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Research Article

Utilizing the Optimal Auxiliary Function Method for the Approximation of a Nonlinear Long Wave System considering Caputo Fractional Order

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In this article, we explore the utilization of the Caputo derivative and the Riemann–Liouville (R–L) fractional integral to analyze the optimal auxiliary function method for approximating fractional nonlinear long waves. Approximate long wave equation with a distinct dispersion relation offers the most accurate description of shallow water wave properties. Various methods, including the Adomian decomposition technique, the variational iteration method, the optimum homotopy asymptotic method, and the new iterative technique, have been employed and compared to those obtained using the fractional-order approximate long wave equation. The results of our study indicate that the optimal auxiliary function method is highly successful and practically simple, achieving better and more rapid convergence after just one repetition. This method is recognized as an efficient approach, demonstrating high precision in solving intriguing and intricate problems. Furthermore, it proves to be more time and resource efficient than other relevant analytical techniques, leading to significant savings in both volume and time. Compared to the Adomian decomposition technique, the new iterative technique, the variational iteration method, and the optimum homotopy asymptotic method, the suggested technique is extremely accurate computationally. It is also easy to analyze and solve fractionally linked nonlinear complex phenomena that arise in science and technology. We present the numerical and graphical findings that support these conclusions.

1. Introduction

A variation of classical calculus known as fractional calculus (FC) deals with noninteger (fractional)-order integration and differentiation procedures. Fractional operator theory was introduced nearly simultaneously with the creation of classical ones. The subject of the semiderivative meaning was brought up in a correspondence between G. W. Leibniz and Marquis de l'Hospital in 1695, and here is where the earliest instance of this may be discovered [1–3]. As a result, this problem attracted the attention of several eminent mathematicians, including Euler, Liouville, Laplace, Riemann, Grünwald, Letnikov, and many more [4]. The rapid development of the theory of fractional calculus throughout the eighteenth century has been extremely beneficial to fractional differential equations (FDEs), fractional dynamics, and other practical domains. FC is employed in many different applications these days. It is accurate to state that fractional calculus's techniques and tools have an influence on almost every area of modern engineering and science in general.

The versatility of fractional calculus is evident in its applications across diverse fields such as bioengineering, rheology, viscoelasticity, acoustics, optics, robotics, control theory, chemical and statistical physics, and electrical and mechanical engineering [5–8]. One may even claim that fractional-order systems in general explain real-world occurrences. The major reason for the success of FC applications is that these new fractional-order models are often more accurate than integer-order models; that is, the fractional-order model has more degrees of freedom than the similar classical one. Fractional derivatives (and integrals) are not local (or point) variables, which is one of the subject's fascinating aspects. In order to simulate the nonlocal and dispersed effects frequent in technological and natural events, all fractional operators take into account the whole history of the process being studied.

In practice, fractional calculus serves as a valuable tool for understanding the memory and hereditary characteristics of materials and processes. There are several available methods in the literature for approximating problem-related differential equations, both linear and nonlinear. A linear problem's solution is easier to approach than a nonlinear one. For these issues, several numerical and analytical methods have been proposed, including the control volume scheme, the Laplace transformation method, the finite element method (FVM), the Adomain decomposition method (ADM), the variation iteration method (VIM), and the homotopy analysis method (HAM) [9–12]. Although these techniques offer many advantages, not all issues can be solved with them. The strategy put out by Vasile Marinca uses a potent method called optimum auxiliary function. For fractional-order equations containing the Caputo operator, we introduce OAFM in this study as a unique variation of the recently created semianalytical technique known as the optimal auxiliary function method (OAFM). It is explained how OAFM works mathematically, and its efficacy is demonstrated by applying it to the well-known ALW. To show the OAFM's validity, tables and charts are used to contrast the results of the OAFM with those of other approaches and their precise answers. A quick convergence series solution from OAFM is verified by contrasting it with other outputs.

The study shows that our approach is straightforward to use, needs minimal computing effort, and quickly converges to the precise solution within the first iteration. A variety of solutions to the issue are found using OAFM. By contrasting the OAFM results with those from the literature, the validity of the results is confirmed. OAFM is discovered to be quickly convergent, less computationally intensive, and easily adaptable. OAFM involves less computing work than other approaches, and even a low-spec machine can easily complete it. The Optimal Auxiliary Function Method, despite its advantages, has certain limitations. One key limitation is its applicability to specific types of problems. The method may not be suitable for all types of fractional nonlinear long wave equations or for problems with certain boundary conditions. Additionally, the method's effectiveness can depend on the choice of the auxiliary function, which may not always be straightforward to determine, especially for complex problems. Overall, while the Optimal Auxiliary Function Method is a powerful tool, its limitations should be considered when applying it to solve fractional nonlinear long wave equations.

The future direction of this work could involve further exploring the capabilities of the adapted Optimal Auxiliary Function Method (OAFM) for solving a wider range of complex fractional differential equations. One direction could be to investigate its applicability to systems of equations or to problems with more intricate boundary conditions. Additionally, the method could be refined or extended to handle problems in different domains or with different types of fractional orders.

The novelty of this work lies in its successful application of the adapted OAFM to a wide range of linear and nonlinear fractional differential equations, showing its potential for solving real-world problems with simplicity, speed, and efficiency. This study's contribution is its innovative approach to improving an existing method and its demonstration of the method's applicability to various complex models, highlighting its usefulness in solving fractional-order integro-differential equations arising from physical processes. It is strongly advised that you carry out the suggested system research in order to understand continuous quantum measurement and estimation. We are dealing with the ALW system, which is given as follows:

$$\begin{aligned}\frac{\partial^{\eta}\phi(\alpha,\beta)}{\partial\beta^{\eta}}+\phi(\alpha,\beta)\frac{\partial\phi(\alpha,\beta)}{\partial\alpha}+\frac{\partial\phi(\alpha,\beta)}{\partial\alpha}+\frac{1}{2}\frac{\partial^{2}\phi(\alpha,\beta)}{\partial\alpha^{2}}&=0,\\ \frac{\partial^{\omega}\theta(\alpha,\beta)}{\partial\beta^{\omega}}+\frac{\partial\phi(\alpha,\beta)}{\partial\alpha}\frac{\partial\theta(\alpha,\beta)}{\partial\beta}-\frac{1}{2}\frac{\partial^{2}\theta(\alpha,\beta)}{\partial\alpha^{2}}&=0,\end{aligned}\tag{1}$$

where $0<\eta,\ \omega\leq 1$.

Subject to the subsidiary conditions,

$$\begin{aligned}\phi(\alpha,0)&=\lambda-2k\coth(k(\alpha+h)),\\ \theta(\alpha,0)&=-2\left(k^{2}\cos h^{2}(k(\alpha+h))\right).\end{aligned}\tag{2}$$

This article's sections are organised as follows. Basic terms are given in Section 2. In Section 3, the proposed method for solving the current model is covered in detail. Several challenges are tried in Section 4, and the results and conclusions of the tests are provided in Section 5.

2. Basic Terminologies

To understand the OAFM concept, the following is a list of basic terms.

Definition 1. The fractional integral of Riemann–Liouville (R–L) is specified as [13–15]

$$\mathrm{I}_{\varpi}^{\mathfrak{F}}=\begin{cases}\dfrac{1}{\Gamma(\mathfrak{F})}\displaystyle\int_{o}^{\varpi}(\varpi-\nu)^{\mathfrak{F}-1}f(\nu)\mathrm{d}r, & \text{if } \mathfrak{F}>0,\nu>0,\\ f(r), & \text{if } \mathfrak{F}=0,\end{cases}\quad(3)$$

where the special function symbolised by Γ is the gamma function.

Definition 2. The Riemann–Liouville order function $f's$ fractional derivative is defined as

$$D_{\varpi}^{\mathfrak{F}}f(\varpi)=\frac{1}{\Gamma(p-\mathfrak{F})}\frac{d^{p}}{d\varpi^{p}}\int_{o}^{\varpi}(\varpi-\nu)^{q-\mathfrak{F}-1}f(\nu)\mathrm{d}r,\quad \text{if } \mathfrak{F}>0,\nu>0.\quad(4)$$

In this case, p is a positive integer that satisfied

$$p-1<\mathfrak{F}\leq p.\quad(5)$$

Definition 3. Caputo states that a fractional derivative of order is as follows [16].

For

$$p\in\mathbb{N},\varpi>0,r\geq-1 \text{ and } \varphi\in C_{r};$$

$$D_{r}^{\mathfrak{F}}f(\varpi)=\begin{cases}I^{p-\mathfrak{F}}\left[\dfrac{\partial^{p}}{\partial r^{p}}f(\varpi)\right], & \text{if } p-1<\mathfrak{F}\leq p,p\in\mathbb{N},\\ \dfrac{d^{\mathfrak{F}}}{dr^{\mathfrak{F}}}f(\varpi), & \text{if } \mathfrak{F}\in\mathbb{N}.\end{cases}\quad(6)$$

Definition 4. A numerical technique for resolving an integral, partial, or ordinary differential equation is the collocation method. A finite-dimensional space of polynomials up to a specified degree and a specific number of points (collocation points) in the domain is chosen, and the solution that solves the provided equation is then chosen at the collocation points.

Definition 5. Auxiliary functions are not predetermined types of functions; instead, they are functions that are either expressly established or at the very least proved to exist, present a contradiction to some presumptive notion, or otherwise prove the desired outcome.

Definition 6. A mistake often manifests as a difference between an estimated value and an exact mathematical value. Therefore, absolute error refers to the size of the discrepancy between the precise value and the approximate value.

$$\varepsilon=|(a-\tilde{a})|,\quad(7)$$

where a is exact solution and $\tilde{a}$ is approximate solution.

3. Formulation of Mathematical Models

A partial differential equation of fractional order is expressed in general form as

$$\frac{\partial^{\eta}\phi(\alpha,\beta)}{\partial\beta^{\eta}}=\wp(\alpha,\beta)+L_{2}(\phi(\alpha,\beta)).\quad(8)$$

Subject to the boundary conditions,

$$\begin{aligned}&D_{0}^{\eta-k}\alpha(\chi,0)=h_{k}(\chi),(k=0,1,\ldots\ldots\ldots,j-1),\\&D_{0}^{\eta-n}\alpha(\chi,0)=0,j=[\eta],\\&D_{0}^{k}\alpha(\chi,0)=g_{k}(\eta),(k=0,1,\ldots\ldots\ldots,j-1)\\&D_{0}^{n}\psi(\eta,0)=0,j=[\eta],\end{aligned}\quad(9)$$

where $(\partial^{\eta}/\partial\beta^{\eta})$ denotes the Caputo or R–L operator, an unknown function is denoted by $\phi(\alpha,\beta)$, whereas a known statistical function is denoted by $\theta(\alpha,\beta)$.

Step 7. Two-component form of equation (8) will be taken into account in order to get the estimated answer of the equation and is presented as

$$\begin{aligned}\tilde{\phi}(\alpha,\beta)&=\phi_{0}(\alpha,\beta)+\phi_{1}(\alpha,\beta,C_{i}),\\ n&=1,2,3,4,5\ldots\ldots\rho.\end{aligned}\quad(10)$$

Step 8. We obtain the zero- and first-order solution by substituting equation (10) into (8), which is given as

$$\begin{aligned}&\frac{\partial^{\eta}\phi_{0}(\alpha,\beta)}{\partial\beta^{\eta}}+\frac{\partial^{\eta}\phi_{1}(\alpha,\beta)}{\partial\beta^{\eta}}+\theta(\alpha,\beta)\\&\quad+L_{2}\left[\frac{\partial^{\eta}\phi_{0}(\alpha,\beta)}{\partial\beta^{\eta}}+\frac{\partial^{\eta}\phi_{1}(\alpha,\beta),C_{i}}{\partial\beta^{\eta}}\right]=0.\end{aligned}\quad(11)$$

Step 9. Because the nonlinear equation is complicated and has a difficult time being solved, we utilise the linear equation to generate an initial approximation of the kind shown below. Lastly, we use the result to inform our first forecast.

$$\frac{\partial^{\eta}\phi_{0}(\alpha,\beta)}{\partial\beta^{\eta}}+\theta(\alpha,\beta)=0.\quad(12)$$

With the help of the inverse operator, we arrive to $\phi_{0}(\alpha,\beta)$ as follows:

$$\phi_{0}(\alpha,\beta)=\wp(\alpha,\beta).\quad(13)$$

Step 10. The expanding version of the nonlinear component in equation (11) is

$$L_2\left[\frac{\partial^e ta\phi_0(\alpha,\beta)}{\partial\beta^\eta}+\frac{\partial^\eta\phi_1(\alpha,\beta,C_i)}{\partial\beta^\eta}\right]=L_2[\phi_0(\alpha,\beta)]+\sum_{k=1}^{\infty}\frac{\mu_1^k}{k!}\mathrm{N}^{(k)}[\phi_0(\alpha,\beta)]. \tag{14}$$

Step 11. Let us suggest an equation to simplify equation (14), smooth its convergence, and accelerate the first-order approximation. The expression is shown below $\tilde{\phi}(\alpha,\beta)$:

$$\frac{\partial^\eta\phi_1(\alpha,\beta,C_i)}{\partial\beta^\eta}=-G_1[\phi_0(\alpha,\beta)]L_2[\phi_0(\alpha,\beta)]-G_2\left[\phi_0(\alpha,\beta),C_j\right], \tag{15}$$

where G_1 and G_2 are the auxiliary functions depending upon $\phi_0(\alpha,\beta)$ and convergence control parameter C_i and $C_j\, n=1,2,3,4,\ldots,j=\Omega+1,\Omega+2,\ldots$

Remark 12. G_1 and G_2 are in the form of $\phi_0(\alpha,\beta)$ or $L_2[\alpha_0(\chi,\xi)]$ in the combination of both $\phi_0(\alpha,\beta)$ and $L_2[\theta_0(\alpha,\beta)]$ but they are not particular.

Step 13. After substituting an auxiliary function into equation (15), we use the inverse operator (Definition 1) to arrive at the first-order solution $\phi_1(\alpha,\beta)$ using OAFM.

Step 14. There are many methods in the literature such as Galerkins method, Ritz method, and collocation method, for the values of C_i and C_j, using which one must compute the square of the residual error.

$$G\left(C_i,C_j\right)=\int_0^\beta\int_\Omega R^2\left(\alpha,\beta;C_i,C_j\right)\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{16}$$

In this context, the residual R is defined.

$$R\left(\alpha,\beta,C_i,C_j\right)=\frac{\partial^\eta\tilde{\phi}(\alpha,\beta),C_i,C_j}{\partial t}+\theta(\alpha,\beta)+L_2\left[\alpha\left(\alpha,\beta,C_i,C_j\right)\right]$$
$$i=1.2.3,\ldots,\Omega,\ j=\Omega+1,\Omega+2,\Omega+3\ldots,\ell. \tag{17}$$

The following system will function as the convergence control parameter:

$$\frac{\partial C}{\partial C_1}=\frac{\partial C}{\partial C_2}=\frac{\partial C}{\partial C_3}=\frac{\partial C}{\partial C_i}=0,\quad i=1,2,\ldots \tag{18}$$

4. Applications

In this part of the article, a few instances are given to demonstrate the precision and intensity of the method that was previously explained.

4.1. Problem. Utilizing the time-fractional ALW equation, assume the following structure [17–20]:

$$\frac{\partial^\eta\phi(\alpha,\beta)}{\partial\beta^\eta}+\phi(\alpha,\beta)\frac{\partial\phi(\alpha,\beta)}{\partial\alpha}+\frac{\partial\phi(\alpha,\beta)}{\partial\alpha}+\frac{1}{2}\frac{\partial^2\phi(\alpha,\beta)}{\partial\alpha^2}=0,$$
$$\frac{\partial^\omega\theta(\alpha,\beta)}{\partial\beta^\omega}+\frac{\partial\phi(\alpha,\beta)}{\partial\alpha}\frac{\partial\theta(\alpha,\beta)}{\partial\beta}-\frac{1}{2}\frac{\partial^2\theta(\alpha,\beta)}{\partial\alpha^2}=0, \tag{19}$$

where $0<\eta,\ \omega\leq 1$.

Subject to the supplementary conditions,

$$\phi(\alpha,0)=\lambda-2k\coth(k(\alpha+h)),$$
$$\theta(\alpha,0)=-2\left(k^2\cos h^2(k(\alpha+h))\right). \tag{20}$$

Exact solution of equation (19) when $\eta=\omega=1$ is

$$\phi(\alpha,\beta)=\lambda-k\coth(k(h+\alpha-\lambda\beta)),$$
$$\theta(\alpha,\beta)=\lambda-k\csc h^2(k(h+\alpha-\lambda\beta)). \tag{21}$$

In equation (19), we define the terms "linear" and "nonlinear" as

$$\begin{cases} L_1(\phi(\alpha,\beta))=\dfrac{\partial^\eta\alpha(\chi,\xi)}{\partial\chi^\eta}, \\ L_2(\phi(\alpha,\beta))=\phi(\alpha,\beta)\dfrac{\partial\phi(\alpha,\beta)}{\partial\alpha}+\dfrac{\partial\phi(\alpha,\beta)}{\partial\alpha}+\dfrac{1}{2}\dfrac{\partial^2\phi(\alpha,\beta)}{\partial\alpha^2}, \end{cases}$$
$$\begin{cases} L_1(\theta(\alpha,\beta))=\dfrac{\partial^\omega\theta(\alpha,\beta)}{\partial\alpha^\omega}, \\ L_2(\theta(\alpha,\beta))=\dfrac{\partial\phi(\alpha,\beta)}{\partial\alpha}\dfrac{\partial\theta(\alpha,\beta)}{\partial\alpha}-\dfrac{1}{2}\dfrac{\partial\theta^2(\alpha,\beta)}{\partial\alpha^2}. \end{cases} \tag{22}$$

According to OAFM, the following is the answer to the zero-order problem:

$$\frac{\partial^\eta\phi_1(\alpha,\beta)}{\partial\beta^\eta}=0,\phi_1(\alpha,0)=\lambda-k\coth(k(\alpha+h)),$$
$$\frac{\partial^\omega\theta_1(\alpha,\beta)}{\partial\alpha^\omega}=0,\theta_1(\alpha,0)=-k^2\csc h^2(k(\alpha+h)). \tag{23}$$

By applying the R–L operator to (23), the answer is

Table 1: Configuring convergence control settings for varying η values in given problem.

	$\eta = 1.0$	$\eta = 0.7$	$\eta = 0.8$
C_1	0.99999999999	1.11072369965	1.57080328204
C_2	0.99999999998	1.11073032374	1.57085426364

Table 2: The time-fractional ALW equation's $\phi\ (\alpha,\beta)$ has an OAFM solution at various η values.

(α,β)	$\phi\ (\alpha,\beta)$ $\eta = 0.5$	$\phi\ (\alpha,\beta)$ $\eta = 0.75$	$\phi\ (\alpha,\beta)$ $\eta = 1.0$	Exact solution
(0.1, 0.1)	−0.1255978967929	−0.125595289954	−0.1255924013294	−0.125592401329
(0.1, 0.3)	−0.125608344314	−0.125604326552	−0.1255994566257	−0.125599456626
(0.1, 0.5)	−0.125616717337	−0.125612129860	−0.1256065137643	−0.125606513765
(0.2, 0.1)	−0.124901434598	−0.124898894786	−0.1248960804333	−0.124896080433
(0.2, 0.3)	−0.124911613481	−0.124907699026	−0.1249029543175	−0.124902954317
(0.2, 0.5)	−0.124919771193	−0.124915301675	−0.1249098299871	−0.124909829988
(0.3, 0.1)	−0.124222834069	−0.124220359220	−0.1242176168533	−0.124217616853
(0.3, 0.3)	−0.124232752585	−0.124228938256	−0.1242243149110	−0.124224314911
(0.3, 0.5)	−0.124240701617	−0.124236346426	−0.1242310146993	−0.124231014700
(0.4, 0.1)	−0.123561547366	−0.123559135496	−0.1235564629148	−0.123556462914
(0.4, 0.3)	−0.123571213469	−0.123567496208	−0.1235629905178	−0.123562990518
(0.4, 0.5)	−0.123578960196	−0.123574715840	−0.1235695197987	−0.123569519800
(0.5, 0.1)	−0.122917047497	−0.122914696696	−0.1229120917853	−0.122912091785
(0.5, 0.3)	−0.122926468842	−0.122922845704	−0.1229184541020	−0.122918454102
(0.5, 0.5)	−0.122934019399	−0.122929882516	−0.1229248180459	−0.122924818047

Table 3: The time-fractional ALW equation's $\theta(\alpha,\beta)$ has an OAFM solution at various η values.

(α,β)	$\theta\ (\alpha,\beta)$ $\eta = 0.5$	$\theta\ (\alpha,\beta)$ $\eta = 0.75$	$\theta\ (\alpha,\beta)$ $\eta = 1.0$	Exact solution
(0.1, 0.1)	−0.0070558106346	−0.0070551297422	−0.00705443752849	−0.00705437528500
(0.1, 0.3)	−0.0070585395122	−0.0070574900451	−0.0070562180709	−0.00705621807103
(0.1, 0.5)	−0.0070607266792	−0.0070595283452	−0.0070580614374	−0.00705806143806
(0.2, 0.1)	−0.0068743826998	−0.0068737228515	−0.0068729917119	−0.00687299171193
(0.2, 0.3)	−0.0068770272312	−0.0068760102007	−0.0068747775403	−0.00687477754051
(0.2, 0.5)	−0.0068791467890	−0.0068779854942	−0.0068765639270	−0.00687656392766
(0.3, 0.1)	−0.0066985408355	−0.0066979012228	−0.0066971925052	−0.00669719250527
(0.3, 0.3)	−0.0067011042607	−0.0067001184208	−0.0066989235642	−0.00669892356431
(0.3, 0.5)	−0.0067031588067	−0.0067020331283	−0.0067006551600	−0.00670065516057
(0.4, 0.1)	−0.0065280714529	−0.0065274513059	−0.0065267641571	−0.00652676415716
(0.4, 0.3)	−0.0065305568585	−0.0065296010224	−0.0065284425308	−0.00652844253096
(0.4, 0.5)	−0.0065325488672	−0.0065314574498	−0.0065301214210	−0.00653012142161
(0.5, 0.1)	−0.0063627710333	−0.0063621696181	−0.0063615032248	−0.00636150322489
(0.5, 0.3)	−0.0063651813610	−0.00636425439974	−0.0063631308998	−0.00636313089993
(0.5, 0.5)	−0.0063671131906	−0.0063660547423	−0.0063647590719	−0.00636475907243

$$\phi_0(\alpha,0) = \lambda - k\coth(k(\alpha+h)), \tag{24}$$

$$\theta_0(\alpha,0) = -k^2 \csc h^2(k(\alpha+h)). \tag{25}$$

By using (24) into (23), the nonlinear operator becomes

$$\begin{aligned} L_2(\phi_1(\alpha,\beta)) &= \phi_1(\alpha,\beta)\frac{\partial\phi_1(\alpha,\beta)}{\partial\alpha} + \frac{\partial\phi_0(\alpha,\beta)}{\partial\alpha} + \frac{1}{2}\frac{\partial^2\phi_1(\alpha,\beta)}{\partial\alpha^2}, \\ L_2(\theta_1(\alpha,\beta)) &= \frac{\partial\phi_1(\alpha,\beta)}{\partial\alpha}\frac{\partial\theta_0(\alpha,\beta)}{\partial\alpha} - \frac{1}{2}\frac{\partial\theta_0^2(\alpha,\beta)}{\partial\alpha^2}. \end{aligned} \tag{26}$$

Table 4: The ALW equation's absolute errors for $\phi(\alpha, \beta)$ were compared to the solutions obtained by ADM, VIM, OHAM, and NIM at $\eta = 1$.

(α, β)	Absl error ADM [26]	Absl error VIM [24]	Absl error OHAM [24]	Absl error NIM	Absl error OAFM
(0.1, 0.1)	8.029×10^{-6}	3.176×10^{-6}	3.176×10^{-6}	1.203×10^{-13}	1.210×10^{-14}
(0.1, 0.3)	7.382×10^{-6}	9.542×10^{-6}	9.542×10^{-6}	3.250×10^{-12}	3.267×10^{-13}
(0.1, 0.5)	6.799×10^{-6}	1.592×10^{-6}	1.592×10^{-6}	1.504×10^{-11}	1.512×10^{-12}
(0.2, 0.1)	3.232×10^{-5}	3.094×10^{-5}	3.094×10^{-5}	1.138×10^{-13}	1.168×10^{-14}
(0.2, 0.3)	2.971×10^{-5}	9.297×10^{-5}	9.297×10^{-5}	3.074×10^{-12}	3.140×10^{-13}
(0.2, 0.5)	2.736×10^{-5}	1.551×10^{-4}	1.551×10^{-4}	1.423×10^{-11}	1.454×10^{-12}
(0.3, 0.1)	7.320×10^{-5}	3.015×10^{-5}	3.015×10^{-5}	1.436×10^{-14}	1.119×10^{-14}
(0.3, 0.3)	6.730×10^{-5}	9.059×10^{-5}	9.059×10^{-5}	2.909×10^{-13}	3.021×10^{-13}
(0.3, 0.5)	6.197×10^{-5}	1.512×10^{-4}	1.512×10^{-4}	1.346×10^{-12}	1.398×10^{-12}
(0.4, 0.1)	1.310×10^{-4}	2.936×10^{-5}	2.936×10^{-5}	1.020×10^{-14}	1.079×10^{-14}
(0.4, 0.3)	1.204×10^{-4}	8.828×10^{-5}	8.828×10^{-5}	2.754×10^{-13}	2.906×10^{-13}
(0.4, 0.5)	1.109×10^{-4}	1.473×10^{-4}	1.473×10^{-4}	1.275×10^{-12}	1.345×10^{-12}
(0.5, 0.1)	2.061×10^{-4}	2.864×10^{-5}	2.864×10^{-5}	9.660×10^{-14}	1.036×10^{-14}
(0.5, 0.3)	1.895×10^{-4}	8.605×10^{-5}	8.605×10^{-5}	2.608×10^{-13}	2.797×10^{-13}
(0.5, 0.5)	1.745×10^{-4}	1.436×10^{-4}	1.436×10^{-4}	1.207×10^{-12}	1.295×10^{-12}

Table 5: The ALW equation's absolute errors for $\theta(\alpha, \beta)$ were compared to the solutions obtained by ADM, VIM, OHAM, and NIM at $\eta = 1$.

(α, β)	Absl error ADM [17]	Absl error VIM [18]	Absl error OHAM [19]	Absl error NIM	Absl error OAFM
(0.1, 0.1)	4.819×10^{-4}	8297×10^{-6}	8.297×10^{-6}	6.719×10^{-14}	4.719×10^{-15}
(0.1, 0.3)	4.508×10^{-4}	9.542×10^{-6}	2.493×10^{-6}	1.814×10^{-12}	1.292×10^{-13}
(0.1, 0.5)	4.222×10^{-4}	1.592×10^{-6}	4.162×10^{-6}	8.399×10^{-12}	5.982×10^{-13}
(0.2, 0.1)	9.766×10^{-4}	3.094×10^{-5}	8.040×10^{-5}	6.308×10^{-14}	4.572×10^{-15}
(0.2, 0.3)	9.135×10^{-4}	9.297×10^{-5}	2.416×10^{-5}	1.703×10^{-12}	1.230×10^{-13}
(0.2, 0.5)	8.554×10^{-4}	1.551×10^{-4}	4.043×10^{-4}	7.885×10^{-12}	5.696×10^{-13}
(0.3, 0.1)	1.484×10^{-3}	3.015×10^{-5}	7.794×10^{-5}	5.925×10^{-14}	4.349×10^{-15}
(0.3, 0.3)	1.388×10^{-3}	9.059×10^{-5}	2.342×10^{-5}	1.599×10^{-12}	1.172×10^{-13}
(0.3, 0.5)	1.300×10^{-3}	1.512×10^{-4}	3.910×10^{-4}	7.407×10^{-12}	5.426×10^{-13}
(0.4, 0.1)	2.007×10^{-3}	7.536×10^{-5}	7.556×10^{-5}	5.569×10^{-14}	4.151×10^{-15}
(0.4, 0.3)	1.876×10^{-3}	2.270×10^{-5}	2.270×10^{-5}	1.509×10^{-12}	1.117×10^{-13}
(0.4, 0.5)	1.756×10^{-3}	3.791×10^{-4}	3.791×10^{-4}	6.691×10^{-12}	5.771×10^{-13}
(0.5, 0.1)	2.543×10^{-3}	7.328×10^{-5}	7.328×10^{-5}	5.236×10^{-14}	3.957×10^{-15}
(0.5, 0.3)	2.378×10^{-3}	2.202×10^{-5}	2.202×10^{-5}	1.413×10^{-12}	1.065×10^{-13}
(0.5, 0.5)	2.225×10^{-3}	3.676×10^{-4}	3.676×10^{-4}	6.545×10^{-12}	4.931×10^{-13}

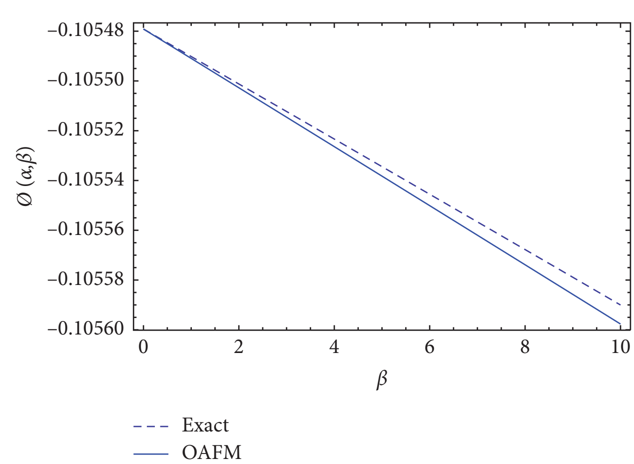

Figure 1: 2D charts displaying the precise and OAFM solution $\phi(\alpha, \beta)$ of the problem.

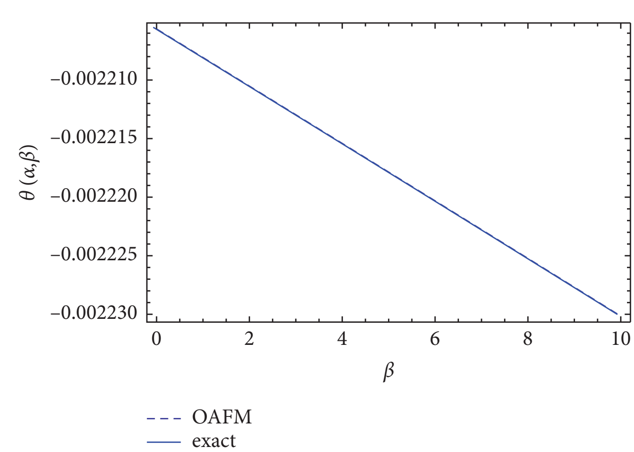

Figure 2: 2D charts displaying the precise and OAFM solution $\theta(\alpha, \beta)$ of the problem.

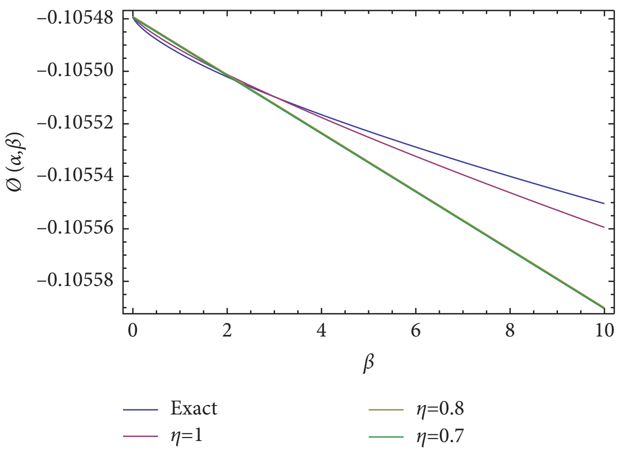

FIGURE 3: Effect of η on the solution for OAFM to the problem.

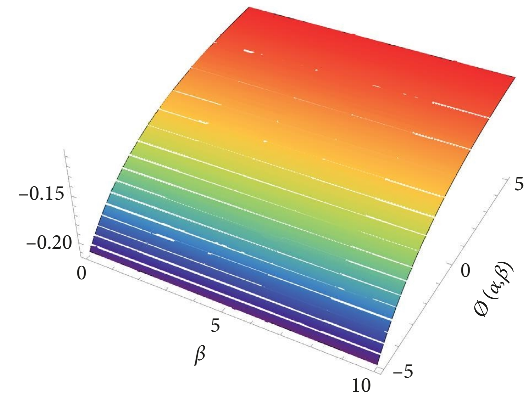

FIGURE 5: 3D charts displaying the precise and OAFM solution $\phi(\alpha,\beta)$ of the problem.

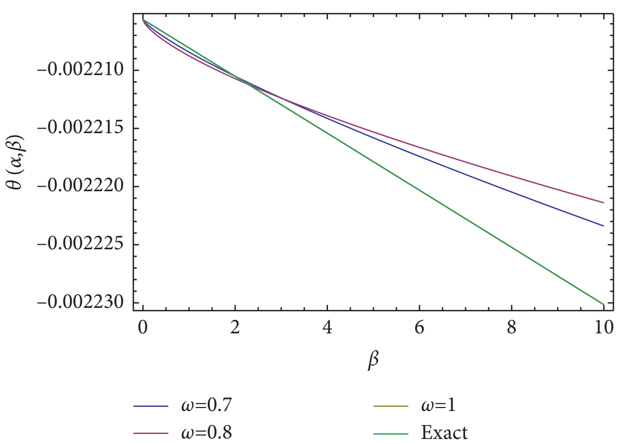

FIGURE 4: Effect of ω on the solution for OAFM to the problem.

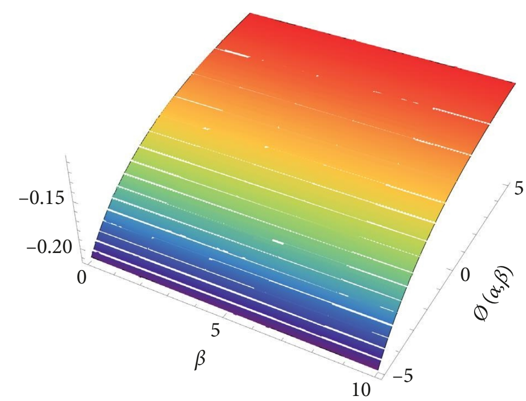

FIGURE 6: Effect of β on the solution for OAFM to the problem $\phi(\alpha,\beta)$.

The first approximation as $\phi_1(\alpha,\beta)$ and $\theta_1(\alpha,\beta)$ is obtained as

$$\begin{aligned}
\frac{\partial^{\eta}\phi_2(\alpha,\beta)}{\partial\beta^{\eta}} &= -G_1\left[\phi_1(\alpha,\beta),C_\iota\right]L_2\left[\phi_1(\alpha,\beta)\right]-G_2\left[\phi_1(\alpha,\beta),C_j\right],\\
\frac{\partial^{\omega}\theta_2(\alpha,\beta)}{\partial\beta^{\omega}} &= -G_3\left[\theta_1(\alpha,\beta),C_\iota\right]L_2\left[\theta_1(\alpha,\beta)\right]-G_4\left[\theta_1(\alpha,\beta),C_j\right],
\end{aligned} \tag{27}$$

where we get G_1, G_2, G_3, G_4, using initial approximation

$$\begin{aligned}
G_1 &= -C_1\left(k^2\lambda-k^3\lambda\coth^2(hk+k\alpha))+k^2\lambda^2\coth(hk+k\alpha))\beta\right),\\
G_2 &= 0,\\
G_3 &= -\left(C_2\left(-k^3\lambda\coth(hk+k\alpha)\operatorname{csc}h^2(hk+k\alpha)+-k^4\lambda^2\operatorname{csc}h--k^4\lambda^2\coth^2(hk+k\alpha)\operatorname{csc}h^2(hk+k\alpha)\beta\right)\right),\\
G_4 &= 0.
\end{aligned} \tag{28}$$

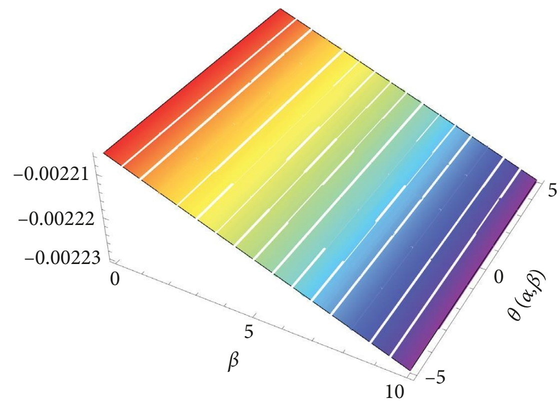

FIGURE 7: 3D charts displaying the precise and OAFM solution $\theta(\alpha, \beta)$ of the problem.

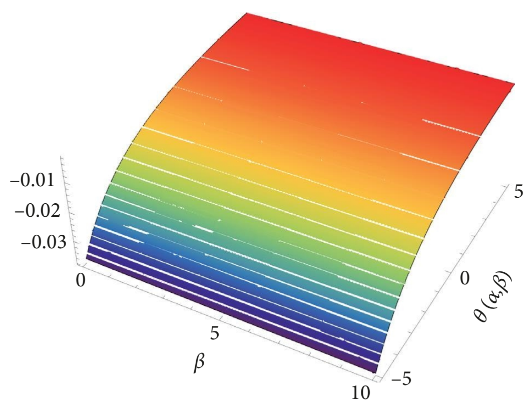

FIGURE 8: Effect of β on the solution for OAFM to the problem $\theta(\alpha, \beta)$.

Using equations (26) and (28) into (27), we obtained the first approximation as

$$\begin{aligned}
\tilde{\phi}(\alpha, \beta) &= -\frac{C_1 k^2 \beta^\eta \lambda\big(1 + kt\lambda \coth(k(\alpha + h)) \operatorname{csc} h^2 (k(\alpha + h))\big)}{\Gamma(1 + \eta)}, \\
\tilde{\theta}(\alpha, \beta) &= -\frac{C_2 k^3 \beta^\eta \operatorname{csc} h^4 (k(\alpha + h)) (kt\lambda (2 + \cosh(2k(\alpha + h)) + \sinh(2k(\alpha + h))}{\Gamma(1 + \eta)}.
\end{aligned} \tag{29}$$

We derive an approximation of the first-order solution as equations (28) and (29).

$$\begin{aligned}
\tilde{\phi}(\alpha, \beta) &= \phi_1(\alpha, \beta) + \phi_2(\alpha, \beta, C_1), \\
\tilde{\theta}(\alpha, \beta) &= \theta_1(\alpha, \beta) + \theta_2(\alpha, \beta, C_2),
\end{aligned} \tag{30}$$

$$\begin{aligned}
\tilde{\phi}(\alpha, \beta) &= \lambda - k \coth(k(\alpha + h)) - \frac{C_1 k^2 \beta^\eta \lambda (1 + kt\lambda \coth(k(\alpha + h)) \operatorname{csc} h^2 (k(\alpha + h))}{\Gamma(1 + \eta)}, \\
\tilde{\theta}(\alpha, \beta) &= -k^2 \operatorname{csc} h^2 (k(\alpha + h)) - \frac{C_2 k^3 \beta^\eta \operatorname{csc} h^4 (k(\alpha + h)) (kt\lambda (2 + \cosh(2k(\alpha + h)) + \sinh(2k(\alpha + h))}{\Gamma(1 + \eta)}.
\end{aligned} \tag{31}$$

5. Discussion

OAFM was used to resolve the nonlinear ALW system's fractional-order equations. Section 4, tables, and figures for the ALW system give the results of OAFM for the fractional-order equation using ADM, VIM, and OHAM.

In the problem, the absolute errors of the variational iteration method (VIM) solution, the Adomian decomposition method (ADM) solution, the optimum homotopy asymptotic method (OHAM) solution, and the second-order new iterative method (NIM) solution for the fractional-order approximate long wave (ALW) equation's $\phi(\alpha,\beta)$ and $\theta(\alpha,\beta)$ variables are compared with $\eta = \omega = 1$. Table 1 displays the various values of C_1 and C_2 used in the calculations. The values of C_1 and C_2 are crucial parameters in the solutions obtained by the different methods. Tables 2–5 provide insight into how the methods perform with different choices of these parameters, showing which combinations lead to more accurate solutions for $\phi(\alpha,\beta)$ and $\theta(\alpha,\beta)$. The tables likely contain numerical values showing the errors for each method and parameter combination, allowing for a detailed comparison of their performance.

Figures 1 and 2 compare the precise and Optimal Auxiliary Function Method (OAFM) solutions in 2D plots of $\phi(\alpha,\beta)$ and $\theta(\alpha,\beta)$ at $\beta = 0.1$. These figures likely illustrate how well the OAFM approximates the precise solution for different values of α. In Figures 3 and 4, the first-order OAFM solution of $\phi(\alpha,\beta)$ and $\theta(\alpha,\beta)$ is shown for various values of the parameter ω. These figures likely demonstrate how the OAFM solution changes with different values of ω and how it compares to the precise solution. Figures 5–8 display the accurate answer and the OAFM 3D graphic of $\phi(\alpha,\beta)$ and $\theta(\alpha,\beta)$ for the problem at $\beta = 1$. These figures likely provide a detailed comparison between the accurate solution and the OAFM solution in a 3D graphical format, showing the behavior of ϕ and θ in the α-β plane and highlighting any discrepancies between the two solutions.

6. Conclusion

In this work, we instituted a systematic adaptation in employing the OAFM to approximate the fractional nonlinear long wave equation with the application of Caputo fractional order. Several problems, both linear and nonlinear, that entail fractional differential equations were investigated, and it is shown that the suggested adjustment to the Optimal Auxiliary Function Method has increased the method's effectiveness compared to it's the previous iteration. The OAFM takes less computational work than previous methods, and even a machine with smaller space may successfully finish the operation. This method is currently unrestricted, allowing us to utilize it in the future for more intricate models drawn from real-world difficulties. Moreover, it is observed in this paper that OAFM is simple, quick, and efficient. Thus, fractional-order integro-differential equations that arise from physical processes can be solved using the suggested approach based on our mathematical findings. However, certain limitations, such as scope, assumptions, and numerical stability, should be considered when applying the method to real-world problems. Future research can address these limitations to further improve the method's applicability and accuracy.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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