66)–(74), we have for class $\mathbf{S}_h$:

$$
\begin{aligned}
\mathbf{S}_{h_{(k+1)}} = {} & \mathbf{S}_h(0) + \frac{1}{\Gamma(\rho)} \sum_{i=2}^{k} \mathscr{K}_1\left(t_{i-2}, \mathbf{S}_h^{i-2}, \mathbf{E}^{i-2}, \mathbf{I}_m^{i-2}, \mathbf{I}_s^{i-2}, \mathbf{H}^{i-2}, \mathbf{R}^{i-2}, \mathbf{V}^{i-2}, \mathbf{S}_r^{i-2}, \mathbf{I}_r^{i-2}\right) \\
& \times \int_{t_i}^{t_{i+1}} (t_{k+1} - \zeta)^{\rho-1} \mathrm{d}\zeta \\
& + \frac{1}{\Gamma(\rho)} \sum_{i=2}^{k} \frac{1}{\Delta t} \left\{ \mathscr{K}_1\left(t_{i-1}, \mathbf{S}_h^{i-1}, \mathbf{E}^{i-1}, \mathbf{I}_m^{i-1}, \mathbf{I}_s^{i-1}, \mathbf{H}^{i-1}, \mathbf{R}^{i-1}, \mathbf{V}^{i-1}, \mathbf{S}_r^{i-1}, \mathbf{I}_r^{i-1}\right) \right. \\
& \left. - \mathscr{K}_1\left(t_{i-2}, \mathbf{S}_h^{i-2}, \mathbf{E}^{i-2}, \mathbf{I}_m^{i-2}, \mathbf{I}_s^{i-2}, \mathbf{H}^{i-2}, \mathbf{R}^{i-2}, \mathbf{V}^{i-2}, \mathbf{S}_r^{i-2}, \mathbf{I}_r^{i-2}\right) \right\} \\
& \times \int_{t_i}^{t_{i+1}} (\zeta - t_{i-2})(t_{k+1} - \zeta)^{\rho-1} \mathrm{d}\zeta \\
& + \frac{1}{\Gamma(\rho)} \sum_{i=2}^{k} \frac{1}{2\Delta t^2} \left\{ \mathscr{K}_1\left(t_i, \mathbf{S}_h^{i}, \mathbf{E}^{i}, \mathbf{I}_m^{i}, \mathbf{I}_s^{i}, \mathbf{H}^{i}, \mathbf{R}^{i}, \mathbf{V}^{i}, \mathbf{S}_r^{i}, \mathbf{I}_r^{i}\right) \right. \\
& - 2\mathscr{K}_1\left(t_{i-1}, \mathbf{S}_h^{i-1}, \mathbf{E}^{i-1}, \mathbf{I}_m^{i-1}, \mathbf{I}_s^{i-1}, \mathbf{H}^{i-1}, \mathbf{R}^{i-1}, \mathbf{V}^{i-1}, \mathbf{S}_r^{i-1}, \mathbf{I}_r^{i-1}\right) \\
& \left. + \mathscr{K}_1\left(t_{i-2}, \mathbf{S}_h^{i-2}, \mathbf{E}^{i-2}, \mathbf{I}_m^{i-2}, \mathbf{I}_s^{i-2}, \mathbf{H}^{i-2}, \mathbf{R}^{i-2}, \mathbf{V}^{i-2}, \mathbf{S}_r^{i-2}, \mathbf{I}_r^{i-2}\right) \right\} \\
& \times \int_{t_i}^{t_{i+1}} (\zeta - t_{i-2})(\zeta - t_{i-1})(t_{k+1} - \zeta)^{\rho-1} \mathrm{d}\zeta.
\end{aligned} \tag{76}
$$

The following calculations can be made for the integral mentioned in equation (76):

$$
\begin{aligned}
\int_{t_i}^{t_{i+1}} (t_{k+1} - \zeta)^{\rho-1} \mathrm{d}\zeta &= \frac{(\Delta t)^{\rho}}{\rho} \left[(k-i+1)^{\rho} - (k-i)^{\rho} \right], \\
\int_{t_i}^{t_{i+1}} (\zeta - t_{i-2})(t_{k+1} - \zeta)^{\rho-1} \mathrm{d}\zeta &= \frac{(\Delta t)^{\rho+1}}{\rho(\rho+1)} \left[(k-i+1)^{\rho}(k-i+3+2\rho) - (k-i)^{\rho}(k-i+3+3\rho) \right], \\
\int_{t_i}^{t_{i+1}} (\zeta - t_{i-2})(\zeta - t_{i-1})(t_{k+1} - \zeta)^{\rho-1} \mathrm{d}\zeta &= \frac{(\Delta t)^{\rho+2}}{\rho(\rho+1)(\rho+2)} \times \left[(k-i+1)^{\rho} \left\{ 2(k-i)^2 + (3\rho+10)(k-i) + 2\rho^2 + 9\rho + 12 \right\} \right. \\
& \left. - (k-i)^{\rho} \left\{ 2(k-i)^2 + (5\rho+10)(k-i) + 6\rho^2 + 18\rho + 12 \right\} \right].
\end{aligned} \tag{77}
$$

Hence, we get finally

$$\begin{aligned}\mathbf{S}_h(t_{k+1}) = \mathbf{S}_h(0) &+ \frac{(\Delta t)^\rho}{\Gamma(\rho+1)}\sum_{i=2}^{k}\mathscr{K}_1\left(t_{i-2}, \mathbf{S}_h^{i-2}, \mathbf{E}^{i-2}, \mathbf{I}_m^{i-2}, \mathbf{I}_s^{i-2}, \mathbf{H}^{i-2}, \mathbf{R}^{i-2}, \mathbf{V}^{i-2}, \mathbf{S}_r^{i-2}, \mathbf{I}_r^{i-2}\right)\times\Lambda_1\\ &+ \frac{(\Delta t)^\rho}{\Gamma(\rho+2)}\sum_{i=2}^{k}\left[\mathscr{K}_1\left(t_{i-1}, \mathbf{S}_h^{i-1}, \mathbf{E}^{i-1}, \mathbf{I}_m^{i-1}, \mathbf{I}_s^{i-1}, \mathbf{H}^{i-1}, \mathbf{R}^{i-1}, \mathbf{V}^{i-1}, \mathbf{S}_r^{i-1}, \mathbf{I}_r^{i-1}\right)\right.\\ &\left.- \mathscr{K}_1\left(t_{i-2}, \mathbf{S}_h^{i-2}, \mathbf{E}^{i-2}, \mathbf{I}_m^{i-2}, \mathbf{I}_s^{i-2}, \mathbf{H}^{i-2}, \mathbf{R}^{i-2}, \mathbf{V}^{i-2}, \mathbf{S}_r^{i-2}, \mathbf{I}_r^{i-2}\right)\right]\times\Lambda_2\\ &+ \frac{\rho(\Delta t)^\rho}{2\Gamma(\rho+3)}\sum_{i=2}^{k}\left[\mathscr{K}_1\left(t_i, \mathbf{S}_h^{i}, \mathbf{E}^{i}, \mathbf{I}_m^{i}, \mathbf{I}_s^{i}, \mathbf{H}^{i}, \mathbf{R}^{i}, \mathbf{V}^{i}, \mathbf{S}_r^{i}, \mathbf{I}_r^{i}\right)\right.\\ &- 2\mathscr{K}_1\left(t_{i-1}, \mathbf{S}_h^{i-1}, \mathbf{E}^{i-1}, \mathbf{I}_m^{i-1}, \mathbf{I}_s^{i-1}, \mathbf{H}^{i-1}, \mathbf{R}^{i-1}, \mathbf{V}^{i-1}, \mathbf{S}_r^{i-1}, \mathbf{I}_r^{i-1}\right)\\ &\left.+ \mathscr{K}_1\left(t_{i-2}, \mathbf{S}_h^{i-2}, \mathbf{E}^{i-2}, \mathbf{I}_m^{i-2}, \mathbf{I}_s^{i-2}, \mathbf{H}^{i-2}, \mathbf{R}^{i-2}, \mathbf{V}^{i-2}, \mathbf{S}_r^{i-2}, \mathbf{I}_r^{i-2}\right)\right]\times\Lambda_3,\end{aligned} \tag{78}$$

where

$$\begin{aligned}\Lambda_1 &= (k-i+1)^\rho - (k-i)^\rho,\\ \Lambda_2 &= (k-i+1)^\rho(k-i+3+2\rho) - (k-i)^\rho(k-i+3+3\rho),\\ \Lambda_3 &= (k-i+1)^\rho\left[2(k-i)^2 + (3\rho+10)(k-i) + 2\rho^2 + 9\rho + 12\right]\\ &\quad - (k-i)^\rho\left[2(k-i)^2 + (5\rho+10)(k-i) + 6\rho^2 + 18\rho + 12\right].\end{aligned} \tag{79}$$

Similarly, we get

$$\begin{aligned}\mathbf{E}(t_{k+1}) = \mathbf{E}(0) &+ \frac{(\Delta t)^\rho}{\Gamma(\rho+1)}\sum_{i=2}^{k}\mathscr{K}_2\left(t_{i-2}, \mathbf{S}_h^{i-2}, \mathbf{E}^{i-2}, \mathbf{I}_m^{i-2}, \mathbf{I}_s^{i-2}, \mathbf{H}^{i-2}, \mathbf{R}^{i-2}, \mathbf{V}^{i-2}, \mathbf{S}_r^{i-2}, \mathbf{I}_r^{i-2}\right)\times\Lambda_1\\ &+ \frac{(\Delta t)^\rho}{\Gamma(\rho+2)}\sum_{i=2}^{k}\left[\mathscr{K}_2\left(t_{i-1}, \mathbf{S}_h^{i-1}, \mathbf{E}^{i-1}, \mathbf{I}_m^{i-1}, \mathbf{I}_s^{i-1}, \mathbf{H}^{i-1}, \mathbf{R}^{i-1}, \mathbf{V}^{i-1}, \mathbf{S}_r^{i-1}, \mathbf{I}_r^{i-1}\right)\right.\\ &\left.- \mathscr{K}_2\left(t_{i-2}, \mathbf{S}_h^{i-2}, \mathbf{E}^{i-2}, \mathbf{I}_m^{i-2}, \mathbf{I}_s^{i-2}, \mathbf{H}^{i-2}, \mathbf{R}^{i-2}, \mathbf{V}^{i-2}, \mathbf{S}_r^{i-2}, \mathbf{I}_r^{i-2}\right)\right]\times\Lambda_2\\ &+ \frac{\rho(\Delta t)^\rho}{2\Gamma(\rho+3)}\sum_{i=2}^{k}\left[\mathscr{K}_2\left(t_i, \mathbf{S}_h^{i}, \mathbf{E}^{i}, \mathbf{I}_m^{i}, \mathbf{I}_s^{i}, \mathbf{H}^{i}, \mathbf{R}^{i}, \mathbf{V}^{i}, \mathbf{S}_r^{i}, \mathbf{I}_r^{i}\right)\right.\\ &- 2\mathscr{K}_2\left(t_{i-1}, \mathbf{S}_h^{i-1}, \mathbf{E}^{i-1}, \mathbf{I}_m^{i-1}, \mathbf{I}_s^{i-1}, \mathbf{H}^{i-1}, \mathbf{R}^{i-1}, \mathbf{V}^{i-1}, \mathbf{S}_r^{i-1}, \mathbf{I}_r^{i-1}\right)\\ &\left.+ \mathscr{K}_2\left(t_{i-2}, \mathbf{S}_h^{i-2}, \mathbf{E}^{i-2}, \mathbf{I}_m^{i-2}, \mathbf{I}_s^{i-2}, \mathbf{H}^{i-2}, \mathbf{R}^{i-2}, \mathbf{V}^{i-2}, \mathbf{S}_r^{i-2}, \mathbf{I}_r^{i-2}\right)\right]\times\Lambda_3,\end{aligned}$$

$$\begin{aligned}\mathbf{I}_m(t_{k+1}) = \mathbf{I}_m(0) &+ \frac{(\Delta t)^\rho}{\Gamma(\rho+1)}\sum_{i=2}^{k}\mathscr{K}_3\left(t_{i-2}, \mathbf{S}_h^{i-2}, \mathbf{E}^{i-2}, \mathbf{I}_m^{i-2}, \mathbf{I}_s^{i-2}, \mathbf{H}^{i-2}, \mathbf{R}^{i-2}, \mathbf{V}^{i-2}, \mathbf{S}_r^{i-2}, \mathbf{I}_r^{i-2}\right)\times\Lambda_1\\ &+ \frac{(\Delta t)^\rho}{\Gamma(\rho+2)}\sum_{i=2}^{k}\left[\mathscr{K}_3\left(t_{i-1}, \mathbf{S}_h^{i-1}, \mathbf{E}^{i-1}, \mathbf{I}_m^{i-1}, \mathbf{I}_s^{i-1}, \mathbf{H}^{i-1}, \mathbf{R}^{i-1}, \mathbf{V}^{i-1}, \mathbf{S}_r^{i-1}, \mathbf{I}_r^{i-1}\right)\right.\\ &\left.- \mathscr{K}_3\left(t_{i-2}, \mathbf{S}_h^{i-2}, \mathbf{E}^{i-2}, \mathbf{I}_m^{i-2}, \mathbf{I}_s^{i-2}, \mathbf{H}^{i-2}, \mathbf{R}^{i-2}, \mathbf{V}^{i-2}, \mathbf{S}_r^{i-2}, \mathbf{I}_r^{i-2}\right)\right]\times\Lambda_2\end{aligned}$$

$$+\frac{\rho(\Delta t)^{\rho}}{2\Gamma(\rho+3)}\sum_{i=2}^{k}\Big[\mathscr{K}_3\big(t_i,\mathbf{S}_h^i,\mathbf{E}^i,\mathbf{I}_m^i,\mathbf{I}_s^i,\mathbf{H}^i,\mathbf{R}^i,\mathbf{V}^i,\mathbf{S}_r^i,\mathbf{I}_r^i\big)$$

$$-2\mathscr{K}_3\big(t_{i-1},\mathbf{S}_h^{i-1},\mathbf{E}^{i-1},\mathbf{I}_m^{i-1},\mathbf{I}_s^{i-1},\mathbf{H}^{i-1},\mathbf{R}^{i-1},\mathbf{V}^{i-1},\mathbf{S}_r^{i-1},\mathbf{I}_r^{i-1}\big)$$

$$+\mathscr{K}_3\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\Big]\times\Lambda_3,$$

$$\mathbf{I}_s(t_{k+1})=\mathbf{I}_s(0)+\frac{(\Delta t)^{\rho}}{\Gamma(\rho+1)}\sum_{i=2}^{k}\mathscr{K}_4\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\times\Lambda_1$$

$$+\frac{(\Delta t)^{\rho}}{\Gamma(\rho+2)}\sum_{i=2}^{k}\Big[\mathscr{K}_4\big(t_{i-1},\mathbf{S}_h^{i-1},\mathbf{E}^{i-1},\mathbf{I}_m^{i-1},\mathbf{I}_s^{i-1},\mathbf{H}^{i-1},\mathbf{R}^{i-1},\mathbf{V}^{i-1},\mathbf{S}_r^{i-1},\mathbf{I}_r^{i-1}\big)$$

$$-\mathscr{K}_4\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\Big]\times\Lambda_2$$

$$+\frac{\rho(\Delta t)^{\rho}}{2\Gamma(\rho+3)}\sum_{i=2}^{k}\Big[\mathscr{K}_4\big(t_i,\mathbf{S}_h^i,\mathbf{E}^i,\mathbf{I}_m^i,\mathbf{I}_s^i,\mathbf{H}^i,\mathbf{R}^i,\mathbf{V}^i,\mathbf{S}_r^i,\mathbf{I}_r^i\big)$$

$$-2\mathscr{K}_4\big(t_{i-1},\mathbf{S}_h^{i-1},\mathbf{E}^{i-1},\mathbf{I}_m^{i-1},\mathbf{I}_s^{i-1},\mathbf{H}^{i-1},\mathbf{R}^{i-1},\mathbf{V}^{i-1},\mathbf{S}_r^{i-1},\mathbf{I}_r^{i-1}\big)$$

$$+\mathscr{K}_4\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\Big]\times\Lambda_3,$$

$$\mathbf{H}(t_{k+1})=\mathbf{H}(0)+\frac{(\Delta t)^{\rho}}{\Gamma(\rho+1)}\sum_{i=2}^{k}\mathscr{K}_5\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\times\Lambda_1$$

$$+\frac{(\Delta t)^{\rho}}{\Gamma(\rho+2)}\sum_{i=2}^{k}\Big[\mathscr{K}_5\big(t_{i-1},\mathbf{S}_h^{i-1},\mathbf{E}^{i-1},\mathbf{I}_m^{i-1},\mathbf{I}_s^{i-1},\mathbf{H}^{i-1},\mathbf{R}^{i-1},\mathbf{V}^{i-1},\mathbf{S}_r^{i-1},\mathbf{I}_r^{i-1}\big)$$

$$-\mathscr{K}_5\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\Big]\times\Lambda_2$$

$$+\frac{\rho(\Delta t)^{\rho}}{2\Gamma(\rho+3)}\sum_{i=2}^{k}\Big[\mathscr{K}_5\big(t_i,\mathbf{S}_h^i,\mathbf{E}^i,\mathbf{I}_m^i,\mathbf{I}_s^i,\mathbf{H}^i,\mathbf{R}^i,\mathbf{V}^i,\mathbf{S}_r^i,\mathbf{I}_r^i\big)$$

$$-2\mathscr{K}_5\big(t_{i-1},\mathbf{S}_h^{i-1},\mathbf{E}^{i-1},\mathbf{I}_m^{i-1},\mathbf{I}_s^{i-1},\mathbf{H}^{i-1},\mathbf{R}^{i-1},\mathbf{V}^{i-1},\mathbf{S}_r^{i-1},\mathbf{I}_r^{i-1}\big)$$

$$+\mathscr{K}_5\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\Big]\times\Lambda_3,$$

$$\mathbf{R}(t_{k+1})=\mathbf{R}(0)+\frac{(\Delta t)^{\rho}}{\Gamma(\rho+1)}\sum_{i=2}^{k}\mathscr{K}_6\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\times\Lambda_1$$

$$+\frac{(\Delta t)^{\rho}}{\Gamma(\rho+2)}\sum_{i=2}^{k}\Big[\mathscr{K}_6\big(t_{i-1},\mathbf{S}_h^{i-1},\mathbf{E}^{i-1},\mathbf{I}_m^{i-1},\mathbf{I}_s^{i-1},\mathbf{H}^{i-1},\mathbf{R}^{i-1},\mathbf{V}^{i-1},\mathbf{S}_r^{i-1},\mathbf{I}_r^{i-1}\big)$$

$$-\mathscr{K}_6\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\Big]\times\Lambda_2$$

$$+\frac{\rho(\Delta t)^{\rho}}{2\Gamma(\rho+3)}\sum_{i=2}^{k}\Big[\mathscr{K}_6\big(t_i,\mathbf{S}_h^i,\mathbf{E}^i,\mathbf{I}_m^i,\mathbf{I}_s^i,\mathbf{H}^i,\mathbf{R}^i,\mathbf{V}^i,\mathbf{S}_r^i,\mathbf{I}_r^i\big)$$

$$-2\mathscr{K}_6\big(t_{i-1},\mathbf{S}_h^{i-1},\mathbf{E}^{i-1},\mathbf{I}_m^{i-1},\mathbf{I}_s^{i-1},\mathbf{H}^{i-1},\mathbf{R}^{i-1},\mathbf{V}^{i-1},\mathbf{S}_r^{i-1},\mathbf{I}_r^{i-1}\big)$$

$$+\mathscr{K}_6\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\Big]\times\Lambda_3,$$

$$\mathbf{V}(t_{k+1})=\mathbf{V}(0)+\frac{(\Delta t)^{\rho}}{\Gamma(\rho+1)}\sum_{i=2}^{k}\mathscr{K}_7\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\times\Lambda_1$$

$$+\frac{(\Delta t)^{\rho}}{\Gamma(\rho+2)}\sum_{i=2}^{k}\Big[\mathscr{K}_7\big(t_{i-1},\mathbf{S}_h^{i-1},\mathbf{E}^{i-1},\mathbf{I}_m^{i-1},\mathbf{I}_s^{i-1},\mathbf{H}^{i-1},\mathbf{R}^{i-1},\mathbf{V}^{i-1},\mathbf{S}_r^{i-1},\mathbf{I}_r^{i-1}\big)$$
$$-\mathscr{K}_7\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\Big]\times\Lambda_2$$
$$+\frac{\rho(\Delta t)^{\rho}}{2\Gamma(\rho+3)}\sum_{i=2}^{k}\Big[\mathscr{K}_7\big(t_i,\mathbf{S}_h^{i},\mathbf{E}^{i},\mathbf{I}_m^{i},\mathbf{I}_s^{i},\mathbf{H}^{i},\mathbf{R}^{i},\mathbf{V}^{i},\mathbf{S}_r^{i},\mathbf{I}_r^{i}\big)$$
$$-2\mathscr{K}_7\big(t_{i-1},\mathbf{S}_h^{i-1},\mathbf{E}^{i-1},\mathbf{I}_m^{i-1},\mathbf{I}_s^{i-1},\mathbf{H}^{i-1},\mathbf{R}^{i-1},\mathbf{V}^{i-1},\mathbf{S}_r^{i-1},\mathbf{I}_r^{i-1}\big)$$
$$+\mathscr{K}_7\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\Big]\times\Lambda_3,$$

$$\mathbf{S}_r(t_{k+1})=\mathbf{S}_r(0)+\frac{(\Delta t)^{\rho}}{\Gamma(\rho+1)}\sum_{i=2}^{k}\mathscr{K}_8\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\times\Lambda_1$$
$$+\frac{(\Delta t)^{\rho}}{\Gamma(\rho+2)}\sum_{i=2}^{k}\Big[\mathscr{K}_8\big(t_{i-1},\mathbf{S}_h^{i-1},\mathbf{E}^{i-1},\mathbf{I}_m^{i-1},\mathbf{I}_s^{i-1},\mathbf{H}^{i-1},\mathbf{R}^{i-1},\mathbf{V}^{i-1},\mathbf{S}_r^{i-1},\mathbf{I}_r^{i-1}\big)$$
$$-\mathscr{K}_8\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\Big]\times\Lambda_2$$
$$+\frac{\rho(\Delta t)^{\rho}}{2\Gamma(\rho+3)}\sum_{i=2}^{k}\Big[\mathscr{K}_8\big(t_i,\mathbf{S}_h^{i},\mathbf{E}^{i},\mathbf{I}_m^{i},\mathbf{I}_s^{i},\mathbf{H}^{i},\mathbf{R}^{i},\mathbf{V}^{i},\mathbf{S}_r^{i},\mathbf{I}_r^{i}\big)$$
$$-2\mathscr{K}_8\big(t_{i-1},\mathbf{S}_h^{i-1},\mathbf{E}^{i-1},\mathbf{I}_m^{i-1},\mathbf{I}_s^{i-1},\mathbf{H}^{i-1},\mathbf{R}^{i-1},\mathbf{V}^{i-1},\mathbf{S}_r^{i-1},\mathbf{I}_r^{i-1}\big)$$
$$+\mathscr{K}_8\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\Big]\times\Lambda_3,$$

$$\mathbf{I}_r(t_{k+1})=\mathbf{I}_r(0)+\frac{(\Delta t)^{\rho}}{\Gamma(\rho+1)}\sum_{i=2}^{k}\mathscr{K}_9\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\times\Lambda_1$$
$$+\frac{(\Delta t)^{\rho}}{\Gamma(\rho+2)}\sum_{i=2}^{k}\Big[\mathscr{K}_9\big(t_{i-1},\mathbf{S}_h^{i-1},\mathbf{E}^{i-1},\mathbf{I}_m^{i-1},\mathbf{I}_s^{i-1},\mathbf{H}^{i-1},\mathbf{R}^{i-1},\mathbf{V}^{i-1},\mathbf{S}_r^{i-1},\mathbf{I}_r^{i-1}\big)$$
$$-\mathscr{K}_9\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\Big]\times\Lambda_2$$
$$+\frac{\rho(\Delta t)^{\rho}}{2\Gamma(\rho+3)}\sum_{i=2}^{k}\Big[\mathscr{K}_9\big(t_i,\mathbf{S}_h^{i},\mathbf{E}^{i},\mathbf{I}_m^{i},\mathbf{I}_s^{i},\mathbf{H}^{i},\mathbf{R}^{i},\mathbf{V}^{i},\mathbf{S}_r^{i},\mathbf{I}_r^{i}\big)$$
$$-2\mathscr{K}_9\big(t_{i-1},\mathbf{S}_h^{i-1},\mathbf{E}^{i-1},\mathbf{I}_m^{i-1},\mathbf{I}_s^{i-1},\mathbf{H}^{i-1},\mathbf{R}^{i-1},\mathbf{V}^{i-1},\mathbf{S}_r^{i-1},\mathbf{I}_r^{i-1}\big)$$
$$+\mathscr{K}_9\big(t_{i-2},\mathbf{S}_h^{i-2},\mathbf{E}^{i-2},\mathbf{I}_m^{i-2},\mathbf{I}_s^{i-2},\mathbf{H}^{i-2},\mathbf{R}^{i-2},\mathbf{V}^{i-2},\mathbf{S}_r^{i-2},\mathbf{I}_r^{i-2}\big)\Big]\times\Lambda_3. \tag{80}$$

5. Numerical Simulation

Fractional differential equations were used in the mathematical investigation of the dynamics of Lassa fever transmission from rodents to people and from person to person and in infected areas. Several ρ values are taken into consideration in order to conduct a reliable study. For solutions created using the Caputo fractional operator, we give simulation using MATLAB in this section. The suggested system's parameter values and initial circumstances from [2] are $\mathbf{S}_h(0)=1.13\times10^8$, $\mathbf{E}(0)=164$, $\mathbf{I}_m(0)=20$, $\mathbf{I}_s(0)=9$, $\mathbf{H}(0)=6$, $\mathbf{R}(0)=2$, $\mathbf{V}(0)=10\times10^3$, $\mathbf{S}_r(0)=\mathbf{S}_h(0)\times10^{-2}$, $\mathbf{I}_r(0)=76$, $\Phi_h=2500$, $\Phi_r=0.1$, $\varphi=0.0067$, $\xi_{\mathbf{I}_m}=0.22$, $\xi_{\mathbf{I}_s}=0.19$, $\xi_{\mathbf{I}_{vh}}=0.12$, $\xi_{\mathbf{I}_r}=0.142$, $\xi_{\mathbf{I}_{vr}}=0.15$, $\lambda=0.52$, $\chi=0.32$, $\beta_{\mathbf{I}_m}=0.0517$, $\beta_{\mathbf{I}_s}=0.031$, $\beta_h=0.035$, $\kappa_{\mathbf{I}_m}=0.0123$, $\kappa_{\mathbf{I}_s}=0.012$, $\mu_{\mathbf{I}_s}=0.2$, $\mu_h=0.19$, $\alpha_h=10^2-10^4$, and $\alpha_r=10^3-10^5$.

We address the simulation of susceptible people $\mathbf{S}_h(t)$ under various fractional orders $\rho=0.85,0.90,0.95,1$ as shown in Figure 1. Humans who are susceptible are becoming more prevalent for lower fractional-order values. We describe the simulation of the exposed humans $\mathbf{E}$ as shown in Figure 2, which demonstrates how the exposed individuals are growing faster as fractional-order values rise.

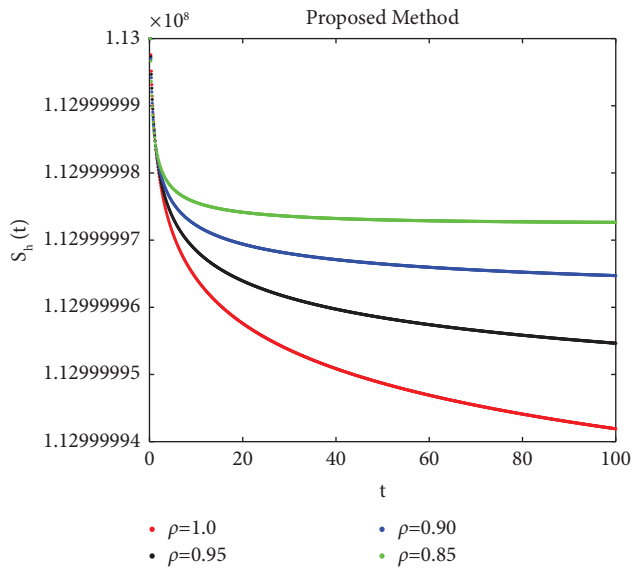

Figure 1: Simulation of $\mathbf{S}_h(t)$.

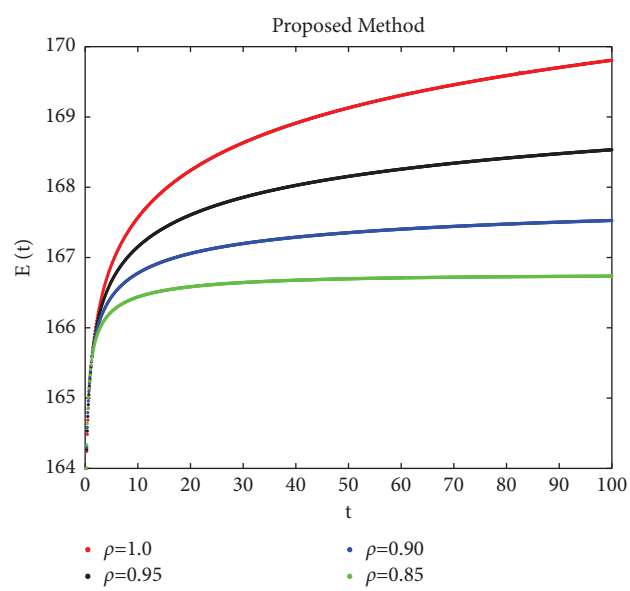

Figure 2: Simulation of $\mathbf{E}(t)$.

The simulations of those with mild-to-moderate symptoms $\mathbf{I}_m$ and $\mathbf{I}_s$ those with severe symptoms are shown in Figures 3 and 4, respectively. These infected classes increase in number as fractional order increases. As seen in Figure 5, the number of hospitalized individuals $\mathbf{H}(t)$ is rising by lowering fractional order and reducing by increasing fractional order. Additionally, the number of people who have recovered, $\mathbf{R}$, is rising by high fractional orders, as illustrated in Figure 6. As fractional order is increased, the concentration of LASV in the environment $\mathbf{V}$ increases more quickly, as seen in Figure 7. Figures 8 and 9 depict the simulation of susceptible and infected rodents, respectively. While the number of infected rats increases as fractional order increases, the number of susceptible rodents swiftly decreases.

The significant chance of infection stems from the availability of a large number of susceptible individuals, which causes the initial quick rise of the disease. However, as a result of fewer contacts and fewer vulnerable hosts or vectors, the disease's self-limitation causes an infection rate to go down. There will be fewer susceptible people, which lowers the chance of new infections. The memory effect increases in the epidemiological system as the fractional order decreases, leading to a gradual increase but significant long-term equilibrium value. Although the fractional-order model forecasts smaller epidemic peak levels, it also predicts a prolonged period of elevated disease prevalence in the general population. Simulations of the proposed design show that the total

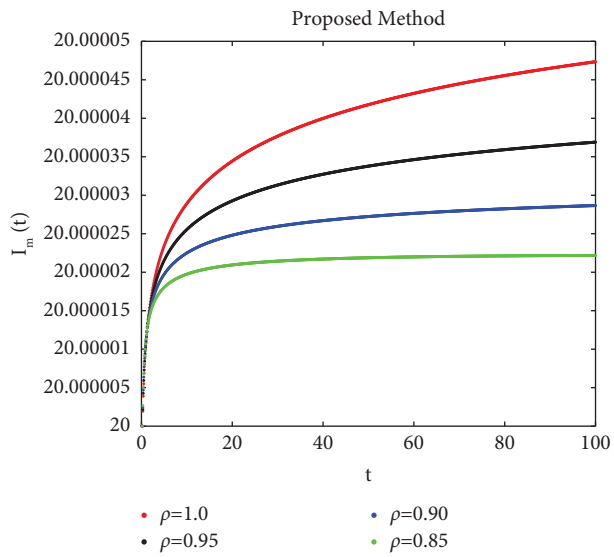

Figure 3: Simulation of $\mathbf{I}_m(t)$.

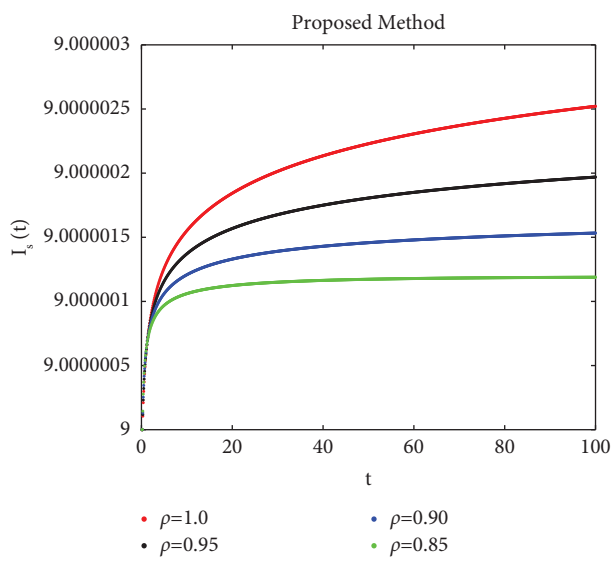

Figure 4: Simulation of $\mathbf{I}_s(t)$.

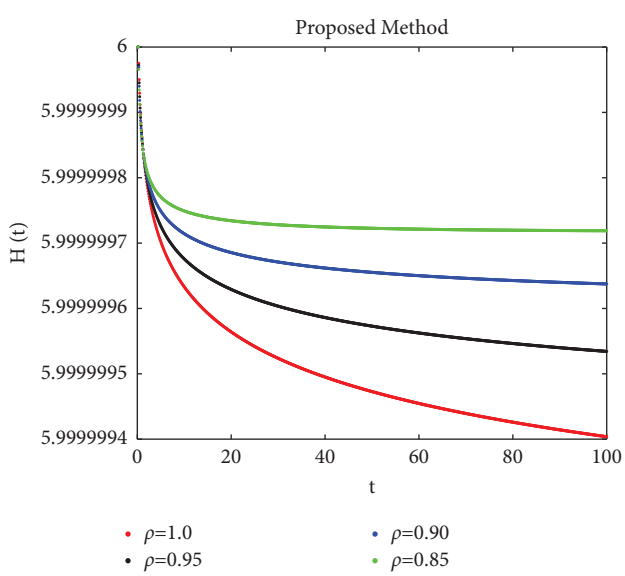

Figure 5: Simulation of $\mathbf{H}(t)$.

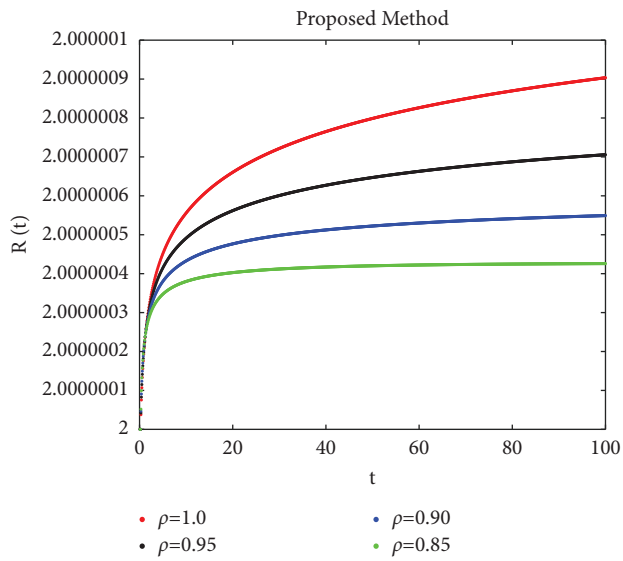

Figure 6: Simulation of $\mathbf{R}(t)$.

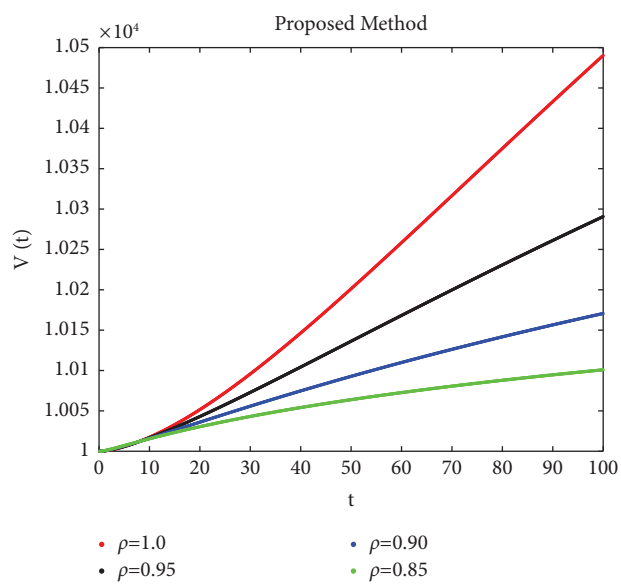

Figure 7: Simulation of $\mathbf{V}(t)$.

density of all the compartments suitable for analyzing internal behavior will lie between 0 and 1. Graphical results show how the fractional-order model works, with protracted phrases greatly increasing its efficiency. The fractional-order Lassa fever model is more flexible and can be changed to get different answers from each of the model's compartments. Every compartment's initial zero-slope curve has a considerable increase. In understanding physical processes, fractional-order derivations outperform classical integer-order models.

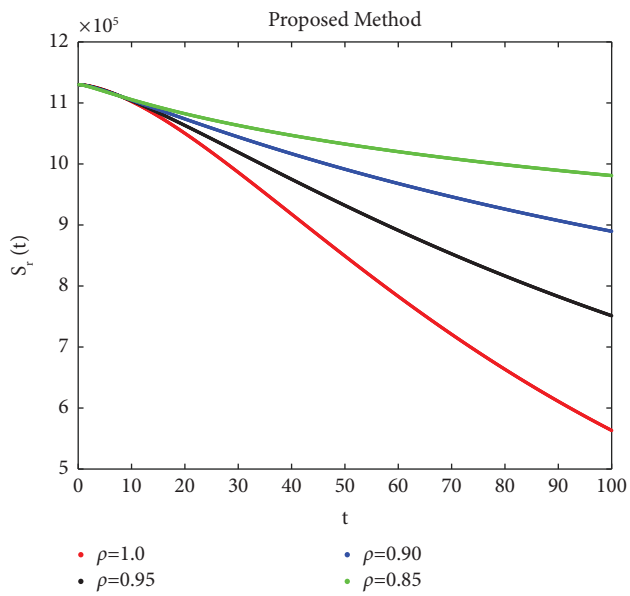

Figure 8: Simulation of $\mathbf{S}_r(t)$.

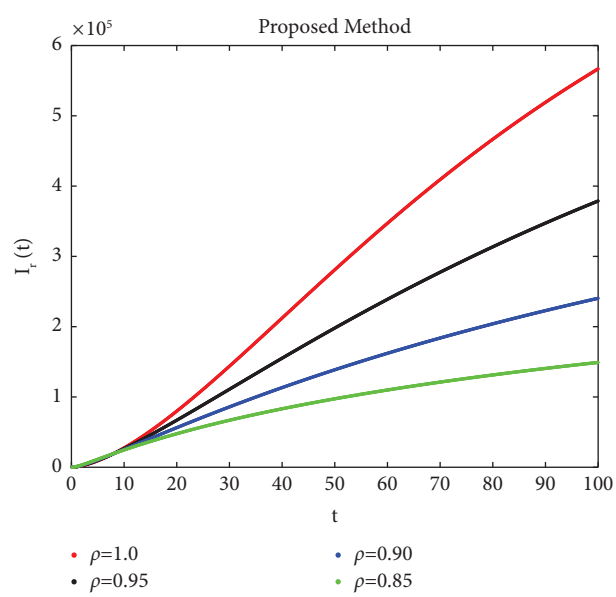

Figure 9: Simulation of $\mathbf{I}_r(t)$.

6. Conclusion

This study proposes a fractional-order model to study the transmission of Lassa fever, considering both mild and severe illness and the environment's impact. The model uses nine compartments and uses the Caputo fractional operator to obtain solutions. The model addresses both qualitative and quantitative concerns, satisfying biological feasibility. It also includes global stability analysis using the Lyapunov function and a singular solution discovered using fixed point theory. A numerical approach based on Newton's polynomial interpolation solves the model. Results show significant changes in the Lassa virus's behavior, useful for understanding the condition and developing effective control measures. The Caputo fractional derivative allows for more accurate and effective modeling of the Lassa fever model, enabling a better understanding of the spread of illness and transmission. Policymakers and public health experts can use the study's findings to prevent the spread of the Lassa virus. When the results were compared with different fractional orders, the study discovered that the results were stronger when the order was equal to one. Every propagation route has an impact on the development of Lassa fever, according to our system modeling, with some pathways having a noticeably greater impact than others. Therefore, when creating health strategies, measures in these fields should be avoided or disregarded. This research can be expanded to examine the role that disease awareness plays in

improving comprehension of the condition and facilitating the widespread application of preventative measures.

Data Availability

The information that justifies the study's conclusions is included in the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed equally to the manuscripts.

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