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Research Article

Problem Involving the Riemann–Liouville Derivative and Implicit Variable-Order Nonlinear Fractional Differential Equations

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The problem of boundary values for implicit differential equations with nonlinear fractions involving the variable order and the Riemann–Liouville derivative is examined in this article along with its existence and stability. Specifically, the locally solvability, which is equivalent to the existence of solutions, is related to the symmetry of a transformation of a nonlinear equations system. To demonstrate the reliability of the found results, we design an example.

1. Introduction

The subject of fractional calculus has gained much attention and importance among the society of researchers. The existing differential equations in this theory are determined by generalizing integer-order derivatives to arbitrary order ones. For the sake of the effective memory of the fractional derivation operator, such classes of equations have been widely used in mathematical modeling including parameter identification in the 2D fractional system and modeling of heat distribution in porous aluminum (see [1–5]).

Recently, several researchers contributed in this field with many published papers that are concerned with the study to many different problems of fractional differential equations, e.g., Borisut et al. [6] presented the ψ-Hilfer fractional differential equation with nonlocal multipoint condition, Ahmad et al. [7] investigated a fractional-order compartmental HIV and malaria coinfection epidemic model using the Caputo derivative, and Özer et al. [8–12] established the existence and uniqueness of the common fixed point theorem in c^*-algebra valued b-metric spaces, see [6, 13–26].

The existence of the solutions variable-order problems is rarely discussed in literature [27–37]; specifically, Souid et al. [38, 39] presented the existence, uniqueness, and stability of solutions to many different problems (implicit, thermostat, and resonance).

Ulam-type stability is often studied in the context of various types of differential equations, including ordinary, partial, fractional, and integrodifferential equations, among others. This concept has broad applications in various fields of science and engineering, such as the control theory, signal processing, and physics, where the stability of solutions under small perturbations is a crucial consideration [40].

In [41, 42], the presence of implicit nonlinear differential equations involving fractional of constant order is studied by Benchohra et al.

$$\begin{cases} {}^{c}D_{0^+}^{u}\kappa(s) = \hbar\left(s, \kappa(s), {}^{c}D_{0^+}^{u}\kappa(s)\right), s \in \gamma := [0, \Upsilon], 0 < \Upsilon < +\infty, 1 < u \leq 2, \\ \kappa(0) = \kappa_0, \kappa(\Upsilon) = \kappa_1. \end{cases} \tag{1}$$

When $\hbar: \gamma \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined, $\kappa_0, \kappa_1 \in \mathbb{R}$, and ${}^{c}D_{0^+}^{u}$ is the Caputo fractional derivative.

From [41, 42] and [32, 43–45], we solved the boundary value problem (PVB) as follows:

$$\begin{cases} D_{\eta^+}^{u(s)}\kappa(s) + \hbar\left(s, \kappa(s), D_{\eta^+}^{u(s)}\kappa(s)\right) = 0, & s \in \gamma := [\eta, \Upsilon], \\ \kappa(\eta) = 0, \quad \kappa(\Upsilon) = 0, \end{cases} \tag{2}$$

where $0 < \eta < \Upsilon < +\infty$, $1 < u(s) \leq 2$, $\hbar: \gamma \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, and $D_{\eta^+}^{u(s)}$ and $I_{\eta^+}^{u(s)}$ are the Riemann–Liouville fractional derivative and variable-order integral $u(s)$.

The organization of this paper is outlined as follows. Basic and crucial topics, including definitions and theorems, are covered in Section 2. The main results, which offer two main theorems of uniqueness and existence, are found in Section 3. Section 4 discusses stability in the sense of Ulam and Hyres. One example is presented in Section 5 to show the efficiency and validity of the proposed results. Finally, some conclusion notes are given in Section 6.

2. Preliminaries

This section introduces a few crucial, essential definitions that are necessary to understand in order to get our outcomes in the next sections.

The representation of the space of continuous Banach functions $C(\gamma, \mathbb{R})$ is $y: \gamma \longrightarrow \mathbb{R}$ using the norm

$$\|y\| = \sup\{|y(s)|: s \in \gamma\}. \tag{3}$$

In the case of $-\infty < \sigma_1 < \sigma_2 < +\infty$, we take into account $u(s): [\sigma_1, \sigma_2] \longrightarrow (0, +\infty)$ and $v(s): [\sigma_1, \sigma_2] \longrightarrow (n-1, n)$. The function $\Psi_1(s)$ is, therefore, represented by the variable-order left Riemann–Liouville fractional integral (FIRL) $u(s)$ (see [32, 44, 46]).

$$I_{\sigma_1^+}^{u(s)}\Psi_1(s) = \int_{\sigma_1}^{s} \frac{(s-\pi)^{u(s)-1}}{\Gamma(u(s))}\Psi_1(\pi)\mathrm{d}\pi, \quad s > \sigma_1, \tag{4}$$

and the variable-order $v(s)$ of the function $\Psi_1(s)$'s fractional derivative of Riemann–Liouville on the left (FDRL) (see [32, 44, 46]) is

$$D_{\sigma_1^+}^{u(s)}\Psi_1(s) = \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^n I_{\sigma_1^+}^{n-u(s)}\Psi_1(s) = \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^n \int_{\sigma_1}^{s} \frac{(s-\pi)^{n-u(s)-1}}{\Gamma(n-u(s))}\Psi_1(\pi)\mathrm{d}\pi, \quad s > \sigma_1, \tag{5}$$

where $\Gamma(.)$ is the gamma function.

As expected, FDRL and FIRL correspond to the usual Riemann–Liouville fractional derivative and integral, together, in the case where $u(s)$ and $v(s)$ are constant; for example, see [43, 44, 46].

Remember the succeeding crucial finding.

Lemma 1 (see [43]). *Let $\alpha, \beta > 0$, $\sigma_1 > 0$, $\Psi_1 \in L(\sigma_1, \sigma_2)$, and $D_{\sigma_1^+}^{\alpha}\Psi_1 \in L(\sigma_1, \sigma_2)$. Consequently, the differential equation*

$$D_{\sigma_1^+}^{\alpha}\Psi_1 = 0 \tag{6}$$

has a singular solution

$$\begin{aligned} \Psi_1(s) &= \Pi_1(s-\sigma_1)^{\alpha-1} + \Pi_2(s-\sigma_1)^{\alpha-2} + \cdots + \Pi_n(s-\sigma_1)^{\alpha-n}, \\ I_{\sigma_1^+}^{\alpha}D_{\sigma_1^+}^{\alpha}\Psi_1(s) &= \Psi_1(s) + \Pi_1(s-\sigma_1)^{\alpha-1} + \Pi_2(s-\sigma_1)^{\alpha-2} + \cdots + \Pi_n(s-\sigma_1)^{\alpha-n}, \end{aligned} \tag{7}$$

where $\Pi_m \in \mathbb{R}$, $m = 1, 2, \ldots, n$; here, $n-1 < \alpha \leq n$.

Furthermore,

$$\begin{aligned} D_{\sigma_1^+}^{\alpha}I_{\sigma_1^+}^{\alpha}\Psi_1(s) &= \Psi_1(s), \\ I_{\sigma_1^+}^{\alpha}I_{\sigma_1^+}^{\beta}\Psi_1(s) &= I_{\sigma_1^+}^{\beta}I_{\sigma_1^+}^{\alpha}\Psi_1(s) = I_{\sigma_1^+}^{\alpha+\beta}\Psi_1(s). \end{aligned} \tag{8}$$

Remark 2 (see [47, 48]). Be aware that the general functions $u(s)$ and $v(s)$ do not satisfy the semigroup property, i.e.,

$$I_{\sigma_1^+}^{u(s)}I_{\sigma_1^+}^{v(s)}\Psi_1(s) \neq I_{\sigma_1^+}^{u(s)+v(s)}\Psi_1(s). \tag{9}$$

Example 1. Assuming that

$$\begin{aligned} u(s) &= \begin{cases} 1, & s \in [0, 1], \\ 1, & s \in \,]1, 2], \end{cases} \\ v(s) &= \begin{cases} 2, & s \in [0, 1], \\ 3, & s \in \,]1, 2], \end{cases} \end{aligned} \tag{10}$$

and $\Psi_1(s) = s/3, s \in [0,2]$. Then, we get

$$\begin{aligned}
I_{0^+}^{u(s)} I_{0^+}^{v(s)} \Psi_1(s) &= \int_0^1 \frac{(s-\nu)^{u(s)-1}}{\Gamma(u(s))} \int_0^\nu \frac{(\nu-\tau)^{v(\nu)-1}}{\Gamma(v(\nu))} \Psi_1(\tau) \mathrm{d}\tau \mathrm{d}\nu \\
&\quad + \int_1^s \frac{(s-\nu)^{u(s)-1}}{\Gamma(u(s))} \int_0^\nu \frac{(\nu-\tau)^{v(\nu)-1}}{\Gamma(v(\nu))} \Psi_1(\tau) \mathrm{d}\tau \mathrm{d}\nu \\
&= \int_0^1 \frac{(s-\nu)^0}{\Gamma(1)} \int_0^\nu \frac{(\nu-\tau)^1}{\Gamma(2)} \frac{\tau}{3} \mathrm{d}\tau \mathrm{d}\nu \\
&\quad + \int_1^s \frac{(s-\nu)^0}{\Gamma(1)} \left[\int_0^1 \frac{(\nu-\tau)^1}{\Gamma(2)} \frac{\tau}{3} \mathrm{d}\tau + \int_1^\nu \frac{(\nu-\tau)^2}{\Gamma(3)} \frac{\tau}{3} \mathrm{d}\tau \right] \mathrm{d}\nu \\
&= \frac{1}{3} \int_0^1 \left(\frac{\nu^3}{2} - \frac{\nu^3}{3} \right) \mathrm{d}\nu + \int_1^s \left[\frac{1}{3} \left(\frac{\nu^3}{2} - \frac{\nu^3}{3} \right) + \frac{1}{6} \left(\frac{\nu^4}{12} - \frac{\nu^2}{2} + \frac{2}{3}\nu - \frac{1}{4} \right) \right] \mathrm{d}\nu.
\end{aligned} \tag{11}$$

It can be seen that the following equations are satisfied:

$$\begin{aligned}
I_{0^+}^{u(s)} I_{0^+}^{v(s)} \Psi_1(s)|_{s=2} &= \frac{1}{72} + \frac{1}{3} \int_1^2 \left(\frac{\nu^4}{24} + \frac{\nu^3}{6} - \frac{\nu^2}{4} + \frac{s}{3} - \frac{1}{24} \right) \mathrm{d}\nu \\
&= \frac{96}{360}, \\
I_{0^+}^{u(s)+v(s)} \Psi_1(s)|_{s=2} &= \int_0^1 \frac{(2-\nu)^{1+2-1}}{\Gamma(1+2)} \frac{\nu}{3} \mathrm{d}\nu + \int_1^2 \frac{(2-\nu)^{1+3-1}}{\Gamma(1+3)} \frac{\nu}{3} \mathrm{d}\nu \\
&= \frac{11}{72} + \frac{3}{180} = \frac{61}{360}.
\end{aligned} \tag{12}$$

Therefore, we obtain

$$I_{0^+}^{u(s)} I_{0^+}^{v(s)} \Psi_1(s)|_{s=2} \neq I_{0^+}^{u(s)+v(s)} \Psi_1(s)|_{s=2}. \tag{13}$$

Lemma 3 (see [45]). *Let $u\colon \gamma \longrightarrow (1,2]$ be a continuous function, then the variable order fractional integral $I_{0^+}^{u(s)}$ $y(s)$ exists for any points on gamma if $y \in C_\rho(\gamma, X) = \{y(s) \in C(\gamma, X),\ s^\rho y(s) \in C(\gamma, X)\}$, $(0 \le \rho \le \min_{s\in\gamma} |u(s)|)$.*

Lemma 4 (see [45]). *Let $u\colon \gamma \longrightarrow (1,2]$ be a continuous function, then $I_{0^+}^{u(s)} y(s) \in C(\gamma, X)$ for $y \in C(\gamma, X)$.*

Definition 5 (see [49–51]). If I of $\mathbb{R}$ is either an interval, a point $\{a_1\}$, or the empty set, it is referred to as a generalized interval. A partition of I is a finite set $\mathscr{P}$ if every x in I falls in precisely one of the generalized intervals E in $\mathscr{P}$. If a function $g\colon I \longrightarrow X$ is constant on E for every $E \in \mathscr{P}$, it is referred to as a piecewise constant with regards to the partition of I.

Theorem 6 (see [43]). *Let G be a Banach space and P be a convex subset of Gand $L\colon P \longrightarrow P$ is compact and is the continuous map. Then, L has at least one fixed point in P.*

Definition 7 (see [52]). If there is a real number $c_\hbar > 0$ such that (for each) $\epsilon > 0$ and (for each) solution $\chi \in C(\gamma_m, \mathbb{R})$, then problem (1) is Ulam–Hyers stable. For the price of the inequality

$$\left| D^{u_m} \chi(s) + \hbar(s, \chi(s), D^{u_m} \chi(s)) \right| \le \epsilon, s \in \gamma_m, \tag{14}$$

there exists a solution $\kappa \in C(\gamma_m, \mathbb{R})$ of PVB (2) with

$$|\chi(s) - \kappa(s)| \le c_\hbar \epsilon, s \in \gamma_m. \tag{15}$$

3. Main Results

Let us state the underlying presumptions:

(P1) Let n be an integer in $\mathbb{N}$, $\mathscr{P} = \{\gamma_1 := [\eta, \Upsilon_1], \gamma_2 := (\Upsilon_1, \Upsilon_2], \gamma_3 := (\Upsilon_2, \Upsilon_3], \ldots \gamma_n := (\Upsilon_{n-1}, \Upsilon]\}$ be a partition of γ, and $u(s)\colon \gamma \longrightarrow (1,2]$ be a piecewise constant function, i.e.,

$$u(s) = \sum_{m=1}^{n} u_m I_m(s) = \begin{cases} u_1, & if\ s \in \gamma_1, \\ u_2, & if\ s \in \gamma_2, \\ \cdot & \\ \cdot & \\ \cdot & \\ u_n, & if\ s \in \gamma_n, \end{cases} \quad (16)$$

where $1 < u_m \leq 2$ are constants, and I_m is the indicator of the interval $\gamma_m := (\Upsilon_{m-1}, \Upsilon_m], m = 1, 2, \ldots, n$ (with $\Upsilon_0 = \eta, \Upsilon_n = \Upsilon$) such that

$$I_m(s) = \begin{cases} 1, & \text{for}\ s \in \gamma_m, \\ 0, & \text{for elsewhere}. \end{cases} \quad (17)$$

(P2) Let $s^\rho \hbar$: $\gamma \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function $(0 \leq \rho < 1)$ and there exists $\varpi_1, \varpi_2 > 0$, with $0 < s^{-\rho}\varpi_2 < 1$, such that $s^\gamma |\hbar(s, u_1, v_1) - \hbar(s, u_2, v_2)| \leq \varpi_1 |u_1 - u_2| + \varpi_2 |v_1 - v_2|$, for any $u_1, u_2, v_1, v_2 \in \mathbb{R}$, and $s \in \gamma$.

The Banach space of continuous functions from γ_m into $\mathbb{R}$ is denoted by $E_m = C(\gamma_m, \mathbb{R})$, where $m \in \{1, 2, \ldots, n\}$ with the norm

$$\|\kappa\|_{E_m} = \sup_{s \in \gamma_m} |\kappa(t)|. \quad (18)$$

We first provide an essential study concerning (2) in order to arrive at our main findings.

For any $s \in (\Upsilon_{m-1}, \Upsilon_m], m = 1, \ldots, n$, the FDRL of the variable order $u(s)$ for $\kappa(s) \in C(\gamma, \mathbb{R})$, given by (5), is the sum of the FDRLs of the constant orders $u_1, \ldots, u_m$, i.e.,

$$\frac{d^2}{ds^2} \int_\eta^s \frac{(s - \nu)^{1-u(\nu)}}{\Gamma(2 - u(\nu))} \kappa(\nu) d\nu = \frac{d^2}{ds^2} \left(\int_\eta^{\Upsilon_1} \frac{(s - \nu)^{1-u_1}}{\Gamma(2 - u_1)} \kappa(\nu) d\nu + \cdots + \int_{\Upsilon_{m-1}}^s \frac{(s - \nu)^{1-u_m}}{\Gamma(2 - u_m)} \kappa(\nu) d\nu \right). \quad (19)$$

Thus, according to (19), the (2) of the variable order can be written for any $s \in (\Upsilon_{m-1}, \Upsilon_m]$ in the form

$$\frac{d^2}{ds^2} \left(\int_\eta^{\Upsilon_1} \frac{(s - \nu)^{1-u_1}}{\Gamma(2 - u_1)} \kappa(\nu) d\nu + \cdots + \int_{\Upsilon_{m-1}}^s \frac{(s - \nu)^{1-u_m}}{\Gamma(2 - u_m)} \kappa(\nu) d\nu \right) + \hbar\left(s, \kappa(s), D_{\eta^+}^{u_m} \kappa(s)\right) = 0. \quad (20)$$

Here is a definition of the solution to PVB (2).

Definition 8. If there are functions $\kappa_m (m = 1, 2, \ldots, n)$, then we can say that the boundary value problem (1) has a solution such that $\kappa_m \in C([\eta, \Upsilon_m], \mathbb{R})$ satisfying equation (5) and $\kappa_m(\eta) = 0 = \kappa_m(\Upsilon_m)$.

According to the abovementioned analysis, the equation of PVB (2) can be expressed as (??), which is translated as (20) in $\gamma_m, m \in \{1, 2, \ldots, n\}$.

For $\eta \leq s \leq T_{m-1}$, we take $\kappa(s) \equiv 0$, then (20) is written as follows:

$$D_{\Upsilon_{m-1}^+}^{u_m} \kappa(s) + \hbar\left(s, \kappa(s), D_{\Upsilon_{m-1}^+}^{u_m} \kappa(s)\right) = 0, \quad s \in \gamma_m. \quad (21)$$

Now, we consider the following boundary value problem:

$$\begin{cases} D_{\Upsilon_{m-1}^+}^{u_m} \kappa(s) + \hbar\left(s, \kappa(s), D_{\Upsilon_{m-1}^+}^{u_m} \kappa(s)\right) = 0, & s \in \gamma_m \\ \kappa(\Upsilon_{m-1}) = 0, \quad \kappa(\Upsilon_m) = 0. \end{cases} \quad (22)$$

We utilize auxiliary lemma to prove that there are solutions to problem (22).

Lemma 9. *If and only if a function $\kappa \in E_m$ holds the following integral equation:*

$$\kappa(s) = \int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s, s) \hbar\left(s, \int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s, \tau) \vartheta(\tau) d\tau, \vartheta(s)\right) ds, \quad (23)$$

it forms the solution to problem (22), where $D_{\Upsilon_{m-1}^+}^{u_m} \kappa(s) = \vartheta(s)$ and $G_m(s, \nu)$ is the Green's function defined as follows:

$$G_m(s,\nu)=\begin{cases}\dfrac{1}{\Gamma(u_m)}\left[(\Upsilon_m-\Upsilon_{m-1})^{1-u_m}(s-\Upsilon_{m-1})^{u_m-1}(\Upsilon_m-\nu)^{u_m-1}-(s-\nu)^{u_m-1}\right],\\ \Upsilon_{m-1}\leq\nu\leq s\leq\Upsilon_m,\\ \dfrac{1}{\Gamma(u_m)}(\Upsilon_m-\Upsilon_{m-1})^{1-u_m}(s-\Upsilon_{m-1})^{u_m-1}(\Upsilon_m-\nu)^{u_m-1},\\ \Upsilon_{m-1}\leq s\leq\nu\leq\Upsilon_m.\end{cases}\tag{24}$$

Proof. Let $\kappa\in E_m$ be a solution of the PVB (22). Now, we take $D^{u_m}_{\Upsilon^+_{m-1}}\kappa(s)=\vartheta(s)$ and use the $I^{u_m}_{\Upsilon^+_{m-1}}$ operator to each of these sides of (22). According to Lemma 1, we obtain

$$\kappa(s)=\Pi_1(s-\Upsilon_{m-1})^{u_m-1}+\Pi_2(s-\Upsilon_{m-1})^{u_m-2}-I^{u_m}_{\Upsilon^+_{m-1}}\vartheta(s),\quad s\in\gamma_m.\tag{25}$$

By $\kappa(T_{m-1})=0$ and the assumption of function $\hbar$, we could get $\Pi_2=0$.

If $\kappa(s)$ is satisfying $\kappa(\Upsilon_m)=0$, thus we can get $\Pi_1=(\Upsilon_m-\Upsilon_{m-1})^{1-u_m}I^{u_m}_{\Upsilon^+_{m-1}}\vartheta(\Upsilon_m)$. So, we obtain

$$\kappa(s)=(\Upsilon_m-\Upsilon_{m-1})^{1-u_m}(s-\Upsilon_{m-1})^{u_m-1}I^{u_m}_{\Upsilon^+_{m-1}}\vartheta(\Upsilon_m)-I^{u_m}_{\Upsilon^+_{m-1}}\vartheta(s),\quad s\in\gamma_m.\tag{26}$$

The problem's solution (22) is then provided by

$$\begin{aligned}\kappa(s)&=(\Upsilon_m-\Upsilon_{m-1})^{1-u_m}(s-\Upsilon_{m-1})^{u_m-1}\frac{1}{\Gamma(u_m)}\int_{\Upsilon_{m-1}}^{\Upsilon_m}(\Upsilon_m-\nu)^{u_m-1}\vartheta(\nu)\mathrm{d}\nu\\&\quad-\frac{1}{\Gamma(u_m)}\int_{\Upsilon_{m-1}}^{s}(s-\nu)^{u_m-1}\vartheta(\nu)\mathrm{d}\nu\\&=\frac{1}{\Gamma(u_m)}\bigg[\int_{\Upsilon_{m-1}}^{s}\left[(\Upsilon_m-\Upsilon_{m-1})^{1-u_m}(s-\Upsilon_{m-1})^{u_m-1}(\Upsilon_m-\nu)^{u_m-1}-(s-\nu)^{u_m-1}\right]\vartheta(\nu)\mathrm{d}\nu\\&\quad+\int_{s}^{\Upsilon_m}(\Upsilon_m-\Upsilon_{m-1})^{1-u_m}(s-\Upsilon_{m-1})^{u_m-1}(\Upsilon_m-\nu)^{u_m-1}\vartheta(\nu)\mathrm{d}\nu\bigg].\end{aligned}\tag{27}$$

For the implied continuity of Green's function,

$$\kappa(s)=\int_{\Upsilon_{m-1}}^{T_m}G_m(s,\nu)\hbar\left(\nu,\int_{\Upsilon_{m-1}}^{\Upsilon_m}G_m(s,\tau)\vartheta(\tau)\mathrm{d}\tau,\vartheta(\nu)\right)\mathrm{d}\nu.\tag{28}$$

On the other hand, consider $\kappa\in E_m$ as the integral equation's solution (23). Then, it is evident that κ is the solution to problem (22) due to the continuity of the function $t^{\rho}\hbar$ and Lemma 1.

There will be a need for the following proposition. □

Proposition 10 (see [51]). *Considering that* $s^{\rho}\hbar\colon\gamma\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$, $(0\leq\rho<1)$ *is a continuous function and* $u(s)\colon\gamma\longrightarrow(1,2]$ *meets P1, the following criteria are met by Green's functions of problem (6):*

(1) $\forall \Upsilon_{m-1} \leq s, \nu \leq \Upsilon_m$, $G_m(s,\nu) \geq 0$,

(2) $\max_{s\in\gamma_m} G_m(s,\nu) = G_m(\nu,\nu), \nu \in \gamma_m$,

(3) $G_m(\nu,\nu)$ *has one unique maximum given by*

$$\max_{s\in\gamma_m} G_m(\nu,\nu) = \frac{1}{\Gamma(u_m)}\left(\frac{T_m - \Upsilon_{m-1}}{4}\right)^{u_m-1}, \quad (29)$$

for $m = 1, 2, \ldots, n$.

According to Theorem 6, our first existence result is as follows:

Theorem 11. *Suppose the hypotheses* $(P1)$ *and* $(P2)$ *are valid and the following inequality is satisfied*

$$\frac{\varpi_1 (\Upsilon_m - \Upsilon_{m-1})^{u_m-1}\left(\Upsilon_m^{1-\rho} - \Upsilon_{m-1}^{1-\rho}\right)}{(1-\rho)(1-\varpi_2\Upsilon_{m-1}^{-\rho})\Gamma(u_m)} < \frac{1}{4^{1-u_m}}. \quad (30)$$

Then, at least one solution for the PVB (6) exists on E_m.

Proof. Consider the operator

$$S_1\colon E_m \longrightarrow E_m, \quad (31)$$

defined by

$$S_1\kappa(s) = \int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s,\nu)\vartheta(\nu)\mathrm{d}\nu, s \in \gamma_m, \quad (32)$$

where

$$\vartheta(s) = \hbar(s, \kappa(s), \vartheta(s)). \quad (33)$$

The operator $S_1\colon E_m \longrightarrow E_m$ defined in (32) is well defined, as evidenced by the characteristics of fractional integrals and the continuity of the function $s^\rho \hbar$.

Now, let us consider

$$R_m \geq \frac{\hbar^* (\Upsilon_m - \Upsilon_{m-1})^{u_m}/4^{u_m-1}(1-\varpi_2\Upsilon_{m-1}^{-\rho})\Gamma(u_m)}{1 - \varpi_1(\Upsilon_m - \Upsilon_{m-1})^{u_m-1}\left(\Upsilon_m^{1-\rho} - \Upsilon_{m-1}^{1-\rho}\right)/4^{u_m-1}(1-\rho)(1-\varpi_2 T_{m-1}^{-\rho})\Gamma(u_m)}, \quad (34)$$

where

$$\hbar^* = \sup_{s\in\gamma_m} |\hbar(s,0,0)|. \quad (35)$$

We pay regard to the set

$$B_{R_m} = \left\{\kappa \in E_m, \|\kappa\|_{E_m} \leq R_m\right\}. \quad (36)$$

There is no doubt that B_{R_m} is closed, convex, bounded, and nonempty.

We will now demonstrate that S_1 meets the Theorem 11's fundamental assumption. Three steps will be taken to provide the proof. □

Step 12. $S_1(B_{R_m}) \subseteq (B_{R_m})$.

Let $\kappa \in B_{R_m}$ and $s \in \gamma_m$. From Proposition 10, we have

$$\begin{aligned}\left|S_1\kappa(s)\right| &= \left|\int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s,\nu)\vartheta(\nu)\mathrm{d}\nu\right| \leq \int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s,\nu)|\vartheta(\nu)|\mathrm{d}\nu \\ &\leq \frac{1}{\Gamma(u_m)}\left(\frac{\Upsilon_m - \Upsilon_{m-1}}{4}\right)^{u_m-1}\int_{\Upsilon_{m-1}}^{\Upsilon_m}|\vartheta(\nu)|\mathrm{d}\nu,\end{aligned} \quad (37)$$

where

$$\vartheta(s) = \hbar(s, \kappa(s), \vartheta(s)). \quad (38)$$

By P2, we have

$$\begin{aligned}|\vartheta(s)| &= |\hbar(s,\kappa(s),\vartheta(s))| \\ &\leq |\hbar(s,\kappa(s),\vartheta(s)) - \hbar(s,0,0)| + |\hbar(s,0,0)| \\ &\leq s^{-\rho}\left(\varpi_1|\kappa(s)| + \varpi_2|\vartheta(s)|\right) + \hbar^* \\ &\leq s^{-\rho}\left(\varpi_1 R_m + \varpi_2|\vartheta(s)|\right) + \hbar^*.\end{aligned} \quad (39)$$

Then,

$$|\vartheta(s)| \leq \frac{\varpi_1 R_m s^{-\rho} + \hbar^*}{1 - \varpi_2 s^{-\rho}}. \quad (40)$$

Thus,

$$\begin{aligned}\left|S_1\kappa(s)\right| &\leq \frac{1}{\Gamma(u_m)}\left(\frac{\Upsilon_m - \Upsilon_{m-1}}{4}\right)^{u_m-1}\int_{\Upsilon_{m-1}}^{\Upsilon_m}\left(\frac{\varpi_1 R_m \nu^{-\rho} + \hbar^*}{1-\varpi_2\nu^{-\rho}}\right)\mathrm{d}\nu \\ &\leq \frac{\varpi_1(\Upsilon_m - \Upsilon_{m-1})^{u_m-1}\left(\Upsilon_m^{1-\rho} - \Upsilon_{m-1}^{1-\rho}\right)}{4^{u_m-1}(1-\rho)(1-\varpi_2\Upsilon_{m-1}^{-\rho})\Gamma(u_m)}R_m + \frac{\hbar^*(\Upsilon_m - \Upsilon_{m-1})^{u_m}}{4^{u_m-1}(1-\varpi_2\Upsilon_{m-1}^{-\rho})\Gamma(u_m)} \\ &\leq R_m.\end{aligned} \quad (41)$$

It means that $S_1(B_{R_m}) \subseteq B_{R_m}$.

Step 13. S_1 is continuous.

The sequence (κ_n) is assumed to converge to κ in E_m and $s \in \gamma_m$. Due to Proposition 10, we have

$$\left|(S_1\kappa_n)(s) - (S_1\kappa)(s)\right| = \left|\int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s,\nu)\vartheta_n(\nu) - \int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s,\nu)\vartheta(\nu)\mathrm{d}\nu\right|, \tag{42}$$

where

$$\begin{aligned}\vartheta_n(s) &= \hbar(s,\kappa_n(s),\vartheta_n(s)),\\ \vartheta(s) &= \hbar(s,\kappa(s),\vartheta(s)).\end{aligned} \tag{43}$$

Then,

$$\begin{aligned}\left|(S_1\kappa_n)(s) - (S_1\kappa)(s)\right| &\le \int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s,\nu)\left|\vartheta_n(\nu) - \vartheta(\nu)\right|\mathrm{d}\nu\\ &\le \frac{1}{\Gamma(u_m)}\left(\frac{\Upsilon_m - \Upsilon_{m-1}}{4}\right)^{u_m-1}\int_{\Upsilon_{m-1}}^{\Upsilon_m}\left|\vartheta_n(\nu) - \vartheta(\nu)\right|\mathrm{d}\nu.\end{aligned} \tag{44}$$

P2 gives us

$$\begin{aligned}\left|\vartheta_n(s) - \vartheta(s)\right| &= \left|\hbar(s,\kappa_n(s),\vartheta_n(s)) - \hbar(s,\kappa(s),\vartheta(s))\right|\\ &\le s^{-\rho}\left(\varpi_1\left|\kappa_n(s) - \kappa(s)\right| + \varpi_2\left|\vartheta_n(s) - \vartheta(s)\right|\right).\end{aligned} \tag{45}$$

Thus,

$$\left|\vartheta_n(s) - \vartheta(s)\right| \le \frac{\varpi_1 s^{-\rho}}{1 - \varpi_2 s^{-\rho}}\left|\kappa_n(s) - \kappa(s)\right|. \tag{46}$$

Hence,

$$\begin{aligned}\left|(S_1\kappa_n)(s) - (S_1\kappa)(s)\right| &\le \frac{1}{\Gamma(u_m)}\left(\frac{\Upsilon_m - \Upsilon_{m-1}}{4}\right)^{u_m-1}\int_{\Upsilon_{m-1}}^{\Upsilon_m}\frac{\varpi_1\nu^{-\rho}}{1 - \varpi_2\nu^{-\rho}}\left|\kappa_n(\nu) - \kappa(\nu)\right|\mathrm{d}\nu\\ &\le \frac{1}{\Gamma(u_m)}\left(\frac{\Upsilon_m - \Upsilon_{m-1}}{4}\right)^{u_m-1}\left(\frac{\varpi_1}{1 - \varpi_2\Upsilon_{m-1}^{-\rho}}\right)\left(\frac{\Upsilon_m^{1-\rho} - \Upsilon_{m-1}^{1-\rho}}{1-\rho}\right)\left\|\kappa_n - \kappa\right\|_{E_m}\\ &\le \frac{\varpi_1(\Upsilon_m - \Upsilon_{m-1})^{u_m-1}\left(\Upsilon_m^{1-\rho} - \Upsilon_{m-1}^{1-\rho}\right)}{4^{u_m-1}(1-\rho)\left(1 - \varpi_2\Upsilon_{m-1}^{-\rho}\right)\Gamma(u_m)}\left\|\kappa_n - \kappa\right\|_{E_m},\end{aligned} \tag{47}$$

i.e., we obtain

$$\left\|(S_1\kappa_n) - (S_1\kappa)\right\|_{E_m} \longrightarrow 0\, as\, n \longrightarrow \infty. \tag{48}$$

The operator S_1 is hence a continuous on E_m.

Step 14. $S_1(B_{R_m})$ is relatively compact.

We must now demonstrate that $S_1(B_{R_m})$ is relatively compact. Due to step 13, it is obvious that $S_1(B_{R_m})$ is uniformly bounded. As a result, we obtain $S_1(B_{R_m}) = \{S_1(\kappa): \kappa \in B_{R_m}\} \subseteq B_{R_m}$, and for any $\kappa \in B_{R_m}$, we have $\|S_1(\kappa)\|_{E_m} \le R_m$, indicating that $S_1(B_{R_m})$ is uniformly bounded. It needs to be demonstrated that $S_1(B_{R_m})$ is equicontinuous.

For $s_1, s_2 \in \gamma_m, s_1 < s_2$ and $\kappa \in B_{R_m}$, we have

$$\left|(S_1\kappa)(s_2) - (S_1\kappa)(s_1)\right| = \left|\int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s_2,\nu)\vartheta(\nu)\mathrm{d}\nu - \int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s_1,\nu)\vartheta(\nu)\mathrm{d}\nu\right|, \tag{49}$$

where

$$\vartheta(s) = \hbar(s, \kappa(s), \vartheta(s)). \tag{50}$$

Then,

$$\begin{aligned} &\left|(S_1\kappa)(s_2) - (S_1\kappa)(s_1)\right| \\ &\leq \int_{\Upsilon_{m-1}}^{\Upsilon_m} \left|(G_m(s_2,\nu) - G_m(s_1,\nu))\vartheta(\nu)\right| d\nu \\ &\leq \int_{\Upsilon_{m-1}}^{\Upsilon_m} \left|G_m(s_2,\nu) - G_m(s_1,\nu)\right| \left|\left(\frac{\varpi_1 R_m \nu^{-\rho} + \hbar^*}{1-\varpi_2\nu^{-\rho}}\right)\right| d\nu \\ &\leq \frac{\varpi_1 R_m (\Upsilon_{m-1})^{-\rho} + \hbar^*}{1-\varpi_2(\Upsilon_{m-1})^{-\rho}} \int_{\Upsilon_{m-1}}^{\Upsilon_m} \left|G_m(s_2,\nu) - G_m(s_1,\nu)\right| d\nu, \end{aligned} \tag{51}$$

considering Green's functions' continuity G_m. Thus, $|(S_1\kappa)(s_2) - (S_1\kappa)(s_1)| \longrightarrow 0$ as $|s_2 - s_1| \longrightarrow 0$, which indicates that $S_1(B_{R_m})$ is equicontinuous.

We arrived at the conclusion that S_1 is completely continuous due to steps 12 through 14 and the Arzela–Ascoli theorem.

Problem (22) has at least one solution ($\widetilde{\kappa_m}$ in B_{R_m}) according to Theorem 11.

We let

$$\kappa_m = \begin{cases} 0, & s \in [\eta, T_{m-1}], \\ \widetilde{\kappa}_m, & t \in \gamma_m. \end{cases} \tag{52}$$

Thus, we know that $\kappa_m \in C([\eta, \Upsilon_m], \mathbb{R})$ satisfies equation

$$\frac{d^2}{ds^2}\left(\int_\eta^{\Upsilon_1} \frac{(s-\nu)^{1-u_1}}{\Gamma(2-u_1)}\kappa_m(\nu)d\nu + \cdots + \int_{\Upsilon_{m-1}}^{s} \frac{(s-\nu)^{1-u_m}}{\Gamma(2-u_m)}\kappa_m(\nu)d\nu\right) + \hbar\left(\nu, \kappa_m(\nu), D_{\eta^+}^{u_m}\kappa_m(\nu)\right) = 0, \tag{53}$$

for which $s \in \gamma_m$, indicating that κ_m is a solution of (20) with $\kappa_m(\eta) = 0, \kappa_m(T_m) = \widetilde{\kappa_m}(\Upsilon_m) = 0$.

Then, we obtain

$$\kappa(s) = \begin{cases} \kappa_1(s), \; s \in \gamma_1, \\ \kappa_2(s) = \begin{cases} 0, \; s \in \gamma_1, \\ \widetilde{\kappa_2}, \; s \in \gamma_2 \end{cases} \\ \cdot \\ \cdot \\ \cdot \\ \kappa_n(s) = \begin{cases} 0, \; s \in [\eta, \Upsilon_{m-1}], \\ \widetilde{\kappa_m}, \; s \in \gamma_m, \end{cases} \end{cases} \tag{54}$$

which is a form of the solution of PVB (2).

Now, the following result can be defined using the Banach contraction principle.

Theorem 15. *Let us consider the assumptions* $(P1)$, $(P2)$ *hold and*

$$\frac{\varpi_1(\Upsilon_m - \Upsilon_{m-1})^{u_m-1}\left(\Upsilon_m^{1-\rho} - \Upsilon_{m-1}^{1-\rho}\right)}{4^{u_m-1}(1-\rho)(1-\varpi_2\Upsilon_{m-1}^{-\rho})\Gamma(u_m)} < 1. \tag{55}$$

Problem (22) then has a solution on E_m.

Proof. The Banach contraction principle demonstrates that S_1 has a singular fixed point as stated in (32).

We have for $\kappa, y \in E_m$, and $s \in \gamma_m$.

$$\begin{aligned} \left|(S_1\kappa)(s) - (S_1 y)(s)\right| = &\left|\int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s,\nu)\vartheta(\nu)d\nu \right. \\ &\left. - \int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s,\nu)\psi(\nu)d\nu\right|, \end{aligned} \tag{56}$$

where

$$\begin{aligned} \vartheta(s) &= \hbar(s, \kappa(s), \vartheta(s)), \\ \psi(s) &= \hbar(s, y(s), \psi(s)). \end{aligned} \tag{57}$$

By P2, we have

$$\begin{aligned} |\vartheta(s) - \psi(s)| &= |\hbar(s, \kappa(s), \vartheta(s)) - \hbar(s, y(s), \psi(s))| \\ &\leq s^{-\rho}(\varpi_1|\kappa(s) - y(s)| + \varpi_2|\vartheta(s) - \psi(s)|). \end{aligned} \tag{58}$$

Then,

$$|\vartheta(s) - \psi(s)| \leq \frac{\varpi_1 s^{-\rho}}{1-\varpi_2 s^{-\rho}}|\kappa(s) - y(s)|. \tag{59}$$

Thus,

$$\begin{aligned}\left|(S_1\kappa)(s)-(S_1 y)(s)\right| &\le \int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s,\nu)|\vartheta(\nu)-\psi(\nu)|\mathrm{d}\nu \\ &\le \frac{1}{\Gamma(u_m)}\left(\frac{\Upsilon_m-\Upsilon_{m-1}}{4}\right)^{u_m-1}\int_{T_{m-1}}^{T_m}\left(\frac{\varpi_1\nu^{-\rho}}{1-\varpi_2\nu^{-\rho}}\right)|\kappa(\nu)-y(\nu)|\mathrm{d}\nu \\ &\le \frac{1}{\Gamma(u_m)}\left(\frac{\Upsilon_m-\Upsilon_{m-1}}{4}\right)^{u_m-1}\left(\frac{\varpi_1}{1-\varpi_2\Upsilon_{m-1}^{-\rho}}\right)\|\kappa-y\|_{E_m}\int_{\Upsilon_{m-1}}^{\Upsilon_m} s^{-\rho}\mathrm{d}\nu \\ &\le \frac{1}{\Gamma(u_m)}\left(\frac{\Upsilon_m-\Upsilon_{m-1}}{4}\right)^{u_m-1}\left(\frac{\varpi_1}{1-\varpi_2\Upsilon_{m-1}^{-\rho}}\right)\left(\frac{\Upsilon_m^{1-\rho}-\Upsilon_{m-1}^{1-\rho}}{1-\rho}\right)\|\kappa-y\|_{E_m} \\ &\le \frac{\varpi_1(\Upsilon_m-\Upsilon_{m-1})^{u_m-1}\left(\Upsilon_m^{1-\rho}-\Upsilon_{m-1}^{1-\rho}\right)}{4^{u_m-1}(1-\rho)(1-\varpi_2\Upsilon_{m-1}^{-\rho})\Gamma(u_m)}\|\kappa-y\|_{E_m}.\end{aligned} \tag{60}$$

Consequently, by (55), the operator S_1 is condensed. As a result, according to the Banach contraction principle, S_1 has a singular fixed point $\widetilde{\kappa_m}\in E_m$, which is a singular solution to problem (22).

We let

$$\kappa_m=\begin{cases}0, & s\in[\eta,\Upsilon_{m-1}],\\ \widetilde{\kappa}_m, & s\in\gamma_m.\end{cases} \tag{61}$$

We are clear that the following equation, which is defined by (61), is satisfied by $\kappa_m\in C([\eta,\Upsilon_m],\mathbb{R})$:

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\int_\eta^{\Upsilon_1}\frac{(s-\nu)^{1-u_1}}{\Gamma(2-u_1)}\kappa_m(\nu)\mathrm{d}\nu+\cdots+\int_{\Upsilon_{m-1}}^{s}\frac{(s-\nu)^{1-u_m}}{\Gamma(2-u_m)}\kappa_m(\nu)\mathrm{d}\nu\right)+\hbar\left(\nu,\kappa_m(\nu),D_{\eta^+}^{u_m}\kappa_m(\nu)\right)=0, \tag{62}$$

for $s\in\gamma_m$, which denotes that κ_m is a singular solution of (20) with $\kappa_m(\eta)=0,\kappa_m(\Upsilon_m)=\widetilde{\kappa_m}(\Upsilon_m)=0$.

Then, it is seen that

$$\kappa(s)=\begin{cases}\kappa_1(s), & s\in\gamma_1,\\ \kappa_2(s)=\begin{cases}0, & s\in\gamma_1,\\ \widetilde{\kappa}_2, & s\in\gamma_2,\end{cases}\\ \cdot\\ \cdot\\ \cdot\\ \kappa_n(s)=\begin{cases}0, & s\in[\eta,\Upsilon_{m-1}],\\ \widetilde{\kappa_m}, & s\in\gamma_m\end{cases}\end{cases} \tag{63}$$

is the form of unique solution of PVB (1). □

4. Stability of Ulam–Hyers

Theorem 16. *Assuming P1, P2, and (10) are valid. Then, PVB (1) is Ulam–Hyers stable.*

Proof. Let $\chi\in E_m$ be a solution of the inequality. So, we have

$$\left|D^{u_m}\chi(s)+\hbar(s,\chi(s),D^{u_m}\chi(s))\right|\le\epsilon, s\in\gamma_m. \tag{64}$$

Let us use $\kappa\in E_m$ to represent the problem's singular solution (22). By using Lemma 9, we have

$$\kappa(s)=\int_{\Upsilon_{m-1}}^{\Upsilon_m}G_m(s,\nu)\vartheta(\nu)\mathrm{d}\nu, \tag{65}$$

where

$$\vartheta(s)=\hbar(s,\kappa(s),\vartheta(s)). \tag{66}$$

By integration of (64), we also obtain

$$\left|\chi(s)+\int_{\Upsilon_{m-1}}^{\Upsilon_m}G_m(s,\nu)\psi(\nu)\mathrm{d}\nu\right|\le\epsilon\frac{(\Upsilon_m-\Upsilon_{m-1})^{u_m}}{\Gamma(u_m+1)}, \tag{67}$$

where

$$\psi(s)=\hbar(s,\chi(s),\psi(s)). \tag{68}$$

However, we also have

$$\begin{aligned}
|\chi(s)-\kappa(s)| &\leq |\chi(s)-\int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s,\nu)\psi(\nu)\mathrm{d}\nu| + |\int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s,\nu)\psi(\nu)\mathrm{d}\nu \\
&\quad - \int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s,\nu)\vartheta(\nu)\mathrm{d}\nu| \\
&\leq |\chi(s)+\int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s,\nu)\psi(\nu)\mathrm{d}\nu| + |\int_{\Upsilon_{m-1}}^{\Upsilon_m} G_m(s,\nu)\psi(\nu)\mathrm{d}\nu \\
&\quad - \int_{\Upsilon_{m-1}}^{\Upsilon_i} G_m(s,\nu)\vartheta(\nu)\mathrm{d}\nu| \\
&\leq \frac{\epsilon(\Upsilon_m-\Upsilon_{m-1})^{u_m}}{\Gamma(u_m+1)} + \frac{1}{\Gamma(u_m)}\left(\frac{\Upsilon_m-\Upsilon_{m-1}}{4}\right)^{u_m-1}\int_{\Upsilon_{m-1}}^{\Upsilon_m} |\psi(\nu)-\vartheta(\nu)|.
\end{aligned} \tag{69}$$

According to P2, for any $s \in \gamma_m$, we have

$$\begin{aligned}
|\psi(s)-\vartheta(s)| &= |\hbar(s,\chi(s),\psi(s)) - \hbar(s,\kappa(s),\vartheta(s))| \\
&\leq s^{-\rho}\left(\varpi_1|\chi(s)-\kappa(s)| + \varpi_2|\psi(s)-\vartheta(s)|\right).
\end{aligned} \tag{70}$$

Then,

$$|\psi(s)-\vartheta(s)| \leq \frac{\varpi_1 s^{-\rho}}{1-\varpi_2 s^{-\rho}}|\chi(s)-\kappa(s)|. \tag{71}$$

Thus,

$$\begin{aligned}
|\chi(s)-\kappa(s)| &\leq \frac{\epsilon(\Upsilon_m-\Upsilon_{m-1})^{u_m}}{\Gamma(u_m+1)} + \frac{1}{\Gamma(u_m)}\left(\frac{\Upsilon_m-\Upsilon_{m-1}}{4}\right)^{u_m-1}\int_{\Upsilon_{m-1}}^{\Upsilon_m} \frac{\varpi_1 \nu^{-\rho}}{1-\varpi_2\nu^{-\rho}}|\chi(\nu)-\kappa(\nu)|\mathrm{d}\nu \\
&\leq \frac{\epsilon(\Upsilon_m-\Upsilon_{m-1})^{u_m}}{\Gamma(u_m+1)} \\
&\quad + \frac{1}{\Gamma(u_m)}\left(\frac{\Upsilon_m-\Upsilon_{m-1}}{4}\right)^{u_m-1}\left(\frac{\varpi_1}{1-\varpi_2\Upsilon_{m-1}^{-\rho}}\right)\int_{\Upsilon_{m-1}}^{\Upsilon_m} \nu^{-\rho}|\chi(\nu)-\kappa(\nu)|\mathrm{d}\nu \\
&\leq \frac{\epsilon(\Upsilon_m-\Upsilon_{m-1})^{u_m}}{\Gamma(u_m+1)} + \frac{\varpi_1(\Upsilon_m-\Upsilon_{m-1})^{u_m-1}\left(\Upsilon_m^{1-\rho}-\Upsilon_{m-1}^{1-\rho}\right)}{(1-\rho)4^{u_m-1}\Gamma(u_m)\left(1-\varpi_2\Upsilon_{m-1}^{-\rho}\right)}\|\chi-\kappa\|_{E_m}.
\end{aligned} \tag{72}$$

Hence, we get

$$\|\chi-\kappa\|_{E_m}\left[1-\frac{\varpi_1(\Upsilon_m-\Upsilon_{m-1})^{u_m-1}\left(\Upsilon_m^{1-\rho}-\Upsilon_{m-1}^{1-\rho}\right)}{(1-\rho)4^{u_m-1}(1-\varpi_2\Upsilon_{m-1}^{-\rho})\Gamma(u_m)}\right] \leq \frac{\epsilon(\Upsilon_m-\Upsilon_{m-1})}{\Gamma(u_m+1)}. \tag{73}$$

Then, it is obtained that

$$\|\chi-\kappa\|_{E_m} \leq \left[1-\frac{\varpi_1(\Upsilon_m-\Upsilon_{m-1})^{u_m-1}\left(\Upsilon_m^{1-\rho}-\Upsilon_{m-1}^{1-\rho}\right)}{(1-\rho)4^{u_m-1}(1-\varpi_2\Upsilon_{m-1}^{-\rho})\Gamma(u_m)}\right]^{-1}\frac{(\Upsilon_m-\Upsilon_{m-1})}{\Gamma(u_m+1)}\epsilon := c_\hbar\epsilon, \tag{74}$$

for each $t \in \gamma_m$. So, problem (22) is SUH. Consequently, the equation of (2) is SUH. □

5. Example

Let us look at the following fractional boundary problem:

$$\begin{cases} D_{1/2^+}^{u(t)}\kappa(s) + \dfrac{1}{(s+1)^{1/2}e^{s+1/2}\left(1+|\kappa(s)|+\left|D_{0^+}^{u(s)}\kappa(s)\right|\right)} = 0, \quad s \in \gamma \coloneqq \left[\dfrac{1}{2}, 2\right], \\ \kappa\left(\dfrac{1}{2}\right) = 0, \quad \kappa(2) = 0, \end{cases} \tag{75}$$

with

$$\hbar(s, y, z) = \frac{1}{(s+1)^{1/2}e^{s+1/2}(1+|y|+|z|)}, (t, y, z) \in \left[\frac{1}{2}, 2\right] \times [0, +\infty) \times [0, +\infty), \tag{76}$$

$$u(s) = \begin{cases} \dfrac{3}{2}, & s \in \gamma_1 \coloneqq \left[\dfrac{1}{2}, 1\right], \\ \dfrac{7}{4}, & s \in \gamma_2 \coloneqq]1, 2]. \end{cases} \tag{77}$$

Then, we get

$$\begin{aligned} s^{1/2}|\hbar(s, u_1, v_1) - \hbar(s, u_2, v_2)| &= \left|s^{1/2}\left(\frac{1}{(s+1)^{1/2}e^{s+1/2}(1+|u_1|+|v_1|)} - \frac{1}{(s+1)^{1/2}e^{s+1/2}(1+|u_2|+|v_2|)}\right)\right| \\ &\le \frac{1}{e^{s+1/2}}\left||u_2|-|u_1|\right| + \frac{1}{e^{s+1/2}}\left||v_2|-|v_1|\right| \\ &\le \frac{1}{e}|u_1 - u_2| + \frac{1}{e}|v_1 - v_2|. \end{aligned} \tag{78}$$

As a result, condition P2 is satisfied when $\gamma = 1/2$ and $\varpi_1 = \varpi_2 = 1/2$ are used.

The equation for problem (75) is split into two parts as follows by (77):

$$\begin{cases} D_{1/2^+}^{3/2}\kappa(s) + \dfrac{1}{(s+1)^{1/2}e^{s+1/2}\left(1+|\kappa(s)|+\left|D_{0^+}^{3/2}\kappa(s)\right|\right)} = 0, \quad s \in \gamma_1, \\ D_{1^+}^{7/4}\kappa(s) + \dfrac{1}{(s+1)^{1/2}e^{s+1/2}\left(1+|\kappa(s)|+\left|D_{1^+}^{7/4}\kappa(s)\right|\right)} = 0, \quad s \in \gamma_2. \end{cases} \tag{79}$$

For $s \in \gamma_1$, problem (75) is equivalent to the following problem:

$$\begin{cases} D_{1/2^+}^{3/2}\kappa(s) + \dfrac{1}{(s+1)^{1/2}e^{s+1/2}\left(1+|\kappa(s)|+\left|D_{0^+}^{3/2}\kappa(s)\right|\right)} = 0, \quad s \in \gamma_1, \\ \kappa\left(\dfrac{1}{2}\right) = 0, \quad \kappa(1) = 0. \end{cases} \tag{80}$$

We will ascertain whether or not condition (55) is met.

$$\frac{\varpi_1 (\Upsilon_1 - \Upsilon_0)^{u_1-1}\left(\Upsilon_1^{1-\rho} - \Upsilon_0^{1-\rho}\right)}{4^{u_1-1}(1-\rho)(1-\varpi_2\Upsilon_0^{-\rho})\Gamma(u_1)} = \frac{1/e}{4^{0.5}1/2\Gamma(1.5)} \simeq 0.4151 < 1. \tag{81}$$

According to Theorem 15, problem (80) has a solution $\kappa_1 \in E_1$, and from Theorem 16, problem (80) is SUH.

Problem (75) can be expressed as the following piecewise function for $s \in \gamma_2$.

$$\begin{cases} D_{1^+}^{7/4}\kappa(s) + \dfrac{1}{(s+1)^{1/2}e^{s+1/2}\left(1+|\kappa(s)|+\left|D_{1^+}^{7/4}\kappa(s)\right|\right)} = 0, \quad s \in \gamma_2, \\ \kappa(1) = 0, \quad \kappa(2) = 0. \end{cases} \tag{82}$$

It can be seen that

$$\frac{\varpi_1 (\Upsilon_2 - \Upsilon_1)^{u_2-1}\left(\Upsilon_2^{1-\rho} - \Upsilon_1^{1-\rho}\right)}{4^{u_2-1}(1-\rho)(1-\varpi_2\Upsilon_1^{-\rho})\Gamma(u_2)} = \frac{1/e\left(2^{0.5} - 1^{0.5}\right)}{4^{0.75}1/2(1-1/e)\Gamma(1.75)} \simeq 0.1854 < 1. \tag{83}$$

Thus, condition (55) is satisfied.

By Theorem 15, problem (82) has a solution $\widetilde{\kappa}_2 \in E_2$, and from Theorem 16, problem (82) is SUH.

As is well known,

$$\kappa_2(s) = \begin{cases} 0, & s \in \gamma_1, \\ \widetilde{\kappa}_2(s), & s \in \gamma_2. \end{cases} \tag{84}$$

So, there is a solution to problem (75) that takes the form

$$\kappa(s) = \begin{cases} \kappa_1(s), & s \in \gamma_1, \\ \kappa_2(s) = \begin{cases} 0, & s \in \gamma_1, \\ \widetilde{\kappa}_2(s), & s \in \gamma_2. \end{cases} \end{cases} \tag{85}$$

According to Theorem 16, problem (75) is SUH.

6. Conclusion

The semigroup properties of the Riemann–Liouville fractional integral have played a key role in dealing with the existence of solutions to differential equations of fractional order. Based on some results of some experts, we know that the Riemann–Liouville variable order fractional integral does not have semigroup property, thus bringing us extreme difficulties in considering the existence of solutions of variable order fractional differential equations. In this work, we presented results about the existence and the uniqueness of solutions for implicit nonlinear fractional differential equations of variable order $u(t)$, where $u(t)$: $[\eta, \Upsilon] \longrightarrow (1,2]$ is a piecewise constant function.

All our results are based on the Schauder's fixed-point theorem and the Banach contraction principle. Lastly, we conducted a research on SUH, our problem's stability; finally, we illustrated the theoretical findings by an example.

All the results in this work show a great potential to be applied in various applications of sciences. Moreover, we will extend our studies in reducing chaos and stabilising the system of the utilisation of a Chua oscillator in the future.

Data Availability

The data used to support the findings of this work are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally in this study.

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