ON PRIGOGINE'S APPROACHES TO IRREVERSIBILITY: A CASE STUDY BY THE BAKER MAP

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The baker map is investigated by two different theories of irreversibility by Prigogine and his colleagues, namely, the Λ-transformation and complex spectral theories, and their structures are compared. In both theories, the evolution operator $U^\dagger$ of observables (the Koopman operator) is found to acquire dissipativity *by restricting* observables to an appropriate subspace Φ of the Hilbert space L^2 of square integrable functions. Consequently, its spectral set contains an annulus in the unit disc. However, the two theories are not equivalent. In the Λ-transformation theory, a bijective map $\Lambda^{\dagger-1} : \Phi \to L^2$ is looked for and the evolution operator U of densities (the Frobenius-Perron operator) is transformed to a dissipative operator $W = \Lambda U \Lambda^{-1}$. In the complex spectral theory, the class of densities is restricted further so that most values in the interior of the annulus are removed from the spectrum, and the relaxation of expectation values is described in terms of a few point spectra in the annulus (Pollicott-Ruelle resonances) and faster decaying terms.

1. Introduction

Consistent description of macroscopic irreversibility in terms of reversible microscopic dynamics is one of the long standing problems in statistical mechanics. Prigogine and his colleagues have studied this problem since 1960s [1, 2, 3, 4, 7, 8, 14, 15, 16, 17, 18, 24, 25, 26, 27, 28, 31, 32, 33, 34, 35] and proposed two answers: the Λ-transformation theory [1, 7, 8, 15, 24, 25, 26] and the complex spectral theory [2, 3, 4, 14, 16, 17, 18, 27, 28, 33] (the complex spectral decomposition of references [27, 28, 33] is equivalent to the one-dimensional subdynamics decomposition of reference [14]). In the former, the reversible evolution operator is related to a dissipative evolution in a bijective way (via the Λ-transformation) and, in the latter, the reversible evolution restricted to certain classes of initial densities and observables is represented as a superposition of decaying eigenmodes. Both theories came out from their earlier work, the "subdynamics theory" [14, 34, 35], which was developed as a generalization of the van Hove's $\lambda^2 t$-approximation [45, 46] to all orders with respect to the coupling strength λ.

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However, the relation between the two approaches is not transparent. In particular, one may have an impression that the irreversible evolution is derived in the two approaches through quite different mechanisms. Indeed, the structural property (the K-property [6]) of the Kolmogorov systems is used in the Λ-transformation theory, while it is not in the complex spectral theory. Fortunately, there is an example, the baker map, to which both approaches were applied [3, 18, 26]. Since the baker map is a typical Kolmogorov system, the application of the Λ-transformation theory is straightforward. On the other hand, the complex spectral theory leads to a generalized spectral decomposition in the sense of Gelfand [12, 13], Maurin [23], and Lindblad and Nagel [22], where the decay rates are given by the Pollicott-Ruelle resonances [29, 30, 37, 38, 39, 40, 41]. Note that the results of [3, 18] can be obtained by different methods [10, 11] (see the appendix).

In this paper, the two approaches are compared for the baker map. In Section 2, we review the Λ-transformation theory as applied to the baker map and study the spectral property of the transformed evolution operator. Then, the properties of the transformed operator are characterized as those of the original operator restricted to a subspace Φ of the Hilbert space L^2, and a new interpretation of the Λ-transformation is given. In Section 3, we explicitly derive the first two generalized decaying eigenmodes and decompose the expectation values of a certain class of observables with respect to a certain class of initial densities into a sum of the decaying eigenmodes and a residual faster decaying term. This decomposition (hereafter, it will be referred to as the Pollicott-Ruelle decomposition) is a special case of the results on axiom-A systems by Pollicott [29, 30] and Ruelle [37, 38, 39, 40, 41], and is a precursor of the complex spectral decomposition. Then, the spectral properties of the restricted evolution operator $U^\dagger$ of observables (the Koopman operator) are investigated and the mechanism of the emergence of Pollicott-Ruelle decomposition is discussed. The last section is devoted to the discussions.

As a common feature of the two approaches, we find that the Koopman operator $U^\dagger$ acquires dissipativity by restricting observables to an appropriate subspace Φ of L^2, and that the spectral set of the restricted operator contains an annulus in the unit disc. However, the two approaches are not equivalent. In the Λ-transformation theory, one looks for a bijective map $\Lambda^{\dagger-1} : \Phi \to L^2$ so that the evolution operator U of densities (the Frobenius-Perron operator) is transformed to a dissipative operator $W = \Lambda U \Lambda^{-1}$. In the complex spectral theory, one further restricts the class of densities so that most values in the interior of the annulus are removed from the spectrum, and the relaxation of expectation values is described by the Pollicott-Ruelle decomposition.

Now, we begin with the description of the model. The baker map is one of the first examples of reversible mixing transformations and was introduced by Hopf [21]. It is defined on the unit square $[0,1)^2$ as a two-step operation: (1) squeeze the unit square to a $2 \times 1/2$-rectangle and (2) cut the rectangle into two $1 \times 1/2$-squares and pile them up to recover the unit square:

$$B(x,y) = \begin{cases} \left(2x, \dfrac{y}{2}\right), & 0 \le x < \dfrac{1}{2}, \\ \left(2x-1, \dfrac{y+1}{2}\right), & \dfrac{1}{2} \le x < 1. \end{cases} \tag{1.1}$$

It admits the Lebesgue measure as an ergodic invariant measure and has Kolmogorov-Sinai entropy $\log 2$ [6]. Also, it is a typical Kolmogorov system [6]. The time evolution of the probability densities $\rho(x,y)$ is governed by the Frobenius-Perron operator

$$U\rho(x,y) \equiv \rho(B^{-1}(x,y)) = \begin{cases} \rho\left(\dfrac{x}{2}, 2y\right), & 0 \le y < \dfrac{1}{2}, \\ \rho\left(\dfrac{x+1}{2}, 2y-1\right), & \dfrac{1}{2} \le y < 1. \end{cases} \tag{1.2}$$

The operator U is unitary on the Hilbert space L^2 of square integrable functions, equipped with the standard inner product $\langle f,g\rangle \equiv \int_{[0,1]^2} dx\,dy\, f^*(x,y)g(x,y)$ and the norm $\|f\|_2 \equiv \sqrt{\langle f,f\rangle}$ [6]. Therefore, the spectrum of U on L^2 is a unit circle $\{z : |z| = 1\}$.

2. Λ-transformation approach

2.1. Summary of the previous work. Here we review the work by Misra et al. [26] in the case of the baker map. The map B^t is called "intrinsically random" if there exists a bounded operator Λ on L^2 and a contraction semigroup W_t for $t \ge 0$ such that

(a) Λ preserves positivity;
(b) $\int_{[0,1)^2} dx\,dy\,\Lambda\rho(x,y) = \int_{[0,1)^2} dx\,dy\,\rho(x,y)$;
(c) $\Lambda 1 = 1$, where 1 stands for the unit function;
(d) Λ has a densely defined inverse Λ^{-1};
(e) $\Lambda U^t \rho = W_t^\dagger \Lambda\rho$ (for $t \ge 0$), where $W_t^\dagger$ is the hermitian conjugate of W_t,

where (i) W_t preserves positivity, (ii) $W_t 1 = 1$, (iii) $W_t^\dagger 1 = 1$, and (iv) $\|W_t^\dagger(\rho - 1)\|_2$ decreases strictly monotonically to 0 as $t \to +\infty$.

For the baker map, the Λ-transformation is constructed as follows [26]. Let χ_0 be a function such that $\chi_0(x,y) = -1$ if $0 \le x < 1/2$ and $\chi_0(x,y) = 1$ if $1/2 \le x < 1$. And, for each finite set $S = (n_1,\dots,n_r)$ of integers, we set

$$\chi_S(x,y) \equiv U^{n_1}\chi_0(x,y)U^{n_2}\chi_0(x,y)\cdots U^{n_r}\chi_0(x,y), \tag{2.1}$$

then the family of functions $\{\chi_S\}$ together with the unit function 1 form a complete orthonormal set of L^2. Note that $U\chi_S = \chi_{S+1}$ where $S+1 = (n_1+1,\dots,n_r+1)$ if $S = (n_1,\dots,n_r)$. Now, for each integer n $(= 0, \pm 1, \pm 2, \dots)$, define an operator E_n to be a projection operator onto the subspace spanned by χ_S such that $\max\{n_j \in S\} = n$, then the Λ-transformation is defined by

$$\Lambda = \sum_{n=-\infty}^{+\infty} \lambda_n E_n + P_0, \tag{2.2}$$

where P_0 is the one-dimensional projection onto the unit function and $\{\lambda_n\}_{-\infty<n<+\infty}$ is a positive monotonically decreasing sequence bounded by 1 such that λ_{n+1}/λ_n also decreases monotonically as n increases. This leads to the following semigroup $W_t^\dagger$:

$$W_t^\dagger = (W^\dagger)^t, \tag{2.3}$$

$$W^\dagger = \Lambda U \Lambda^{-1} = \sum_{n=-\infty}^{+\infty} \frac{\lambda_{n+1}}{\lambda_n} U E_n + P_0. \tag{2.4}$$

Before closing, we give a spectral characterization of the semigroup $W^\dagger$.

Proposition 2.1. *The spectral set $\sigma(W^\dagger)$ of $W^\dagger$ satisfies*

$$\{z : c < |z| < 1\} \subset \sigma(W^\dagger) \subset \{z : |z| \le 1\}, \tag{2.5}$$

where $c = \lim_{n\to+\infty} \lambda_{n+1}/\lambda_n$. Moreover, the eigenfunction of $W^\dagger$ corresponding to an eigenvalue $z \in \{z : c < |z| < 1\}$ is

$$\varphi_z^S(x,y) = \sum_{n=-\infty}^{+\infty} \frac{\lambda_{m+n}}{z^n} \chi_{S+n}(x,y), \tag{2.6}$$

where S is a finite set of integers and $m = \max\{n_j \in S\}$. Since $\sigma(W) = \overline{\sigma(W^\dagger)}$, the spectral set $\sigma(W)$ of W satisfies the same relation as (2.5).

Proof. From (2.4), $E_n E_m = \delta_{nm} E_n$, $E_n P_0 = 0$, and $P_0^2 = P_0$, one has

$$W W^\dagger = \sum_{n=-\infty}^{+\infty} \left(\frac{\lambda_{n+1}}{\lambda_n}\right)^2 E_n + P_0 \tag{2.7}$$

and, as $\lambda_{n+1}/\lambda_n \le 1$,

$$\begin{aligned} \|W^\dagger \rho\|_2^2 &= \langle \rho, W W^\dagger \rho \rangle \\ &= \sum_{n=-\infty}^{+\infty} \left(\frac{\lambda_{n+1}}{\lambda_n}\right)^2 \|E_n \rho\|_2^2 + \|P_0 \rho\|_2^2 \\ &\le \sum_{n=-\infty}^{+\infty} \|E_n \rho\|_2^2 + \|P_0 \rho\|_2^2 \\ &= \sum_{n=-\infty}^{+\infty} \langle \rho, E_n \rho \rangle + \langle \rho, P_0 \rho \rangle = \|\rho\|_2^2, \end{aligned} \tag{2.8}$$

or $\|W^\dagger\| \le 1$. Thus, because of the spectral radius formula [36], the spectral radius is less than or equal to unity and this implies the second inclusion of (2.5). The first inclusion is a consequence of (2.6).

Now we show the convergence of (2.6) when $|z| < 1$, $\|\chi_{S+n}\|_2 = 1$, and $\lambda_{m-n} \le 1$ lead to

$$\begin{aligned} \|\varphi_z^S\|_2 &\le \sum_{n=-\infty}^{+\infty} \frac{\lambda_{m+n}}{|z|^n} = \sum_{n=1}^{+\infty} \lambda_{m-n}|z|^n + \sum_{n=0}^{+\infty} \frac{\lambda_{m+n}}{|z|^n} \\ &\le \sum_{n=1}^{+\infty} |z|^n + \sum_{n=0}^{+\infty} \frac{\lambda_{m+n}}{|z|^n} = \frac{|z|}{1-|z|} + \sum_{n=0}^{+\infty} \frac{\lambda_{m+n}}{|z|^n}, \end{aligned} \tag{2.9}$$

where the second power series converges if $1/|z| < 1/c$ or $c < |z|$ with $1/c$ the convergence radius. The well-known formula gives

$$c = \lim_{n\to+\infty} \frac{\lambda_{m+n+1}}{\lambda_{m+n}} = \lim_{n\to+\infty} \frac{\lambda_{n+1}}{\lambda_n}, \tag{2.10}$$

which converges as the positive sequence λ_{n+1}/λ_n is monotonically decreasing. □

2.2. Properties in the original representation. We reinvestigate the above results in terms of the original variables. For the new representation to give the same prediction as the original one, the average of an observable should take the same value in the original and new representations, namely,

$$\langle A_\Lambda, \Lambda\rho\rangle = \langle A, \rho\rangle, \tag{2.11}$$

where A and ρ are an observable and a density in the original representation and A_Λ is an observable in the new representation. Thus, A_Λ should be $\Lambda^{\dagger-1}A$ and observables should be in the domain $\mathcal{D}(\Lambda^{\dagger-1})$ of $\Lambda^{\dagger-1}$; or the following observation holds.

Observation 2.2. The Λ-transformation theory implicitly assumes the restriction of a class of observables in the original representation to $\mathcal{D}(\Lambda^{\dagger-1})$.

Then, it is natural to study the evolution $U^{\dagger t}$ of observables in the restricted space $\mathcal{D}(\Lambda^{\dagger-1}) = \Lambda^\dagger L^2 \subset L^2$.

Proposition 2.3. *Define a norm $\|\cdot\|_\Lambda$ in $\Lambda^\dagger L^2$ by $\|A\|_\Lambda \equiv \|\Lambda^{\dagger-1}A\|$ (for all $A \in \Lambda^\dagger L^2$), then*

(i) *with respect to $\|\cdot\|_\Lambda$, the space $\Lambda^\dagger L^2$ is a Banach space. It is dense in the Hilbert space L^2 and its norm topology is stronger than the Hilbert space topology. Then there exist a triple $\Lambda^\dagger L^2 \subset L^2 \subset (\Lambda^\dagger L^2)^\dagger$, where $(\Lambda^\dagger L^2)^\dagger$ is the space of continuous conjugate linear functionals over (i.e., the dual space of) $\Lambda^\dagger L^2$;*

(ii) *the space $\Lambda^\dagger L^2$ is invariant under $U^\dagger$ and $\|U^\dagger A\|_\Lambda \le \|A\|_\Lambda$. Then, U can be extended to the dual space $(\Lambda^\dagger L^2)^\dagger$;*

(iii) *for $z \in \{z : c < |z| < 1\}$ with $c = \lim_{n\to+\infty} \lambda_{n+1}/\lambda_n$, let ψ_z^S be a conjugate linear functional defined by*

$$\psi_z^S(A) = \langle \Lambda^{\dagger-1}A, \varphi_z^S\rangle, \quad A \in \Lambda^\dagger L^2, \tag{2.12}$$

where φ_z^S is an eigenfunction of $W^\dagger$ given in Proposition 2.1, then, $\psi_z^S \in (\Lambda^\dagger L^2)^\dagger$ and, for any $A \in \Lambda^\dagger L^2$, the relation

$$\psi_z^S(U^\dagger A) = z\psi_z^S(A), \tag{2.13}$$

holds or ψ_z^S is an eigenfunction of the extension of U to $(\Lambda^\dagger L^2)^\dagger$ with eigenvalue z;

(iv) *the spectral set $\sigma(U^\dagger|_{\Lambda^\dagger L^2})$ of $U^\dagger$ restricted to the space $\Lambda^\dagger L^2$ satisfies*

$$\{z : c < |z| < 1\} \subset \sigma(U^\dagger|_{\Lambda^\dagger L^2}) \subset \{z : |z| \le 1\}, \tag{2.14}$$

where $c = \lim_{n\to+\infty} \lambda_{n+1}/\lambda_n$.

Proof. (i) To show that $\Lambda^\dagger L^2$ is a Banach space, it is enough to check its completeness. Let $\{A_n\}_{n\ge1}$ be a Cauchy sequence in $\Lambda^\dagger L^2$ with respect to the norm $\|\cdot\|_\Lambda$, or

$$\|A_n - A_m\|_\Lambda = \|\Lambda^{\dagger-1}A_n - \Lambda^{\dagger-1}A_m\|_2 \longrightarrow 0 \quad (n,m \longrightarrow +\infty). \tag{2.15}$$

Then, $\{\Lambda^{\dagger-1}A_n\}_{n\ge1}$ is a Cauchy sequence in L^2 and there exists $B \in L^2$ such that

$$0 = \lim_{n\to+\infty} \|\Lambda^{\dagger-1}A_n - B\|_2 = \lim_{n\to+\infty} \|A_n - \Lambda^\dagger B\|_\Lambda, \tag{2.16}$$

or $\{A_n\}_{n\ge1}$ has the limit $\Lambda^\dagger B \in \Lambda^\dagger L^2$ and, thus, $\Lambda^\dagger L^2$ is complete.

Since $\Lambda^\dagger = \Lambda$, $\Lambda^\dagger L^2 = \Lambda L^2$ is the domain of Λ^{-1} and, thus, is dense in L^2 by the property (d) of Λ. In addition, the boundedness of $\Lambda^\dagger$ leads to $\|A\|_2 = \|\Lambda^\dagger\Lambda^{\dagger-1}A\|_2 \le \|\Lambda^\dagger\|_2\|\Lambda^{\dagger-1}A\|_2 = \|\Lambda^\dagger\|_2\|A\|_\Lambda$, or the topology of $\Lambda^\dagger L^2$ is stronger than the Hilbert space topology.

(ii) Let $A \in \Lambda^\dagger L^2$, then there exists $B \in L^2$ such that $A = \Lambda^\dagger B$. On the other hand, the property (e) of Λ implies $\Lambda^\dagger W = U^\dagger\Lambda^\dagger$ and, thus, $U^\dagger A = U^\dagger\Lambda^\dagger B = \Lambda^\dagger WB \in \Lambda^\dagger L^2$. Moreover, as $\|W\|_2 = \|W^\dagger\|_2 \le 1$,

$$\|U^\dagger A\|_\Lambda = \|\Lambda^{\dagger-1}U^\dagger A\|_2 = \|WB\|_2 \le \|W\|_2\|B\|_2 = \|W\|_2\|A\|_\Lambda \le \|A\|_\Lambda. \tag{2.17}$$

(iii) When $A \in \Lambda^\dagger L^2$, $\Lambda^{\dagger-1}A \equiv B \in L^2$, and $\psi_z^S(A)$ is well defined and bounded, $|\psi_z^S(A)| \le \|\varphi_z^S\|_2\|A\|_\Lambda$, or $\psi_z^S \in (\Lambda^\dagger L^2)^\dagger$. Moreover, because of $U^\dagger\Lambda^\dagger = \Lambda^\dagger W$ and Proposition 2.1, one has the desired result:

$$\psi_z^S(U^\dagger A) = \langle \Lambda^{\dagger-1}U^\dagger\Lambda^\dagger B, \varphi_z^S\rangle = \langle WB, \varphi_z^S\rangle = \langle B, W^\dagger\varphi_z^S\rangle = z\langle B, \varphi_z^S\rangle = z\psi_z^S(A). \tag{2.18}$$

(iv) The second inclusion is a consequence of (ii) and the first inclusion follows from (iii) and the next lemma.

Lemma 2.4. *Let $U : X \to X$ be a bounded operator on a Banach space X and suppose, for $\lambda \in \mathbb{C}$, there exists an element y^* $(\neq 0)$ of the dual space $X^\dagger$ (or y^* is a conjugate linear functional over X) such that $y^*(Ux) = \lambda^* y^*(x)$ holds for every $x \in X$. Then λ is in the spectrum of U: $\lambda \in \sigma(U)$.*

This follows immediately. Suppose λ is in the resolvent set of U, then, for every $x \in X$, there exists $x' = (\lambda\mathbf{1} - U)^{-1}x \in X$ with $\mathbf{1}$ the identity operator. But, by assumption, $y^*(x) = y^*((\lambda\mathbf{1} - U)(\lambda\mathbf{1} - U)^{-1}x) = \lambda^* y^*(x') - y^*(Ux') = 0$. This contradicts $y^* \neq 0$, or λ is in the spectrum. □

2.3. Λ-transformation revisited. We have observed that the Λ-transformed operator W and the restricted operator $U^\dagger|_{\Lambda^\dagger L^2}$ have similar spectral sets, and that $\Lambda^{\dagger-1}$ maps their domains with each other: $\Lambda^{\dagger-1} : \Lambda^\dagger L^2 \to L^2$. This is not a mere coincidence. Indeed, one has the following proposition.

Proposition 2.5. (i) *The map $\Lambda^{\dagger-1} : \Lambda^\dagger L^2 \to L^2$ is isometric and onto.*

(ii) *For every $\psi \in (\Lambda^\dagger L^2)^\dagger$, there exists a unique $g \in L^2$ such that $\psi(A) = \langle \Lambda^{\dagger-1}A, g\rangle$ (for all $A \in \Lambda^\dagger L^2$). Since $\Lambda^\dagger(\Lambda^\dagger L^2) \subset \Lambda^\dagger L^2$, Λ can be extended continuously to $(\Lambda^\dagger L^2)^\dagger$ and $\Lambda\psi = g$.*

(iii) *As a dual space of the Banach space, $(\Lambda^\dagger L^2)^\dagger$ is again a Banach space with its norm $\|\psi\|_{(\Lambda^\dagger L^2)^\dagger} = \|\Lambda\psi\|_2$.*

(iv) *The map $\Lambda : (\Lambda^\dagger L^2)^\dagger \to L^2$ is isometric and onto.*

(v) *Let $\psi_z^S \in (\Lambda^\dagger L^2)^\dagger$ be an eigenfunction of U defined by (2.13) and let $\varphi_z^S \in L^2$ be an eigenfunction of W defined by (2.12), then $\Lambda\psi_z^S = \varphi_z^S$.*

This observation may provide a new interpretation of the Λ-transformation. By restricting the evolution operator $U^\dagger$ to a subspace $\Phi \equiv \Lambda^\dagger L^2 \subset L^2$ with a stronger topology, it becomes dissipative. Note that its adjoint U is simultaneously extended to the dual space $\Phi^\dagger \supset L^2$. Now the Λ-transformation is introduced so that $\Lambda : \Phi^\dagger \to L^2$ and $\Lambda^{\dagger-1} : \Phi \to L^2$ are isometric and onto. And, as a consequence, the evolution U is transformed into the semigroup $W^\dagger$. In short, Λ manifests the dissipative nature of $U^\dagger|_{\Lambda^\dagger L^2}$ as a Hilbert space property.

Proof of Proposition 2.5. (i) This immediately follows from the definition of $\|\cdot\|_\Lambda$.

(ii) For each $\psi \in (\Lambda^\dagger L^2)^\dagger$, $\psi(\Lambda^\dagger B)$ $(B \in L^2)$ defines a conjugate linear functional on L^2 and the Riesz theorem [36] implies the existence of a unique $g \in L^2$ such that $\psi(\Lambda^\dagger B) = \langle B, g\rangle$, or $\psi(A) = \langle \Lambda^{\dagger-1}A, g\rangle$. Moreover, it implies $\Lambda\psi(A) \equiv \psi(\Lambda^\dagger A) = \langle A, g\rangle$ or $\Lambda\psi = g$ as well.

(iii) This follows from $|\psi(A)| = |\langle \Lambda^{\dagger-1}A, g\rangle| \leq \|A\|_\Lambda \|g\|_2$ and $|\psi(\Lambda^\dagger g)| = \|g\|_2^2$ with $g = \Lambda\psi$.

(iv) This is a consequence of (ii) and (iii).

(v) This immediately follows from the definition. □

3. Complex spectral theory

The complex spectral theory gives the generalized spectral decomposition consisting of point spectra in the following sense [3, 10, 11, 18]:

$$\begin{aligned}\langle A,\rho\rangle &= F_0(A)\tilde{F}_0^*(\rho)+\sum_{\nu=1}^{\infty}\sum_{r=0}^{\nu}F_{\nu,r}(A)\tilde{F}_{\nu,r}^*(\rho),\\ \langle A,U\rho\rangle &= F_0(A)\tilde{F}_0^*(\rho)+\sum_{\nu=1}^{\infty}\left\{\sum_{r=0}^{\nu}\frac{1}{2^\nu}F_{\nu,r}(A)\tilde{F}_{\nu,r}^*(\rho)+\sum_{r=0}^{\nu-1}F_{\nu,r+1}(A)\tilde{F}_{\nu,r}^*(\rho)\right\},\end{aligned}\tag{3.1}$$

where F_0, $\tilde{F}_0$, $F_{\nu,r}$, and $\tilde{F}_{\nu,r}$ are conjugate linear functionals and A and ρ are appropriate functions. This decomposition, however, requires narrower classes of observables A and initial densities ρ, and is not appropriate for studying the general structure. Thus, we adopt wider classes of A and ρ so that one has

$$\begin{aligned}\langle A,U^t\rho\rangle &= F_0(A)\tilde{F}_0^*(\rho)+\frac{1}{2^t}\sum_{r=0}^{1}F_{1,r}(A)\tilde{F}_{\nu,r}^*(\rho)\\ &\quad+\frac{t}{2^{t-1}}F_{1,1}(A)\tilde{F}_{1,0}^*(\rho)+\mathrm{O}\left(\frac{t^2}{4^t}\right).\end{aligned}\tag{3.2}$$

This formula can be regarded as a special case of the Pollicott-Ruelle theorem [29, 30, 37, 38, 39, 40, 41] and is referred to as the Pollicott-Ruelle decomposition.

In this section, we construct the Pollicott-Ruelle decomposition for the baker map. We begin with the description of subspaces of L^2 corresponding to the classes of observables and initial densities.

3.1. Functional spaces. Let $C_x^2\subset L^2$ be a space of functions $f(x,y)$ such that

(C1) for almost every y, $f(x,y)$ is twice continuously differentiable in x,
(C2) $\sup_{0\le x<1}|f(x,y)|^2$ and $\sup_{0\le x<1}|\partial^j f(x,y)/\partial x^j|^2$ $(j=1,2)$ are integrable in y.

The space is equipped with the norm

$$\|f\|_{Cx}\equiv\sqrt{\int_0^1 dy\sup_{0\le x<1}|f(x,y)|^2}+\sum_{j=1}^{2}\sqrt{\int_0^1 dy\sup_{0\le x<1}\left|\frac{\partial^j f(x,y)}{\partial x^j}\right|^2}.\tag{3.3}$$

The other subspace C_y^2 of twice y-differentiable functions is defined by interchanging x and y in the definition of C_x^2, and is equipped with the norm

$$\|f\|_{Cy}\equiv\sqrt{\int_0^1 dx\sup_{0\le y<1}|f(x,y)|^2}+\sum_{j=1}^{2}\sqrt{\int_0^1 dx\sup_{0\le y<1}\left|\frac{\partial^j f(x,y)}{\partial y^j}\right|^2}.\tag{3.4}$$

For these spaces, we have the following proposition.

Proposition 3.1. (i) *The space C_x^2 is a Banach space with respect to the norm $\|\cdot\|_{Cx}$.*

(ii) *The subspace C_x^2 is dense in the Hilbert space L^2 and its norm topology is stronger than the Hilbert space topology. Thus an inclusion $C_x^2\subset L^2\subset C_x^{2\dagger}$ holds, where $C_x^{2\dagger}$ is the dual space of C_x^2.*

(iii) *The space C_x^2 is invariant with respect to the evolution operator U: $UC_x^2 \subset C_x^2$ and is bounded: $\|Uf\|_{Cx} \le \|f\|_{Cx}$, but it is not invariant under the adjoint operator $U^\dagger$.*

The space C_y^2 satisfies the above statements (i) and (ii), and

(iii′) *$U^\dagger C_y^2 \subset C_y^2$ and $\|U^\dagger f\|_{Cy} \le \|f\|_{Cy}$, but C_y^2 is not invariant under U.*

Proof. (i) It is enough to show that C_x^2 is complete. The proof is almost parallel to the standard proof of the completeness of L^p [20]. Let $\{f_n\}_{n\ge1} \subset C_x^2$ be a Cauchy sequence. Then, one can find a subsequence $\{f_{n_j}\}_{j\ge1}$ of $\{f_n\}_{n\ge1}$ which satisfies, for almost every y,

$$\sup_{0\le|x|<1} \left|\partial_x^s f_{n_j}(x,y) - \partial_x^s f_{n_k}(x,y)\right| \longrightarrow 0 \quad (j,k \longrightarrow \infty;\ \text{for } s=0,1,2), \tag{3.5}$$

where ∂_x is the x-derivative. Hence, for each fixed y, the sequence of functions $\{f_{n_j}(x,y)\}_{j\ge1}$ of x converges uniformly to a limit $g(x,y)$, which is twice continuously differentiable with respect to x. Moreover, one has, for almost every y,

$$\lim_{j\to+\infty} \sup_{0\le x<1} \left|\partial_x^s f_{n_j}(x,y) - \partial_x^s g(x,y)\right| = 0 \quad (\text{for } s=0,1,2). \tag{3.6}$$

Combining this equality,

$$\int_0^1 dy \sup_{0\le x<1} \left|\partial_x^s f_{n_j}(x,y)\right|^2 \le \left\|f_{n_j}\right\|_{Cx}{}^2 < +\infty, \tag{3.7}$$

and Fatou's lemma [20], one finds that the limit $g(x,y)$ satisfies the condition (C2) and, thus, $g \in C_x^2$. Finally, $\lim_{n\to+\infty} \|f_n - g\|_{C_x} = 0$ can be shown immediately.

(ii) The space $\mathcal{P}$ of polynomials of x and y is dense in L^2. Then, since $\mathcal{P} \subset C_x^2$, C_x^2 is dense as well. Moreover, for $f \in C_x^2$, one has

$$\|f\|_2^2 = \int_{[0,1)} dx\,dy\,|f(x,y)|^2 \le \int_0^1 dy \sup_{0\le x<1} |f(x,y)|^2 \le \|f\|_{Cx}{}^2, \tag{3.8}$$

or the topology of C_x^2 is stronger than that of L^2.

(iii) The twice continuous differentiability of $Uf(x,y)$ in x immediately follows from definition (1.2) of U. And we have

$$\sum_{j=0}^{2} \sqrt{\int_0^1 dy \sup_{0\le x<1} \left|\partial_x^j Uf(x,y)\right|^2} \le \sum_{j=0}^{2} \sqrt{\frac{1}{2^j}\int_0^1 dy \sup_{0\le x<1} \left|\partial_x^j f(x,y)\right|^2} \le \|f\|_{Cx}, \tag{3.9}$$

which implies $Uf \in C_x^2$ and $\|Uf\|_{Cx} \le \|f\|_{Cx}$. Since $U^\dagger$ introduces a discontinuity at $x = 1/2$ in general, $U^\dagger C_x^2 \not\subset C_x^2$.

The proof for C_y^2 is the same as above. □

3.2. Pollicott-Ruelle decomposition. From Proposition 3.1, the adjoint $U^\dagger$ of the evolution operator can be continuously extended to the dual space $C_x^{2\dagger}$, and U to $C_y^{2\dagger}$. These extensions admit decaying eigenfunctions, which control the decay of expectation values. More precisely, we have the following proposition.

Proposition 3.2. *Suppose $A \in C_y^2$ and $\rho \in C_x^2$, then there exist conjugate linear functionals $F_0, F_1^a, F_1^b \in C_y^{2\dagger}$ and $\tilde{F}_0, \tilde{F}_1^a, \tilde{F}_1^b \in C_x^{2\dagger}$ defined by*

$$\tilde{F}_0(\rho) \equiv \int_{[0,1)^2} dx\,dy\rho(x,y)^*, \tag{3.10}$$

$$\tilde{F}_1^a(\rho) \equiv \int_{[0,1)^2} dx\,dy\partial_x\rho(x,y)^*, \tag{3.11}$$

$$\tilde{F}_1^b(\rho) \equiv \int_{[0,1)^2} dx\,dy\left(y-\frac{1}{2}\right)\rho(x,y)^* - \int_{[0,1)^2} dg_1(x)dy\partial_x\rho(x,y)^*, \tag{3.12}$$

$$F_0(A) \equiv \int_{[0,1)^2} dx\,dyA(x,y)^*, \tag{3.13}$$

$$F_1^a(A) \equiv \int_{[0,1)^2} dx\,dy\left(x-\frac{1}{2}\right)A(x,y)^* - \int_{[0,1)^2} dx\,dg_1(y)\partial_y A(x,y)^*, \tag{3.14}$$

$$F_1^b(A) \equiv \int_{[0,1)^2} dx\,dy\partial_y A(x,y)^*, \tag{3.15}$$

where the function g_1 is continuous and is defined as a unique solution of

$$g_1(x) = \begin{cases} \dfrac{1}{2}g_1(2x) - \dfrac{x^2}{4} + \dfrac{x}{8}, & 0 \le x \le \dfrac{1}{2}, \\ \dfrac{1}{2}g_1(2x-1) + \dfrac{x^2}{4} - \dfrac{3x}{8} + \dfrac{1}{8}, & \dfrac{1}{2} \le x \le 1. \end{cases} \tag{3.16}$$

Note that the integrals involving dg_1 are the Riemann-Stieltjes integrals, which are well defined [47] since g_1 is continuous and the integrands are of finite variation with respect to the integration variables.

(i) *Those functionals are principal vectors of the extensions (i.e., generalized principal vectors) of U and $U^\dagger$, respectively:*

$$UF_0(A) \equiv F_0(U^\dagger A) = F_0(A), \tag{3.17}$$

$$UF_1^a(A) = \frac{1}{2}F_1^a(A) + \frac{1}{16}F_1^b(A), \tag{3.18}$$

$$UF_1^b(A) = \frac{1}{2}F_1^b(A), \tag{3.19}$$

$$U^\dagger\tilde{F}_0(\rho) \equiv \tilde{F}_0(U\rho) = \tilde{F}_0(\rho), \tag{3.20}$$

$$U^\dagger\tilde{F}_1^a(\rho) = \frac{1}{2}\tilde{F}_1^a(\rho), \tag{3.21}$$

$$U^\dagger\tilde{F}_1^b(\rho) = \frac{1}{2}\tilde{F}_1^b(\rho) + \frac{1}{16}\tilde{F}_1^a(\rho). \tag{3.22}$$

(ii) *The time evolution of the expectation value of A at time t is given by*

$$\begin{aligned}\langle A, U^t\rho\rangle &= F_0(A)\tilde{F}_0(\rho)^* \\ &\quad + \frac{1}{2^t}\left[F_1^a(A)\tilde{F}_1^a(\rho)^* + F_1^b(A)\tilde{F}_1^b(\rho)^* + \frac{t}{8}F_1^b(A)\tilde{F}_1^a(\rho)^*\right] + R_t(A,\rho),\end{aligned} \tag{3.23}$$

where $R_t(A,\rho)$ is a sesquilinear form satisfying

$$|R_t(A,\rho)| \le \frac{\|A\|_{Cy}\|\rho\|_{Cx}}{4^t}\{K_2t^2 + K_1t + K_0\}, \tag{3.24}$$

and K_j's are positive constants.

Proof. First we discuss the properties of $g_1(x)$. Its defining equation is similar to de Rham's functional equation [9, 19, 42, 43, 44] and is a fixed point equation of the following map $\mathcal{T}$:

$$\mathcal{T}g(x) = \begin{cases} \dfrac{1}{2}g(2x) - \dfrac{x^2}{4} + \dfrac{x}{8}, & 0 \le x \le \dfrac{1}{2}, \\ \dfrac{1}{2}g(2x-1) + \dfrac{x^2}{4} - \dfrac{3x}{8} + \dfrac{1}{8}, & \dfrac{1}{2} \le x \le 1. \end{cases} \tag{3.25}$$

As easily seen, $\mathcal{T}$ is a contraction on a Banach space of bounded functions equipped with the supremum norm and, because of Banach's fixed point theorem [20], it admits a unique fixed point g_1. If g is continuous, $\mathcal{T}g$ is continuous and, thus, an approximate sequence $\{f_n\}_{n\ge0}$ of g_1 defined by $f_n = \mathcal{T}f_{n-1}$ and $f_0 \equiv 0$ is a sequence of continuous functions uniformly converging to g_1. As a result, the limit g_1 is continuous. Note that g_1 has a fractal graph and $g_1(0) = g_1(1) = 0$.

Next we show $F_0, F_1^a, F_1^b \in C_y^{2\dagger}$ and $\tilde{F}_0, \tilde{F}_1^a, \tilde{F}_1^b \in C_x^{2\dagger}$. As the arguments are similar, we only consider $\tilde{F}_1^b$. As $\rho \in C_x^2$, $\tilde{F}_1^b(\rho)$ is well defined and its conjugate linearity is obvious. The continuity follows from a straightforward calculation. Indeed, integration by parts and $g_1(0) = g_1(1) = 0$ lead to

$$\tilde{F}_1^b(\rho) = \int_{[0,1)^2} dx\,dy\left\{\left(y - \frac{1}{2}\right)\rho(x,y)^* + g_1(x)\partial_x^2\rho(x,y)^*\right\}, \tag{3.26}$$

and, thus,

$$\begin{aligned}|\tilde{F}_1^b(\rho)| &\le \int_0^1 dy\left\{\left|y - \frac{1}{2}\right| \sup_{0\le x<1}|\rho(x,y)| + \sup_{0\le x<1}|g_1(x)| \sup_{0\le x<1}|\partial_x^2\rho(x,y)|\right\} \\ &\le \left\|y - \frac{1}{2}\right\|_2 \sqrt{\int_0^1 dy \sup_{0\le x<1}|\rho(x,y)|^2} \\ &\quad + \sup_{0\le x<1}|g_1(x)| \sqrt{\int_0^1 dy \sup_{0\le x<1}|\partial_x^2\rho(x,y)|^2} \\ &\le \left(\left\|y - \frac{1}{2}\right\|_2 + \sup_{0\le x<1}|g_1(x)|\right)\|\rho\|_{Cx} \le \left(\frac{1}{2\sqrt{3}} + \frac{1}{32}\right)\|\rho\|_{Cx},\end{aligned} \tag{3.27}$$

where we have used $\|y-1/2\|_2 = 1/(2\sqrt{3})$ and $\sup_x |g_1(x)| \le 1/32$ (cf. (3.25)). Then, $\tilde{F}_1^b(\rho)$ is bounded and, thus, continuous.

(i) Equation (3.22) is shown as follows. Because $\partial_x U\rho = (1/2)U\partial_x\rho$ and

$$\begin{aligned} dg_1(2x) &= 2dg_1(x) + \left(x - \frac{1}{4}\right)dx, \\ dg_1(2x-1) &= 2dg_1(x) + \left(-x + \frac{3}{4}\right)dx \end{aligned} \tag{3.28}$$

(cf. (3.16)), we obtain the desired relation (3.22)

$$\begin{aligned} \tilde{F}_1^b(U\rho) &= \int_0^1 dy\left[\int_0^{1/2} dx\frac{y-1}{2}\rho(x,y)^* + \int_{1/2}^1 dx\frac{y}{2}\rho(x,y)^*\right] \\ &\quad - \frac{1}{4}\int_0^1 dy\left[\int_0^{1/2} dg_1(2x)\partial_x\rho(x,y)^* + \int_{1/2}^1 dg_1(2x-1)\partial_x\rho(x,y)^*\right] \\ &= \frac{1}{2}\int_0^1 dy\left[\int_0^{1/2} dx(y-1)\rho(x,y)^* - \int_0^{1/2}\left\{dg_1(x) + \left(\frac{x}{2} - \frac{1}{8}\right)dx\right\}\partial_x\rho(x,y)^*\right] \\ &\quad + \frac{1}{2}\int_0^1 dy\left[\int_{1/2}^1 dxy\rho(x,y)^* - \int_{1/2}^1\left\{dg_1(x) + \left(-\frac{x}{2} + \frac{3}{8}\right)dx\right\}\partial_x\rho(x,y)^*\right] \\ &= \frac{1}{2}\tilde{F}_1^b(\rho) + \frac{1}{16}\int_0^1\{\rho(1,y)^* - \rho(0,y)^*\} = \frac{1}{2}\tilde{F}_1^b(\rho) + \frac{1}{16}\tilde{F}_1^a(\rho). \end{aligned} \tag{3.29}$$

The proofs of the other relations are the same as above.

(ii) The derivation is given in the appendix. □

3.3. Spectrum of the restricted evolution operator. At first sight, the decay property as expressed by (3.23) seems to be an operator property of U restricted to the subspace C_x^2 or $U^\dagger$ restricted to C_y^2. However, it is not the case and (3.23) is the property of a triple C_y^2, C_x^2, and U (or $U^\dagger$). Indeed, for the operator $U^\dagger$ restricted to C_y^2, we have the following proposition.

Proposition 3.3. (i) *The spectral set $\sigma(U^\dagger|_{C_y^2})$ of $U^\dagger$ restricted to the space C_y^2 satisfies*

$$\{z : 1/4 < |z| < 1\} \subset \sigma(U^\dagger|_{C_y^2}) \subset \{z : |z| \le 1\}. \tag{3.30}$$

(ii) *Let $h(y)$ be a function satisfying $\int_0^1 dyh(y) = \int_0^1 dyyh(y) = 0$ and let η_z with $1/4 < |z| < 1$ be a conjugate linear functional defined by*

$$\begin{aligned} \eta_z(A) &= \sum_{n=1}^{+\infty} z^n \int_{[0,1)^2} dx\,dyh(y)U^{\dagger-n}A(x,y)^* \\ &\quad + \sum_{n=0}^{+\infty}\left(\frac{1}{4z}\right)^n \int_{[0,1)^2} dx\,dyh(y)J_yU^{\dagger n}\partial_y^2 A(x,y)^*, \end{aligned} \tag{3.31}$$

where $J_y f(x,y) \equiv \int_0^y dy' \int_0^{y'} dy'' f(x,y'')$, *then* $\eta_z \in C_y^{2\dagger}$ *and* $U\eta_z = z\eta_z$. *Examples of* $h(y)$ *are* $y^2 - y + 1/6$ *or* $\cos(2\pi m y)$ $(m \neq 0)$.

Moreover, when $h(y) = y^2 - y + 1/6$,

$$\eta_z(A) = \int_{[0,1)^2} dx\,dy\{h_z^{(1)}(x,y)A(x,y)^* + h_z^{(2)}(y)\partial_y^2 A(x,y)^*\}, \tag{3.32}$$

where $h_z^{(1)}(x,y) = 4y^2/(4-z) + \{a_z(x) - 2/(2-z)\}y + b_z(x) + 1/(6-6z)$, *and* a_z, b_z, *and* $h_z^{(2)}$ *are unique solutions of the following equations:*

$$a_z(x) = \begin{cases} \dfrac{z}{2}a_z(2x), & 0 \le x < \dfrac{1}{2}, \\ \dfrac{z}{2}a_z(2x-1) + \dfrac{2z}{4-z}, & \dfrac{1}{2} \le x < 1, \end{cases} \tag{3.33}$$

$$b_z(x) = \begin{cases} zb_z(2x), & 0 \le x < \dfrac{1}{2}, \\ zb_z(2x-1) + a_z(x) - \dfrac{2z(3-z)}{(2-z)(4-z)}, & \dfrac{1}{2} \le x < 1, \end{cases} \tag{3.34}$$

$$h_z^{(2)}(y) = \begin{cases} \dfrac{1}{4z}h_z^{(2)}(2y) + \dfrac{y^2(1-2y)^2}{12z}, & 0 \le y < \dfrac{1}{2}, \\ \dfrac{1}{4z}h_z^{(2)}(2y-1) + \dfrac{(1-y)^2(1-2y)^2}{12z}, & \dfrac{1}{2} \le y < 1. \end{cases} \tag{3.35}$$

Proof. (i) The second inclusion follows from Proposition 3.1(iii′) and the spectral radius formula [36]. The first inclusion holds because of (ii) and Lemma 2.4.

(ii) Each term of (3.31) is well defined for $A \in C_y^2$ and, for every $z \in \{z : 1/4 < |z| < 1\}$, we have $\eta_z \in C_y^{2\dagger}$ because (3.31) converges absolutely and

$$|\eta_z(A)| \le \sqrt{\int_0^1 dy\,|h(y)|^2}\left[\frac{|z|}{1-|z|} + \frac{4|z|}{4|z|-1}\right]\|A\|_{Cy}. \tag{3.36}$$

Now we consider $\eta_z(U^\dagger A)$. Because of $\partial_y^2 U^\dagger A = (1/4)U^\dagger \partial_y^2 A$ and the properties of h, we have

$$\begin{aligned} \eta_z(U^\dagger A) - z\eta_z(A) &= z\int_{[0,1)^2} dx\,dy\,h(y)\{A(x,y)^* - J_y\partial_y^2 A(x,y)^*\} \\ &= z\int_0^1 dx\left\{A(x,0)\int_0^1 dy\,h(y) + \partial_y A(x,0)^*\int_0^1 dy\,y\,h(y)\right\} = 0. \end{aligned} \tag{3.37}$$

This shows the first half.

The second half is shown as follows. By noting $\int_0^1 dy\,h(y)J_y f(y) = \int_0^1 dy\,H(y)f(y)$ with $H(y) = \int_y^1 dy'(y'-y)h(y')$, one has (3.32) with

$$h_z^{(1)} = \sum_{n=0}^{+\infty} z^n U^{\dagger n} h, \qquad h_z^{(2)} = \sum_{n=1}^{+\infty}\left(\frac{1}{4z}\right)^n U^n H, \tag{3.38}$$

which satisfy

$$h_z^{(1)} = zU^\dagger h_z^{(1)} + h, \tag{3.39}$$

$$h_z^{(2)} = \frac{1}{4z} U\{h_z^{(2)} + H\}. \tag{3.40}$$

The solution of (3.39) is quadratic in y^2 and can be cast into the expression just after (3.32) with a_z and b_z given, respectively, by (3.33) and (3.34). Equation (3.40) is nothing but (3.35). □

3.4. Pollicott-Ruelle decomposition revisited. Now we reinvestigate the Pollicott-Ruelle decomposition. In case of the one-dimensional Bernoulli map [5] defined on the unit interval $[0,1)$ by

$$S(x) = \begin{cases} 2x, & 0 \le x < \dfrac{1}{2}, \\ 2x-1, & \dfrac{1}{2} \le x < 1, \end{cases} \tag{3.41}$$

the spectrum of the Frobenius-Perron operator on the Hilbert space of square integrable functions is the unit disk $\{z : |z| \le 1\}$ and the corresponding eigenfunctions are mostly represented by nonsmooth Weierstrass functions. By restricting densities to m-times continuously differentiable functions, eigenfunctions without this smoothness are not allowed and the corresponding eigenvalues are removed from the spectrum. In this way, the spectrum of the Frobenius-Perron operator changes from the unit disk to a set $\{1, 1/2, \dots, 1/2^{m-1}\} \cup \{z : |z| \le 1/2^m\}$, and the Pollicott-Ruelle decomposition is derived.

Propositions 3.2 and 3.3 suggest that the Pollicott-Ruelle decomposition (3.23) is obtained in a similar way. To see this in detail, a projection π_x onto the dilating direction is defined.

For $\psi \in (C_y^2)^\dagger$, if there exists a function $f(x)$ such that $\psi(A) = \int_0^1 dx f(x) A(x)^*$ holds for every $A \in C_y^2$ which does not depend on y, then $\pi_x \psi(x) \equiv f(x)$.

With this definition, one has

$$\begin{gathered} \pi_x F_1^a = x - \frac{1}{2}, \qquad \pi_x F_1^b = 0, \\ \pi_x \eta_z = \int_0^1 dy h_z^{(1)}(x,y) = \frac{1}{2} a_z(x) + b_z(x) + \frac{z^2}{2(4-z)(2-z)(1-z)}. \end{gathered} \tag{3.42}$$

Hence the projection $\pi_x \eta_z$ of the generalized eigenfunction of U is singular with respect to x. Indeed, for nonreal z, the function $a_z(x)$ is nondifferentiable, of infinite variation, and has a fractal graph [43]. On the other hand, the projections $\pi_x F_1^a$ and $\pi_x F_1^b$ of the generalized eigenfunctions involved in the Pollicott-Ruelle decomposition are smooth in x. This suggests the following. First, by restricting the class of observables A to C_y^2, the spectrum of $U^\dagger$ changes so that it contains an annulus $\{z : 1/4 < |z| < 1\}$. Next, by restricting densities to functions smooth along the x-direction, there remains only the eigenfunctions smooth in the x-direction (which corresponds to the eigenvalues 1 and 1/2), and

the other values in the annulus $\{z : 1/4 < |z| < 1\}$ are removed from the spectrum. Then, one obtains the Pollicott-Ruelle decomposition (3.23).

4. Discussions

We have studied the baker map by two different theories of irreversibility by Prigogine and his colleagues: the Λ-transformation theory and the complex spectral theory. In both approaches, by restricting the class of observables to a subset Φ of L^2 ($\Phi = \Lambda^\dagger L^2$ or C_y^2), the evolution operator $U^\dagger$ becomes dissipative as expressed by the spectral set containing an annulus in the unit disc. However, the two approaches are not equivalent. In the former, one looks for a surjective isometric transformation $\Lambda^{\dagger -1} : \Phi \to L^2$ (the conditions (a), (b), (c), (d), and (e) are imposed as well). Then, the transformed evolution $W = \Lambda U \Lambda^{-1}$ of the densities becomes a dissipative Markov operator. In the latter, one further restricts the class of densities so that most values in the interior of the annulus are removed from the spectrum, and the relaxation of expectation values is described by point spectra in the annulus and faster decaying terms. One thus obtains the Pollicott-Ruelle decomposition.

The dissipativity of the restricted operator $U^\dagger|_\Phi$ ($\Phi = \Lambda^\dagger L^2, C_y^2$) can be seen easily by considering the averages. First we note that $A \in \Phi$ is "smooth" along the contracting y-direction. (When $\Phi = C_y^2$, this is obvious.) When $\Phi = \Lambda^\dagger L^2$, this can be seen as follows. For large enough n, E_n-subspace consists of functions χ_S which are constant along the x-direction and highly oscillatory along the y-direction. Because every element of $\Lambda^\dagger L^2$ can be expressed as $A = \Lambda^\dagger B$ ($B \in L^2$), A contains less and less E_n-components as n increases, or A contains less and less highly oscillatory component along the y-direction. Next we remark that, as time goes on, the density $U^t\rho$ becomes highly oscillatory along the y-direction. Hence, the fine structure of the density $U^t\rho$ cannot be "probed" by the average value of the restricted class of observables $A \in \Phi$. In other words, dissipation arises in the evaluation of observables which are smooth along the y-direction with respect to densities $U^t\rho$ which are highly oscillatory along the y-direction. In this sense, the restriction of the operator acts as a kind of coarse graining. However, as Φ is dense in L^2, no information is lost in this procedure.

Appendix

A. Functional equation method

So far, the subdynamics decomposition and the resolvent method have been used to derive the generalized spectral decomposition and, thus, the Pollicott-Ruelle decomposition (3.23). Although they are systematic, it is not easy to obtain explicit expressions of the generalized eigenfunctions, which may involve Stieltjes integrals with respect to fractal functions. In this appendix, the Pollicott-Ruelle decomposition is derived via a set of functional equations of de Rham type [9, 43, 44].

First, we note that the expectation value of an observable A at time t is rewritten as

$$\begin{aligned}\langle A, U^t\rho\rangle = &\int_0^1 dx\{h_t(x)A(x,1)^* - G_t(x,1)\partial_y A(x,1)\} \\ &+ \int_{[0,1)^2} dx\,dy\,G_t(x,y)\partial_y^2 A(x,y)^*,\end{aligned} \tag{A.1}$$

where ρ is the initial density, U is the Frobenius-Perron operator, $h_t(x) \equiv \int_0^1 dy U^t \rho(x,y)$ and $G_t(x,y) \equiv \int_0^y dy' \int_0^{y'} dy'' U^t \rho(x,y'')$ are auxiliary functions, and ∂_y stands for the partial derivative with respect to y.

From the definition of the Frobenius-Perron operator (1.2), the recursion relations of h_t and G_t are easily derived:

$$h_{t+1}(x) = \frac{1}{2}\left\{h_t\left(\frac{x}{2}\right) + h_t\left(\frac{x+1}{2}\right)\right\} \equiv V h_t(x), \tag{A.2}$$

where V is a linear operator defined by the preceding expression, and

$$G_{t+1}(x,y) = \Lambda G_t(x,y) + F_t(x,y), \tag{A.3}$$

where a linear operator Λ and a function F_t are given by

$$\begin{aligned} \Lambda G(x,y) &= \begin{cases} \frac{1}{4} G\left(\frac{x}{2}, 2y\right), & 0 < y \le \frac{1}{2}, \\ \frac{1}{4} G\left(\frac{x+1}{2}, 2y-1\right), & \frac{1}{2} < y \le 1, \end{cases} \\ F_t(x,y) &= \begin{cases} 0, & 0 < y \le \frac{1}{2}, \\ \frac{1}{4} G_t\left(\frac{x}{2}, 1\right) + \frac{1}{2}\left(y - \frac{1}{2}\right) h_t\left(\frac{x}{2}\right), & \frac{1}{2} < y \le 1. \end{cases} \end{aligned} \tag{A.4}$$

A.1. Auxiliary function $h_t(x)$. Let I_x be an integral operator

$$I_x f(x) \equiv \int_0^x dx' f(x') - \int_0^1 dx \int_0^x dx' f(x'), \tag{A.5}$$

then $I_x V = 2 V I_x$ and

$$h_0(x) = \int_0^1 dx' h_0(x') + \left(x - \frac{1}{2}\right) \int_0^1 dx' \partial_{x'} h_0(x') + I_x^2 \partial_x^2 h_0(x). \tag{A.6}$$

Since $V^t 1 = 1$ and $V^t(x - 1/2) = 1/2^t (x - 1/2)$, one gets

$$\begin{aligned} h_t(x) &= V^t h_0(x) \\ &= \int_0^1 dx' h_0(x') + \frac{1}{2^t}\left(x - \frac{1}{2}\right) \int_0^1 dx' \partial_{x'} h_0(x') + \frac{1}{4^t} I_x^2 V^t \partial_x^2 h_0(x) \\ &= \tilde{F}_0(\rho)^* + \frac{1}{2^t}\left(x - \frac{1}{2}\right) \tilde{F}_1^a(\rho)^* + \frac{1}{4^t} I_x^2 V^t \partial_x^2 h_0(x), \end{aligned} \tag{A.7}$$

where $\tilde{F}_0$ and $\tilde{F}_1^a$ are functionals given by (3.10) and (3.11), and we have used

$$\begin{aligned} \int_0^1 dx' h_0(x') &= \int_{[0,1)^2} dx\, dy \rho(x,y), \\ \int_0^1 dx' \partial_{x'} h_0(x') &= \int_{[0,1)^2} dx\, dy \partial_x \rho(x,y). \end{aligned} \tag{A.8}$$

A.2. Auxiliary function $G_t(x,1)$. From (A.3), (A.4), $G_t(x,1)$ is found to obey

$$G_{t+1}(x,1) = \frac{1}{2} V G_t(x,1) + a_t(x), \tag{A.9}$$

where

$$a_t(x) \equiv \frac{1}{4} h_t\left(\frac{x}{2}\right) = \frac{1}{4}\int_0^1 dx' h_0(x') + \frac{1}{2^t}\frac{x-1}{8}\int_0^1 dx' \partial_{x'} h_0(x') + \frac{1}{4^{t+1}}\delta h_t(x) \tag{A.10}$$

with $\delta h_t(x) = I_x^2 V^t \partial_x^2 h_0(x/2)$. Because of $f(x) = \int_0^1 dx' f(x') + I_x \partial_x f(x)$ and $I_x V = 2VI_x$, this leads to

$$\begin{aligned} G_t(x,1) &= \frac{1}{2^t} V^t G_0(x,1) + \sum_{s=1}^{t} \frac{1}{2^{s-1}} V^{s-1} a_{t-s}(x) \\ &= \frac{1}{2}\int_0^1 dx' h_0(x') - \frac{t}{2^t}\frac{1}{8}\int_0^1 dx' \partial_{x'} h_0(x') \\ &\quad + \frac{1}{2^t}\left[\int_0^1 dx' \left\{G_0(x',1) - \frac{1}{2} h_0(x')\right\} + \left(\frac{x}{2} - \frac{1}{4}\right)\int_0^1 dx' \partial_{x'} h_0(x') \right. \\ &\qquad \left. + \sum_{s=0}^{\infty}\int_0^1 dx' \frac{\delta h_s(x')}{2^{s+1}}\right] + \frac{1}{4^t} r_t(x), \end{aligned} \tag{A.11}$$

where

$$\begin{aligned} r_t(x) &= \sum_{s=1}^{t} I_x V^{s-1} \partial_x \delta h_{t-s}(x) - \sum_{s=0}^{\infty}\int_0^1 dx' \frac{\delta h_{s+t}(x')}{2^{s+1}} \\ &\quad + I_x V^t \partial_x G_0(x,1) - \left(\frac{x}{2} - \frac{1}{4}\right)\int_0^1 dx' \partial_{x'} h_0(x'). \end{aligned} \tag{A.12}$$

Because of $I_x V = 2VI_x$ and $\int_0^1 dx' \partial_{x'} h_0(x') = \int_0^1 dx' V^s \partial_{x'} h_0(x')$, one has

$$\begin{aligned} \int_0^1 dx' \frac{\delta h_s(x')}{2^{s+1}} &= \int_0^{1/2} dx I_x \left\{V^s \partial_x h_0(x) - \int_0^1 dx'' V^s \partial_{x''} h_0(x'')\right\} \\ &= \int_0^{1/2} dx I_x V^s \partial_x h_0(x) + \frac{1}{8}\int_0^1 dx V^s \partial_x h_0(x) \\ &= \int_0^{1/2} dx \left(\frac{1}{8} - \frac{x}{2}\right) V^s \partial_x h_0(x) + \int_{1/2}^1 dx \left(\frac{x}{2} - \frac{3}{8}\right) V^s \partial_x h_0(x) \\ &= \int_0^1 d\gamma(x) V^s \partial_x h_0(x), \end{aligned} \tag{A.13}$$

where

$$\gamma(x) = \begin{cases} \dfrac{x}{8} - \dfrac{x^2}{4}, & 0 \le x < \dfrac{1}{2}, \\ \dfrac{x^2}{4} - \dfrac{3x}{8} + \dfrac{1}{8}, & \dfrac{1}{2} \le x < 1. \end{cases} \tag{A.14}$$

With the aid of an equality

$$\int_0^1 dg(x)Vf(x) = \int_0^1 d(\mathcal{T}g(x))f(x), \tag{A.15}$$

with

$$\widehat{\mathcal{T}}g(x) \equiv \begin{cases} \frac{1}{2}g(2x), & 0 \le x < \frac{1}{2}, \\ \frac{1}{2}g(2x-1), & \frac{1}{2} \le x < 1, \end{cases} \tag{A.16}$$

the above expression leads to

$$\begin{aligned} \sum_{s=0}^{\infty}\int_0^1 dx' \frac{\delta h_s(x')}{2^{s+1}} &= \sum_{s=0}^{\infty}\int_0^1 d(\widehat{\mathcal{T}}^s\gamma(x))\partial_x h_0(x) = \int_0^1 dg_1(x)\partial_x h_0(x) \\ &= \int_{[0,1)^2} dg_1(x)dy\partial_x\rho(x,y), \end{aligned} \tag{A.17}$$

where $g_1(x) \equiv \sum_{s=0}^{\infty}\widehat{\mathcal{T}}^s\gamma(x)$ is the solution of the functional equation (3.16).

With the aid of

$$\int_0^1 dx'\left\{G_0(x',1) - \frac{1}{2}h_0(x')\right\} = -\int_{[0,1)^2} dx\,dy\left(y-\frac{1}{2}\right)\rho(x,y), \tag{A.18}$$

$G_t(x,1)$ casts into

$$G_t(x,1) = \frac{1}{2}\tilde{F}_0(\rho)^* - \frac{1}{2^t}\left[\tilde{F}_1^b(\rho)^* - \left(\frac{x}{2}-\frac{1}{4}\right)\tilde{F}_1^a(\rho)^*\right] - \frac{t}{2^{t+3}}\tilde{F}_1^a(\rho)^* + \frac{r_t(x)}{4^t}, \tag{A.19}$$

where $\tilde{F}_0$, $\tilde{F}_1^a$, and $\tilde{F}_1^b$ are functionals given, respectively, by (3.10), (3.11), and (3.12).

A.3. Auxiliary function $G_t(x,y)$ and Pollicott-Ruelle decomposition. According to the results of the previous subsections, $F_t(x,y)$ in (A.3) reads as

$$\begin{aligned} F_t(x,y) = &\left[\left(\frac{y}{2}-\frac{1}{8}\right)\tilde{F}_0(\rho)^* - \frac{t}{2^{t+5}}\tilde{F}_1^a(\rho)^*\right]\theta\left(y-\frac{1}{2}\right) \\ &- \frac{1}{2^{t+2}}\left[\tilde{F}_1^b(\rho)^* - \left(\frac{y}{2}-\frac{1}{8}\right)(x-1)\tilde{F}_1^a(\rho)^*\right]\theta\left(y-\frac{1}{2}\right) + \frac{1}{4^{t+1}}R_t^{(0)}(x,y), \end{aligned} \tag{A.20}$$

where θ is a step function and

$$R_t^{(0)}(x,y) = \left[r_t\left(\frac{x}{2}\right) + (2y-1)I_x^2V^t\partial_x^2 h_0\left(\frac{x}{2}\right)\right]\theta\left(y-\frac{1}{2}\right). \tag{A.21}$$

Then, the solution of the recursion relation (A.3) is given by

$$\begin{aligned} G_t(x,y) &= \Lambda G_0(x,y) + \sum_{s=1}^{t} \Lambda^{s-1} F_{t-s}(x,y) \\ &= \tilde{F}_0(\rho)^* \alpha_0(y) - \frac{\tilde{F}_1^a(\rho)^*}{2^{t+4}}\left[t\alpha_1(y) - \alpha_2(x,y)\right] \\ &\quad - \frac{\tilde{F}_1^b(\rho)^*}{2^{t+1}}\alpha_1(y) + R_t^{(1)}(x,y), \end{aligned} \tag{A.22}$$

where the residual term is given by

$$\begin{aligned} R_t^{(1)}(x,y) &= \Lambda^t\left[G_0(x,y) + \frac{1}{2}\tilde{F}_1^b(\rho)^*\alpha_1(y) - \tilde{F}_0(\rho)^*\alpha_0(y) - \frac{1}{16}\tilde{F}_1^a(\rho)^*\alpha_2(x,y)\right] \\ &\quad + \frac{1}{4^t}\sum_{s=1}^{t}(4\Lambda)^{s-1}R_{t-s}^{(0)}(x,y), \end{aligned} \tag{A.23}$$

and the functions α_j $(j = 0,1,2)$ are the solutions of the de Rham-type functional equations

$$\begin{aligned} \alpha_0(y) &= \begin{cases} \dfrac{1}{4}\alpha_0(2y), & 0 \le y < \dfrac{1}{2}, \\ \dfrac{1}{4}\alpha_0(2y-1) + \dfrac{y}{2} - \dfrac{1}{8}, & \dfrac{1}{2} \le y < 1, \end{cases} \\ \alpha_1(y) &= \begin{cases} \dfrac{1}{2}\alpha_1(2y), & 0 \le y < \dfrac{1}{2}, \\ \dfrac{1}{2}\alpha_1(2y-1) + 1, & \dfrac{1}{2} \le y < 1, \end{cases} \\ \alpha_2(x,y) &= \begin{cases} \dfrac{1}{2}\alpha_2\left(\dfrac{x}{2}, 2y\right) + \alpha_1(y), & 0 \le y < \dfrac{1}{2}, \\ \dfrac{1}{2}\alpha_2\left(\dfrac{x+1}{2}, 2y-1\right) + \alpha_1(y) + (8y-2)(x-1), & \dfrac{1}{2} \le y < 1. \end{cases} \end{aligned} \tag{A.24}$$

As easily seen, the solutions of the above equations are

$$\alpha_0(y) = \frac{y^2}{2}, \qquad \alpha_1(y) = 2y, \qquad \alpha_2(x,y) = 4\{y^2(2x-1) + 4g_1(y)\}, \tag{A.25}$$

and thus,

$$\begin{aligned} G_t(x,y) &= \frac{y^2}{2}\tilde{F}_0(\rho)^* - \frac{1}{2^t}\left[\frac{y^2}{2}\left(x - \frac{1}{2}\right) + g_1(y)\right]\tilde{F}_1^a(\rho)^* \\ &\quad - \frac{y}{2^t}\tilde{F}_1^b(\rho)^* - \frac{t}{2^t}\frac{y}{8}\tilde{F}_1^a(\rho)^* + R_t^{(1)}(x,y). \end{aligned} \tag{A.26}$$

By substituting (A.7) and (A.26) into (A.1), one obtains the desired decomposition (3.23) with

$$\begin{aligned} R_t(A,\rho) = \int_0^1 dx \left[\frac{1}{4^t} I_x^2 V^t \partial_x^2 h_0(x) - R_t^{(1)}(x,1) \right] \partial_y A(x,1)^* \\ + \int_{[0,1)^2} dx\, dy R_t^{(1)}(x,y) \partial_y^2 A(x,y)^*. \end{aligned} \tag{A.27}$$

Then, by noting $|\Lambda f(x,y)| \le \sup_{x,y} |f(x,y)|/4$ and the boundedness of I_x and V, one obtains the desired estimate (3.24) of $R_t(A,\rho)$.

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