ON MONOTONE SOLUTIONS OF SOME CLASSES OF DIFFERENCE EQUATIONS

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We describe a method for finding monotone solutions of some classes of difference equations converging to the corresponding equilibria. The method enables us to confirm three conjectures posed by the present author in a talk, which are extensions of three conjectures by M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations. With Open Problems and Conjectures*. Chapman and Hall/CRC, 2002. It is interesting that the method, in some cases, can be applied also when the parameters are variable.

1. Introduction

Recently there has been a great interest in studying nonlinear difference equations of order greater than one. Many of these equations stem from mathematical biology, economy, population dynamics , and so forth (see, e.g., [5, 7–9, 11, 14] and the references therein). An interesting problem in the theory of difference equations is finding monotone solutions. This paper is devoted to this problem.

Motivated by [8, Conjectures 5.4.6 and 6.10.3] in a talk (see, [16]) we posed the following three conjectures. The first one concerns a generalization of (1.2).

Conjecture 1.1. *Show that for every $p > -1$, the following equation:*

$$x_{n+1} = p + \frac{x_{n-k}}{\sum_{i=0}^{k-1} \alpha_i x_{n-i}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $k \in \mathbb{N}$, $\alpha_i \geq 0$, $i = 0, \ldots, k-1$, and $\sum_{i=0}^{k-1} \alpha_i = 1$, has a positive solution which remains above the equilibrium $\bar{x}_1 = p + 1$ for all $n \geq -k$.

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In [6] DeVault et al. investigate the behavior of the positive solutions of the difference equation

$$x_{n+1} = p + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where $p > 0$ and $k \in \mathbb{N}$ is fixed. Among other things they have proved that all nonoscillatory solutions of (1.2) converge to the positive equilibrium $\bar{x} = p + 1$.

Based on this observation they have posed the following open problem.

Open problem 1.2. Do there exist nonoscillatory solutions of (1.2)?

The following conjectures are generalizations of [8, Conjectures 5.4.6 and 6.10.3].

Conjecture 1.3. *Show that the following equation:*

$$x_{n+1} = \frac{1 + x_{n-k}}{\sum_{i=0}^{k-1} \alpha_i x_{n-i}}, \quad n = 0, 1, \ldots, \tag{1.3}$$

where $k \in \mathbb{N}$, $\alpha_i \geq 0$, $i = 0, \ldots, k-1$, and $\sum_{i=0}^{k-1} \alpha_i = 1$, has a nontrivial positive solution which decreases to the equilibrium $x_2 = (1+\sqrt{5})/2$.

Conjecture 1.4. *Show that the following equation:*

$$x_{n+1} = \frac{\alpha + x_{n-k}}{1 + \sum_{i=0}^{k-1} \alpha_i x_{n-i}}, \quad n = 0, 1, \ldots, \tag{1.4}$$

where $k \in \mathbb{N}$, $\alpha > 0$, $\alpha_i \geq 0$, $i = 0, \ldots, k-1$, and $\sum_{i=0}^{k-1} \alpha_i = 1$, has a positive solution which decreases to the equilibrium $x_3 = \sqrt{\alpha}$.

Our aim in this paper is to confirm the above mentioned conjectures.

The linearized equation for (1.1), respectively, (1.3) and (1.4), about the corresponding positive equilibrium $\bar{x}_i$, $i \in \{1, 2, 3\}$, is

$$(p+1)y_{n+1} + \alpha_0 y_n + \cdots + \alpha_{k-1} y_{n-k+1} - y_{n-k} = 0, \tag{1.5}$$

$$\bar{x}_2 (y_{n+1} + \alpha_0 y_n + \cdots + \alpha_{k-1} y_{n-k+1}) - y_{n-k} = 0, \tag{1.6}$$

$$(1+\sqrt{\alpha}) y_{n+1} + \sqrt{\alpha}(\alpha_0 y_n + \cdots + \alpha_{k-1} y_{n-k+1}) - y_{n-k} = 0. \tag{1.7}$$

The characteristic polynomial associated with (1.5), respectively, (1.6) and (1.7), is

$$p_1(t) = (p+1)t^{k+1} + \alpha_0 t^k + \cdots + \alpha_{k-1} t - 1 = 0, \tag{1.8}$$

$$p_2(t) = \bar{x}_2 (t^{k+1} + \alpha_0 t^k + \cdots + \alpha_{k-1} t) - 1 = 0, \tag{1.9}$$

$$p_3(t) = (1+\sqrt{\alpha}) t^{k+1} + \sqrt{\alpha}(\alpha_0 t^k + \cdots + \alpha_{k-1} t) - 1 = 0. \tag{1.10}$$

Since $p_1(0) = -1 < 0$, $p_1(1) = p+1$, and $p_1'(t) = (p+1)(k+1)t^k + \alpha_0 k t^{k-1} + \cdots + \alpha_{k-1} > 0$ for $t \in (0, 1]$, it follows that for each $p > -1$, there is a unique positive root t_1 of the polynomial (1.8) belonging to the interval $(0, 1)$.

Similarly, it can be shown that (1.9) and (1.10) have also a unique positive roots t_2 and t_3 in the interval $(0,1)$.

This fact motivated us to believe that there are solutions of (1.1), (1.3), and (1.4) which have the following asymptotics:

$$x_n = \bar{x} + at_i^n + o(t_i^n), \tag{1.11}$$

where $a \in \mathbb{R}$ and t_i, $i \in \{1,2,3\}$, are the above mentioned roots of polynomials (1.5), (1.6), and (1.7), respectively .

We solve the open problem, showing that such solutions exist, developing Berg's idea in [2] which are based on asymptotics. Asymptotics for solutions of difference equations has been investigated for a long time by L. Berg and S. Stević, see, for example, [1–4, 10–15] and the reference therein. We solve it by constructing two appropriate sequences y_n and z_n with

$$y_n \leq x_n \leq z_n \tag{1.12}$$

for sufficiently large n. In [1, 2], some methods can be found for the construction of these bounds, see, also [3, 4].

From (1.11) and results in Berg's paper [2], we expect that for $k \geq 2$ such solutions have the first four members in their asymptotics in the following form:

$$\varphi_n = \bar{x} + at^n + bt^{2n} + ct^{3n}. \tag{1.13}$$

2. The inclusion theorem

We need the following result in the proof of the main theorem. The proof of the result is similar to that of [2, Theorem 1].

Theorem 2.1. *Let $f : I^{k+2} \to I$ be a continuous and nondecreasing function in each argument on the interval $I \subset \mathbb{R}$, and let (y_n) and (z_n) be sequences with $y_n < z_n$ for $n \geq n_0$ and such that*

$$y_{n-k} \leq f(n, y_{n-k+1}, \dots, y_{n+1}), \quad f(n, z_{n-k+1}, \dots, z_{n+1}) \leq z_{n-k}, \tag{2.1}$$

for $n > n_0 + k - 1$.

Then there is a solution of the following difference equation:

$$x_{n-k} = f(n, x_{n-k+1}, \dots, x_{n+1}), \tag{2.2}$$

with property (1.12) for $n \geq n_0$.

Proof. Let N be an arbitrary integer such that $N > n_0 + k - 1$. The solution (x_n) of (2.2) with given initial values $x_N, x_{N+1}, \dots, x_{N+k}$ satisfying (1.12) for $n \in \{N, N+1, \dots, N+k\}$ can be continued by (2.2) to all $n < N$. Inequalities (2.1) and the monotonic character of f imply that (1.12) holds for all $n \in \{n_0, \dots, N+k\}$. Let A_N be the set of all $(k+1)$-tuples $(x_{n_0}, \dots, x_{n_0+k})$ such that there exist solutions (x_n) of (2.2) with these initial values satisfying (1.12) for all $n \in \{n_0, \dots, N+k\}$. It is clear that A_N is a closed nonempty set for every $N > n_0 + k - 1$, and that $A_{N+1} \subset A_N$. It follows that the set $A = \cap_{n=n_0+k}^{\infty} A_N$ is a nonempty subset of $\mathbb{R}^{k+1}$ and that if $(x_{n_0}, \ldots, x_{n_0+k}) \in A$, then the corresponding solutions of (2.2) satisfy (1.12) for all $n \geq n_0$, as desired. □

3. The main result

In this section we prove the main result of this paper, which confirms Conjectures 1.1, 1.3, and 1.4.

Theorem 3.1. *The following statements are true:*

(a) *let $\alpha_i \geq 0$, $i = 0, \ldots, k-1$, $\sum_{i=0}^{k-1} \alpha_i = 1$, and $p > -1$. Then (1.1) has a positive solution which remains above the equilibrium $\bar{x}_1 = p + 1$;*

(b) *let $\alpha_i \geq 0$, $i = 0, \ldots, k-1$, $\sum_{i=0}^{k-1} \alpha_i = 1$. Then (1.3) has a nontrivial positive solution which decreases to the equilibrium $\bar{x}_2$;*

(c) *let $\alpha > 0$, $\alpha_i \geq 0$, $i = 0, \ldots, k-1$, $\sum_{i=0}^{k-1} \alpha_i = 1$. Then (1.4) has a nontrivial positive solution which decreases to the equilibrium $\bar{x}_3 = \sqrt{\alpha}$.*

Proof. (a) Note that (1.2) can be written in the following equivalent form:

$$F(x_{n-k}, \ldots, x_n, x_{n+1}) = (x_{n+1} - p)(\alpha_0 x_n + \cdots + \alpha_{k-1} x_{n-k+1}) - x_{n-k} = 0. \tag{3.1}$$

We expect that solutions of (1.2) have asymptotic approximation (1.13). Thus, we calculate $F(\varphi_{n-k}, \ldots, \varphi_n, \varphi_{n+1})$. We have

$$\begin{aligned}
F &= \left(1 + at^{n+1} + bt^{2(n+1)} + ct^{3(n+1)}\right) \\
&\quad \times \left(p + 1 + a\alpha_0 t^n + \cdots + a\alpha_{k-1} t^{n-k+1} + b\alpha_0 t^{2n} + \cdots + b\alpha_{k-1} t^{2(n-k+1)} + \mathbb{O}(t^{3n})\right) \\
&\quad - \left(p + 1 + at^{n-k} + bt^{2(n-k)} + ct^{3(n-k)}\right) \\
&= at^n \left((p+1)t + \alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}} - t^{-k}\right) \\
&\quad + t^{2n} \left(b\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{2(k-1)}}\right) + a^2 t\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}}\right) + b(p+1)t^2 - bt^{-2k}\right) + \mathbb{O}(t^{3n}).
\end{aligned} \tag{3.2}$$

Let

$$D_1(t) = (p+1)t + \alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}} - \frac{1}{t^k}. \tag{3.3}$$

Choose $t \in (0,1)$ such that $D_1(t) = 0$, and $a, b \in \mathbb{R}$, $a \neq 0$, such that the coefficients in (3.2) are equal to zero. $D_1(t) = 0$ implies that $t = t_1$ (see, Section 1). Further we obtain

$$b = -\frac{a^2 t_1 \left(\alpha_0 + \cdots + \alpha_{k-1} t_1^{-k+1}\right)}{(p+1)t_1^2 + \alpha_0 + \cdots + (\alpha_{k-1})/t_1^{2(k-1)} - t_1^{-2k}} = -\frac{a^2 t_1 \left(\alpha_0 + \cdots + \alpha_{k-1} t_1^{-k+1}\right)}{D_1(t_1^2)}. \tag{3.4}$$

If $\widehat{\varphi}_n = p+1+at_1^n+qt_1^{2n}$, we obtain

$$F(\widehat{\varphi}_{n-k},\ldots,\widehat{\varphi}_n,\widehat{\varphi}_{n+1}) \sim \left(qD_1(t_1^2)+a^2t_1(\alpha_0+\cdots+\alpha_{k-1}t_1^{-k+1})\right)t_1^{2n}. \tag{3.5}$$

Let

$$H_{t_1}(q) = qD_1(t_1^2)+a^2t_1(\alpha_0+\cdots+\alpha_{k-1}t_1^{-k+1}). \tag{3.6}$$

We have

$$D_1'(t) = p+1+\frac{k}{t^{k+1}}-\frac{\alpha_1}{t^2}-\cdots-\frac{(k-1)\alpha_{k-1}}{t^k}. \tag{3.7}$$

Hence, when $t\in(0,1)$, it follows that

$$\begin{aligned} D_1'(t) &> p+1+\frac{k}{t^{k+1}}-\frac{\alpha_1+\cdots+(k-1)\alpha_{k-1}}{t^{k+1}} \\ &> p+1+\frac{k}{t^{k+1}}-\frac{(k-1)\sum_{i=1}^{k_1}\alpha_i}{t^{k+1}} > p+1+\frac{1}{t^{k+1}} > 0. \end{aligned} \tag{3.8}$$

From this, since $D_1(t_1)=0$, and $t_1^2<t_1$, we have that $D_1(t_1^2)<0$. Thus, we obtain that there are $q_1<b$ and $q_2>b$ such that $H_{t_1}(q_1)>0$ and $H_{t_2}(q_2)<0$.

With the notations

$$y_n = p+1+at_1^n+q_1t_1^{2n}, \qquad z_n = p+1+at_1^n+q_2t_1^{2n}, \tag{3.9}$$

we get

$$\begin{aligned} F(y_{n-k},\ldots,y_n,y_{n+1}) &\sim \left(q_1D_1(t_1^2)+a^2t_1(\alpha_0+\cdots+\alpha_{k-1}t_1^{-k+1})\right)t_1^{2n}>0, \\ F(z_{n-k},\ldots,z_n,z_{n+1}) &\sim \left(q_2D_1(t_1^2)+a^2t_1(\alpha_0+\cdots+\alpha_{k-1}t_1^{-k+1})\right)t_1^{2n}<0. \end{aligned} \tag{3.10}$$

These relations show that the inequalities in (1.12) are satisfied for sufficiently large n, where $f=F+x_{n-k}$ and F is given by (3.1). Applying Theorem 2.1 it follows that there is a solution of (1.1) with the asymptotics $x_n=\widehat{\varphi}_n+o(t_1^{2n})$, in particular, the solution of (1.1) converges monotonically to the positive equilibrium $\bar{x}_1=p+1$, when $p>-1$ and $n\ge n_0$. Hence, the solution x_{n+n_0+k} converges monotonically for $n\ge -k$.

(b) Equation (1.3) can be written in the following equivalent form:

$$F(x_{n-k},\ldots,x_n,x_{n+1}) = x_{n+1}(\alpha_0x_n+\cdots+\alpha_{k-1}x_{n-k+1})-(1+x_{n-k})=0. \tag{3.11}$$

Now we calculate $F(\varphi_{n-k},\dots,\varphi_n,\varphi_{n+1})$. We have

$$
\begin{aligned}
F &= \left(\bar{x}_2 + at^{n+1} + bt^{2(n+1)} + ct^{3(n+1)}\right)\\
&\quad\times \left(\bar{x}_2 + a\alpha_0 t^n + \cdots + a\alpha_{k-1}t^{n-k+1} + b\alpha_0 t^{2n} + \cdots + b\alpha_{k-1}t^{2(n-k+1)} + \mathbb{O}(t^{3n})\right)\\
&\quad- \left(1 + \bar{x}_2 + at^{n-k} + bt^{2(n-k)} + ct^{3(n-k)}\right)\\
&= at^n\left(\bar{x}_2 t + \bar{x}_2\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}}\right) - \frac{1}{t^k}\right)\\
&\quad+ t^{2n}\left(\bar{x}_2 b\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{2(k-1)}}\right) + a^2 t\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}}\right) + \bar{x}_2 bt^2 - bt^{-2k}\right) + \mathbb{O}(t^{3n}).
\end{aligned}
\tag{3.12}
$$

Let

$$
D_2(t) = \bar{x}_2 t + \bar{x}_2\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}}\right) - t^{-k} = \frac{p_2(t)}{t^k}. \tag{3.13}
$$

Choose $t \in (0,1)$ such that $D_2(t) = 0$, and $a,b \in \mathbb{R}, a \neq 0$, such that the coefficients in (3.12) are equal to zero. Since $D_2(t) = 0$ is equivalent to $p_2(t) = 0$, we have that $t = t_2$, and consequently

$$
b = -\frac{a^2 t_2\left(\alpha_0 + \cdots + \alpha_{k-1}t_2^{-k+1}\right)}{\bar{x}_2 t_2^2 + \bar{x}_2\left(\alpha_0 + \cdots + (\alpha_{k-1})/t_2^{2(k-1)}\right) - t_2^{-2k}} = -\frac{a^2 t_2\left(\alpha_0 + \cdots + \alpha_{k-1}t_2^{-k+1}\right)}{D_2(t_2^2)}. \tag{3.14}
$$

If $\widehat{\varphi}_n = \bar{x}_2 + at_2^n + qt_2^{2n}$, we obtain

$$
F(\widehat{\varphi}_{n-k},\dots,\widehat{\varphi}_n,\widehat{\varphi}_{n+1}) \sim \left(qD_2(t_2^2) + a^2 t_2\left(\alpha_0 + \cdots + \alpha_{k-1}t_2^{-k+1}\right)\right)t_2^{2n}. \tag{3.15}
$$

Let

$$
H_{t_2}(q) = qD_2(t_2^2) + a^2 t_2\left(\alpha_0 + \cdots + \alpha_{k-1}t_2^{-k+1}\right). \tag{3.16}
$$

Since

$$
p_2'(t) = \bar{x}_2\left((k+1)t^k + k\alpha_0 t^{k-1} + \cdots + \alpha_{k-1}\right) > 0, \tag{3.17}
$$

when $t \in (0,1)$, and since $p_2(t_2) = 0$, and $t_2^2 < t_2$, we have that $p_2(t_2^2) < 0$, which implies $D_2(t_2^2) < 0$. Thus, we obtain that there are $q_3 < b$ and $q_4 > b$ such that $H_{t_2}(q_3) > 0$ and $H_{t_2}(q_4) < 0$.

With the notations

$$
y_n = \bar{x}_2 + at_2^n + q_3 t_2^{2n}, \qquad z_n = \bar{x}_2 + at_2^n + q_4 t_2^{2n}, \tag{3.18}
$$

we get

$$
\begin{aligned}
F(y_{n-k},\dots,y_n,y_{n+1}) &\sim H_{t_2}(q_3)t_2^{2n} > 0,\\
F(z_{n-k},\dots,z_n,z_{n+1}) &\sim H_{t_2}(q_4)t_2^{2n} < 0.
\end{aligned}
\tag{3.19}
$$

These relations show that the inequalities in (1.12) are satisfied for sufficiently large n, where $f = F + x_{n-k}$ and F is given by (3.11). Applying Theorem 2.1 it follows that there is a solution of (1.3) with the asymptotics $x_n = \widehat{\varphi}_n + o(t_2^{2n})$. This solution obviously converges monotonically to the positive equilibrium $\bar{x}_2 = (\sqrt{5}+1)/2$, for $n \geq n_1$. A suitable shift of x_n is decreasing for all $n \geq -k$.

(c) Equation (1.4) can be written in the following equivalent form:

$$F(x_{n-k},\dots,x_n,x_{n+1}) = x_{n+1}(1+\alpha_0 x_n + \cdots + \alpha_{k-1}x_{n-k+1}) - (\alpha + x_{n-k}) = 0. \tag{3.20}$$

We have

$$\begin{aligned} F &= (\bar{x}_3 + at^{n+1} + bt^{2(n+1)} + ct^{3(n+1)}) \\ &\quad \times (1 + \bar{x}_3 + a\alpha_0 t^n + \cdots + a\alpha_{k-1}t^{n-k+1} + b\alpha_0 t^{2n} + \cdots + b\alpha_{k-1}t^{2(n-k+1)} + \mathcal{O}(t^{3n})) \\ &\quad - (\alpha + \bar{x}_3 + at^{n-k} + bt^{2(n-k)} + ct^{3(n-k)}) \\ &= at^n\left((1+\bar{x}_3)t + \bar{x}_3\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}}\right) - t^{-k}\right) \\ &\quad + t^{2n}\left(\bar{x}_3 b\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{2(k-1)}}\right) + a^2 t\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}}\right) + (1+\bar{x}_3)bt^2 - bt^{-2k}\right) \\ &\quad + \mathcal{O}(t^{3n}). \end{aligned} \tag{3.21}$$

Let

$$D_3(t) = (1+\bar{x}_3)t + \bar{x}_3\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}}\right) - t^{-k} = \frac{p_3(t)}{t^k}. \tag{3.22}$$

Choose $t \in (0,1)$ such that $D_3(t) = 0$, and $a, b \in \mathbb{R}$, $a \neq 0$, such that the coefficients in (3.21) are equal to zero.

Since

$$p_3'(t) = (1+\sqrt{\alpha})(k+1)t^k + \sqrt{\alpha}(k\alpha_0 t^{k-1} + \cdots + \alpha_{k-1}) > 0, \tag{3.23}$$

when $t \in (0,1]$, and $D_3(t) = 0$ is equivalent to $p_3(t) = 0$, we have that $t = t_3$. From this and (3.21) it follows that

$$b = -\frac{a^2 t_3(\alpha_0 + \cdots + \alpha_{k-1}t_3^{-k+1})}{(1+\sqrt{\alpha})t_3^2 + \sqrt{\alpha}(\alpha_0 + \cdots + (\alpha_{k-1})/t_3^{2(k-1)}) - t_3^{-2k}} = -\frac{a^2 t_3(\alpha_0 + \cdots + \alpha_{k-1}t_3^{-k+1})}{D_3(t_3^2)}. \tag{3.24}$$

If $\widehat{\varphi}_n = \sqrt{\alpha} + at_3^n + qt_3^{2n}$, we obtain

$$F(\widehat{\varphi}_{n-k},\dots,\widehat{\varphi}_n,\widehat{\varphi}_{n+1}) \sim (qD_3(t_3^2) + a^2 t_3(\alpha_0 + \cdots + \alpha_{k-1}t_3^{-k+1}))t_3^{2n}. \tag{3.25}$$

Let

$$H_{t_3}(q) = qD_3(t_3^2) + a^2 t_3(\alpha_0 + \cdots + \alpha_{k-1}t_3^{-k+1}). \tag{3.26}$$

Since $D_3(t_3) = 0$, and $t_3^2 < t_3$, we have that $D_3(t_3^2) < 0$. Thus, we obtain that there are $q_5 < b$ and $q_6 > b$ such that $H_{t_3}(q_5) > 0$ and $H_{t_3}(q_6) < 0$.

With the notations

$$y_n = \sqrt{\alpha} + at_3^n + q_5 t_3^{2n}, \qquad z_n = \sqrt{\alpha} + at_3^n + q_6 t_3^{2n}, \tag{3.27}$$

we get

$$F(y_{n-k},\dots,y_n,y_{n+1}) \sim H_{t_3}(q_5)t_3^{2n} > 0, \qquad F(z_{n-k},\dots,z_n,z_{n+1}) \sim H_{t_3}(q_6)t_3^{2n} < 0. \tag{3.28}$$

These relations show that the inequalities in (1.12) are satisfied for sufficiently large n, where $f = F + x_{n-k}$ and F is given by (3.20). Hence, there is a solution of (1.4) with the asymptotics $x_n = \hat{\varphi}_n + o(t_3^{2n})$. The result follows similarly to the above mentioned cases. $\square$

From Theorem 3.1(a) with $\alpha_0 = 1$ and $\alpha_i = 0$, $i \neq 0$, we get the following corollary.

Corollary 3.2. *There is a nonoscillatory solution of (1.2).*

Remark 3.3. Since $a \in \mathbb{R} \setminus \{0\}$ is an arbitrary parameter, by Theorem 3.1 we find a set of nonoscillatory solutions of (1.1), (1.3), and (1.4) converging to the corresponding positive equilibria.

Remark 3.4. Note that using (1.13) better asymptotics for these solutions can be obtained, that is, $x_n = \varphi_n + o(t_i^{3n})$, $i \in \{1,2,3\}$, where b is given by (3.4), (3.14), or (3.24), and c can be found equating to zero the coefficient nearby t^{3n}.

Remark 3.5. From the proof of Theorem 3.1, we see that we can assume that the parameter p in (1.1) can be replaced by a nondecreasing sequence with the following asymptotics: $p_n = p + o(t_1^{2n})$.

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