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Research Article

Stability Results for a Class of Fractional Itô–Doob Stochastic Integral Equations

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In this paper, we study the Hyers–Ulam stability of Hadamard fractional Itô–Doob stochastic integral equations by using the Banach fixed point method and some mathematical inequalities. Finally, we exhibit three theoretical examples to apply our theory.

1. Introduction

The concept of fractional derivative first appeared in a correspondence between L'Hôpital and Leibnitz in 1695. Many scientists have explored this idea. To illustrate this notion, we will give an overview overall history of work in this area. We can cite the study of Euler in 1730. Also, one should not overlook the applications of J. L. Lagrange in 1772, nor forget the proposed notion of fractal derivative by Laplace in 1812. Additionally, the work of Abel in the field of fractional calculus significantly contributes to this area. For more details about the extension of fractional calculus, one can refer to the works of Atangana, Baleanu, and other scientists (see [1–9]).

Over the last decades, fractional calculus had played an interesting role. Its importance appears in various areas such as chemistry, physics, economics, biology, and other fields. Over the past decade, fractional calculus has been applied for describing long-memory processes. Many classical techniques are difficult to apply directly to fractional differential equations. It is, therefore, necessary to develop especially new theories and methods whose analysis becomes more difficult. Compared with the classical properties of differential equations, research on the concept of fractional differential equations is still in its initial stage of development.

In stability concept, the Ulam stability was first introduced by Ulam (see [10]) and then was generalized by Hyers and Rassias (see [11, 12]). Many scientists generalized the Ulam–Hyers–Rassias results in various systems; for Hadamard fractional Itô–Doob stochastic integral equations and Caputo-derivative, we can refer to [13–16], and for fractional stochastic differential equation with fractional Brownian motion and pantograph differential equations, see [17–21].

One of the most important classes of fractional differential equations are the fractional Itô–Doob stochastic differential equations which had many applications in describing many phenomena of real life, and the nonlocal conditions describe numerous problems in physics (see [13, 22, 23]), finance (see [24, 25]), and mechanical problem (see [26, 27]). To the best of our knowledge, there is no existing work on the Hyers–Ulam stability of fractional Itô–Doob stochastic integral equations. Motivated by the previous works, in this paper, we will cover this gap. The main contributions of the paper are as follows:

(i) Study the existence and uniqueness of the solution of Hadamard fractional Itô–Doob stochastic integral equations.

(ii) Investigate the Hyers–Ulam stability of Hadamard fractional Itô–Doob stochastic integral equations.

(iii) Extend the work on [13] to summed Hadamard fractional Itô–Doob stochastic integral equations.

The organization of the paper is as follows. We exhibit some preliminaries and basic notions in Section 2. Section 3 is devoted to the fundamental results. In Section 4, we present three examples to show the effectiveness of our results.

2. Basic Notions

Set $\Theta > 1$ and $\mathcal{S} = \{\mathbb{Y}, \widetilde{\mathfrak{M}}, \widetilde{\mathbb{M}} = (\widetilde{\mathfrak{M}}_\omega)_{1\le\omega\le\Theta}, \widetilde{\mathscr{P}}\}$ as a complete probability space and $\mathbb{W}(\omega)$ as a standard Brownian motion.

For $q \geq 2$, set $\mathbb{Y}^q_\omega = L^q(\mathbb{Y}, \mathfrak{M}_\omega, \widetilde{\mathscr{P}})$ space of all $\mathbb{M}_\omega$-measurable and q-th integrable functions $\phi = (\phi_1, \ldots, \phi_n)^T : \mathbb{Y} \longrightarrow \mathbb{R}$ with

$$\|\phi\|_q = (\mathbb{E}(|\phi|^q))^{1/q}. \quad (1)$$

Definition 1 (see [1]). Set $\beta \in (0,1)$ and $f(\omega)$ as a continuous function and thus the fractional Hadamard integral of $f(\omega)$ takes the form

$$I^\beta g(\omega) = \frac{1}{\Gamma(\beta)} \int_1^\omega \left(\log\frac{\omega}{s}\right)^{\beta-1} \frac{f(s)}{s} ds. \quad (2)$$

Consider the Hadamard fractional Itô–Doob stochastic integral equation

$$\xi(\omega) = \psi + \int_1^\omega v_1(s,\xi(s))ds + \sum_{i=2}^n \beta_i \int_1^\omega \left(\ln\left(\frac{\omega}{s}\right)\right)^{\beta_i-1} \frac{v_i(s,\xi(s))}{s} ds + \int_1^\omega g(s,\xi(s))d\mathbb{W}(s), \quad (3)$$

where $\psi \in \mathbb{R}$, $0 < \beta_i < 1$ for $i \in \{2,3,\ldots,n\}$, $\omega \in [1,\Theta]$, and $v_i, g : [1,\Theta] \times \mathbb{R} \longrightarrow \mathbb{R}$, $i \in \{2,3,\ldots,n\}$ are measurable functions.

As we proceed, we take into account $q > \max_{1\le i\le n}\{1/\beta_i\}$.

Now, we consider the following assumptions which are important criteria to prove the main results of the next sections:

$\mathscr{H}_1$: there exist $\bar{D} > 0$ such that

$$|g(\omega,\gamma_1) - g(\omega,\gamma_2)| \vee |v_1(\omega,\gamma_1) - v_1(\omega,\gamma_2)| \vee, \ldots, \vee |v_n(\omega,\gamma_1) - v_n(\omega,\gamma_2)| \le \bar{D}|\gamma_1 - \gamma_2|, \quad (4)$$

for all $(\omega, \gamma_1, \gamma_2) \in [1,\Theta] \times \mathbb{R} \times \mathbb{R}$.

$\mathscr{H}_2$: there exist $d > 0$ such that

$\text{ess sup}_{\omega\in[1,\Theta]}|g(\omega,0)| \le d$, $\text{ess sup}_{\omega\in[1,\Theta]}|v_i(\omega,0)| \le d$ for $i \in \{1,2,\ldots,n\}$.

3. Main Results

Let $\mathcal{S}^q([1,\Theta])$ be the family of all processes ξ which are measurable and $\widetilde{\mathbb{M}}$ adapted satisfying $\sup_{\omega\in[1,\Theta]}\|\xi(\omega)\|_q < \infty$. Let $\|\|\|_{\mathcal{S}^q}$ be the norm on $\mathcal{S}^q([1,\Theta])$ given by

$$\|\xi\|\mathcal{S}^q = \sup_{\omega\in[1,\Theta]} \|\xi(\omega)\|_q. \quad (5)$$

We have, $(\mathcal{S}^q([1,\Theta]), \|\cdot\|_{\mathcal{S}^q})$ is a Banach space.

Definition 2. Let

$$S(\omega) = y(1) + \int_1^\omega v_1(s,y(s))ds + \sum_{i=2}^n \beta_i \int_1^\omega \left(\ln\left(\frac{\omega}{s}\right)\right)^{\beta_i-1} \frac{v_i(s,y(s))}{s} ds + \int_1^\omega g(s,y(s))d\mathbb{W}(s), \quad (6)$$

where $y(1) = \psi \in \mathbb{R}$, $0 < \beta_i < 1$ for $i \in \{2,3,\ldots,n\}$, $\omega \in [1,\Theta]$, and $v_i, g : [1,\Theta] \times \mathbb{R} \longrightarrow \mathbb{R}$, $i \in \{2,3,\ldots,n\}$ are measurable functions.

Equation (3) is Ulam–Hyers stable with respect to ϵ if there exists $\Delta > 0$ such that for each $\epsilon > 0$ and for each solution $y \in \mathcal{S}^q([1,\Theta])$, with

$$\|y(\omega) - S(\omega)\|_q^q \le \epsilon, \quad \forall\, \omega \in [1,\Theta], \quad (7)$$

there is a solution $\varphi \in \mathcal{S}^q([1,\Theta])$ of (3), with $\varphi(1) = y(1)$, and

$$\|y(\omega) - \varphi(\omega)\|_q^q \le C\epsilon, \quad \forall\, \omega \in [1,\Theta]. \quad (8)$$

Theorem 3. *Assume that $\mathscr{H}_1$ and $\mathscr{H}_2$ hold. Let $z \in \mathscr{S}^q([1,\Theta])$ with, satisfying*

$$\left\| y(\omega) - S(\omega) \right\|_q^q \leq \epsilon, \quad \forall\, \omega \in [1,\Theta], \tag{9}$$

$$S(\omega) = y(1) + \int_1^\omega v_1(s, y(s))\mathrm{d}s + \sum_{i=2}^n \beta_i \int_1^\omega \left(\ln\left(\frac{\omega}{s}\right)\right)^{\beta_i - 1} \frac{v_i(s, y(s))}{s} ds + \int_1^\omega g(s, y(s))\mathrm{d}\mathbb{W}(s), \tag{10}$$

where $\epsilon > 0$. Then, there is a solution $\varphi \in \mathscr{S}^q([1,\Theta])$ of (3), with

$$\left\| y(\omega) - \varphi(\omega) \right\|_q^q \leq M\epsilon, \quad \forall\, \omega \in [1,\Theta,] \tag{11}$$

where

$$\begin{aligned} M &= \frac{\Theta^\rho}{(1-c)^q}, \\ \lambda &= (n+1)^{q-1}\left[\bar{D}^q \Theta^{q-1} + \sum_{i=2}^n \beta_i^q \bar{D}^q \ln(\Theta)^{\beta_i q - 1}\left(\frac{q-1}{\beta_i q - 1}\right)^{q-1} + \bar{D}^q M_q \Theta^{q-2/2}\right], \\ M_q &= \left(\frac{(q-1)q}{2}\right)^{q/2} \Theta^{q-2/2}, \end{aligned} \tag{12}$$

and ρ is some positive constant such that

$$c = \left(\frac{\lambda\Theta}{\rho+1}\right)^{1/q} < 1. \tag{13}$$

Proof. Consider $\|.\|_\rho$ norm on $\mathscr{S}^q([1,\Theta])$ defined by

$$\|\xi\|_\rho^q = \operatorname{esssup}_{\omega\in[1,\Theta]}\left(\frac{\mathbb{E}|\xi(\omega)|^q}{\omega^\rho}\right). \tag{14}$$

We have, $\|.\|_{\mathscr{S}^q}$ and $\|.\|_\rho$ are equivalent.

Let $\mathscr{K}\colon \mathscr{S}^q([1,\Theta]) \longrightarrow \mathscr{S}^q([1,\Theta])$ given by

$$(\mathscr{K}\xi)(\omega) = y(1) + \int_1^\omega v_1(s, \xi(s)) ds + \sum_{i=2}^n \beta_i \int_1^\omega \left(\ln\left(\frac{\omega}{s}\right)\right)^{\beta_i - 1} \frac{v_i(s, \xi(s))}{s} ds + \int_1^\omega g(s, \xi(s)) d\mathbb{W}(s), \tag{15}$$

for every $\omega \in [1,\Theta]$. We will split our proof into the following three steps:

Step 1: $\mathscr{K}$ is well defined.

Let $\xi \in \mathscr{S}^q([1,\Theta])$, we get for $\omega \in [1,\Theta]$,

$$\begin{aligned} \|(\mathscr{K}\xi)(\omega)\|_q^q \leq (n+2)^{q-1}&\left[\left\|y(1)\right\|_q^q + \left\|\int_1^\omega v_1(s,\xi(s))\mathrm{d}s\right\|_q^q + \left\|\int_1^\omega g(s,\xi(s))\mathrm{d}\mathbb{W}(s)\right\|_q^q\right. \\ &\left. + \sum_{i=2}^n \beta_i^q \left\|\int_1^\omega \left(\ln\left(\frac{\omega}{s}\right)\right)^{\beta_i-1} \frac{v_i(s,\xi(s))}{s}\mathrm{d}s\right\|_q^q\right]. \end{aligned} \tag{16}$$

By the Hölder inequality, we obtain

$$\begin{aligned}\left\|\int_1^\omega v_1(s,\xi(s))\mathrm{d}s\right\|_q^q &\le \mathbb{E}\left(\int_1^\omega \left|v_1(s,\xi(s))\right|\mathrm{d}s\right)^q\\ &\le \omega^{q-1}\mathbb{E}\int_1^\omega \left|v_1(s,\xi(s))\right|^q\mathrm{d}s\\ &\le \Theta^{q-1}\mathbb{E}\int_1^\omega \left|v_1(s,\nu(s))\right|^q\mathrm{d}s\\ &\le \Theta^{q-1}\mathbb{E}\int_1^\omega \left|v_1(s,\nu(s))-v_1(s,0)+v_1(s,0)\right|^q\mathrm{d}s\\ &\le (2\Theta)^{q-1}\mathbb{E}\int_1^\omega\left(\overline{D}^q|\xi(s)|^q+\left|v_1(s,0)\right|^q\right)\mathrm{d}s\\ &\le (2\Theta)^{q-1}\overline{D}^q\Theta\|\xi\|_{\mathcal{S}^q}^q+(2\Theta)^{q-1}\int_1^\Theta \left|v_1(s,0)\right|^q\mathrm{d}s,\end{aligned} \tag{17}$$

$$\begin{aligned}\left\|\int_1^\omega\left(\ln\left(\frac{\omega}{s}\right)\right)^{\beta_i-1}\frac{v_i(s,\xi(s))}{s}\mathrm{d}s\right\|_q^q &\le \mathbb{E}\left(\int_1^\omega\frac{1}{r}\left(\ln\left(\frac{\omega}{r}\right)\right)^{\beta_i-1}\left|v_i(r,\xi(r))\right|\mathrm{d}r\right)^q\\ &\le \mathbb{E}\left(\left(\int_1^\omega\frac{1}{r}\left(\ln\left(\frac{\omega}{r}\right)\right)^{((\beta_i-1)q/q-1)}\mathrm{d}r\right)^{q-1}\int_1^\omega\left|v_i(r,\xi(r))\right|^q\mathrm{d}r\right)\\ &\le \frac{(\ln\Theta)^{\beta_i q-1}(q-1)^{q-1}}{(\beta_i q-1)^{q-1}}\int_1^\omega\left\|v_i(s,\xi(s))\right\|_q^q\mathrm{d}s\\ &\le \frac{(\ln\Theta)^{\beta_i q-1}(2q-2)^{q-1}}{(\beta_i q-1)^{q-1}}\left(\overline{D}^q\Theta\|\xi\|_{\mathcal{S}^q}^q+\int_1^\Theta\left|v_i(s,0)\right|^q\mathrm{d}s\right).\end{aligned}$$

Using the Hölder inequality and Theorem 7.1 in [28], we get

$$\begin{aligned}\left\|\int_1^\omega g(s,\xi(s))\mathrm{d}\mathbb{W}(s)\right\|_q^q &\le M_q\mathbb{E}\left|\int_1^\omega |g(s,\xi(s))|^2\mathrm{d}s\right|^{q/2}\\ &\le M_q\mathbb{E}\left(\int_1^\omega |g(s,\xi(s))|^q\mathrm{d}s\right)\left(\int_1^\omega\mathrm{d}s\right)^{q-2/2}\\ &\le M_q\Theta^{q-2/2}\mathbb{E}\left(\int_1^\omega |g(s,\xi(s))|^q\mathrm{d}s\right)\\ &\le M_q 2^{q-1}\Theta^{q/2}\left(\overline{D}^q\|\xi\|_{\mathcal{S}^q}^q+d^q\right).\end{aligned} \tag{18}$$

Therefore, $\|\mathscr{K}\xi(\omega)\|_{\mathcal{S}^q} < \infty$.

Step 2: $\mathscr{K}$ is contractive.

We have for all $\omega \in [1, \Theta]$,

$$\begin{aligned}\left\|\mathscr{K}\xi(\omega) - \mathscr{K}\widehat{\xi}(\omega)\right\|_q^q \leq (n+1)^{q-1}\Bigg[\left\|\int_1^\omega \left(v_1(w,\xi(w)) - v_1(w,\widehat{\xi}(w))\right)dw\right\|_q^q + \left\|\int_1^\omega (g(w,\nu(w)) - g(w,\widehat{\nu}(w)))d\mathbb{W}(w)\right\|_q^q \\ + \sum_{i=2}^n \beta_i^q \left\|\int_1^\omega \left(\ln\left(\frac{\omega}{w}\right)\right)^{\beta_i - 1}\left(v_i(w,\xi(w)) - v_i(w,\widehat{\xi}(w))\right)\frac{dw}{w}\right\|_q^q\Bigg].\end{aligned} \tag{19}$$

Using $\mathscr{H}_1$ and the Hölder inequality, we obtain

$$\begin{aligned}\left\|\int_1^\omega \left(v_1(l,\xi(l)) - v_1(l,\widehat{\xi}(l))\right)\mathrm{d}l\right\|_q^q &\leq \omega^{q-1}\mathbb{E}\left(\int_1^\omega \left|v_1(s,\xi(s)) - v_1(s,\widehat{\xi}(s))\right|^q \mathrm{d}s\right) \\ &\leq \overline{D}^q \Theta^{q-1}\int_1^\omega \left\|\xi(s) - \widehat{\xi}(s)\mathrm{d}s\right\|_q^q \\ &\left\|\int_1^\omega \frac{1}{l}\left(\ln\left(\frac{\omega}{l}\right)\right)^{\beta_i - 1}\left(v_i(l,\xi(l)) - v_i(l,\widehat{\xi}(l))\right)\mathrm{d}l\right\|_q^q \\ &\leq \mathbb{E}\left(\left(\int_1^\omega \frac{1}{s^{q/q-1}}\left(\ln\left(\frac{\omega}{s}\right)\right)^{(q(\beta_i - 1)/q-1)}\mathrm{d}s\right)^{q-1}\int_1^\omega \left|v_i(s,\xi(s)) - v_i(s,\widehat{\xi}(s))\right|^q \mathrm{d}s\right) \\ &\leq \frac{\overline{D}^q \ln(\Theta)^{\beta_i q - 1}(q-1)^{q-1}}{(\beta_i q - 1)^{q-1}}\int_1^\omega \left\|\xi(s) - \widehat{\xi}(s)\right\|_q^q \mathrm{d}s.\end{aligned} \tag{20}$$

Using $\mathscr{H}_1$, the Hölder inequality and Theorem 7.1 in [28], we get

$$\begin{aligned}\left\|\int_1^\omega (g(\varpi,\xi(\varpi)) - g(\varpi,\widehat{\xi}(\varpi)))d\mathbb{W}(\varpi)\right\|_q^q &\leq M_q\mathbb{E}\left|\int_1^\omega |g(\varpi,\xi(\varpi)) - g(\varpi,\widehat{\xi}(\varpi))|^2 d\varpi\right|^{q/2} \\ &\leq M_q \overline{D}^q \Theta^{q-2/2}\int_1^\omega \left\|\xi(\varpi) - \widehat{\xi}(\varpi)\right\|_q^q d\varpi.\end{aligned} \tag{21}$$

Therefore, we get

$$\left\|\mathscr{K}\xi(\omega)-\mathscr{K}\widehat{\xi}(\omega)\right\|_q^q\leq\lambda\int_1^\omega\left\|\xi(s)-\widehat{\xi}(s)\right\|_q^q\mathrm{d}s. \tag{22}$$

Then,

$$\begin{aligned}\left\|\mathscr{K}\xi(\omega)-\mathscr{K}\widehat{\xi}(\omega)\right\|_q^q&\leq\lambda\int_1^\omega\left\|\xi(s)-\widehat{\xi}(s)\right\|_q^q\mathrm{d}s\\&\leq\lambda\int_1^\omega\frac{\left\|\xi(s)-\widehat{\xi}(s)\right\|_q^q}{s^\varrho}s^\varrho\mathrm{d}s\\&\leq\lambda\left\|\xi-\widehat{\xi}\right\|_\varrho^q\int_1^\omega s^\varrho\mathrm{d}s\\&\leq\frac{\lambda\omega^{\varrho+1}}{(\varrho+1)}\left\|\xi-\widehat{\xi}\right\|_\varrho^q\\&\leq\frac{\lambda\Theta}{\varrho+1}\left\|\xi-\widehat{\xi}\right\|_\varrho^q\omega^\varrho.\end{aligned} \tag{23}$$

Thus,

$$\left\|\mathscr{K}\xi-\mathscr{K}\hat{\xi}\right\|_\varrho\leq c\|\nu-\widehat{\nu}\|_\varrho. \tag{24}$$

Therefore, $\mathscr{K}$ is contractive.

Step 3: An estimation of $\left\|y(\omega)-\varphi(\omega)\right\|_q^q$. It follows from (9) that

$$\frac{\mathbb{E}|y(\omega)-\mathscr{K}y(\omega)|^q}{\omega^\varrho}\leq\epsilon. \tag{25}$$

Then,

$$\left\|\mathscr{K}y-y\right\|_\varrho\leq\epsilon. \tag{26}$$

In view of Theorem 2.1 in [13], there is a unique solution $\varphi\in\mathcal{S}^q([1,\Theta])(\varphi(1)=y(1))$ such that

$$\left\|\varphi-y\right\|_\varrho\leq\frac{\epsilon^{1/q}}{1-c}. \tag{27}$$

Consequently, $\forall\omega\in[1,\Theta]$, and we have

$$\mathbb{E}|\varphi(\omega)-y(\omega)|^q\leq M\epsilon. \tag{28}$$

□

4. Illustrative Examples

Three examples are given to show the effectiveness of our results.

Example 1. Let equation (3) be with $\Theta=2$, $n=3$, $\beta_2=0.6$, and $\beta_3=0.4$.

$$\begin{aligned}v_1(\omega,\xi)&=\sin(2\xi),\\v_2(\omega,\xi)&=\frac{\cos(2\xi)}{1+\omega^2},\\v_3(\omega,\xi)&=\omega\sin(\xi),\\g(\omega,\xi(\omega))&=\omega^2\sin(\xi).\end{aligned} \tag{29}$$

We have,

$$\begin{aligned}\left|v_1(\omega,\xi_1)-v_1(\omega,\xi_2)\right|&\leq2\left|\xi_1-\xi_2\right|,\\\left|v_2(\omega,\xi_1)-v_2(\omega,\xi_2)\right|&\leq2\left|\xi_1-\xi_2\right|,\\\left|v_3(\omega,\xi_1)-v_3(\omega,\xi_2)\right|&\leq2\left|\xi_1-\xi_2\right|,\\\left|g(\omega,\xi_1)-g(\omega,\xi_2)\right|&\leq4\left|\xi_1-\xi_2\right|.\end{aligned} \tag{30}$$

Thus, assumption $\mathscr{H}_1$ holds for $\bar{D}=4$. Moreover, for $i\in\{1,2,3\}$, we have

$$\begin{aligned}\operatorname{ess}\sup_{\omega\in[1,2]}\left|v_i(\omega,0)\right|&\leq\frac{1}{2},\\\operatorname{ess}\sup_{\omega\in[1,2]}|g(\omega,0)|&\leq\frac{1}{2}.\end{aligned} \tag{31}$$

Therefore, assumption $\mathscr{H}_2$ holds for d = 1/2.

Assume that y satisfies

$$\mathbb{E}\left|y(\omega)-y(1)-\int_1^\omega\sin(2y(s))\mathrm{d}s-0.6\int_1^\omega\left(\ln\left(\frac{\omega}{s}\right)\right)^{-0.4}\frac{\cos(2y(s))}{s(1+s^2)}\mathrm{d}s-0.4\int_1^\omega\left(\ln\left(\frac{\omega}{s}\right)\right)^{-0.6}\sin(y(s))\mathrm{d}s-\int_1^\omega s^2\sin(y(s))\mathrm{d}\mathbb{W}(s)\right|^3\leq\epsilon, \tag{32}$$

for every $\omega\in[1,2]$, where $\epsilon>0$.

Then, it follows from Theorem 3 that there is $\varphi\in\mathcal{S}^3([1,2])$,

$$\begin{aligned}\varphi(\omega)=y(1)&+\int_1^\omega\sin(2\varphi(s))\mathrm{d}s+0.6\int_1^\omega\left(\ln\left(\frac{\omega}{s}\right)\right)^{-0.4}\frac{\cos(2\varphi(s))}{s\left(1+s^2\right)}\mathrm{d}s\\&+0.4\int_1^\omega\left(\ln\left(\frac{\omega}{s}\right)\right)^{-0.6}\sin(\varphi(s))\mathrm{d}s+\int_1^\omega s^2\sin(\varphi(s))d\mathbb{W}(s),\end{aligned} \tag{33}$$

such that

$$\mathbb{E}|\varphi(\omega) - y(\omega)|^3 \leq \frac{2^{2\lambda}}{\left(1 - (2\lambda/2\lambda + 1)^{1/3}\right)^3}\epsilon, \quad (34)$$

for every $\omega \in [1,2]$, where $\lambda = 4^5[4 + \sqrt{108} + 1.35\ln(2)^{0.8} + 6.4\ln(2)^{0.2}]$.

Example 2. Let equation (3) be with $\Theta = 3$, $n = 3$, $\beta_2 = 0.5$, and $\beta_3 = 0.5$.

$$\begin{aligned} v_1(\omega,\xi) &= 3\xi,\\ v_2(\omega,\xi) &= 3\sin(\xi)e^{-\omega},\\ v_3(\omega,\xi) &= \frac{1}{3}\omega\cos(\xi),\\ g(\omega,\xi(\omega)) &= 2\omega\arctan(\xi). \end{aligned} \quad (35)$$

We have,

$$\begin{aligned} |v_1(\omega,\xi_1) - v_1(\omega,\xi_2)| &\leq 3|\xi_1 - \xi_2|,\\ |v_2(\omega,\xi_1) - v_2(\omega,\xi_2)| &\leq 3|\xi_1 - \xi_2|,\\ |v_3(\omega,\xi_1) - v_3(\omega,\xi_2)| &\leq 3|\xi_1 - \xi_2|,\\ |g(\omega,\xi_1) - g(\omega,\xi_2)| &\leq 6|\xi_1 - \xi_2|. \end{aligned} \quad (36)$$

Thus, assumption $\mathscr{H}_1$ holds for $\bar{D} = 6$. Moreover, for $i \in \{1,2,3\}$, we have

$$\begin{aligned} \operatorname*{ess\,sup}_{\omega\in[1,3]} |v_i(\omega,0)| &\leq 1,\\ \operatorname*{ess\,sup}_{\omega\in[1,3]} |g(\omega,0)| &\leq 1. \end{aligned} \quad (37)$$

Therefore, assumption $\mathscr{H}_2$ holds for $d = 1$.

Assume that y satisfies

$$\mathbb{E}\left|y(\omega) - y(1) - 3\int_1^\omega y(s)\mathrm{d}s - 1.5\int_1^\omega \left(\ln\left(\frac{\omega}{s}\right)\right)^{-0.5}\frac{\sin(y(s))e^{-s}}{s}\mathrm{d}s - \frac{0.5}{3}\int_1^\omega \left(\ln\left(\frac{\omega}{s}\right)\right)^{-0.5}\cos(y(s))\mathrm{d}s - 2\int_1^\omega s\arctan(y(s))\mathrm{d}\mathbb{W}(s)\right|^3 \leq \epsilon, \quad (38)$$

for every $\omega \in [1,3]$, where $\epsilon > 0$.

Then, it follows from Theorem 3 that there is $\varphi \in \mathcal{S}^3([1,3])$,

$$\begin{aligned} \varphi(\omega) = y(1) &+ 3\int_1^\omega \varphi(s)\mathrm{d}s + 1.5\int_1^\omega \left(\ln\left(\frac{\omega}{s}\right)\right)^{-0.5}\frac{\sin(\varphi(s))e^{-s}}{s}\mathrm{d}s\\ &+ \frac{0.5}{3}\int_1^\omega \left(\ln\left(\frac{\omega}{s}\right)\right)^{-0.5}\cos(\varphi(s))ds + 2\int_1^\omega s\arctan(\varphi(s))\mathrm{d}\mathbb{W}(s), \end{aligned} \quad (39)$$

such that

$$\mathbb{E}|\varphi(\omega) - y(\omega)|^3 \leq \frac{3^{3\lambda}}{\left(1 - (3\lambda/3\lambda + 1)^{1/3}\right)^3}\epsilon, \quad (40)$$

for every, where $\lambda = 24^3[18 + 4\ln(3)^{0.5}]$.

Example 3. Let equation (3) be with $\Theta = 4$, $n = 3$, and $\beta_2 = \beta_3 = 0.5$.

$$\begin{aligned} v_1(\omega,\xi) &= \omega\arctan(\xi),\\ v_2(\omega,\xi) &= 4\sin(\xi)\cos(\omega),\\ v_3(\omega,\xi) &= \omega\xi,\\ g(\omega,\xi(\omega)) &= \omega\cos(\xi). \end{aligned} \quad (41)$$

We have,

$$\begin{aligned} |v_1(\omega,\xi_1) - v_1(\omega,\xi_2)| &\leq 4|\xi_1 - \xi_2|,\\ |v_2(\omega,\xi_1) - v_2(\omega,\xi_2)| &\leq 4|\xi_1 - \xi_2|,\\ |v_3(\omega,\xi_1) - v_3(\omega,\xi_2)| &\leq 4|\xi_1 - \xi_2|,\\ |g(\omega,\xi_1) - g(\omega,\xi_2)| &\leq 4|\xi_1 - \xi_2|. \end{aligned} \quad (42)$$

Thus, assumption $\mathscr{H}_1$ holds for $\bar{D} = 4$. Moreover, for $i \in \{1,2,3\}$, we have

$$\begin{aligned} \operatorname*{ess\,sup}_{\omega\in[1,4]} |v_i(\omega,0)| &\leq 4,\\ \operatorname*{ess\,sup}_{\omega\in[1,4]} |g(\omega,0)| &\leq 4. \end{aligned} \quad (43)$$

Therefore, assumption $\mathscr{H}_2$ holds for d = 4.

Assume that y satisfies

$$\mathbb{E}\left|y(\omega)-y(1)-\int_1^\omega s\arctan(y(s))\mathrm{d}s-2\int_1^\omega\left(\ln\left(\frac{\omega}{s}\right)\right)^{-0.5}\frac{\sin(y(s))\cos(s)}{s}\mathrm{d}s-0.5\int_1^\omega\left(\ln\left(\frac{\omega}{s}\right)\right)^{-0.5}y(s)\mathrm{d}s-\int_1^\omega s\cos(y(s))\mathrm{d}\mathbb{W}(s)\right|^3\leq\epsilon, \tag{44}$$

for every $\omega\in[1,4]$, where $\epsilon>0$.

Then, it follows from Theorem 3 that there is $\varphi\in\mathcal{S}^3([1,4])$,

$$\begin{aligned}\varphi(\omega)=y(1)&+\int_1^\omega s\arctan(\varphi(s))\mathrm{d}s+2\int_1^\omega\left(\ln\left(\frac{\omega}{s}\right)\right)^{-0.5}\frac{\sin(\varphi(s))\cos(s)}{s}\mathrm{d}s\\&+0.5\int_1^\omega\left(\ln\left(\frac{\omega}{s}\right)\right)^{-0.5}\varphi(s)\mathrm{d}s+\int_1^\omega s\cos(\varphi(s))\mathrm{d}\mathbb{W}(s),\end{aligned} \tag{45}$$

such that

$$\mathbb{E}|\varphi(\omega)-y(\omega)|^3\leq\frac{4^{4\lambda}}{\left(1-(4\lambda/4\lambda+1)^{1/3}\right)^3}\epsilon, \tag{46}$$

for every $\omega\in[1,4]$, where $\lambda=4^6[4+\ln(4)^{0.5}+3\sqrt{3}]$.

5. Conclusion

This paper focuses on the Hyers–Ulam stability of Hadamard fractional Itô–Doob stochastic integral equations by employing the Banach fixed point method, some stochastic analysis, and mathematically useful techniques. The applicability of the obtained results is proved through three illustrative examples. Combining with some related research in the literature about the fractional stochastic pantograph equations, we can explore various extensions and stability problems for pantograph Hadamard fractional Itô–Doob stochastic integral equations.

Data Availability

No underlying data were collected or produced to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] B. Ahmad, A. Alsaedi, S. K. Ntouyas, and J. Tariboon, *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, Springer, Switzerland, 2017.
[2] R. Almeida, "Caputo-Hadamard fractional derivatives of variable order," *Numerical Functional Analysis and Optimization*, vol. 38, pp. 1–19, 2017.
[3] T. M. Atanackovic, S. Pilipovic, B. Stankovic, and D. Zorica, *Fractional Calculus with Applications in Mechanics*, Wiley-ISTE, London, UK, 2014.
[4] D. Baleanu, J. A. Machado, and A. C. Luo, *Fractional Dynamics and Control*, Springer Science and Business Media, New York, NY, USA, 2011.
[5] Y. Gambo, F. Jarad, D. Baleanu, and T. Abdeljawad, "On Caputo modification of the Hadamard fractional derivatives," *Advances in Difference Equations*, vol. 2014, no. 1, p. 10, 2014.
[6] J. Hadamard, "Essai sur l'étude des fonctions données par leur développement de Taylor," *Journal de Mathematiques Pures et Appliquees*, vol. 8, no. 1892, pp. 101–186.
[7] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, Netherlands, 2006.
[8] C. P. Li and F. H. Zeng, *Numerical Methods for Fractional Calculus*, Chapman and Hall/CRC Press, Boca Raton, FL, USA, 2015.
[9] I. Podlubny, *Fractional Differential Equations*, Academic Press, Cambridge, MA, USA, 1999.
[10] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, NY, USA, 1968.
[11] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences*, vol. 27, no. 4, pp. 222–224, 1941.
[12] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[13] O. Kahouli, A. Ben Makhlouf, L. Mchiri, and H. Rguigui, "Hyers–Ulam stability for a class of Hadamard fractional Itô–Doob stochastic integral equations," *Chaos, Solitons and Fractals*, vol. 166, Article ID 112918, 2023.

[14] A. Ahmadova and N. I. Mahmudov, "Ulam-Hyers stability of Caputo type fractional stochastic neutral differential equations," *Statistics and Probability Letters*, vol. 168, Article ID 108949, 2021.

[15] A. Ben Makhlouf and L. Mchiri, "Some results on the study of Caputo-Hadamard fractional stochastic differential equations," *Chaos, Solitons and Fractals*, vol. 155, Article ID 111757, 2022.

[16] T. Caraballo, L. Mchiri, and M. Rhaima, "Ulam-Hyers-Rassias stability of neutral stochastic functional differential equations," *Stochastics*, vol. 94, no. 6, pp. 959–971, 2022.

[17] Y. Guo, X. B. Shu, Y. Li, and F. Xu, "The existence and Hyers–Ulam stability of solution for an impulsive Riemann–Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$ $1<\beta<2$," *Boundary Value Problems*, vol. 2019, no. 1, p. 59, 2019.

[18] Y. Guo, M. Chen, X.-B. Shu, and F. Xu, "The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm," *Stochastic Analysis and Applications*, vol. 39, no. 4, pp. 643–666, 2021.

[19] S. Li, L. Shu, X. B. Shu, and F. Xu, "Existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays," *Stochastics*, vol. 91, no. 6, pp. 857–872, 2019.

[20] M. Houas, F. Martinez, M. E. Samei, and M. K. A. Kaabar, "Uniqueness and Ulam-Hyers-Rassias stability results for sequential fractional pantograph q-differential equations," *Journal of Inequalities and Applications*, vol. 2022, pp. 93–24, 2022.

[21] M. K. A. Kaabar, V. Kalvandi, N. Eghbali, M. E. Samei, Z. Siri, and F. Martinez, "A generalized ML-Hyers-Ulam Stability of quadratic fractional integral equation," *Nonlinear Engineering*, vol. 10, no. 1, pp. 414–427, 2021.

[22] M. Abouagwa, J. Liu, and J. Li, "Caratheodory approximations and stability of solutions to non-Lipschitz stochastic fractional differential equations of Itô-Doob type," *Applied Mathematics and Computation*, vol. 329, pp. 143–153, 2018.

[23] M. Abouagwa and J. Li, "Approximation properties for solutions to Itô–Doob stochastic fractional differential equations with non-Lipschitz coefficients," *Stochastics and Dynamics*, vol. 19, no. 04, Article ID 1950029, 2019.

[24] G. Jumarie, "On the representation of fractional Brownian motion as an integral with respect to dta," *Applied Mathematics Letters*, vol. 18, no. 7, pp. 739–748, 2005.

[25] J. C. Pedjeu and G. S. Ladde, "Stochastic fractional differential equations: modeling, method and analysis," *Chaos, Solitons & Fractals*, vol. 45, no. 3, pp. 279–293, 2012.

[26] W. Wang and Z. Guo, "Optimal index and averaging principle for Itô-Doob stochastic fractional differential equations," *Stochastics and Dynamics*, vol. 22, no. 06, 2022.

[27] A. B. Makhlouf, L. Mchiri, H. Arfaoui, S. Dhahri, E. S. El-Hady, and B. Cherif, "Hadamard itô-doob stochastic fractional order systems," *Discrete and Continuous Dynamical Systems-S*, vol. 16, no. 8, pp. 2060–2074, 2023.

[28] X. Mao, *Stochastic Differential Equations and Applications*, Ellis Horwood, Chichester, U.K, 1997.