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Research Article

A Stochastic Discrete Fractional Cournot Duopoly Game: Modeling, Stability, and Optimal Control

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A stochastic discrete fractional Cournot duopoly game model with a unique interior Nash equilibrium is developed in this study. Some sufficient criteria of the Lyapunov stability in probability for the proposed model at the interior Nash equilibrium are derived using the Lyapunov theory. The proposed model's finite time stability in probability is then investigated using a nonlinear feedback control approach at the interior Nash equilibrium. The stochastic Bellman theory is also used to explore the locally optimum control problem. Furthermore, bifurcation diagrams, time series, and the 0-1 test are used to investigate the chaotic dynamics of this model. Finally, numerical examples are given to illustrate the obtained results.

1. Introduction

Differential or difference equations are usually used as a powerful analytical tool to study complex behavior in economics [1–17]. However, this kind of model just considers what occurs in the present state but ignores past phases of the process, or what has occurred in earlier states [18]. In reality, in economic processes, these variables are affected not only by their current values but also by their previous values. As a result, the effect of memory on history needs to be considered when building models [19]. For example, the dynamical behavior of the fractional differentiated Cournot triopoly game was investigated by Al-khedhairi [20]. In addition, Al-khedhairi [21] explored the complex dynamics of the discrete version of the fractional differentiated duopoly game. Notwithstanding the memory benefits of the continuous fractional Cournot triopoly game, we are unable to directly employ its numerical discretization strategy since it will rapidly amass numerical mistakes [22, 23]. The difference equation is more useful for modeling on-discrete time scales. To capture the memory effect, a fractional version of it is now introduced.

Recently, discrete fractional calculus [24–28] has attracted more and more attention, which is particularly suitable for building discrete models with memory effects. Wu and Baleanu [29] proposed a discrete fractional logistic map and studied its chaos. Then, Wu et al. [30] studied discrete chaos in fractional sine and standard maps. Later, the Lyapunov functions are used to study in the context of nabla discrete fractional systems by Wei et al. [31]. In [32], Du and Jia discussed the finite-time stability of a family of nonlinear fractional delay difference systems. Furthermore, Yang et al. [33] studied the mean square asymptotic stability of discrete fractional stochastic neural networks with multiple time-varying delays. After that, some researchers proposed various fractional models and investigated their dynamical behaviors [34–41].

As an application, Xin et al. [23] proposed a discrete fractional Cournot duopoly game to overcome the error caused by the discretization of the continuous fractional models. Moreover, based on the above model, some discrete fractional economic models had been proposed and studied successively [18, 42–46].

However, the current literature mainly focuses on deterministic Cournot oligopoly games and little focus on stochastic settings. It is known that stochastic perturbation is an important factor in the economy. For example, Xin and Wang [47] proposed a stochastic Cournot duopoly game in a block chain cloud services market driven by Brownian motion. However, there is still a lack of research on fractional Cournot oligopoly games with stochastic perturbation.

In light of this, discrete fractional systems can better characterize the dependence of discrete systems on past information throughout their evolution. The discrete fractional operator provides us with a powerful tool to study the evolution of games. In our model, we will use a truncated version of the discrete Grünwald-Letnikov fractional difference operator that incorporates a short-term memory process, which can be interpreted as a set of sliding delays or short-term information included in the model. In the real world, this may represent differences in the time it takes to decide between different players of the game or simply be considered as the time it takes for a certain process to occur in a complex gaming system, for example, it may represent the time it takes for negotiation between the management of a firm.

On the other hand, older historical information may not accurately reflect current market demand. Therefore, two finite rational firms try to ignore older historical information and use only recent historical information when formulating their strategies and playing the game, which implies that the finite rational firms will use the short-memory adjustment mechanism against firms competing with them. Therefore, the discrete Grünwald–Letnikov fractional difference operator used in this paper is more flexible and universal compared to the discrete Caputo fractional difference operator with long memory.

In addition, because models in real-world environments are inevitably disturbed by stochastic factors, we shall introduce stochastic noise into the modeling to obtain a more general game model that captures the influence of stochastic factors.

Therefore, this study aims to bring together three areas: game theory, discrete fractional calculus, and stochastic analysis to develop a more general game model and to analyze some of the properties of that model. The main contributions of this paper are as follows:

(1) We develop a new class of discrete fractional models with stochastic perturbations and apply them to the modeling of a Cournot duopoly game, which serves as a theoretical foundation for further research into various economic models, finance models, biological evolution models, and so on.

(2) The Lyapunov stability in probability and finite time stability in probability are proved using the Lyapunov theory, which shows that the Lyapunov function approach to study the stability of the stochastic discrete fractional model is a powerful tool.

(3) The locally optimal control conditions are obtained via the stochastic Bellman theory and feedback control principle.

The rest of this article is structured as follows. Section 2 introduces the definitions and theories of discrete fractional calculus. In Section 3, a new stochastic discrete fractional Cournot duopoly game model is constructed using the discrete Grünwald–Letnikov fractional difference operator. The stability in probability and finite time stability in probability are demonstrated in Section 4. A feedback controller is proposed to study the locally optimal control in Section 5. The stochastic discrete fractional Cournot duopoly game model with 3-period is discussed in Section 6. Section 7 gives several examples to illustrate the validity of the obtained results. Conclusions are given in Section 8.

2. Mathematical Preliminaries

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\overline{\mathbb{N}} = \{0, 1, 2, \cdots\} \cup \{+\infty\}$, $\mathbb{N}^+ = \{1, 2, \cdots\}$, and $\mathbb{N}_{0,h} = \{-h, \ldots, 0\}$ for $h \in \mathbb{N}^+$. Let $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^n$ denote the set of real numbers, the set of positive real numbers, and the set of $n \times 1$ real column vectors, respectively. Let $\|.\|$ denote Euclidean norm. Let $\mathscr{D}$ be an open set of $\mathbb{R}^n$ containing the origin, and $\mathscr{B}_\delta(x)$ is an open ball of radius δ centered at x in $\mathbb{R}^n$ with its closed ball $\overline{\mathscr{B}_\delta(x)}$. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, in which for any $w \in \Omega$, $w \triangleq w(k)(k \in \mathbb{N})$ is a sequence of independent and identically distributed $\mathbb{R}^d$-valued random vectors $(d \in \mathbb{N}^+)$, and $\mathbb{E}[.]$ is the expectation operator. For each $k \in \mathbb{N}^+$, we let $\mathscr{F}_k \subseteq \mathscr{F}$ denote the σ-algebra generated by the random variables $w(0), \ldots, w(k-1)$. The resulting sequence $\{\mathscr{F}_k\}_{k\in\mathbb{N}^+}$ of σ-algebras is a filtration on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Moreover, for every $k \in \mathbb{N}^+$, $w(k)$ is independent of the σ-algebra $\mathscr{F}_l$ for all $0 \le l \le k$ [48].

We first introduce the following α-order Grünwald–Letnikov difference operator.

Definition 1 (see [18, 49, 50]). For a discrete function $x(k)$ on $\mathbb{N}$, the α-order Grünwald–Letnikov difference operator Δ^α is defined as follows:

$$\Delta^\alpha x(k) = \frac{1}{h^\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x(k - jh), \tag{1}$$

where $h(h \in \mathbb{R}^+)$ and $\alpha \in (0, 1)$ denote a sampling period and the fractional order, respectively, and the binomial coefficient $\binom{\alpha}{j}$ can be computed by

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0, \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & \text{for } j > 0. \end{cases} \tag{2}$$

Now, we consider the following stochastic discrete dynamical system [51, 52].

$$x(k+1,w) = f(x(k,w)) + g(x(k,w))w(k) \triangleq F(x(k,w), w(k)), \tag{3}$$

where $x\colon \mathbb{N}\times\Omega \longrightarrow \mathscr{D}$ is a stochastic process with initial condition $x(0,w) \equiv x_0 \in \mathscr{D}$; $f\colon \mathscr{D} \longrightarrow \mathscr{D}$ and $g\colon \mathscr{D} \longrightarrow \mathbb{R}^d$ are continuous functions satisfying $f(0) = 0$ and $g(0) = 0$, respectively.

Similar to [51], we denote the measurable map $s\colon \mathbb{N}\times\mathscr{D}\times\Omega \longrightarrow \mathscr{D}$ as the family of maps of the stochastic dynamical system (3) satisfying

(i) the co-cycle property $s(k, s(l,x,\omega),\omega) = s(k+l, x,\omega)$ for all $k,l \in \mathbb{N}$,

(ii) the identity (on $\mathscr{D}$) property $s(0,x,\omega) = x$ for all $x \in \mathscr{D}, \omega \in \Omega$, and

(iii) the measurable map $s_k = s(k,\cdot,\omega)\colon \mathscr{D} \longrightarrow \mathscr{D}$ is continuous for all $k \in \mathbb{N}$ outside a $\mathbb{P}$-nullset.

Clearly, if the sample path trajectory is denoted by $s^x = s(\cdot,x,\cdot)\colon \mathbb{N}\times\Omega \longrightarrow \mathscr{D}$ for any $x \in \mathscr{D}$, then there exists a trajectory defined for all $k \in \mathbb{N}$, and $\omega \in \Omega$ satisfying the dynamical process (3) with initial condition $x(0) = x$. And the process $x(k) = 0$ a.s. satisfying the system (3) is called the zero solution to the system (3). For convenience, we write $s(k,x)$ for $s(k,x,\omega)$ and $s^x(k)$ for $s^x(k,\omega)$ in the following.

Now, several definitions and theorems about stochastic stability are introduced below.

Definition 2 (see [51, 53])

(i) The zero solution to the system (3) is Lyapunov stable in probability if, for any $\varepsilon > 0$ and $\rho \in (0, 1)$, there exists a $\delta = \delta(\varepsilon,\rho) > 0$ such that for all $x_0 \in \mathscr{B}_\delta(0)$,

$$\mathbb{P}\left(\sup_{k\in\mathbb{N}}\|x(k)\| > \varepsilon\right) < \rho. \tag{4}$$

(ii) The zero solution to the system (3) is asymptotically stable in probability if it is Lyapunov stable in probability and, for any $\rho \in (0, 1)$, there exists a $\delta = \delta(\rho) > 0$ such that for all $x_0 \in \mathscr{B}_\delta(0)$,

$$\mathbb{P}\left(\lim_{k\longrightarrow\infty}\|x(k)\| = 0\right) \geq 1-\rho. \tag{5}$$

(iii) The zero solution to the system (3) is globally asymptotically stable in probability if it is Lyapunov stable in probability and, for all $x_0 \in \mathbb{R}^n$,

$$\mathbb{P}\left(\lim_{k\longrightarrow\infty}\|x(k)\| = 0\right) = 1. \tag{6}$$

(iv) The zero solution of the system (3) is called mean square stable (*resp.* asymptotically mean square stable) if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x_0 \in \mathscr{B}_\delta(0)$,

$$\mathbb{E}\left[x(k)^2\right] < \varepsilon, \tag{7}$$

for all k (*resp.* $\lim_{k\longrightarrow\infty}\mathbb{E}[x(k)^2] = 0$).

Definition 3 (see [52]). The zero solution to the system (1) is finite time stable if there exists a state indexed stochastic process $K\colon \mathscr{D}\times\Omega \longrightarrow \overline{\mathbb{N}}$, called a stochastic settling-time, such that the following statements hold:

(i) Finiteness of the stochastic settling-time. For every $x \in \mathscr{D}$, the stochastic settling-time $K(x,.)$ is finite almost surely.

(ii) Finite time convergence in probability. For every $x(0) = x_0 \in \mathscr{D}\setminus\{0\}$, $s^{x_0}(k,\omega) \in \mathscr{D}\setminus\{0\}$ for $k \in [0, K(x_0,\omega))$ and $\omega \in \Omega$, and

$$\mathbb{P}\left(\left\|s^{x_0}(K(x_0))\right\| = 0\right) = 1. \tag{8}$$

And if $x(0) = x_0 = 0$, then $K(0,\omega) \triangleq 0, \omega \in \Omega$.

(iii) Lyapunov stability in probability. For every $\varepsilon > 0$ and $\rho \in (0, 1)$, there exists $\delta = \delta(\varepsilon,\rho) > 0$ such that, for all $x_0 \in \mathscr{B}_\delta(0)$,

$$\mathbb{P}\left(\sup_{k\in[0,K(x_0)]}\left\|s^{x_0}(k)\right\| > \varepsilon\right) \leq \rho. \tag{9}$$

The zero solution to the system (3) is globally finite time stable in probability if it is finite time stable in probability with $\mathscr{D} = \mathbb{R}^n$.

Definition 4 (see [51]). Consider the stochastic discrete dynamical system (1) and let $V\colon \mathscr{D} \longrightarrow \mathbb{R}$. Then, the difference operator Δ of $x \in \mathscr{D}$ is defined by

$$\Delta V(x) = \mathbb{E}[V(F(x,\omega))] - V(x). \tag{10}$$

Then, the difference operator Δ of the state vector $x(k) \in D$ for $k \in \mathbb{N}$ can be written as follows:

$$\Delta V(x(k)) = \mathbb{E}[V(F(x,\omega))]|_{x=x(k)} - V(x(k)), \tag{11}$$

which is a random variable now.

Theorem 5 (see [51]). *For the system (1), assume that there exists a continuous function $V\colon \mathscr{D} \longrightarrow \mathbb{R}$ such that*

$$\begin{aligned} &V(0) = 0,\\ &V(x) > 0, \qquad x \in \mathscr{D}, x \neq 0,\\ &\Delta V(x) \leq 0. \end{aligned} \tag{12}$$

Then, the zero solution to the system (3) is Lyapunov stable in probability. If, in addition, for every $\varepsilon > 0$ and $\mathscr{B}_\varepsilon(0) \subset \mathscr{D}_r \subset \mathscr{D}$, where $\mathscr{D}_r \subset \mathscr{D}$ is a bounded neighborhood of the origin, there exists $c = c(\varepsilon) > 0$ such that

$$\Delta V(x) \leq -c, x \in \mathscr{D}_r\setminus\overline{\mathscr{B}_\varepsilon(0)}; \tag{13}$$

then, the zero solution to the system (3) is asymptotically stable in probability. Moreover, if $\mathscr{D} = \mathbb{R}^n$, $V(.)$ is radially unbounded, then the zero solution to the system (3) is globally asymptotically stable in probability.

Theorem 6 (see [52]). *For the system (3), assume that there exists a continuous and radially unbounded function $V: \mathscr{D} = \mathbb{R}^n \longrightarrow \mathbb{R}$ such that*

$$\begin{aligned} &V(0) = 0, \\ &V(x) > 0, x \in \mathscr{D}, x \neq 0, \\ &\mathbb{E}[V(F(x,w))] \leq \phi(V(x)), x \in \mathbb{R}^n \backslash \{0\}, \end{aligned} \tag{14}$$

where $\phi: \mathbb{R} \longrightarrow \mathbb{R}$ denotes a nondecreasing function, then the zero solution to the system (3) is globally finite time stable in probability. Moreover, there exists a stochastic settling-time $K(.,.): \mathscr{D} \times \Omega \longrightarrow \mathbb{N}$ such that $\mathbb{E}[K(x_0, w)] \leq C_0$ (C_0 is a finite constant).

Theorem 7 (see [48]). *Consider the following controlled stochastic discrete dynamical system of the system (3) is given by*

$$x(k+1) = f(x(k), u(k)) + g(x(k), u(k))w(k) \triangleq F(x(k), u(k), w(k)), \tag{15}$$

with initial condition $x(0) = x_0 \in \mathscr{D}$ and $u(k) \in U \subseteq \mathbb{R}^m$. Taking the performance measure as

$$J(x_0, u(.)) = \mathbb{E}\left[\sum_{k=0}^{\infty} L(x(k), u(k))\right], \tag{16}$$

assume that there exists a continuous radially unbounded function $V: \mathbb{R}^n \longrightarrow \mathbb{R}$ and a control law $\phi: \mathbb{R}^n \longrightarrow U$ such that

$$\begin{aligned} &V(0) = 0, \\ &\phi(0) = 0, \\ &V(x) > 0, x \in \mathscr{D}, x \neq 0, \\ &\mathbb{E}[V(F(x, \phi(x), w))] < V(x), x \in \mathbb{R}^n \backslash \{0\}, \\ &L(x, \phi(x)) + \mathbb{E}[V(F(x, \phi(x), w))] - V(x) = 0, \\ &L(x, u) + \mathbb{E}[V(F(x, u, w))] - V(x) \geq 0. \end{aligned} \tag{17}$$

Then, zero solution to the system (15) is asymptotically stable in probability, and the feedback control $u(.) = \phi(x)$ minimizes the performance measure, that is, $J(x_0, \phi(x)) = V(x_0) = \min_{u(.) \in U} J(x_0, u(.))$.

Remark 8. For further details on optimal control, including the construction of the Hamiltonian function, see [48].

Theorem 9 (see [53]). *Consider the following stochastic discrete dynamical system is given by*

$$x(i+1) = F(i, x(-h), \ldots, x(i)) + \sum_{k=0}^{i} G(i, k, x(-h), \ldots, x(k))w(k), \quad i \in \mathbb{N}, \tag{18}$$

with the initial condition $x(i) = \varphi_i, i \in \mathbb{N}_{0,h}$. Here, $F, G: \mathbb{N} \times \mathbb{R}^{h+1} \longrightarrow \mathbb{R}$. If there exist a nonnegative functional $V_i = V(i, x(-h), \ldots, x(i))$ and two positive numbers c_1, c_2 such that the following conditions hold, then the zero solution to the system (18) is asymptotically mean square stable.

$$\begin{aligned} &\mathbb{E}[V(0, \varphi_{-h}, \ldots, \varphi_0)] \leq c_1 \sup_{i \in \mathbb{N}_0} \mathbb{E}\left[|\varphi_i|^2\right], \\ &\mathbb{E}[\Delta V_i] \leq -c_2 \mathbb{E}|x(i)|^2, \quad i \in \mathbb{N}, \end{aligned} \tag{19}$$

where $\Delta V_i = V(i+1, x(-h), \ldots, x(i+1)) - V(i, x(-h), \ldots, x(i))$. In addition, if there exist a nonnegative functional $V_{1i} = V(i, x(-h), \ldots, x(i))$ which satisfies condition (19) and the conditions

$$\begin{aligned} &\mathbb{E}[\Delta V_{1i}] \leq a\mathbb{E}\left[|x(i)|^2\right] + \sum_{k=-h}^{i} A_{ik}\mathbb{E}\left[|x(k)|^2\right], \quad i \in \mathbb{N}, A_{ik} \geq 0, \\ &a + b < 0, b = \sup_{i \in \mathbb{N}} \sum_{j=i}^{\infty} A_{ji}, \end{aligned} \tag{20}$$

hold, then the zero solution to the system (18) is asymptotically mean square stable.

3. The Model

In this section, we will construct a stochastic discrete fractional Cournot duopoly game model based on a discrete Cournot duopoly game model in [23].

In the market, there is a standard Cournot rivalry between firms 1 and 2, both of which provide homogenous items that are perfect replacements.

(a) Assume that the market-clearing price is an inverse demand function

$$p(k) = a - b\big(q_i(k) + q_j(k)\big), \quad i, j = 1, 2, i \neq j, \tag{21}$$

where $q_i(k) \in \mathscr{D}$ denotes the quantity supplied by firms $i\,(i = 1, 2)$ in period $k\,(k = 0, 1, 2, 3, \cdots)$, and $a > 0$ and $b > 0$ are constants.

(b) Assume that the cost functions are twice differentiable functions given by

$$C_i(k) = \frac{1}{2}c_i q_i^2(k), \quad i = 1, 2, \tag{22}$$

where $c_i > 0\,(i = 1, 2)$ are constants.

It follows from (21) and (22) that the profits of firms 1 and 2 can be calculated by

$$\begin{aligned} \Pi_1\big(q_1(k), q_2(k)\big) &= p(k)q_1(k) - C_1(k), \\ \Pi_2\big(q_1(k), q_2(k)\big) &= p(k)q_2(k) - C_2(k), \end{aligned} \tag{23}$$

where Π_1 and Π_2 denote the profits of firms 1 and 2, respectively.

Then, the marginal profits of firms 1 and 2 can be obtained by differentiating with respect to $q_1(k)$ and $q_2(k)$, respectively, that is,

$$\begin{aligned} \Phi_1(k) &= \frac{\partial \Pi_1\big(q_1(k), q_2(k)\big)}{\partial q_1(k)} = a - (c_1 + 2b)q_1(k) - bq_2(k), \\ \Phi_2(k) &= \frac{\partial \Pi_2\big(q_2(k), q_1(k)\big)}{\partial q_2(k)} = a - (c_2 + 2b)q_2(k) - bq_1(k). \end{aligned} \tag{24}$$

Note that firms can consider a repeated adjustment game mechanism based on the long-memory effect and the local estimation of the marginal profit, which can be described by

$$\begin{cases} \Delta^\alpha q_1(k+1) = \alpha_1 q_1(k)\Phi_1(k), \\ \Delta^\alpha q_2(k+1) = \alpha_2 q_2(k)\Phi_2(k), \end{cases} \tag{25}$$

where $\alpha_i > 0$ for $i = 1, 2$, and Δ^α denotes the fractional Grünwald–Letnikov difference operator, instead of the α-order left Caputo-like delta difference operator in [23].

However, older historical data may not accurately reflect the current market demand. As a result, the two bounded rational firms attempt to disregard older historical data and use only recent historical data while developing strategies and playing the game, which means that boundedly rational firms will use a short-memory adjustment mechanism to update the amount produced in each period k. In reality, given the markets quick renewal and unpredictability, firms frequently use just short-term historical data for reference while developing their plans. In this study, we explore the truncated version of the fractional Grünwald–Letnikov difference operator to characterize the short-memory effect.

Using the truncated form of the fractional Grünwald–Letnikov difference operator [50], the adjustment game mechanism (25) turns into the following game with the memory length $(M \in \mathbb{N}^+)$.

$$\begin{cases} q_1(k+1) = \alpha_1 q_1(k)\Phi_1(k) + \alpha q_1(k) - \displaystyle\sum_{l=1}^{M} (-1)^{l+1}\binom{\alpha}{l+1} q_1(k-l), \\ q_2(k+1) = \alpha_2 q_2(k)\Phi_2(k) + \alpha q_2(k) - \displaystyle\sum_{l=1}^{M} (-1)^{l+1}\binom{\alpha}{l+1} q_2(k-l). \end{cases} \tag{26}$$

Furthermore, it is inevitable to be disturbed by stochastic factors in the real environment. Therefore, to capture the effect of stochastic factors, we introduce stochastic noise into the above game and study the new stochastic discrete fractional Cournot duopoly game system in this paper as shown below:

$$\begin{cases} q_1(k+1) = \alpha_1 q_1(k)\Phi_1(k) + \alpha q_1(k) - \sum_{l=1}^{M} (-1)^{l+1}\binom{\alpha}{l+1} q_1(k-l) \\ \qquad +D_1(q_1(k), q_2(k))w(k), \\ q_2(k+1) = \alpha_2 q_2(k)\Phi_2(k) + \alpha q_2(k) - \sum_{l=1}^{M} (-1)^{l+1}\binom{\alpha}{l+1} q_2(k-l) \\ \qquad +D_2(q_1(k), q_2(k))w(k), \end{cases} \tag{27}$$

where $D_i\,(i = 1, 2)$: $\mathscr{D} \longrightarrow \mathbb{R}$ are continuous functions, and all stochastic variables $w(k)$ are independent and identically distributed with $\mathbb{E}[w(k)] = 0$ and $\mathbb{E}[w(k)^2] = 1$.

Now, the Nash equilibrium for the system (27) will be explored. Assume that $N^* = (q_1^*, q_2^*)$ is a Nash equilibrium satisfying $D_i(q_1^*, q_2^*) = 0$ for $i = 1, 2$).

Proposition 10. *The system (9) has four Nash equilibria:*

$$\begin{aligned} N_1 &= (0, 0), \\ N_2 &= (p_1, 0), \\ N_3 &= (0, p_2), \\ N_4 &= (p, r), \end{aligned} \tag{28}$$

where

$$\begin{aligned} p_1 &= \frac{\alpha_1 a + \alpha + \beta - 1}{\alpha_1(2b + c_1)}, \\ p &= \frac{\alpha_1\alpha_2 a(b + c_2) + (2\alpha_2 b + \alpha_2 c_2 - \alpha_1 b)(\alpha + \beta - 1)}{\alpha_1\alpha_2\left(3b^2 + 2bc_1 + 2bc_2 + c_1c_2\right)}, \\ p_2 &= \frac{\alpha_2 a + \alpha + \beta - 1}{\alpha_2(2b + c_2)}, \\ r &= \frac{\alpha_1\alpha_2 a(b + c_1) + (2\alpha_1 b + \alpha_1 c_1 - \alpha_2 b)(\alpha + \beta - 1)}{\alpha_1\alpha_2\left(3b^2 + 2bc_1 + 2bc_2 + c_1c_2\right)}, \end{aligned} \tag{29}$$

and $\beta = \sum_{i=1}^{M}\beta_i$, $\beta_i = -(-1)^{i+1}\binom{\alpha}{i+1}$.

Proof. To study the equilibrium points of this system, we first introduce the idea of solving equilibrium point in the following. Without loss of generality, consider a system has the form

$$x(i+1) = f(x(i), x(i-1), \ldots, x(i-j)). \tag{30}$$

The equilibrium point x^* of system (30) satisfies the following form.

$$x^* = f(x^*, x^*, \ldots, x^*). \tag{31}$$

Substituting now, $N^* = (q_1^*, q_2^*)$ into (27) yields

$$\begin{cases} q_1^* = \alpha_1 q_1^*(a - (c_1 + 2b)q_1^* - bq_2^*) + \alpha q_1^* - \sum_{l=1}^{M} (-1)^{l+1}\binom{\alpha}{l+1} q_1^*, \\ q_2^* = \alpha_2 q_2^*(a - (c_2 + 2b)q_2^* - bq_1^*) + \alpha q_2^* - \sum_{l=1}^{M} (-1)^{l+1}\binom{\alpha}{l+1} q_2^*, \end{cases} \tag{32}$$

by straightforward calculation; we can obtain the assertions of Proposition 10 and the proof ends. □

Remark 11. Since a, b, $\alpha_i\,(i = 1, 2)$, and $c_i\,(i = 1, 2)$ are positive constants, then $\alpha_2(c_2 + 2b) \neq 0, \alpha_1(c_1 + 2b) \neq 0$ and $\alpha_2\alpha_1(3b^2 + 2bc_1 + c_2c_1 + 2c_2b) \neq 0$. As a result, the Nash equilibrium points in Proposition 10 have practical value. Furthermore, if the two firms adopt the same learning law (that is $\alpha_1 = \alpha_2$), then we can obtain a positive Nash equilibrium when the memory intensity and length satisfy the condition $\alpha + \beta > 1$.

Similar to [23], it follows that the equilibrium points N_1, N_2, and N_3 are boundary equilibria, and the equilibrium point N_4 is a unique interior Nash equilibrium point.

Considering the meaning of the equilibrium point in real-world problems, we exclusively investigate the stability of the interior equilibrium point $N_4 = (p, r)$ in this study.

4. Stochastic Stability

4.1. Lyapunov Stable in Probability. We present sufficient conditions on the Lyapunov stability in probability by employing the Lyapunov theory for the system (27) in this subsection.

According to the idea of [53], we will consider the asymptotically mean square stability of the linearized system of (27). Putting, first, $x(k) = q_1(k) - p$, $y(k) = q_2(k) - r$, and centering (27) via new variables $x(k)$ and $y(k)$, we have.

$$\begin{cases} x(k+1) = \theta_1 x(k) + \sigma x(k) w(k) + \theta_2 y(k) \\ \qquad\qquad + \sum_{i=1}^{M} \beta_i x(k-i) + H_1(x(k), y(k)), \\ y(k+1) = \varphi_1 y(k) + \sigma y(k) w(k) + \varphi_2 x(k) \\ \qquad\qquad + \sum_{i=1}^{M} \beta_i y(k-i) + H_2(x(k), y(k)), \end{cases} \tag{33}$$

where

$$\begin{aligned} \theta_1 &= -2\alpha_1 c_1 p - 4\alpha_1 bp - \alpha_1 br + \alpha + \alpha_1 a, \theta_2 = -\alpha_1 pb, \\ \varphi_1 &= (\alpha_2 a - 2\alpha_2 c_2 r + \alpha - 4\alpha_2 br - \alpha_2 bp), \varphi_2 = -\alpha_2 rb, \\ H_1(x(k), y(k)) &= (-\alpha_1 c_1 - 2\alpha_1 b)(x(k))^2 - \alpha_1 x(k) b y(k) + \alpha p \\ &\quad + \sum_{i=1}^{M} \beta_i p + \alpha_1 pa - \alpha_1 c_1 p^2 - 2\alpha_1 bp^2 - \alpha_1 pbr - p, \\ H_2(x(k), y(k)) &= (-\alpha_2 c_2 - 2\alpha_2 b)(y(k))^2 - \alpha_2 y(k) b x(k) + \alpha r \\ &\quad + \sum_{i=1}^{M} \beta_i r + \alpha_2 ra - \alpha_2 c_2 r^2 - 2\alpha_2 br^2 - \alpha_2 pbr - r. \end{aligned} \tag{34}$$

Theorem 12. *Let* $\Psi = max\ \{\sigma^2 - 1 + (M+2)\theta_1{}^2 + (M+2)\varphi_2{}^2, \sigma^2 - 1 + (M+2)\theta_2{}^2 + (M+2)\varphi_1{}^2\}$, $D_1 = \sigma(q_1(k) - p)$, *and* $D_2 = \sigma(q_2(k) - r)$. *The system (27) is Lyapunov stable in probability at* $N_4 = (p, r)$ *if*

$$2\theta_1^2 + 2\varphi_2^2 - 1 < 0, 2\theta_2^2 + 2\varphi_1^2 - 1 < 0, \tag{35}$$

and

$$\Psi + (M+2)\left(\beta_1{}^2 + \beta_2{}^2 + \beta_3{}^2\right) < 0. \tag{36}$$

Proof. Note that the linear part of (33) is as follows:

$$X(k+1) = AX(k) + \sum_{i=1}^{M} \beta_i X(k-i) + \sigma w(k) X(k), \tag{37}$$

where $A = \begin{pmatrix} \theta_1 & \theta_2 \\ \varphi_2 & \varphi_1 \end{pmatrix}$, $X(k) = (x(k), y(k))'$, and $(.)'$ denotes the transpose operator. We now consider the auxiliary system of (37).

$$\widehat{X}(k+1) = A\widehat{X}(k), \tag{38}$$

where $\widehat{X}(k) = (\hat{x}(k), \hat{y}(k))'$.

Obviously, it follows from condition (35) and Young's inequality that

$$\widehat{X}(k+1)'\widehat{X}(k+1) - \widehat{X}(k)'\widehat{X}(k) \le 0. \tag{39}$$

So, we can take the Lyapunov function of (38) as follows:

$$V(\widehat{X}(k)) = \widehat{x}(k)^2 + \widehat{y}(k)^2. \tag{40}$$

Therefore, the Lyapunov function of (37) is constructed as the form,

$$V_1(X(k)) = X(k)' X(k) = x(k)^2 + y(k)^2. \tag{41}$$

By (37) and Hölder's inequality, we have

$$\begin{aligned}
\mathbb{E}[\Delta V_1(X(k))] &= \mathbb{E}\left[x(k+1)^2 + y(k+1)^2 - x(k)^2 - y(k)^2\right] \\
&= \mathbb{E}\big[\big(\theta_1 x(k) + \sigma x(k) w(k) + \theta_2 y(k) \\
&\quad + \beta_1 x(k-1) + \cdots + \beta_M x(k-M)\big)^2 \\
&\quad + \big(\varphi_1 y(k) + \sigma y(k) w(k) + \varphi_2 x(k) \\
&\quad + \beta_1 y(k-1) + \cdots + \beta_M y(k-M)\big)^2 \\
&\quad - x(k)^2 - y(k)^2\big] \\
&= \mathbb{E}\big[\sigma^2 x(k)^2 + \sigma^2 y(k)^2 \\
&\quad + \big(\theta_1 x(k) + \theta_2 y(k) + \beta_1 x(k-1) + \cdots + \beta_M x(k-M)\big)^2 \\
&\quad + \big(\varphi_1 y(k) + \varphi_2 x(k) + \beta_1 y(k-1) + \cdots + \beta_M y(k-M)\big)^2 \\
&\quad - x(k)^2 - y(k)^2\big] \\
&\le \left(\sigma^2 - 1 + (M+2){\theta_1}^2 + (M+2){\varphi_2}^2\right)\mathbb{E}\left[x(k)^2\right] \\
&\quad + \left(\sigma^2 - 1 + (M+2){\theta_2}^2 + (M+2){\varphi_1}^2\right)\mathbb{E}\left[y(k)^2\right] \\
&\quad + (M+2)\beta_1^2 \mathbb{E}\left[x(k-1)^2\right] + \cdots + (M+2)\beta_M^2 \mathbb{E}\left[x(k-M)^2\right] \\
&\quad + (M+2)\beta_1^2 \mathbb{E}\left[y(k-1)^2\right] + \cdots + (M+2)\beta_M^2 \mathbb{E}\left[y(k-M)^2\right] \\
&\le \Psi \mathbb{E}\left[X(k)^2\right] + (M+2)\beta_1^2 \mathbb{E}\left[X(k-1)^2\right] + \cdots \\
&\quad + (M+2)\beta_M^2 \mathbb{E}\left[X(k-M)^2\right].
\end{aligned} \tag{42}$$

Therefore, it follows from Theorem 9 that the zero solution to the system (37) is asymptotically mean square stable in probability, which implies that the system (27) is Lyapunov stable in probability at $N_4 = (p, r)$. The proof is completed. □

4.2. Finite Time Stable in Probability. In this subsection, by adopting the feedback control principle, we will study the finite time stability in probability for the system (27).

To address this problem, consider the following controlled stochastic discrete fractional Cournot duopoly game system:

$$\begin{cases}
q_1(k+1) = \alpha_1 q_1(k)\big(a - (c_1 + 2b) q_1(k) - b q_2(k)\big) \\
\qquad\qquad + \alpha q_1(k) - \displaystyle\sum_{l=1}^{M} (-1)^{l+1} \binom{\alpha}{l+1} q_1(k-l) \\
\qquad\qquad + D_1\big(q_1(k), q_2(k)\big) w(k) + u_1(k), \\
q_2(k+1) = \alpha_2 q_2(k)\big(a - (c_2 + 2b) q_2(k) - b q_1(k)\big) \\
\qquad\qquad + \alpha q_2(k) - \displaystyle\sum_{l=1}^{M} (-1)^{l+1} \binom{\alpha}{l+1} q_2(k-l) \\
\qquad\qquad + D_2\big(q_1(k), q_2(k)\big) w(k) + u_2(k),
\end{cases} \tag{43}$$

where $u_i(k)(i = 1, 2)$ are the feedback controllers that are continuous functions with respect to $q_1(k) - p, q_2(k) - r$.

It follows from variable substitution $x(k) = q_1(k) - p$ and $y(k) = q_2(k) - r$ that the system (43) can be reduced to

$$\begin{cases} x(k+1) = \theta_1 x(k) + \sigma x(k) w(k) + \theta_2 y(k) \\ \qquad + \sum_{i=1}^{M} \beta_i x(k-i) + H_1(x(k), y(k)) + u_1(x(k), y(k)), \\ y(k+1) = \varphi_1 y(k) + \sigma y(k) w(k) + \varphi_2 x(k) \\ \qquad + \sum_{i=1}^{M} \beta_i y(k-i) + H_2(x(k), y(k)) + u_2(x(k), y(k)). \end{cases} \tag{44}$$

Furthermore, if we take

$$\begin{aligned} u_1(x(k), y(k)) &= -\theta_2 y(k) - H_1(x(k), y(k)), \\ u_2(x(k), y(k)) &= -\varphi_2 x(k) - H_2(x(k), y(k)), \end{aligned} \tag{45}$$

then substituting the controls $u_i(.)(i = 1, 2)$ into (44) yields

$$\begin{cases} x(k+1) = \theta_1 x(k) + \sigma x(k) w(k) + \sum_{i=1}^{M} \beta_i x(k-i), \\ y(k+1) = \varphi_1 y(k) + \sigma y(k) w(k) + \sum_{i=1}^{M} \beta_i y(k-i). \end{cases} \tag{46}$$

We can observe that the variables $x(k)$ and $y(k)$ are independent in the system (46). Therefore, we only need to consider the finite time stability in probability of the two subsystems of (46), respectively.

Let $(x(k), x(k-1), \ldots, x(k-M))' = (z_{1,1}(k), z_{1,2}(k), \ldots, z_{1,M+1}(k))' \triangleq Z_1(k)$. Then, the first subsystem of (46) can be written as the following matrix form.

$$Z_1(k+1) = \mathrm{J}Z_1(k) + \eta(k), \tag{47}$$

where $J = \begin{pmatrix} \theta_1 & \beta_1 & \cdots & \beta_{M-1} & \beta_M \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$ and $\eta(k) = (\sigma z_1(k) w(k), 0, \ldots, 0)'$. Let $(y(k), y(k-1), \ldots, y(k-M))' = (z_{2,1}(k), z_{2,2}(k), \ldots, z_{2,M+1}(k))' \triangleq Z_2(k)$. The second subsystem of (46) can be discussed similarly. Therefore, the following theorem follows from Theorem 6, with its proof omitted.

Theorem 13. *If there exists a continuous and radially unbounded function $V: \mathbb{R}^{M+1} \longrightarrow \mathbb{R}$ such that*

$$\begin{aligned} &V(0) = 0, \\ &V(Z_i(k)) > 0, Z_i(k) \in \mathscr{D}, Z_i(k) \neq 0, \\ &\mathbb{E}[V(Z_i(k+1))] \leq \phi(V(Z_i(K))), Z_i(k) \in \mathbb{R}^{M+1} \backslash \{0\}, \end{aligned} \tag{48}$$

where $\phi: \mathbb{R} \longrightarrow \mathbb{R}$ denotes a nondecreasing function and $i = 1, 2$, then the system (27) is finite time stable in probability at $N_4 = (p, r)$ with the feedback controllers

$$\begin{aligned} u_1(k) &= -\theta_2(q_2(k) - r) - H_1(q_1(k) - p, q_2(k) - r), \\ u_2(k) &= -\varphi_2(q_1(k) - p) - H_2(q_1(k) - p, q_2(k) - r). \end{aligned} \tag{49}$$

From the study above, it shown that the memory length and fractional order we take into consideration have a significant impact on the occurrence of a nonzero Nash equilibrium point. By taking an acceptable fractional order and memory step, a positive Nash equilibrium point can be reached, indicating that these two firms continue to operate side by side in the market. This helps to maintain the market's stability.

5. Optimal Control

In this section, with the help of the feedback control principle, we will study stochastic locally optimal control for a quadratic performance measure under a controlled version of the system (33).

Denote the vector $(x(k), x(k-1), \ldots, x(k-M), y(k), y(k-1), \ldots, y(k-M))'$ as the vector $(T_1(k), T_2(k), \ldots, T_{2M+2}(k))' \triangleq T(k)$, then the matrix form of (33) is given by

$$T(k+1) = \mathrm{RT}(k) + H + \mathrm{WT}(k) w(k), \tag{50}$$

where $H = (H_1(T_1, T_{M+2}), 0, \ldots, 0, H_2(T_1, T_{M+2}), 0, \ldots, 0)'$,

$$R = \begin{pmatrix} \theta_1 & \beta_1 & \cdots & \beta_M & \theta_2 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \varphi_2 & 0 & \cdots & 0 & \varphi_1 & \beta_1 & \cdots & \beta_M \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$
$$W = \begin{pmatrix} \sigma & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \sigma & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{51}$$

To address the locally optimal control problem, we consider a linear controlled version of the system (50).

$$T(k+1) = \mathrm{RT}(k) + \mathrm{WT}(k)w(k) + u(k), \tag{52}$$

with a performance measure

$$J(T(0), u(.)) = \mathbb{E}\left[\sum_{k=0}^{\infty} L(T(k), u(k))\right], \tag{53}$$

where the initial value $T(0)$ is in $\mathbb{R}^{2M+2}$ and the controller $u(.)$ is feasible and feedback with respect to the state $T(k)$ in (52) satisfying

$$E\left[\sum_{k=0}^{\infty} |L(T(k), u(k))|\right] < \infty, \qquad \lim_{k \longrightarrow \infty} \mathbb{E}[V(T(k))] = 0, \tag{54}$$

for some continuous radially unbounded function V, and the cost function $L(.,.)$: $\mathbb{R}^{2M+2} \times \mathbb{R}^{2M+2} \longrightarrow \mathbb{R}$ is a continuous function [48].

In this paper, consider a quadratic performance measure, which is given by

$$J(T(0), u(.)) = \mathbb{E}\left[\sum_{k=0}^{\infty} T(k)' Y_1 T(k) + u(k)' Y_2 u(k)\right], \tag{55}$$

where $Y_i (i = 1, 2)$ is a positive definite matrix.

Theorem 14. *The system (21) is locally asymptotically stable in probability at zero; furthermore, the feedback controller $\widehat{u}(.) = (Y_2 + I)^{-1} IRT(k)$ minimizes (22) such that*

$$J(T(0), \widehat{u}(.)) = T(0)' \mathrm{IT}(0), \tag{56}$$

if the positive definite matric I satisfies

$$\begin{aligned} I - Y_1 = {} & R' I' \left[(Y_2 + I)^{-1}\right]' Y_2 (Y_2 + I)^{-1} \mathrm{IR} \\ & + R' \mathrm{IR} + R' I (Y_2 + I)^{-1} \mathrm{IR} + R' I' \left[(Y_2 + I)^{-1}\right]' \mathrm{IR} \\ & + R' I' \left[(Y_2 + I)^{-1}\right]' I (Y_2 + I)^{-1} \mathrm{IR} + W' \mathrm{IW}. \end{aligned} \tag{57}$$

Proof. Taking $V(T(k)) = T(k)' \mathrm{IT}(k)$, the Hamiltonian function has the following form:

$$\begin{aligned} \mathscr{H}(T, u) = {} & T(k)' Y_1 T(k) + u(k)' Y_2 u(k) \\ & + \mathbb{E}\Big[(\mathrm{RT}(k) + \mathrm{WT}(k)w(k) + u(k))' I (\mathrm{RT}(k) + \mathrm{WT}(k)w(k) \\ & + u(k)] - T(k)' IT(k). \end{aligned} \tag{58}$$

From (58), it follows that $\partial^2 \mathscr{H} / \partial^2 u = Y_2 + I > 0$; therefore, setting $\partial \mathscr{H} / \partial u = 0$ gives the feedback control as follows:

$$\widehat{u}(.) = (Y_2 + I)^{-1} \mathrm{IRT}(k). \tag{59}$$

Further, it follows from (55) and (57) that

$$\mathscr{H}(T, \widehat{u}(.)) = 0, \tag{60}$$

and $\Delta V(T(k) < 0$, which means that the system (52) is locally asymptotically stable in probability at zero. Moreover, the controller $\widehat{u}(.) = (Y_2 + I)^{-1} \mathrm{IRT}(k)$ minimizes (55), and by Theorem 7,

$$J(T(0), \widehat{u}(.)) = V(T(0)) = T(0)' \mathrm{I} T(0). \tag{61}$$

The proof is completed. □

6. The Stochastic Discrete Fractional Cournot Duopoly Game Model with $M = 3$

Specifically, we present sufficient conditions on the Lyapunov stability in probability and the finite time stability in probability for the system (27) with $M = 3$ at the interior Nash equilibrium point.

Let $M = 3$, by (27), we have

$$\begin{cases} q_1(k+1) = \alpha_1 q_1(k)\left(a - (c_1 + 2b)q_1(k) - bq_2(k)\right) \\ \qquad\qquad + \alpha q_1(k) - \sum_{l=1}^{3} (-1)^{l+1} \binom{\alpha}{l+1} q_1(k-l) \\ \qquad\qquad + D_1\left(q_1(k), q_2(k)\right) w(k), \\ q_2(k+1) = \alpha_2 q_2(k)\left(a - (c_2 + 2b)q_2(k) - bq_1(k)\right) \\ \qquad\qquad + \alpha q_2(k) - \sum_{l=1}^{3} (-1)^{l+1} \binom{\alpha}{l+1} q_2(k-l) \\ \qquad\qquad + D_2\left(q_1(k), q_2(k)\right) w(k). \end{cases} \tag{62}$$

Corollary 15. *The system (62) has a unique interior Nash equilibrium point*

$$N^* = N_4 = (p, r), \tag{63}$$

where

$$\begin{aligned} p &= \frac{A_1\alpha + A_2\alpha_1 + A_3\alpha_2}{\alpha_2\alpha_1\left(3b^2 + 2bc_1 + c_2c_1 + 2c_2b\right)}, \\ r &= \frac{B_1\alpha + B_2\alpha_1 + B_3\alpha_2}{\alpha_2\alpha_1\left(3b^2 + 2bc_1 + c_2c_1 + 2c_2b\right)}, \end{aligned} \tag{64}$$

with parameters

$$\begin{aligned} A_1 &= \alpha_2 c_2 + 2\alpha_2 b - \alpha_1 b, \\ A_2 &= \alpha_2 ab + \alpha_2 c_2 a - \beta_2 b + b - \beta_1 b - \beta_3 b, \\ A_3 &= -c_2 + c_2\beta_1 + 2\beta_3 b + c_2\beta_3 - 2b + 2\beta_2 b + c_2\beta_2 + 2\beta_1 b, \\ B_1 &= \alpha_1 c_1 - \alpha_2 b + 2\alpha_1 b, \\ B_2 &= \alpha_2 ab + c_1\beta_2 + c_1\beta_1 + c_1\beta_3 - c_1 + \alpha_2 ac_1 + 2\beta_2 b - 2b + 2\beta_1 b + 2\beta_3 b, \\ B_3 &= -\beta_3 b - \beta_2 b + b - \beta_1 b, \\ \beta_1 &= -\binom{\alpha}{2}, \\ \beta_2 &= \binom{\alpha}{3}, \\ \beta_3 &= -\binom{\alpha}{4}. \end{aligned} \tag{65}$$

Proof. The result is a direct consequence of Proposition 10 with $M = 3$. □

Corollary 16. *Let* $\Psi = \max\{\sigma^2 - 1 + 5\theta_1{}^2 + 5\varphi_2{}^2, \sigma^2 - 1 + 5\theta_2{}^2 + 5\varphi_1{}^2\}$, $D_1 = \sigma(q_1(k) - p)$, *and* $D_2 = \sigma(q_2(k) - r)$, *the system (62) is Lyapunov stable in probability at* $N_4 = (p, r)$ *if*

$$\begin{aligned} &2\theta_1^2 + 2\varphi_2^2 - 1 < 0,\ 2\theta_2^2 + 2\varphi_1^2 - 1 < 0, \\ &\Psi + 5\left(\beta_1{}^2 + \beta_2{}^2 + \beta_3{}^2\right) < 0. \end{aligned} \tag{66}$$

Proof. It follows from variable substitution $x(k) = q_1(k) - p$ and $y(k) = q_2(k) - r$ that the system (62) can be reduced to

$$\begin{cases} x(k+1) = \theta_1 x(k) + \sigma x(k) w(k) + \theta_2 y(k) \\ \qquad + \beta_1 x(k-1) + \beta_2 x(k-2) + \beta_3 x(k-3) \\ \qquad + \mathbf{H_1}(x(k), y(k)), \\ y(k+1) = \varphi_1 y(k) + \sigma y(k) w(k) + \varphi_2 x(k) \\ \qquad + \beta_1 y(k-1) + \beta_2 y(k-2) + \beta_3 y(k-3) \\ \qquad + \mathbf{H_2}(x(k), y(k)), \end{cases} \tag{67}$$

where

$$\begin{aligned} \theta_1 &= \left(\alpha_1 a - 2\alpha_1 c_1 p + \alpha - 4\alpha_1 b p - \alpha_1 b r\right), \theta_2 = -\alpha_1 p b, \\ \varphi_1 &= \left(\alpha_2 a - 2\alpha_2 c_2 r + \alpha - 4\alpha_2 b r - \alpha_2 b p\right), \varphi_2 = -\alpha_2 r b, \\ \mathbf{H_1}(x(k), y(k)) &= \left(-\alpha_1 c_1 - 2\alpha_1 b\right)(x(k))^2 - \alpha_1 x(k) b y(k) + \alpha p \\ &\quad + \beta_1 p + \beta_2 p + \beta_3 p + \alpha_1 p a - \alpha_1 c_1 p^2 - 2\alpha_1 b p^2 - \alpha_1 \mathrm{pbr} - p, \\ \mathbf{H_2}(x(k), y(k)) &= \left(-\alpha_2 c_2 - 2\alpha_2 b\right)(y(k))^2 - \alpha_2 y(k) b x(k) + \alpha r \\ &\quad + \beta_1 r + \beta_2 r + \beta_3 r + \alpha_2 r a - \alpha_2 c_2 r^2 - 2\alpha_2 b r^2 - \alpha_2 \mathrm{pbr} - r. \end{aligned} \tag{68}$$

Considering the linearized system as above, the system yields

$$\begin{cases} x(k+1) = \theta_1 x(k) + \sigma x(k) w(k) + \theta_2 y(k) \\ \qquad + \beta_1 x(k-1) + \beta_2 x(k-2) + \beta_3 x(k-3), \\ y(k+1) = \varphi_1 y(k) + \sigma y(k) w(k) + \varphi_2 x(k) \\ \qquad + \beta_1 y(k-1) + \beta_2 y(k-2) + \beta_3 y(k-3). \end{cases} \tag{69}$$

Clearly, for the system (69), we can obtain the result of Corollary 16 by using a proof similar to Theorem 12. The proof is completed. □

Remark 17. Here, we focus on the stability in probability of the equilibrium point of the system (62). In Corollary 15, we give sufficient conditions for the existence of the equilibrium point. The stability in probability of the equilibrium point of the original system (62) is transferred to the stability in probability of the zero equilibrium point of the new system (67) using a translational transformation. Furthermore, we can find that the zero solution of the linearized system (69) of the system (67) exists.

To address the finite time stability in probability for the system (62), consider the controlled stochastic discrete fractional Cournot duopoly game system below.

$$\begin{cases} q_1(k+1) = \alpha_1 q_1(k)\left(a - (c_1 + 2b) q_1(k) - b q_2(k)\right) \\ \qquad + \alpha q_1(k) - \sum_{l=1}^{3} (-1)^{l+1} \binom{\alpha}{l+1} q_1(k-l) \\ \qquad + D_1\left(q_1(k), q_2(k)\right) w(k) + u_1(k), \\ q_2(k+1) = \alpha_2 q_2(k)\left(a - (c_2 + 2b) q_2(k) - b q_1(k)\right) \\ \qquad + \alpha q_2(k) - \sum_{l=1}^{3} (-1)^{l+1} \binom{\alpha}{l+1} q_2(k-l) \\ \qquad + D_2\left(q_1(k), q_2(k)\right) w(k) + u_2(k), \end{cases} \tag{70}$$

where $u_i(k)\,(i = 1, 2)$ are the feedback controllers that are continuous functions with respect to $q_1(k) - p, q_2(k) - r$.

Corollary 18. *Let $\lambda \in \mathbb{R}^+$. If*

$$0 < \Phi < 1,\ 0 < \Upsilon < 1, \tag{71}$$

where $\Phi = \max\{\Phi_1, \Phi_2, \Phi_3, \Phi_4\}$ and $\Upsilon = \max\{\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4\}$ with parameters

$$\begin{aligned}
\Phi_1 &= \lambda\,\theta_1\beta_2 + \lambda\,\theta_1\beta_1 + \theta_1{}^2 + 1 + \lambda\,\theta_1\beta_3 + \sigma^2,\\
\Phi_2 &= \frac{\theta_1\beta_1}{\lambda} + \beta_1{}^2 + 1 + \lambda\,\beta_1\beta_2 + \lambda\,\beta_1\beta_3,\\
\Phi_3 &= \lambda\,\beta_2\beta_3 + 1 + \frac{\theta_1\beta_2}{\lambda} + \frac{\beta_1\beta_2}{\lambda} + \beta_2{}^2,\\
\Phi_4 &= \frac{\beta_1\beta_3}{\lambda} + \frac{\beta_2\beta_3}{\lambda} + \frac{\theta_1\beta_3}{\lambda} + \beta_3{}^2,\\
\Upsilon_1 &= \lambda\,\varphi_1\beta_2 + \lambda\,\varphi_1\beta_1 + \varphi_1{}^2 + 1 + \lambda\,\varphi_1\beta_3 + \sigma^2,\\
\Upsilon_2 &= \frac{\varphi_1\beta_1}{\lambda} + \beta_1{}^2 + 1 + \lambda\,\beta_1\beta_2 + \lambda\,\beta_1\beta_3,\\
\Upsilon_3 &= \lambda\,\beta_2\beta_3 + 1 + \frac{\varphi_1\beta_2}{\lambda} + \frac{\beta_1\beta_2}{\lambda} + \beta_2{}^2,\\
\Upsilon_4 &= \frac{\beta_1\beta_3}{\lambda} + \frac{\beta_2\beta_3}{\lambda} + \frac{\varphi_1\beta_3}{\lambda} + \beta_3{}^2.
\end{aligned} \tag{72}$$

Then, the system (62) is finite time stable in probability at the $N_4 = (p, r)$ under the two controllers

$$\begin{aligned}
u_1(k) &= -\theta_2(q_2(k) - r) - \mathbf{H_1}(q_1(k) - p, q_2(k) - r),\\
u_2(k) &= -\varphi_2(q_1(k) - p) - \mathbf{H_2}(q_1(k) - p, q_2(k) - r).
\end{aligned} \tag{73}$$

Proof. It follows from variable substitution $x(k) = q_1(k) - p$ and $y(k) = q_2(k) - r$ that

$$\begin{cases}
x(k+1) = \theta_1 x(k) + \sigma x(k) w(k)\\
\qquad\qquad + \beta_1 x(k-1) + \beta_2 x(k-2) + \beta_3 x(k-3),\\
y(k+1) = \varphi_1 y(k) + \sigma y(k) w(k)\\
\qquad\qquad + \beta_1 y(k-1) + \beta_2 y(k-2) + \beta_3 y(k-3).
\end{cases} \tag{74}$$

Note that the variables $x(k)$ and $y(k)$ are independent in the system (74); therefore, we only need to consider the finite time stability of the subsystem of (74). Let $(x(k), x(k-1), x(k-2), x(k-3))' = (z_1(k), z_2(k), z_3(k), z_4(k))' \triangleq Z(k)$, so we can obtain the matrix form of first subsystem of (74) as follows:

$$Z(k+1) = JZ(k) + \eta(k), \tag{75}$$

where $J = \begin{pmatrix} \theta_1 & \beta_1 & \beta_2 & \beta_3\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $\eta(k) = (\sigma z_1(k) w(k), 0, 0, 0)'$.

Now, taking the Lyapunov function $V(Z(k)) = Z(k)'Z(k)$, it follows from (10) and Young's inequality that

$$\begin{aligned}
\Delta V(Z(k)) &= \mathbb{E}\Big[Z(k+1)'Z(k+1)\Big] - V(Z(k))\\
&= \mathbb{E}\Big[(\mathrm{J}Z(k) + \eta(k))'(\mathrm{J}Z(k) + \eta(k))\Big] - V(Z(k))\\
&= \mathbb{E}\Big[Z(k)'J'\mathrm{J}Z(k) + Z(k)'J'\eta(k) + \eta(k)'\mathrm{J}Z(k) + \eta(k)'\eta(k)\Big] - V(Z(k))\\
&= \mathbb{E}\Big[(z_1(k))^2\theta_1{}^2 + (z_1(k))^2 + 2z_1(k)z_2(k)\theta_1\beta_1 + 2z_1(k)z_3(k)\theta_1\beta_2\\
&\quad + 2z_1(k)z_4(k)\theta_1\beta_3 + (z_2(k))^2\beta_1{}^2 + (z_2(k))^2 + 2z_2(k)z_3(k)\beta_1\beta_2
\end{aligned}$$

$$
\begin{aligned}
&+2z_2(k)z_4(k)\beta_1\beta_3+(z_3(k))^2\beta_2{}^2\\
&+(z_3(k))^2+2z_3(k)z_4(k)\beta_2\beta_3+(z_4(k))^2\beta_3{}^2+\sigma^2(z_1(k))^2\Big]-V(Z(k))\\
\le&\left(\lambda\theta_1\beta_2+\lambda\theta_1\beta_1+\theta_1{}^2+1+\lambda\theta_1\beta_3+\sigma^2\right)(z_1(k))^2\\
&+\left(\frac{\theta_1\beta_1}{\lambda}+\beta_1{}^2+1+\lambda\beta_1\beta_2+\lambda\beta_1\beta_3\right)(z_2(k))^2\\
&+\left(\lambda\beta_2\beta_3+1+\frac{\theta_1\beta_2}{\lambda}+\frac{\beta_1\beta_2}{\lambda}+\beta_2{}^2\right)(z_3(k))^2\\
&+\left(\frac{\beta_1\beta_3}{\lambda}+\frac{\beta_2\beta_3}{\lambda}+\frac{\theta_1\beta_3}{\lambda}+\beta_3{}^2\right)(z_4(k))^2-V(Z(k))\\
\le&\,\Phi V(Z(k))-V(Z(k)).
\end{aligned}
\tag{76}
$$

It follows from (71) that $\Delta V(Z(k))<0$, which means that the system (75) is globally asymptotically stable in probability. Note that (76) further implies the conditions of Theorem 6, thus it follows from Theorem 6 that the zero solution to the system (75) is finite time stable in probability with a stochastic settling-time $K(.,.)\colon \mathscr{D}\times\Omega\longrightarrow\mathbb{N}$ satisfying $\mathbb{E}[K(Z(0),w)]\le C_0$ (C_0 is a finite constant).

Next, let $(y(k),y(k-1),y(k-2),y(k-3))'=(s_1(k),s_2(k),s_3(k),s_4(k))'\triangleq S(k)$, so we can obtain the matrix form of second subsystem of (74) as follows:

$$S(k+1)=QS(k)+\xi(k), \tag{77}$$

where $Q=\begin{pmatrix}\varphi_1 & \beta_1 & \beta_2 & \beta_3\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}$ and $\xi(k)=(\sigma s_1(k)w(k),0,\ 0,\ 0)'$.

Similar to the above proof steps, we can show that the second subsystem is also finite time stable in probability under condition (71). The proof is completed.

With the feedback control principle, next, we will study the stochastic locally optimal control for the system (62). In this case, the cost function is still (55).

First, putting $(x(k),x(k-1),x(k-2),x(k-3),y(k),y(k-1),\ y(k-2),y(k-3))'=(T_1(k),T_2(k),T_3(k),T_4(k),T_5(k),\ T_6(k),T_7(k),T_8(k))'\triangleq T(k)$, we can obtain the matrix form of (67), which is given by

$$T(k+1)=\mathbf{R}T(k)+\mathbf{H}+\mathbf{W}T(k)w(k), \tag{78}$$

where $\mathbf{H}=(\mathbf{H_1}(T_1,T_5),0,\ 0,\ 0,\mathbf{H_2}(T_1,T_5),0,\ 0,\ 0)'$,

$$
\mathbf{R}=\begin{pmatrix}
\theta_1 & \beta_1 & \beta_2 & \beta_3 & \theta_2 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
\varphi_2 & 0 & 0 & 0 & \varphi_1 & \beta_1 & \beta_2 & \beta_3\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
$$
$$
\mathbf{W}=\begin{pmatrix}
\sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \sigma & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\tag{79}
$$

Then, it follows from Theorem 14 that the following Corollary 19 holds. □

Corollary 19. *The system (26) is locally asymptotically stable in probability at the $N_4=(p,r)$; furthermore, the feedback controller $\widehat{u}(.)=(Y_2+I)^{-1}I\mathbf{R}\mathrm{T}(k)$ minimizes (22), that is, $J(T(0),\widehat{u}(.))=T(0)'IT(0)$, if the positive definite matric I satisfies*

$$
\begin{aligned}
I-Y_1=&\,\mathbf{R}'I'\left[(Y_2+I)^{-1}\right]'Y_2(Y_2+I)^{-1}\mathrm{I}\mathbf{R}\\
&+\mathbf{R}'\mathrm{I}\mathbf{R}+\mathbf{R}'I(Y_2+I)^{-1}\mathrm{I}\mathbf{R}+\mathbf{R}'I'\left[(Y_2+I)^{-1}\right]'\mathrm{I}\mathbf{R}\\
&+\mathbf{R}'I'\left[(Y_2+I)^{-1}\right]'I(Y_2+I)^{-1}\mathrm{I}\mathbf{R}+\mathbf{W}'\mathrm{I}\mathbf{W}.
\end{aligned}
\tag{80}
$$

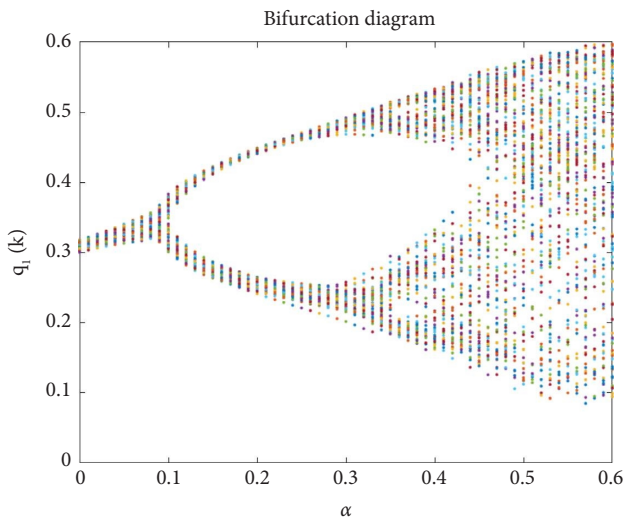

Figure 1: Bifurcation diagram of (62) with $\alpha \in (0, 0.6)$.

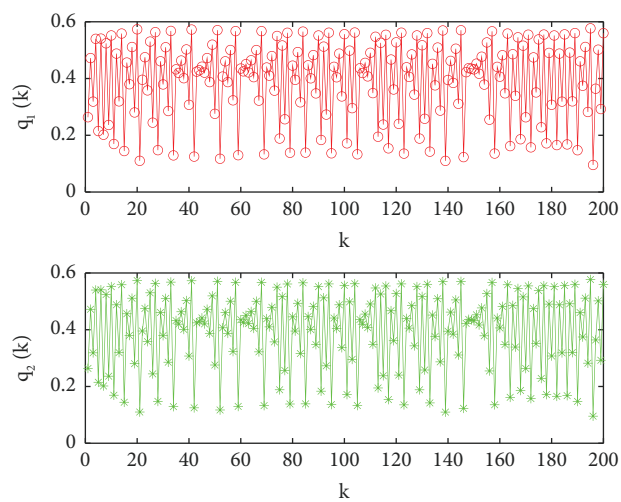

Figure 2: The chaotic solutions of (62) with $\alpha = 0.5$.

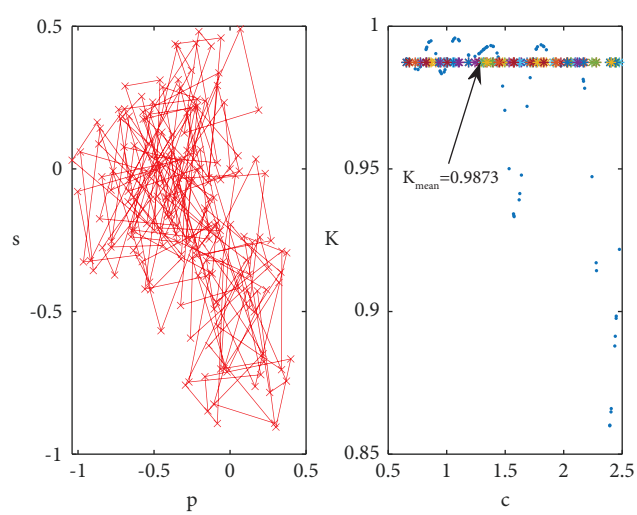

Figure 3: The 0-1 test.

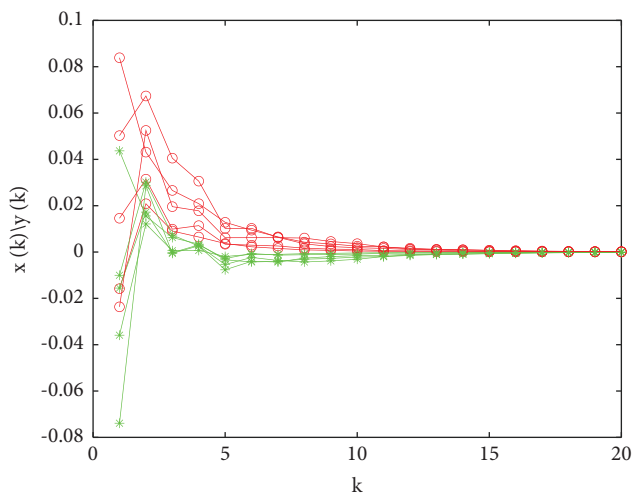

Figure 4: System response of (69) with different initial conditions. The red color indicates $x(k)$ and green color indicates $y(k)$.

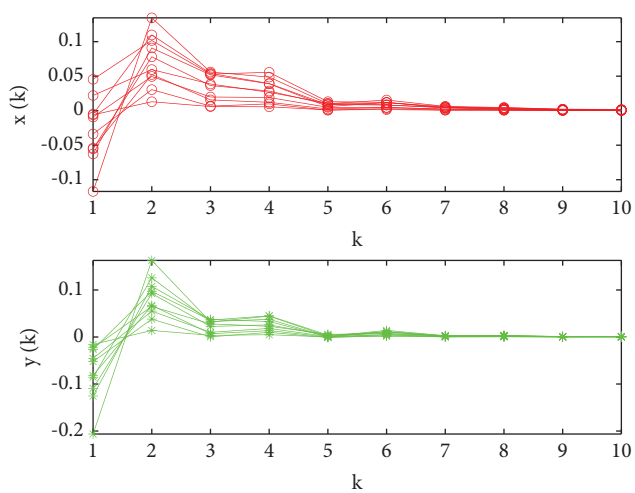

Figure 5: System response of (74) with different initial conditions.

7. Numerical Examples

This section gives several examples to illustrate the validity of the obtained results.

Example 1. Take $a = 6, b = 4, \alpha_1 = 0.45, \alpha_2 = 0.12$, $c_1 = 0.2$, $c_2 = 0.3, \sigma = 0.01$. The bifurcation diagram of (62) in the path sense is shown in Figure 1. Figure 2 depicts the chaotic solutions in the path sense as bifurcation parameter $\alpha = 0.5$, and the corresponding 0-1 test result is shown in Figure 3.

Example 2. Take $a = 2, b = 4, \alpha_1 = 0.45, \alpha_2 = 0.5, c_1 = 0.2, c_2 = 0.3, \alpha = 0.8$. Then, the interior Nash equilibrium point is $N_4 = (88478/585675, 442756/2928375)$. Thus, it follows from Corollary 16 that the system (62) is Lyapunov stable in probability as $0 < \sigma < 0.120165$. When $\sigma = 0.1$, the numerical results are shown in Figure 4.

Example 3. Take $a = 4, b = 2, \alpha_1 = 0.4, \alpha_2 = 0.4, c_1 = 1.2$, $c_2 = 0.5, \alpha = 0.4, \sigma = 0.1$. Then, the interior Nash equilibrium point is $N_4 = (970/383, 6128/12125)$. The numerical results are shown in Figure 5.

8. Conclusion and Discussion

The main contributions of this paper are as follows:

(a) This study proposes a class of stochastic discrete fractional models and develops the Lyapunov function stability theory for such models. Also, the method developed in this paper can be extended to study other stochastic discrete fractional models. The results obtained further enrich the theory of discrete fractional calculus, while also laying the foundations for the application of stochastic discrete fractional calculus to financial models.

(b) We proposed a new stochastic discrete fractional Cournot duopoly game model based on the truncated form of a fractional Grünwald–Letnikov difference operator. This modeling approach introduces a new modeling tool for modeling and analysis in finance. Compared with [23], we considered the effects of stochastic perturbations and sliding memory, and the proposed model is more flexible and general.

(c) By using the Lyapunov theory, we obtained sufficient conditions on the stability in probability and finite time stability in probability for the proposed model at the interior Nash equilibrium point. Furthermore, the locally optimal control conditions are obtained via the stochastic Bellman theory and feedback control principle.

(d) The analysis conducted shows that the new model has a unique interior positive Nash equilibrium point that remains stable under certain conditions by considering appropriate fractional order value and memory step, indicating that these two firms persist and coexist in the market. Both firms can play the game by selecting the memory step and memory strength to achieve a win-win outcome. From the point of view of the evaluation of market stability, these results show that the fractional version has better application prospects.

(e) In addition, the results also show that the memory effect can transform simple games into complex ones. When changing the memory strength, we found that the new short-memory game model has chaotic phenomena through numerical simulations, indicating that the behavior of the game model becomes unpredictable. As a result, the short-memory effect cannot be overlooked as an important driver in economic dynamics. The findings of this study are useful in the development of many dynamical models in economics and finance, biological evolution, and other fields.

It is worth noting that some coupling terms are separated using Young's inequality, which is conservative. Besides, only the stability in probability of the model is investigated, while stronger stability in probability, such as the mean square stability of nonlinear stochastic discrete fractional models, is not addressed in this paper.

The following are future research programs:

(a) Develop new stability conditions based on the Jacobian matrix method.

(b) This paper considers that two bounded rational firms have the same memory length, but the memory length is different due to the difference in individual managers. Therefore, it would be interesting to consider the setting that two bounded rational firms have different memory lengths in future research.

(c) Also, as Xin et al. indicated in [23], there may be a combination of short and long memory, which would be fascinating to investigate.

(d) Because the fractional model is extremely dependent on the fractional order α and the memory step M, it is an open question of how to determine the appropriate step size and fractional order in a real setting.

(e) Combine with real-world financial data to further develop the application of the model.

Data Availability

No additional data are available for the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Jie Ran was responsible for methodology, investigation, conceptualization, writing the original draft, and reviewing and editing the article. Yonghui Zhou was responsible for supervision, conceptualization, reviewing and editing the article, and funding acquisition.

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References

[1] A. Cournot, *Researches into the Principles of the Theory of Wealth*, Macmillan Cancer Support, New York, NY, USA, 1963.

[2] T. Puu, "Complex dynamics with three oligopolists," *Chaos, Solitons & Fractals*, vol. 7, no. 12, pp. 2075–2081, 1996.

[3] M. Kopel, "Simple and complex adjustment dynamics in Cournot duopoly models," *Chaos, Solitons & Fractals*, vol. 7, no. 12, pp. 2031–2048, 1996.

[4] H. N. Agiza and A. A. Elsadany, "Nonlinear dynamics in the Cournot duopoly game with heterogeneous players," *Physica A: Statistical Mechanics and Its Applications*, vol. 320, pp. 512–524, 2003.

[5] A. Agliari, A. K. Naimzada, and N. Pecora, "Nonlinear dynamics of a Cournot duopoly game with differentiated products," *Applied Mathematics and Computation*, vol. 281, pp. 1–15, 2016.

[6] A. A. Elsadany, "Dynamics of a Cournot duopoly game with bounded rationality based on relative profit maximization," *Applied Mathematics and Computation*, vol. 294, pp. 253–263, 2017.

[7] F. Tramontana, A. A. Elsadany, B. Xin, and H. Agiza, "Local stability of the Cournot solution with increasing heterogeneous competitors," *Nonlinear Analysis: Real World Applications*, vol. 26, pp. 150–160, 2015.

[8] J. Andaluz, A. A. Elsadany, and G. Jarne, "Nonlinear Cournot and Bertrand-type dynamic triopoly with differentiated

products and heterogeneous expectations," *Mathematics and Computers in Simulation*, vol. 132, pp. 86–99, 2017.

[9] A. Al-Khedhairi, "Dynamics of a Cournot duopoly game with a generalized bounded rationality," *Complexity*, vol. 2020, Article ID 8903183, 10 pages, 2020.

[10] S. S. Askar and A. Al-Khedhairi, "Analysis of nonlinear duopoly games with product differentiation: stability, global dynamics, and control," *Discrete Dynamics in Nature and Society*, vol. 2017, Article ID 2585708, 13 pages, 2017.

[11] M. Lampart, A. Lampartová, and G. Orlando, "On extensive dynamics of a Cournot heterogeneous model with optimal response," *Chaos*, vol. 32, no. 2, Article ID 023124, 2022.

[12] S. S. Askar, "A competition of duopoly game whose players are public: dynamic investigations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 111, Article ID 106486, 2022.

[13] T. Li, J. Xie, S. Lu, and J. Tang, "Duopoly game of callable products in airline revenue management," *European Journal of Operational Research*, vol. 254, no. 3, pp. 925–934, 2016.

[14] L. Cerboni Baiardi and A. K. Naimzada, "An oligopoly model with rational and imitation rules," *Mathematics and Computers in Simulation*, vol. 156, pp. 254–278, 2019.

[15] J. Ma and L. Xie, "Study on the complexity pricing game and coordination of the duopoly air conditioner market with disturbance demand," *Communications in Nonlinear Science and Numerical Simulation*, vol. 32, pp. 99–113, 2016.

[16] J. Ma and F. Si, "Complex dynamics of a continuous Bertrand duopoly game model with two-stage delay," *Entropy*, vol. 18, no. 7, p. 266, 2016.

[17] A. A. Elsadany and A. M. Awad, "Dynamical analysis of a delayed monopoly game with a log-concave demand function," *Operations Research Letters*, vol. 44, no. 1, pp. 33–38, 2016.

[18] C. Coll, A. Herrero, D. Ginestar, and E. Sánchez, "The discrete fractional order difference applied to an epidemic model with indirect transmission," *Applied Mathematical Modelling*, vol. 103, pp. 636–648, 2022.

[19] V. E. Tarasov and V. V. Tarasova, "Economic dynamics with memory: fractional calculus approach," *Fractional Calculus in Applied Sciences and Engineering*, vol. 8, 2021.

[20] A. Al-khedhairi, "Dynamical analysis of fractional-order differentiated Cournot triopoly game," *Journal of Vibration and Control*, vol. 26, no. 15-16, pp. 1367–1380, 2020.

[21] A. Al-khedhairi, "Differentiated Cournot duopoly game with fractional-order and its discretization," *Engineering Computations*, vol. 2019, 2019.

[22] L. L. Huang, J. H. Park, G. C. Wu, and Z. Mo, "Variable-order fractional discrete-time recurrent neural networks," *Journal of Computational and Applied Mathematics*, vol. 370, Article ID 112633, 2020.

[23] B. Xin, W. Peng, and Y. Kwon, "A discrete fractional-order Cournot duopoly game," *Physica A: Statistical Mechanics and Its Applications*, vol. 558, Article ID 124993, 2020.

[24] F. M. Atici and P. Eloe, "Discrete fractional calculus with the nabla operator," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 3, pp. 1–12, 2009.

[25] T. Abdeljawad, "On Riemann and Caputo fractional differences," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1602–1611, 2011.

[26] F. M. Atici and S. Şengül, "Modeling with fractional difference equations," *Journal of Mathematical Analysis and Applications*, vol. 369, no. 1, pp. 1–9, 2010.

[27] F. Chen, X. Luo, and Y. Zhou, "Existence results for nonlinear fractional difference equation," *Advances in Difference Equations*, vol. 2011, Article ID 713201, 12 pages, 2011.

[28] D. Baleanu, K. Diethelm, E. Scalas, and J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, World Scientific, Singapore, 2012.

[29] G. C. Wu and D. Baleanu, "Discrete fractional logistic map and its chaos," *Nonlinear Dynamics*, vol. 75, no. 1-2, pp. 283–287, 2014.

[30] G. C. Wu, D. Baleanu, and S. D. Zeng, "Discrete chaos in fractional sine and standard maps," *Physics Letters A*, vol. 378, no. 5-6, pp. 484–487, 2014.

[31] Y. Wei, Y. Chen, T. Liu, and Y. Wang, "Lyapunov functions for nabla discrete fractional order systems," *ISA Transactions*, vol. 88, pp. 82–90, 2019.

[32] F. Du and B. Jia, "Finite-time stability of a class of nonlinear fractional delay difference systems," *Applied Mathematics Letters*, vol. 98, pp. 233–239, 2019.

[33] D. Yang, Y. Yu, W. Hu, X. Yuan, and G. Ren, "Mean square asymptotic stability of discrete-time fractional order stochastic neural networks with multiple time-varying delays," *Neural Processing Letters*, vol. 55, no. 7, pp. 9247–9268, 2023.

[34] G. C. Wu and D. Baleanu, "Discrete chaos in fractional delayed logistic maps," *Nonlinear Dynamics*, vol. 80, no. 4, pp. 1697–1703, 2015.

[35] J. Ran, Y. Q. Li, and Y. B. Xiong, "On the dynamics of fractional q-deformation chaotic map," *Applied Mathematics and Computation*, vol. 424, Article ID 127053, 2022.

[36] M. K. Shukla and B. B. Sharma, "Investigation of chaos in fractional order generalized hyperchaotic Hénon map," *AEU-International Journal of Electronics and Communications*, vol. 78, pp. 265–273, 2017.

[37] M. F. Danca, M. Feckan, and N. Kuznetsov, "Chaos control in the fractional order logistic map via impulses," *Nonlinear Dynamics*, vol. 98, no. 2, pp. 1219–1230, 2019.

[38] A. D. Oğuz, J. Alzabut, A. Özbekler, and J. Jonnalagadda, "Lyapunov and Hartman-type inequalities for higher-order discrete fractional boundary value problems," *Miskolc Mathematical Notes*, vol. 24, no. 2, pp. 953–963, 2023.

[39] J. Alzabut, R. Dhineshbabu, A. G. M. Selvam, J. Gómez-Aguilar, and H. Khan, "Existence, uniqueness and synchronization of a fractional tumor growth model in discrete time with numerical results," *Results in Physics*, vol. 54, Article ID 107030, 2023.

[40] W. Shammakh, A. G. M. Selvam, V. Dhakshinamoorthy, and J. Alzabut, "Stability of boundary value discrete fractional hybrid equation of second type with application to heat transfer with fins," *Symmetry*, vol. 14, no. 9, p. 1877, 2022.

[41] C. Luo, B. Q. Liu, and H. S. Hou, "Fractional chaotic maps with q-deformation," *Applied Mathematics and Computation*, vol. 393, Article ID 125759, 2021.

[42] B. Xin, F. Cao, W. Peng, and A. Elsadany, "A bertrand duopoly game with long-memory effects," *Complexity*, vol. 2020, Article ID 2924169, 7 pages, 2020.

[43] A. Al-Khedhairi, A. A. Elsadany, and A. Elsonbaty, "On the dynamics of a discrete fractional-order cournot-bertrand competition duopoly game," *Mathematical Problems in Engineering*, vol. 2022, Article ID 8249215, 13 pages, 2022.

[44] Y. M. Chu, S. Bekiros, E. Zambrano-Serrano et al., "Artificial macro-economics: a chaotic discrete-time fractional-order laboratory model," *Chaos, Solitons & Fractals*, vol. 145, Article ID 110776, 2021.

[45] A. A. Khennaoui, A. O. Almatroud, A. Ouannas, M. M. Al-Sawalha, G. Grassi, and V. Pham, "The effect of caputo fractional difference operator on a novel game theory model," *Discrete & Continuous Dynamical Systems B*, vol. 26, no. 8, p. 4549, 2021.

[46] C. Chen, "Discrete caputo delta fractional economic cobweb models," *Qualitative Theory of Dynamical Systems*, vol. 22, no. 1, p. 8, 2023.

[47] B. Xin and Y. Wang, "Stability and Hopf bifurcation of a stochastic Cournot duopoly game in a blockchain cloud services market driven by brownian motion," *IEEE Access*, vol. 8, pp. 41432–41438, 2020.

[48] M. Lanchares and W. M. Haddad, "Nonlinear–nonquadratic optimal and inverse optimal control for discrete-time stochastic dynamical systems," *International Journal of Robust and Nonlinear Control*, vol. 32, no. 3, pp. 1487–1509, 2022.

[49] A. Dzieliński and D. Sierociuk, "Stability of discrete fractional order state-space systems," *Journal of Vibration and Control*, vol. 14, no. 9-10, pp. 1543–1556, 2008.

[50] O. Megherbi, H. Hamiche, S. Djennoune, and M. Bettayeb, "A new contribution for the impulsive synchronization of fractional-order discrete-time chaotic systems," *Nonlinear Dynamics*, vol. 90, no. 3, pp. 1519–1533, 2017.

[51] W. M. Haddad and J. Lee, "Lyapunov theorems for stability and semistability of discrete-time stochastic systems," *Automatica*, vol. 142, Article ID 110393, 2022.

[52] J. Lee, W. M. Haddad, and M. Lanchares, "Finite time stability and optimal finite time stabilization for discrete-time stochastic dynamical systems," *IEEE Transactions on Automatic Control*, vol. 68, pp. 1–13, 2022.

[53] L. Shaikhet, *Lyapunov Functionals and Stability of Stochastic Difference Equations*, Springer, Berlin, Germany, 2011.