ESTIMATES OF TOPOLOGICAL ENTROPY OF CONTINUOUS MAPS WITH APPLICATIONS

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We present a simple theory on topological entropy of the continuous maps defined on a compact metric space, and establish some inequalities of topological entropy. As an application of the results of this paper, we give a new simple proof of chaos in the so-called N-buffer switched flow networks.

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1. Introduction

Topological entropy can be an indicator of complicated (chaotic) behavior in dynamical systems. Whether the topological entropy of a dynamical system is positive or not is of primary significance, due to the fact that positive topological entropy implies that one can assert that the system is chaotic. As the concept of topological entropy is concerned, it is hard, as remarked by [8], to get a good idea of what entropy means directly from various definitions of entropy. Thus it is enough in this paper to know that topological entropy of a dynamical system is a measure of complexity of dynamic behavior of the system, and it can be seen as a quantitative measurement of how chaotic a dynamical system is. Generally speaking, the larger the entropy of a system is, the more complicated the dynamics of this system would be. For instance, a system on a compact metric space has zero entropy provided its nonwandering set consists of finite number of periodic orbits. For the notions and discussions on entropy of dynamical systems, the reader can refer to [8, Chapter VIII].

In recent years a remarkable progress has been made in topological entropy and chaos in low-dimensional dynamical systems [1, 2, 4, 6, 7], mainly including several methods of estimating the topological entropy in one-dimensional situations. Therefore it is meaningful to present some practical results on estimating topological entropy of arbitrary dimensional dynamical systems that can be applied to real-world problems.

A well-known result in chaos theory is the following theorem.

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Theorem 1.1 [10]. *Let X be a metric space, D a compact subset of X, and $f : D \to X$ a continuous map satisfying the assumption that there exist m mutually disjoint compact subsets $D_1,\dots,D_m$ of D, such that*

$$f(D_j) \supset \bigcup_{i=1}^{m} D_i, \quad j = 1,2,\dots,m. \tag{1.1}$$

Then there exists a compact invariant set $\Lambda \subset D$, such that $f \mid \Lambda$ is semiconjugate to the m-shift map.

In this paper we consider a general situation where the m subsets $D_1,\dots,$ and D_m are not necessarily mutually disjoint or (1.1) does not hold, but satisfy some suitable conditions on the intersection of these subsets. A generalization of the above statement is without doubt of mathematical interest. However, the motivation in investigating this general case in this paper is to try to give a simple proof on chaos in N-buffer switched flow network.

2. Symbolic dynamics and some preliminaries

First we recall some aspects of symbolic dynamics.

Let $S_m = \{0,1,\dots,m-1\}$ be the set of nonnegative successive integer from 0 to $m-1$. Let Σ_m be the collection of all one-sided sequences with their elements belonging to S_m, that is, every element s of Σ_m is of the following form:

$$s = \{s_0, s_1, \dots, s_n, \dots\}, \quad s_i \in S_m. \tag{2.1}$$

Now consider another sequence $\bar{s} = \{\bar{s}_0, \bar{s}_1, \dots, \bar{s}_n, \dots\} \in \Sigma_m$. The distance between s and $\bar{s}$ is defined as

$$d(s,\bar{s}) = \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{|s_i - \bar{s}_i|}{1 + |s_i - \bar{s}_i|}. \tag{2.2}$$

With the distance defined as (2.2), Σ_m is a metric space, and the following fact is well known [8].

The space Σ_m is compact, totally disconnected, and perfect.

A set having these properties is often defined as a Cantor set, such a Cantor set frequently appears in characterization of complex structure of invariant set in a chaotic dynamical system.

Furthermore, define the m-shift map $\sigma : \Sigma_m \to \Sigma_m$ as follows:

$$\sigma(s)_i = s_{i+1}. \tag{2.3}$$

A well-known property of the shift map σ as a dynamical system defined on Σ_m is that the dynamics generated by σ is sensitive to initial conditions, therefore is chaotic.

Next we recall the semiconjugacy in terms of a continuous map and the shift map σ, which is conventionally defined as follows.

Definition 2.1. Let X be a metric space. Consider a continuous map $f : X \to X$. Let Λ be a compact invariant set of f. If there exists a continuous surjective map

$$h : \Lambda \longrightarrow \Sigma_m \tag{2.4}$$

such that $h \circ f = \sigma \circ h$, then the restriction of f to Λ $f \mid \Lambda$ is said to be semiconjugate to σ.

Since Theorem 1.1 in the previous section is useful for the sequel arguments of the main results of this paper, we restate a version of it as a lemma and give a proof for the reader's convenience.

Lemma 2.2. *Let X be a metric space, D a compact subset of X, and $f : D \to X$ a map satisfying the assumption that there exist m mutually disjoint compact subsets $D_1,\dots,D_m$ of D, the restriction of f to each D_i, that is, $f \mid D_i$ is continuous. Suppose that*

$$f(D_j) \supset \bigcup_{i=1}^{m} D_i, \quad j = 1,2,\dots,m, \tag{2.5}$$

then there exists a compact invariant set $\Lambda \subset D$, such that $f \mid \Lambda$ is semiconjugate to the m-shift map.

Proof. The proof is very easy and is a standard argument in horseshoe theory. □

For the concept of topological entropy, the reader can refer to [8]. We just recall the result stated in Lemma 2.3, which will be used in this paper.

Lemma 2.3 [8]. *Let X be a compact metric space, and $f : X \to X$ a continuous map. If there exists an invariant set $\Lambda \subset X$ such that $f \mid \Lambda$ is semiconjugate to the m-shift σ, then*

$$h(f) \geq h(\sigma) = \log m, \tag{2.6}$$

where $h(f)$ denotes the entropy of the map f. In addition, for every positive integer k,

$$h(f^k) = kh(f). \tag{2.7}$$

3. Some heuristic discussions

In the sequel, let X be a metric space, D a compact subset of X, let each D_i, $j = 1,2,\dots,m$, be a compact subset of D, and let $f : D \to X$ be a continuous map. To make the arguments more readable, we first consider the case $m = 2,3$ in this section.

Proposition 3.1. *Let D be a compact subset of X, and each D_i, $j = 1,2$, a subset of D. Suppose a continuous map $f : D \to X$ satisfies the following assumptions:*

(1) $f(D_1 \cap D_2) \cap D_1 \cap D_2 = \emptyset$;

(2) $f(D_j) \supset \cup_{i=1}^{2} D_i$, $j = 1,2$.

Then the entropy of f satisfies $h(f) \geq (1/3)\log 2$.

Proof. Let $D_1^1 \subset D_1$ be the subset such that $f(D_1^1) = D_1$, let $D_{12}^1 \subset D_2$ be the subset such that $f(D_{12}^1) = D_1^1$, and let $D_{121}^1 \subset D_1$ be the subset satisfying $f(D_{121}^1) = D_{12}^1$. It is easy to see that $f^3(D_{121}^1) = f^2(D_{12}^1) = f(D_1^1) = D_1$. Now take a subset $D_2^1 \subset D_2$ such that $f(D_2^1) \subset D_1$, take $D_{21}^1 \subset D_1$ such that $f(D_{21}^1) = D_2^1$, and take $D_{211}^1 \subset D_1$ such that $f(D_{211}^1) = D_{21}^1$, it is easy to see that $f^3(D_{211}^1) = D_1$. Now we show that $D_{211}^1 \cap D_{121}^1 = \varnothing$.

To see this, suppose the contrary holds: let $x \in D_{211}^1 \cap D_{121}^1$, then

$$\begin{gathered} f(x) \in D_{21}^1 \cap D_{12}^1 \subset D_1 \cap D_2, \\ f \circ f(x) \in D_1 \cap D_2. \end{gathered} \tag{3.1}$$

This is in contradiction to the assumption (1), therefore $D_{211}^1 \cap D_{121}^1 = \varnothing$. Now consider the map $F(x) = f^3(x)$. It can be seen in view of Lemma 2.2 that the map $F|_{D_1}: D_1 \to D_1$ is semiconjugate to 2-shift map and the same is true of the restriction map $F|_{D_2}: D_2 \to D_2$. Now Lemma 2.3 implies that $h(F) \geq \log 2$, so

$$h(f) \geq \frac{1}{3}\log 2. \tag{3.2}$$

□

Proposition 3.2. *Let X be a metric space, D a compact subset of X, D_i, $j = 1,2,3$, a subset of D, and let $f : D \to X$ be a continuous map satisfying the following assumptions:*

(1) *there exists a pair $i \neq j$, such that $f(D_i \cap D_j) \cap D_i \cap D_j = \varnothing$;*

(2) *$f(D_j) \supset \cup_{i=1}^3 D_i$, $j = 1,2,3$.*

Then the entropy of f satisfies $h(f) \geq (1/3)\log 2$.

Proof. Without loss of generality, suppose $D_2 \cap D_3 = \varnothing$, or

$$f(D_2 \cap D_3) \cap D_2 \cap D_3 = \varnothing, \tag{3.3}$$

and consider the subset D_1. Let $D_2^1 \subset D_2$ be the subset such that $f(D_2^1) = D_1$, let $D_{23}^1 \subset D_3$ be the subset such that $f(D_{23}^1) = D_2^1$, and let $D_{231}^1 \subset D_1$ be the subset satisfying $f(D_{231}^1) = D_{23}^1$. It is easy to see that $f^3(D_{231}^1) = f^2(D_{23}^1) = f(D_2^1) = D_1$. Similarly, take $D_{2311}^1 \subset D_1$, $D_{231}^1 \subset D_1$, $D_{23}^1 \subset D_3$, $D_2^1 \subset D_2$ such that

$$f^3(D_{321}^1) = f^2(D_{32}^1) = f(D_3^1) = D_1. \tag{3.4}$$

Now we show that $D_{231}^1 \cap D_{321}^1 = \varnothing$. Suppose this is not the case, that is, $D_{231}^1 \cap D_{321}^1 \neq \varnothing$.

Then for $x \in D_{321}^1 \cap D_{231}^1$, it is easy to see that $f(x) \in D_3 \cap D_2$, $f^2(x) \in D_3 \cap D_2$.

Because of condition (1), this is impossible. Now let $F(x) = f^3(x)$, by Lemmas 2.2 and 2.3 we have

$$h(f) \geq \frac{1}{3}\log 2. \tag{3.5}$$

□

It is easy to prove the following result.

Proposition 3.3. *Let X be a metric space, D a compact subset of X, D_i, $j = 1,2,3$, a subset of D, and let $f : D \to X$ be a continuous map satisfying the following assumptions:*

(1) *$D_1 \cap D_2 \cap D_3 = \emptyset$ and there exists a pair $i \neq j$, such that*

$$f(D_i \cap D_j) \subseteq D_i \cap D_j; \tag{3.6}$$

(2) *$f(D_j) \supset \cup_{i=1}^{3} D_i - D_j$, $j = 1,2,3$.*

Then the entropy of f satisfies $h(f) \geq (1/3)\log 2$.

Proof. Without loss of generality, assume that $f(D_2 \cap D_3) \subseteq D_2 \cap D_3$. Then consider the subsets D^1_{321} and D^1_{231} as the same as above. It remains to show that $D^1_{231} \cap D^1_{321} = \emptyset$. Suppose $D^1_{231} \cap D^1_{321} \neq \emptyset$. Then for $x \in D^1_{321} \cap D^1_{231}$, it is easy to see that

$$f(x) \in D_3 \cap D_2, \qquad f^2(x) \in D_3 \cap D_2. \tag{3.7}$$

It follows from the assumption $f(D_2 \cap D_3) \subseteq D_2 \cap D_3$ that $f^3(x) \in D_3 \cap D_2$ and $f^3(x) \in D_1$, contradictory to the assumption $D_1 \cap D_2 \cap D_3 = \emptyset$. □

4. General theorems

In this section we study the topological entropy for arbitrary $m \geq 2$ with assumptions different from those in the above propositions. Although these assumptions are little more stringent, they can apply to a model called switched flow network of arising in manufacturing systems as well as other engineering disciplines.

Theorem 4.1. *Let X be a metric space, D a compact subset of X, D_i, $j = 1,2,\ldots,m$, a subset of D, and $f : D \to X$ a continuous map satisfying the following assumptions:*

(1) *for each pair $i \neq j$, $1 \leq i$, $j \leq m$, $f(D_i \cap D_j) \subseteq D_i \cap D_j$;*

(2) *$D_1 \cap \cdots \cap D_m = \emptyset$;*

(3) *$f(D_j) \supset \cup_{i=1}^{m} D_i - D_j$, $j = 1,2,\ldots,m$;*

then there exists a compact invariant set $\Lambda \subset D$, such that $f \mid \Lambda$ is semiconjugate to $m-1$-shift dynamics. And

$$h(f) \geq \frac{1}{m-1}\log(m-1). \tag{4.1}$$

Proof. Without loss of generality, let us consider the subset D_1 and study the dynamics of the restricted map $f \mid D_1$. We are going to find $m-1$ mutually disjointed subsets contained in D_1, such that Lemma 2.2 can be applied.

For this purpose let $\Pi = \{2,3,\ldots,m\}$, the set of integers from 2 to m. There are many ways to select $m-1$ piecewise different sequences taken from the numbers of Π. Here for convenience we go as follows. Let $Q = q_1q_2,\ldots,q_{m-1}$ be a fixed sequence, with each element q_i appearing once in the set Π.

Let P be a permutation map as follows:

$$\begin{gathered}
P(Q)=q_{m-1}q_1q_2,\dots,q_{m-2},\\
P^2(Q)=q_{m-2}q_{m-1}q_1q_2,\dots,q_{m-3},\\
\cdots\\
P^{m-2}(Q)=q_2,\dots,q_{m-3}q_{m-2}q_{m-1}q_1.
\end{gathered}\tag{4.2}$$

For convenience, let

$$Q_i=P^{i-1}(Q)=q_{m-i+1}q_{m-i+2},\dots,q_1q_2,\dots,q_{m-i},\quad i=1,2,\dots,m-1.\tag{4.3}$$

For each Q_i, let

$$D^1_{Q_i},\ \bar{D}_{Q_i},\ \bar{D}_{q_{m-i+1}q_{m-i+2},\dots,q_1q_2,\dots,q_{m-i-1}},\dots,\ \bar{D}_{q_{m-i+1}}\tag{4.4}$$

be the sequence of subsets satisfying

$$\begin{gathered}
\bar{D}_{q_{m-i+1}}\subset D_{q_{m-i+1}},\\
\cdots\\
\bar{D}_{q_{m-i+1}q_{m-i+2},\dots,q_1q_2,\dots,q_{m-i-1}}\subset D_{q_{m-i-1}},\\
\bar{D}_{Q_i}\subset D_{q_{m-i}},\\
D^1_{Q_i}\subset D_1,
\end{gathered}\tag{4.5}$$

such that

$$\begin{gathered}
f\left(D^1_{Q_i}\right)=\bar{D}_{Q_i},\\
\cdots\\
f\left(\bar{D}_{q_{m-i+1}q_{m-i+2},\dots,q_1q_2,\dots,q_{m-i-1}}\right)=\bar{D}_{q_{m-i+1}q_{m-i+2},\dots,q_1q_2,\dots,q_{m-i-2}},\\
\cdots\\
f\left(\bar{D}_{q_{m-i+1}q_{m-i+2}}\right)=\bar{D}_{q_{m-i+1}},\\
f\left(\bar{D}_{q_{m-i+1}}\right)=D_1.
\end{gathered}\tag{4.6}$$

It is easy to see that

$$f^{m-1}\left(D^1_{Q_i}\right)=D_{1_i},\quad i=1,2,\dots,m-1.\tag{4.7}$$

Now we show that all the $m-1$ subsets $D^1_{Q_i}$, $i=1,2,\dots,m-1$, are mutually disjointed.

To this end, let us suppose that there exists a pair $i\neq j$ such that

$$D^1_{Q_i}\cap D^1_{Q_j}\neq\varnothing.\tag{4.8}$$

Then take a point x from intersection of these two subsets. It can be seen that

$$\begin{gathered} f(x) \in D_{m-i} \cap D_{m-j}, \\ f^2(x) \in D_{m-i-1} \cap D_{m-j-1}, \\ \cdots \\ f^{m-2}(x) \in D_{m-i+1} \cap D_{m-j+1}, \\ f^{m-1}(x) \in D_1. \end{gathered} \tag{4.9}$$

Because of the assumption that $f(D_i \cap D_j) \subseteq D_i \cap D_j$ for each pair $i \neq j$, $1 \leq i, j \leq m$, it is easy to see that

$$\begin{gathered} f^{m-1}(x) \in D_{m-i} \cap D_{m-j}, \\ f^{m-1}(x) \in D_{m-i-1} \cap D_{m-j-1}, \\ \cdots \\ f^{m-1}(x) \in D_{m-i+1} \cap D_{m-j+1}. \end{gathered} \tag{4.10}$$

Note that Q_i is $j-i$ times permutation of Q_j if $j > i$ or vice versa. This implies that

$$f^{m-2}(x) \in D_i \quad \forall i \in \Pi = \{2,3,\dots,m\}, \tag{4.11}$$

together with $f^{m-1}(x) \in D_1$ this again implies that

$$f^{m-1}(x) \in D_1 \cap \cdots \cap D_m, \tag{4.12}$$

which is a contradiction to the assumption $D_1 \cap \cdots \cap D_m = \emptyset$.

Now the theorem follows from Lemmas 2.2 and 2.3. □

Theorem 4.2. *Let X be a metric space, D a compact subset of X, D_i, $j = 1,2,\dots,m$, a subset of D, and $f : D \to X$ a continuous map satisfying the following assumptions:*

(1) *for each pair $i \neq j$, $1 \leq i, j \leq m$, $f(D_i \cap D_j) \subseteq D_i \cap D_j$; or there exists a pair $i \neq j$, such that $D_i \cap D_j = \emptyset$;*
(2) *$D_1 \cap \cdots \cap D_m = \emptyset$;*
(3) *$f(D_j) \supset \cup_{i=1}^m D_i$, $j = 1,2,\dots,m$.*

Then there exists a compact invariant set $\Lambda \subset D$, such that $f \mid \Lambda$ is semiconjugate to m-shift dynamics. And

$$h(f) \geq \frac{1}{m+1} \log m. \tag{4.13}$$

Proof. Without loss of generality, let us consider the subset D_1 and study the dynamics of the restricted map $f \mid D_1$. We are going to find m mutually disjointed subsets contained in D_1 in order to apply Lemma 2.2.

For this purpose let $\Pi = \{1,2,3,\dots,m\}$, the set of integers from 1 to m, and the above theorem can be proved in the similar manner as the proof of Theorem 4.1. □

5. An application: the estimate of entropy of N-buffer switched flow networks

There has been much interests in a model called switched arrival system, often called switched flow network, because of its significance in manufacturing systems and other engineering disciplines. Various dynamical behaviors such as existence and stability of periodic trajectories, bifurcation, and chaos were extensively investigated.

In studying switched flow model for manufacturing system, the work in [3, 5] considered the switched arrival system with one server and three buffers, and obtained an interesting discrete dynamical system via map of equilateral triangle. This dynamical system can be treated by virtue of one-dimensional dynamical systems theory. A mathematical analysis has been given on chaotic dynamics of this system in [5], where it was shown that the map of equilateral triangle is chaotic in terms of sensitive dependence on initial conditions, topological transitivity, and density of periodic orbits. In [1] L. L. Alsedà et al. gave a treatment on topological entropy of the switched arrival system with one server and three buffers. For details, one can see [3, 5]. In case of more than three buffers, [9] gave an elegant rigorous proof on the existence of chaos in terms of positive entropy. However, in [9] a quite deep knowledge about invariant SRB measure and the Markov partition as well as the entropy of ergodic Markov shift makes it not easy to catch on for readers less of good background of ergodic theory in dynamical systems.

In this section, we will revisit this problem and present a thorough but elementary treatment on the topological entropy of the N-buffer fluid networks with one server. We give an estimate formula for topological entropy of N-buffer fluid networks in form of an inequality just by virtue of the result obtained in the previous section. Our arguments are easy to understand even for readers who are not familiar with modern theory of dynamical systems.

Consider a system of N buffers and one server. In such a system, work is removed from buffer i at a fixed rate of $\rho_i > 0$ while the server delivers material to a selected buffer at a unit rate. The control law is applied to the server so that once a buffer empties, the server instantaneously starts to fill the empty buffer. The system is assumed to be close in the sense that

$$\sum_{i=1}^{N} \rho_i = 1. \tag{5.1}$$

Let $x_i(t)$ be the amount of work in buffer i at time $t \geq 0$, and let $x(t) = (x_1(t),\dots,x_N(t))$ denote the state of work of the buffers at time t, then

$$\sum_{i=1}^{N} x_i(t) = 1 \tag{5.2}$$

if it is assumed that $\sum_{i=1}^{N} x_i(0) = 1$.

Consider the sample sequence at clearing time, $\{t_n\}$, which are the times when at least one of these buffers becomes empty. Let $x(n) = (x_1(t_n),\dots,x_N(t_n))$, then the sequence

$\{x(n)\}$ evolves on the $N-2$ dimensional manifold

$$X = \left\{ x : \sum_{i=1}^{N} x_i = 1,\ x_i \geq 0,\ \exists 1 \leq j \leq N,\ x_j = 0 \right\} \tag{5.3}$$

by the following rule $G : X \to X$:

(1) $G(x) = x$ if at least two of the buffers empty at the same time;

(2) $x(n+1) = G(x(n)) = x(n) + \min_{i \neq j}(x_i(n)/\rho_i)(1_j - \rho)$, otherwise,

where 1_j is a vector with all zeros except for one in the jth position and ρ is a vector of work rates ρ_j. It is apparent that X defined above can be regarded as the surface of the standard $(N-1)$ simplex ϕ defined by

$$\phi = \left\{ x \in \mathbb{R}^N : x = \sum_{i=1}^{N} x_i e_i,\ x_i \geq 0,\ \sum_{i=1}^{N} x_i = 1 \right\}, \tag{5.4}$$

where $e_1 = (1,0,\dots,0)$, $e_2 = (0,1,0,\dots,0)$, $\dots$, $e_N = (0,0,\dots,1)$, and they are the vertices of the piecewise linear manifold X.

Let $X_i \subset X$ be the ith face of X,

$$X_i = \{ x \in X : x_i = 0 \}, \quad i = 1,2,\dots,N. \tag{5.5}$$

It is very easy to see that the map G has the following properties.

Proposition 5.1. *The restriction of the map G to every face X_j, that is,*

$$G \mid X_j : X_j \longrightarrow X - \hat{X}_j, \tag{5.6}$$

is a continuous one to one map. Here $\hat{X}_j = X_j - \partial X_j$ (∂X_j is the boundary of X_j), that is, the set consists of interior points of X_j. $G(X_i \cap X_j) = X_i \cap X_j$, $i \neq j$, $1 \leq i, j \leq N$.

In view of the above proposition it is easy to prove the following result as a corollary of Theorem 4.1.

Theorem 5.2. *The map $G : X \to X$ is chaotic and its entropy, h, satisfies the following inequality:*

$$h(G) \geq \frac{1}{N-1} \log(N-1). \tag{5.7}$$

6. Conclusion

In this paper we have discussed the topological entropy of the dynamical system described by continuous maps defined on a compact metric space, and presented several estimates of topological entropy for the continuous maps under some practical conditions. As an application we give a simple proof on the chaos of the so-called N-buffer switched flow networks.

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