Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2007, Article ID 84913, 23 pages
doi:10.1155/2007/84913

Research Article

On the Properties of Reachability, Observability, Controllability, and Constructibility of Discrete-Time Positive Time-Invariant Linear Systems with Aperiodic Choice of the Sampling Instants

M. De la Sen

Received 25 April 2007; Accepted 2 July 2007

This paper investigates the properties of reachability, observability, controllability, and constructibility of positive discrete-time linear time-invariant dynamic systems when the sampling instants are chosen aperiodically. Reachability and observability hold if and only if a relevant matrix defining each of those properties is monomial for the set of chosen sampling instants provided that the continuous-time system is positive. Controllability and constructibility hold globally only asymptotically under close conditions to the above ones guaranteeing reachability/observability provided that the matrix of dynamics of the continuous-time system, required to be a Metzler matrix for the system's positivity, is furthermore a stability matrix while they hold in finite time only for regions excluding the zero vector of the first orthant of the state space or output space, respectively. Some related properties can be deduced for continuous-time systems and for piecewise constant discrete-time ones from the above general framework.

1. Introduction

Nonperiodic sampling has been proved to be a powerful tool to improve error signal adaptation in model-following or signal-tracking problems by adapting the sampling rate to the signal variation through time [1, 2]. It is also useful to reduce relative numerical errors in computations of results from data by appropriately distributing the distribution of such data so that the condition number of the coefficient matrix in linear algebraic problems be minimized as, for instance, when studying controllability, observability and identifiability problems in dynamic systems [1–4]. Other useful applications of nonperiodic sampling techniques arise in multirate sampling problems of usefulness in filtering, flight control of aircrafts and stabilization of stable discrete plant zeros, as well as

in hybrid control, random sampling problems, in compensation to parametrical uncertainties, [1–9] or in algebraic or numerical computation [1, 2, 10–14] as well as in sensor implementation, control of digital systems and estimation of populations, identification, error estimation. A wide range of useful applications related to multirate, nonperiodic, adaptive and random sampling have been investigated in [8, 9, 12–26]. On the other hand, hybrid systems are very common in nature and in technological applications and are composed, in general, of coupled, continuous-time, and discrete-time or digital systems and/or subject to driving events [4, 7, 27–32]. Dynamic hybrid systems have been investigated under nonperiodic or multirate sampling [4, 7], in order to obtain the above-mentioned advantages of such techniques. On the other hand, positive dynamic systems are present in nature, for instance, in biological processes and ecology, and are useful for modelling some queuing models, Markov processes, or electronic circuits, as well as models based on differential, difference, mixed differential—difference (roughly speaking hybrid models) and integral equations, in general (see, for instance, [32–40]). The main characteristics of these systems are that either all the state and output components are nonnegative for all time for nonnegative controls and nonnegative initial conditions. Such a kind of system is said to be internally positive commonly referred to simply as a positive system [32, 33, 38, 39]. A weaker property is the external positivity stated in terms that the system output components are nonnegative for all time under zero initial conditions and nonnegative controls [38, 39]. (Internal) positivity depends on the state-space realization while external positivity is only transfer matrix dependent but not on the state-space description. Positive continuous-time linear time-invariant systems are characterized by the output, control, and direct input-output interconnection matrix being nonnegative (i.e., with all their entries being nonnegative) while the matrix of dynamics is a Metzler matrix (i.e., with all its off-diagonal entries being nonnegative). Linear discrete-time positive systems are characterized by the above four matrices being nonnegative (see, for instance, [32, 33, 38]). The reachability and controllability of positive discrete systems under nonnegative controls have been studied in [39]. In particular, the reachability property of positive systems is quite stringent since it only holds if and only if the controllability Grammian, or equivalently, a square submatrix of the controllability one, is monomial; that is, with no negative entry and only one nonzero and positive entry per row and column and being nonsingular. Controllability is still more stringent since the matrix of dynamics is required to be convergent for asymptotic controllability and, furthermore, nilpotent for controllability in finite time. Those issues are rigorously discussed in detail in [39] for discrete-time positive systems in a pedagogical style easy to follow for readers.

This paper is devoted to investigate the above properties when the sampling period is nonconstant, in general, and to derive sufficient conditions to ensure the maintenance of such properties if the sequence of sampling instants becomes modified. Section 2 contains some definitions about reachability, controllability, observability, and constructibility of linear dynamic systems and preliminaries. It also contains some preliminary technical results. Section 3 is devoted to obtain some formal results for reachability and controllability of positive linear time-invariant systems under, in general, a nonperiodic distribution of the samples. An important feature is that the properties in the discrete-time

framework require their fulfilment in the continuous-time one provided that the sampling instants have appropriate distribution which does not involve very restrictive conditions. Section 5 is concerned with obtaining some parallel results to those of Section 3 related to observability and constructibility. Some examples are discussed in Section 6 and, finally, conclusions end the paper.

Notation 1. $\overline{n} := \{1,2,\ldots,n\} \subset \mathbb{N}$ is a finite subset of the set of the natural numbers $\mathbb{N}$; I_n denotes the n-the order identity matrix.

$\mathbb{R}_+^n := \{z = (z_1,z_2,\ldots,z_n)^T \in \mathbb{R}^n : z_i \geq 0,\ i \in \overline{n}\}$ is the first closed orthant of $\mathbb{R}^n$, where the superscript T stands for transposition

$A = (a_{ij})$ is a real n-Metzler matrix, denoted by $A \in M_E^{n\times n}$, if and only if $a_{ij} \geq 0$, for all $i, j(\neq i) \in \overline{n}$.

The real matrix $P = (p_{ij}) \in \mathbb{R}_+^{n\times m} \Leftrightarrow p_{ij} \geq 0$, for all $i, j \in \overline{n}$. Such a matrix is said to be nonnegative denoted by $P \geq 0$.

The matrix $P \in \mathbb{R}_+^{n\times m}$ is positive, denoted by $P > 0$, if it has at least a positive entry. The matrix $P \in \mathbb{R}_+^{n\times m}$ is strictly positive, denoted by $P \gg 0$, if all its entries are positive.

The square nonsingular matrix $P \in \mathbb{R}_+^{n\times n}$ is monomial if $p_{ik} > 0 \Leftrightarrow p_{ij} = 0$, for all $j(\neq i) \in \overline{n}$, for all $i \in \overline{n}$; that is, each row and column has only a positive entry with all the remaining ones being zero so that P is, in addition, nonsingular. $P \in \mathbb{R}_+^{n\times n}$ monomial $\Leftrightarrow P^{-1} > 0$. A monomial matrix with all its nonzero entries being unity is called a permutation matrix.

2. The system and a set of basic definitions

Consider the linear and time-invariant continuous-time positive system:

$$\dot{x}(t) = Ax(t) + Bu(t); \qquad y(t) = Cx(t) \tag{2.1}$$

subject to $x(0) = x_0$ where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector and $y(t) \in R^p$ is the output vector for any $t \in \mathbb{R}_+$. All the matrices in (2.1) are of compatible orders with the dimensions of the above vectors. The state trajectory solution vector function $x : \mathbb{R}_+ \to \mathbb{R}^n$ and the output trajectory vector function $y : \mathbb{R}_+ \to \mathbb{R}^p$ are unique for each initial condition $x(0) = x_0 \in \mathbb{R}^n$ and each piecewise continuous control vector function $u : \mathbb{R}_+ \to \mathbb{R}^m$ from Picard theorem. The state and output solution trajectories are defined by the subsequent closed formulas:

$$\begin{aligned} x(t) &= \Psi(t)x_0 + \int_0^t \Psi(t-\tau)Bu(\tau)d\tau, \\ y(t) &= C\left(\Psi(t)x_0 + \int_0^t \Psi(t-\tau)Bu(\tau)d\tau\right) \end{aligned} \tag{2.2}$$

for all $t \in \mathbb{R}_+$, where the state-transition matrix $\Psi(t) = e^{At}$ is a C_0-semigroup generated by the infinitesimal generator A which is a fundamental matrix of the differential system of the first (2.1). Assume any totally ordered set of $(q+1)$ sampling instants defined by $\mathrm{SI} := \{t_i : i \in \overline{q} \cup \{0\}\}$, which is then also a finite sequence of real elements, with sampling periods $T_i = t_{i+1} - t_i > 0$; for all $i \in \overline{q-1} \cup \{0\}$ and assume that the control input is constant in-between consecutive sampling instants, that is, generated by a zero-order hold so

that $u(t_i+\tau)=u(t_i)$; $\forall\tau\in[0,T_i)$. Then, one gets from (2.2) that

$$\begin{aligned} x(t_{i+1}) &= \Psi(T_i)x(t_i)+\Gamma(T_i)u(t_i)=\Psi(t_{i+1})x_0+\int_0^{t_{i+1}}\Psi(t_{i+1}-\tau)Bu(\tau)d\tau \\ &= \prod_{j=0}^{i}[\Psi(T_j)]x_0+\sum_{j=0}^{i}\prod_{\ell=j+1}^{i}[\Psi(T_\ell)]Bu(t_j); \quad \forall i\in\overline{q-1}\cup\{0\} \\ y(t_{i+1}) &= Cx(t_i); \quad \forall i\in\overline{q-1}\cup\{0\}; \end{aligned} \tag{2.3}$$

where the control transition matrix is defined by

$$\Gamma(T_i):=\left(\int_0^{T_i}\Psi(T_i-\tau)d\tau\right)B. \tag{2.4}$$

The set of definitions to specify precisely the properties of the discrete system (2.3) to be dealt with are now given as follows.

Definition 2.1 (*Reachability*). The positive system (2.3) is reachable for a given finite set SI of q sampling instants if there is a sequence of controls $u(t_i)\in\mathbb{R}_+^m$; for all $t_i\in\mathrm{SI}(q)$, for all $i\in\overline{q-1}\cup\{0\}$, such that $x(t_q)=x^*$ for any given bounded $x^*\in\mathbb{R}_+^n$ provided that $x_0=0$.

Definition 2.2 (*Global controllability*). The positive system (2.3) is globally controllable for a given finite set $\mathrm{SI}(q)$ of q sampling instants if there is a sequence of controls $u(t_i)\in\mathbb{R}_+^m$; for all $t_i\in\mathrm{SI}$, for all $i\in\overline{q-1}\cup\{0\}$, such that for $x(t_q)=x^*$ each given bounded $x^*\in\mathbb{R}_+^n$ and $x_0\in\mathbb{R}_+^n$.

Definition 2.3 (*Global controllability to a region or to a point*). The positive system (2.3) is controllable to a region $R_S\subset\mathbb{R}_+^n$ for a given finite set $\mathrm{SI}(q)$ of q sampling instants if there is a sequence of controls $u(t_i)\in\mathbb{R}_+^m$; for all $t_i\in\mathrm{SI}$, for all $i\in\overline{q-1}\cup\{0\}$, such that $x(t_q)=x^*$ for each given $x^*\in\mathbb{R}_S$ and any $x_0\in\mathbb{R}_+^n$. If $0\in\mathbb{R}_S$ and the above holds then the system (2.3) is said to be controllable to the origin.

Definition 2.4 (*Global asymptotic controllability to the origin*). The positive system (2.3) is globally asymptotically controllable to the origin for some given infinite set $\mathrm{SI}(q)$ of q sampling instants if there is an infinite sequence of bounded controls $u(t_i)\in\mathbb{R}_+^m$; for all $t_i\in\mathrm{SI}(q)$, for all $i\in\mathbb{N}$, such that $\lim_{i\to\infty}x(t_i)=0$ for each given $x_0\in\mathbb{R}_+^n$.

This last definition might be equivalently referred to as asymptotic stabilizability to the origin and, furthermore, as global asymptotic stabilizability to the origin provided that the only equilibrium is zero.

Definition 2.5 (*Observability*). The positive system (2.3) is observable for a given finite set $\mathrm{SI}(q)$ of q sampling instants if any initial condition $x(0)=x_0\in\mathbb{R}_+^n$ can be calculated uniquely from a finite set of future measured outputs $y(t_i)\in\mathbb{R}_+^p$; for all $t_i\in\mathrm{SI}$, for all $i\in\overline{q}$.

Definition 2.6 (*Global constructibility*). The positive system (2.3) is globally constructible for a given finite set $\mathrm{SI}(q)$ of q sampling instants if any bounded state $x(t_q)=x^*\in\mathbb{R}_+^n$

can be calculated uniquely from a finite set of past measured outputs $y(t_j) \in \mathbb{R}_+^p$; for all $t_j \in \mathrm{SI}$, for all $j \in \overline{q-1} \cup \{0\}$.

Note that since the system is positive, so that $e^{At_i} \in \mathbb{R}_+^{n\times n}$ and $C \in \mathbb{R}_+^{p\times n}$, the unforced response

$$Ce^{At_i}x_0 = y(t_i) - \sum_{j=0}^{i} \prod_{\ell=j+1}^{i} C[\Psi(T_\ell)]u(t_j) \geq 0; \quad \forall t_i \in \mathrm{SI},\ \forall i \in \overline{q-1} \cup \{0\} \tag{2.5}$$

for any set SI of sampling instants. Then, the properties of observability and global constructibility are independent of the controls and can be then tested for the unforced system with no loss in generality. Similar reachability/controllability definitions to Definitions 2.1–2.4 may be given by replacing the state space or a particular state region or point by the output space, a particular output region or a particular output value leading to output reachability/output controllability characterizations.

Definitions 2.7 to 2.10. They are directly referred to the output, replacing the state, concerned with the concepts of output reachability, global output controllability, global output controllability to a region or point and global asymptotic output controllability, respectively, as direct extensions of Definitions 2.1–2.4.

3. Main results on reachability and controllability

In the following, the subsequent matrices are used for then establishing the formulation of the main results of the paper: $C(A,B) := [B, AB, \ldots, A^{n-1}B]$ is the controllability matrix of the continuous-time system (2.1) also often referred to as the controllability matrix of the pair (A,B). Some authors refer to this matrix as the reachability matrix of the system (2.1) or the pair (A,B). $O(C,A) := [C^T, A^TC^T, \ldots, A^{n-1^T}C^T]^T$ is the observability matrix of the continuous-time system (2.1) also often refereed to as the observability matrix of the pair (C,A). $\hat{C}(\Psi,\Gamma,\mathrm{SI}(n)) := [\Gamma(T_{n-1}), \Psi(T_{n-1})\Gamma(T_{n-2}), \ldots, \Psi(\sum_{j=1}^{n-1} T_j)\Gamma(T_0)]$ is the controllability matrix of the discrete-time system (2.3) obtained from the continuous one (2.1) under a zero-order hold for the, in general, sequence of aperiodic sampling instants $\mathrm{SI}(n) := \{t_0, t_1, \ldots, t_n\}$ and corresponding sampling periods $T_i = t_{i+1} - t_i$; for all $i \in \overline{n-1} \cup \{0\}$, and $\hat{O}(C,\Psi,\mathrm{SI}(n)) := [C^T, \Psi^T(T_{n-1})C^T, \ldots, \Psi^T(\sum_{j=1}^{n-1} T_j)C^T]^T$ is the observability matrix of the discrete-time system (2.3) obtained from the continuous one (2.1) under a zero-order hold for the, in general, sequence of aperiodic sampling instants $\mathrm{SI}(n) := \{t_0, t_1, \ldots, t_n\}$ and corresponding sampling periods $T_i = t_{i+1} - t_i$; for all $i \in \overline{n-1} \cup \{0\}$. When the sampling period is constant then the sampling sequence is not included in the notation of the controllability $\hat{C}(\Psi(T),\Gamma(T))$ matrix and the observability $\hat{O}(C,\Psi(T))$ matrix but instead in the discrete state-transition and control matrices since no confusion is expected.

Remarks

Remark 3.1. $\Psi(\sum_{j=1}^{q} T_j) = \prod_{j=1}^{q}[\Psi(T_j)]$ for any sequence of q sampling periods from the properties of the state transition matrix.

Remark 3.2. Define $M(Q,W,q) := [Q, WQ, \ldots, W^{q-1}Q]$ for any $q \in \mathbb{N}$ and any pair of matrices (Q,W) of compatible orders such that the product QW exists then $M(A,B,n) = C(A,B)$. Note that

$$\operatorname{rank}[M(A,B,q)] = \operatorname{rank}[M(A,B,\mu)] = \operatorname{rank}[C(A,B)] \tag{3.1}$$

by construction for any $\mathbb{N} \ni q \geq \mu$ with $\mu \in \overline{n}$ being the degree of the minimal polynomial of A. The meaning is that the rank of the expanded matrix $M(A,B,q)$ in powers of A equalizes for $q \geq \mu$ to the achieved at most for $q = \mu$ by construction. Since the degree of the minimal polynomial of the fundamental matrix of A equalizes that of A then if one defines $\widehat{M}(\Psi,\Gamma,\mathrm{SI}(q),q) := [\Gamma(T_{q-1}), \Psi(T_{q-1})\Gamma(T_{q-2}), \ldots, \Psi(\sum_{j=1}^{q-1} T_j)\Gamma(T_0)]$ for a sequence of q sampling instants $\mathrm{SI}(q) := \{t_1, t_2, \ldots, t_q\}$ and its associate sampling periods $T_i = t_{i+1} - t_i$; for all $i \in \overline{n-1} \cup \{0\}$, then

$$\operatorname{rank}[\widehat{M}(\Psi,\Gamma,\mathrm{SI}(q))] = \operatorname{rank}[\widehat{M}(\Psi,\Gamma,\mathrm{SI}(\mu))] = \operatorname{rank}[\widehat{C}(\Psi,\Gamma,\mathrm{SI}(n))] \tag{3.2}$$

again by construction for any $\mathbf{N}q \geq \mu$ with $\mu \in \overline{n}$ being the degree of the minimal polynomial of A provided that the sampling instants satisfy $\mathrm{SI}(q) \supset \mathrm{SI}(n) \supset \mathrm{SI}(\mu)$.

In the same way, $M(C,A,n) = O(C^T, A^T)$ and for any $\mathbf{N}q \geq \mu$

$$\begin{gathered}\operatorname{rank}[M(C^T,A^T,q)] = \operatorname{rank}[M(C^T,A^T,\mu)] = \operatorname{rank}[O(C,A)],\\ \operatorname{rank}[\widehat{M}(C^T,\Psi^T,\mathrm{SI}(q))] = \operatorname{rank}[\widehat{M}(C^T,\Psi^T,\mathrm{SI}(\mu))] = \operatorname{rank}[\widehat{O}(C,\Psi,\mathrm{SI}(n))].\end{gathered} \tag{3.3}$$

Remark 3.3. $q_c \in \mathbf{N}$ exists such that

$$\operatorname{rank}[M(A,B,q_c)] = \operatorname{rank}[M(A,B,\mu)] = \operatorname{rank}[C(A,B)] \tag{3.4}$$

and $q_c := \mathrm{Max}_{1\leq i\leq m}(q_i)$ is furthermore the controllability index of (2.1), provided that such a system is controllable, for some set of nonnegative integer numbers $q_i \in \overline{q}$; for all $i \in \overline{m}$ such that $\mu = \sum_{i=1}^{m} q_i$, and

$$\begin{aligned}\operatorname{rank}[M(A,B,q)] &= \operatorname{rank}[M(A,B,q_c)]\\ &= \operatorname{rank}[M(A,B,\mu)] = \operatorname{rank}[C(A,B)] = n\end{aligned} \tag{3.5}$$

for all $\mathbb{N}q \geq q_c$ if (2.1) is controllable. In the multi-input case (i.e., $m \geq 2$), it can occur that $q_c < \mu$. In the single-input case (i.e., $m = 1$), $q_c = \mu$. In the same way, we can define the observability index $q_0 := \mathrm{Max}_{1\leq i\leq p}(q_{0i})$ for an observable system (2.1) which can be lesser than the degree of the minimal polynomial of the matrix A in the single-output case. *The controllability, the observability indexes, respectively, are also the number of samples, distributed appropriately, required to keep the respective property under discretization from the continuous-time case*. Note that the controllability index could also be named "reachability index" in the same way as the controllability matrix could be renamed "reachability matrix." Although controllability is sometimes equivalent to reachability (as, for instance, in the case of purely discrete-time systems) they are not coincident, in general. In particular, they are not coincident in the particular case of positive reachable or controllable

systems discussed in this paper. However, the controllability and reachability indexes are anyway identical since they refer to the maximum of the maximum number of appearances of all the columns of the control matrix in the reachability matrix or, equivalently, to the maximum number of samples of any nonzero input component necessary to achieve full rank of the controllability matrix. Similar considerations apply to the coincidence of observability and constructibility indexes. Due to this fact, we refer in the following to the indexes, as usual in the literature, as the controllability or, respectively, the observability index irrespective of the fact that reachability/controllability or, respectively, observability/constructibility property be characterized in the same way as the corresponding matrices will be referred to as the controllability and observability matrices. An appropriate distribution of samples that maintain the respective property always exists for arbitrary linear time-invariant systems as discussed in [1–3]. The integers q_i (resp., q_{0i}) are the sets of samples requested to appropriately generate each control input component (resp., to observe each output component) in order to guarantee the respective property from the continuous-time case for some appropriate distribution of the sampling instants (that in fact might be generically selected [1]).

A set of known results from the literature is summarized as follows for the dynamic continuous-time system (2.1) and its discrete-time counterpart under a fixed sampling period T and a zero-order hold provided that (2.1) is not positive. It is taken into account that the ranks of the controllability and observability matrices equalize by construction those of their submatrices involving powers of the matrix of A only up till the degree μ of its minimal polynomial.

Theorem 3.4. *The following properties hold.*

(i) *The system (2.1) is reachable if and only if* $\operatorname{rank} C(A,B) = \operatorname{rank}[B, AB, \dots, A^{\mu-1}B] = n$. *The discrete-time system (2.3) under constant sampling period sequence* $T_i = T$; $i \in \mathbb{N}$ *is reachable and, equivalently, globally controllable if and only if*

$$\operatorname{rank} \hat{C}(\Psi(T), \Gamma(T)) = \operatorname{rank}\left[\Gamma(T), \Psi(T)\Gamma(T), \dots, \Psi(T)^{\mu-1}\Gamma(T)\right] = n. \tag{3.6}$$

(ii) *The system (2.1) is observable if and only if* $\operatorname{rank} O(C,A) = \operatorname{rank}[C^T, A^T C^T, \dots, A^{\mu-1^T} C^T]^T = n$. *The discrete-time system (2.3) under constant sampling period sequence* $T_i = T$; $i \in \mathbb{N}$ *is observable and, equivalently, globally constructible if and only if*

$$\operatorname{rank} \hat{O}(C, \Psi(T)) = \operatorname{rank}\left[C^T, \Psi(T)^T C^T, \dots, \Psi(T)^{\mu-1^T} C^T\right]^T = n. \tag{3.7}$$

Note that since $\Psi(T)$ is a fundamental matrix and then nonsingular, it follows that if the system (2.3) is reachable under constant sampling then it is controllable-to-the origin and vice-versa since

$$\begin{aligned}
\operatorname{rank} \hat{C}(\Psi(T), \Gamma(T)) &= \operatorname{rank}\left[\Gamma(T), \Psi(T)\Gamma(T), \dots, \Psi(T)^{\mu-1}\Gamma(T)\right] \\
&= \operatorname{rank}\left[\Psi(T)^{-n}\Gamma(T), \Psi(T)^{1-n}\Gamma(T), \dots, \Psi(T)^{\mu-1-n}\Gamma(T)\right] \\
&= \operatorname{rank}\left[\Psi(T)^{-n}\Gamma(T), \Psi(T)^{1-n}\Gamma(T), \dots, \Psi(T)^{-1}\Gamma(T)\right] = n.
\end{aligned} \tag{3.8}$$

In particular, note that any bounded $x(0) = x_0 \in \mathbb{R}^n$ is driven to $x(nT) = 0$ by an existing unique control sequence $\{u_0, u_1, \ldots, u_{n-1}\}$ with $u_i = u(T_i)$; for all $i \in \overline{n-1} \cup \{0\}$ that satisfies the linear algebraic system:

$$-x_0 = \left[\Psi(T)^{-n}\Gamma(T), \Psi(T)^{1-n}\Gamma(T), \ldots, \Psi(T)^{-1}\Gamma(T)\right]\left[u_0^T, u_1^T, \ldots, u_{n-1}^T\right]^T. \tag{3.9}$$

The system is also shown to be equivalently globally controllable. A similar reasoning applies to the equivalence between observability and global constructibility. However, note the following observations.

Observation 1. The equivalences reachability/controllability do not hold for arbitrary digital systems which can possess components which do not involve discretization from the continuous-time system since then the state transition matrix is not a fundamental matrix from a differential system. Also, controllability-to-the origin system in a continuous-time, even in finite time, does not require the controllability matrix to be full rank although this property guarantees both reachability and global controllability. Very close considerations apply to potential equivalences or not of observability/global constructibility [39].

Observation 2. If the system (2.1) is positive then its discrete-time version (2.3) is also positive for discretization under constant sampling period and zero-order hold [38]. It turns out by inspection that (2.3) is also positive for discretization under any arbitrary sampling sequence. The properties of positivity, reachability, observability, and so forth, are established through this paper in an integrated way. Then, the full rank condition which guarantees each property in an arbitrary system are not useful for positive systems without incorporating further constraints since the state and output have to evolve in the first orthant of the state space under nonnegative control sequences for any nonnegative initial condition.

The subsequent technical result holds.

Lemma 3.5. *The following properties hold.*

(i) *The unique solution of the state of the system (2.1) for $x(0) = x_0$ and any piecewise continuous control $u : \mathbb{R}_+ \to \mathbb{R}^m$ is given by*

$$x(t) = \sum_{k=0}^{\mu-1} \alpha_k(t) A^k x_o + M(A, B, \mu)\left[\beta_0(t,u), \beta_1(t,u), \ldots, \beta_{\mu-1}(t,u)\right]^T, \tag{3.10}$$

where $\{\beta_k : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R} : k \in \overline{\mu-1} \cup \{0\}\}$ is a set of m-real vector functions defined uniquely as $\beta_k(t,u) := \int_0^t \alpha_k(t-\tau)u(\tau)d\tau$ being linearly independent functions on $\mathbb{R}_+$ and $\{\alpha_k : \mathbb{R}_+ \to \mathbb{R} : k \in \overline{\mu-1} \cup \{0\}\}$ is a set of unique real linearly independent functions on $\mathbb{R}_+$ which satisfy the linear algebraic system [1, 2]

$$\begin{aligned} &\frac{d^i}{d\lambda^i}\left[1, \lambda, \ldots, \lambda^{\mu_j - 1}\right]\left[\alpha_0(t), \alpha_1(t), \ldots, \alpha_{\mu-1}(t)\right]^T \\ &\quad = \left[e^{\lambda_1 t}, te^{\lambda_1 t}, t^{\mu_1 - 1}e^{\lambda_1 t}, \ldots, e^{\lambda_\sigma t}, te^{\lambda_\sigma t}, t^{\mu_\sigma - 1}e^{\lambda_\sigma t}\right]^T; \quad i \in \overline{\mu_j - 1} \cup \{0\},\ j \in \overline{\sigma}, \end{aligned} \tag{3.11}$$

where $\{\lambda_1,\lambda_2,\ldots,\lambda_\sigma\}$ *is the spectrum of* A *and* μ_i *is the multiplicity of* λ_i*; for all* $i \in \overline{\sigma}$ *in the minimal polynomial of* A.

(ii) *Let* $\mathrm{SI}(q) := \{t_0 = 0, t_1, t_2, \ldots, t_q\}$ *be a set of sampling instants. Then*

$$x(t_i) = \sum_{k=0}^{\mu-1} \alpha_k(t_i) A^k x_o + M(A,B,\mu)\left[\sum_{j=0}^{i-1} \gamma_{0j}(t_j,T_j,t_i)u(t_j),\ldots,\sum_{j=0}^{i-1} \gamma_{\mu-1,j}(t_j,T_j,t_i)u(t_j)\right]^T \tag{3.12}$$

for all $t_i \in \mathrm{SI}(q)$*, provided that the discretization of (2.1) is performed through a zero-order hold, where:* $\gamma_{kj}(t_j,T_j,t_i) := \int_0^{T_j} \alpha_k(t_i - t_j - \tau)u(\tau)d\tau$; $T_j := t_{j+1} - t_j$ *(sampling periods), for all* $t_i \in \mathrm{SI}(q)$*, for all* $j \in \overline{i-1} \cup \{0\}$*, for all* $k \in \overline{\mu-1} \cup \{0\}$.

(iii) *Properties (i)-(ii) might be reformulated by calculating the state-trajectory solution from the formula*

$$x(t) = \sum_{k=0}^{\rho-1} \alpha_{\rho k}(t) A^k x_o + M(A,B,\rho)\left[\beta_{\rho 0}(t,u),\beta_{\rho 1}(t,u),\ldots,\beta_{\rho,\rho-1}(t,u)\right]^T, \tag{3.13}$$

where $\rho(\geq \mu) \in \mathbb{N}$ *is arbitrary, the* $\alpha_{\rho k}(\cdot)$*-real functions are linearly independent on* $\mathbb{R}_+$*, dependent on* ρ *and unique for each given* ρ*, and calculated from a similar linear algebraic system to (3.11) with the replacements* $\alpha_k(\cdot) \to \alpha_{\rho k}(\cdot)$, $\beta_k(\cdot) \to \beta_{\rho k}(\cdot)$, $\mu \to \rho \geq \mu$, $\mu_j \to \rho_j \geq \mu_j$ $(j \in \overline{\sigma},\ k \in \overline{\rho-1} \cup \{0\})$ *being in general nonunique but satisfying* $\rho = \sum_{j=1}^{\sigma} \rho_j$.

Proof. (i) With the given definitions of the functions $\alpha_{(\cdot)}(t)$ and vector functions $\beta_{(\cdot)}(t)$, note that the unforced and forced state-trajectory solutions of (2.1) are $\Psi(t)x_0 = e^{At}x_0 = \sum_{k=0}^{\mu-1} \alpha_k(t)A^k x_0$ and $\int_0^t \Psi(t-\tau)Bu(\tau)d\tau = \sum_{k=0}^{\mu-1} A^k B\beta_k(t)$, respectively. Thus, Property (i) follows trivially by composing both solutions by using the superposition principle while taking into account the definition of the matrix $M(A,B,\mu)$. Uniqueness of the solution for each initial condition and control input is direct from the well-known Picard-Lindëloff theorem for ordinary differential equations.

(ii) The state-trajectory solution of (2.1) by Property (i) and the fact that the input is piecewise constant generated by a zero-order hold becomes at a sampling time $t = t_i \in \mathrm{SI}(q)$:

$$\begin{aligned} x(t_i) &= \Psi(t_i)x_0 + \int_0^{t_i} \Psi(t_i-\tau)Bu(\tau)d\tau = \Psi(t_i)x_0 + \sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} \Psi(t_i-\tau)Bu(\tau)d\tau \\ &= e^{At_i}\left[x_0 + \sum_{j=0}^{i-1}\left(\int_{t_j}^{t_{j+1}} e^{-A\tau}d\tau\right)Bu(t_j)\right] \\ &= \sum_{k=0}^{\mu-1} \alpha_k(t_i)A^k x_0 + \sum_{j=0}^{i-1}\sum_{k=0}^{\mu-1}\left(\int_{t_j}^{t_{j+1}} \alpha_k(t_i-\tau)d\tau\right)A^k Bu(t_j) \end{aligned} \tag{3.14}$$

which coincides with (3.12) by taking into account the definition of the matrix $M(A,B,\mu)$ and defining $\gamma_{kj}(t_j,t_{j+1},t_i) := \int_{t_j}^{t_{j+1}} \alpha_k(t_i-\tau)d\tau = \int_0^{T_j} \alpha_k(t_i-t_j-\tau)d\tau$; for all $t_i \in \mathrm{SI}(q)$, for all $j \in \overline{i-1} \cup \{0\}$, for all $k \in \overline{\mu-1} \cup \{0\}$ and the proof is complete.

(iii) It is proved as (i) by considering e^{At} as a function of the matrix A which might be calculated by sets of linearly independent functions satisfying linear algebraic systems similar to (3.11) [1, 2]. □

4. Main results on reachability and controllability

The subsequent result addresses the facts that a necessary condition for the reachability of a positive system (2.3) is the positivity and reachability of the continuous-time system (2.3) while a necessary and sufficient condition, provided that the above necessary condition holds, is that, furthermore, *the necessary minimum set of sampling instants is non larger than the degree of the minimal polynomial of A*, which can be always fulfilled through a design disposal together with *their appropriate distribution so that the discrete controllability matrix possesses a square monomial submatrix*, namely, each row and column of such a submatrix has only a nonzero positive entry. This second part of sufficiency is, in principle, much more restrictive than the generic condition to maintain reachability under, in general, aperiodic sampling for arbitrary linear time-invariant systems. In general, aperiodic sampling systems are reachable under similar conditions except that the requirement of the existence of a monomial square submatrix of the controllability one is replaced with a weakest condition. Such a weakest requirement is that the sampling instants are distributed so that a certain square matrix associated with the $\gamma_{(\cdot)}$-functions of Lemma 3.5(i) is nonsingular. It has been proved in the literature that this property is (generically) accomplished with by almost any arbitrary distribution of the samples since the linearly independent functions $\alpha_{(\cdot)}$ are, furthermore, a Tchebyshev system on each real interval $[\zeta, \zeta + \pi/\omega)$ where $\zeta \in \mathbb{R}_+$ is arbitrary and ω is an upper bound of the maximum eigenfrequency of (2.1), that is, any upper-bound of the maximum absolute value of the imaginary part of all its complex eigenvalues [1, 2]. *Thus, it suffices to choose*

$$\mathrm{SI}(\mu) := \left\{ t_j \left(\neq t_i \text{ for } i \neq j \right) \in \left[\zeta, \zeta + \frac{\pi}{\omega} \right) : \zeta \in \mathbb{R}_+,\ t_j < t_{j+1} < t_{j+2},\ \forall j \in \overline{\mu} \right\} \tag{4.1}$$

to maintain a reachability property from the continuous-time system by avoiding potential hidden oscillations nondetectable at sampling instants and the associate lost of reachability. If all such eigenvalues are real then admissible intervals to distribute the sampling instants are $[\zeta, \infty)$. (see, for instance, [1, 2] and references therein).

Theorem 4.1. *The system (2.3) is positive and reachable in finite time for a set of sampling instants* $\mathrm{SI}(\mu) := \{t_0 \equiv 0,\ t_1, \ldots, t_\mu\}$ *if and only if the following two conditions hold together.*

(1) The continuous-time system (2.1) is positive and reachable; that is, $A \in M_E^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{p\times n}$ *and* $\operatorname{rank} C(A,B) = n$ *which requires the necessary condition* $\mu \geq n/m$.

(2) An n-square real submatrix $\widehat{C}_S(\Psi, \Gamma, \mathrm{SI}(n))$ *of the controllability matrix* $\widehat{C}(\Psi, \Gamma, \mathrm{SI}(n))$ *of the discrete-time system (2.3) is monomial. A necessary condition for* $\widehat{C}(\Psi, \Gamma, \mathrm{SI}(n))$ *to possess a monomial submatrix* $\widehat{C}_S(\Psi, \Gamma, \mathrm{SI}(n))$*, and, then, to be also full rank, is that the*

real square μm-matrix $T_{\mathrm{SI}(\mu)}$ defined by

$$T_{\mathrm{SI}(\mu)} := \begin{bmatrix} \gamma_{00}(t_0,T_0,t_\mu)I_m & \cdots & \gamma_{0,\mu-1}(t_{\mu-1},T_{\mu-1},t_\mu)I_m \\ \vdots & \vdots & \vdots \\ \gamma_{\mu-1,0}(t_0,T_0,t_\mu)I_m & \cdots & \gamma_{\mu-1,\mu-1}(t_{\mu-1},T_{\mu-1},t_\mu)I_m \end{bmatrix} \tag{4.2}$$

and depending on a set $\mathrm{SI}(\mu) \subset \mathrm{SI}(n)$ of sampling instants, be nonsingular, where the real functions $\gamma_{kj}(t_j,T_j,t_\mu)$; for all $t_i \in \mathrm{SI}(\mu)$, for all $k,j \in \overline{\mu-1} \cup \{0\}$ are defined in Lemma 3.5(ii).

Proof ("Sufficiency part"). Note from Lemma 3.5(ii) that

$$\begin{aligned} x(t_\mu) &= \widehat{M}(\Psi,\Gamma,\mathrm{SI}(\mu))\left[u^T(t_0),u^T(t_1),\ldots,u^T(t_{\mu-1})\right]^T \\ &= M(A,B,\mu)T_{\mathrm{SI}(\mu)}\left[u^T(t_0),u^T(t_1),\ldots,u^T(t_{\mu-1})\right]^T. \end{aligned} \tag{4.3}$$

Note that if Conditions (1)-(2) hold jointly then the discrete system is positive, directly from Condition (1). Furthermore, the coefficient matrix $\widehat{M}(\Psi,\Gamma,\mathrm{SI}(\mu)) = M(A,B,\mu)T_{\mathrm{SI}(\mu)}$ of (4.3) is full rank if and only if $M(A,B,\mu)$ is full rank and the square matrix $T_{\mathrm{SI}(\mu)}$ is nonsingular with $\mu \geq n/m$. In other words, the algebraic system (4.3) is solvable from Rouché-Froebenius theorem from Linear Algebra since

$$\begin{aligned} \operatorname{rank}\left(M(A,B,\mu)T_{\mathrm{SI}(\mu)}\right) &= \operatorname{rank}\left(M(A,B,\mu)T_{\mathrm{SI}(\mu)},x^*\right) \\ &= \operatorname{rank}\widehat{C}(\Psi,\Gamma,\mathrm{SI}(n)) = \operatorname{rank}\widehat{M}(\Psi,\Gamma,\mathrm{SI}(q)) \\ &= \operatorname{rank}\widehat{M}(\Psi,\Gamma,\mathrm{SI}(\mu)) = \operatorname{rank}M(A,B,\mu) \\ &= \operatorname{rank}C(A,B) = n \end{aligned} \tag{4.4}$$

for any given arbitrary bounded prefixed $x(t_\mu) = x^*$ so that a solution $[u^T(t_0),u^T(t_1),\ldots,u^T(t_{\mu-1})]^T$ exists and is able to drive the state-solution trajectory from $x_0 = 0$ to $x(t_\mu) = x^*$ from Theorem 3.4 provided that the continuous-time system is reachable and provided that the sequence of sampling instants satisfies $\operatorname{Det} T_{\mathrm{SI}(\mu)} \neq 0$. Furthermore, if there exists a monomial matrix $\widehat{C}_S(\Psi,\Gamma,\mathrm{SI}(\mu))$, a submatrix of $\widehat{M}(\Psi,\Gamma,\mathrm{SI}(\mu))$, then its inverse $\widehat{C}_S^{-1}(\Psi,\Gamma,\mathrm{SI}(\mu)) \in \mathbb{R}_+^{n\times n}$ exists (since the inverse of a monomial matrix exists and it is positive and a matrix is monomial if and only if its inverse is positive [38]) so that a valid control sequence solution satisfying (4.3), for each bounded prefixed $x(t_\mu) = x^*$, has the form

$$\overline{u}(\mathrm{SI}(\mu)) := \left[u^T(t_0),u^T(t_1),\ldots,u^T(t_{\mu-1})\right]^T = \left(V\widehat{C}_S^{-1}(\Psi,\Gamma,\mathrm{SI}(\mu))x^*\right) \in \mathbb{R}_+^{\mu m} \tag{4.5}$$

provided that $x^* \in \mathbb{R}_+^n$, since $\widehat{C}_S^{-1}(\Psi,\Gamma,\mathrm{SI}(\mu)) \in \mathbb{R}_+^{n\times n}$, where V is a real $n \times \mu m$-matrix whose entries are all either zero or unity so that $(m\mu - n)$ components of the solution $\overline{u}(\mathrm{SI}(\mu))$ are fixed to zero and the remaining ones, which are not all zero if $x^* \neq 0$, are calculated from $\widehat{C}_S^{-1}(\Psi,\Gamma,\mathrm{SI}(\mu))x^*$. Then, there exists a nonnegative control which drives

the state-space trajectory from a zero initial state to any arbitrary bounded $x^* \in \mathbb{R}_+^m$ if Conditions (1)-(2) hold jointly. The sufficiency part has been fully proven.

Necessity part. If Condition (1) fails then either the discrete-time system is not positive, from the lack of positivity of a discrete-time system if the corresponding continuous-time system is not positive, or it is not reachable since $\operatorname{rank} C(A,B) < n \Rightarrow \operatorname{rank} \widehat{C}(\Psi,\Gamma,\mathrm{SI}(n)) < n$ by inspection of (4.3). If Condition (2) fails, then either $\operatorname{rank} \widehat{C}(\Psi,\Gamma,\mathrm{SI}(n)) < n$ and the discrete-time system is not reachable for the given sequence of sampling instants even if Condition (1) holds, or $\operatorname{rank} \widehat{C}(\Psi,\Gamma,\mathrm{SI}(n)) = n$ but there is no subset of columns of $\widehat{C}(\Psi,\Gamma,\mathrm{SI}(n))$ such that the associate square matrix is monomial so that there is no nonnegative control sequence able to drive the equilibrium for all given arbitrary state in the first orthant. Necessity has also been proven. □

The subsequent result follows directly from Theorem 4.1 and Lemma 3.5(iii).

Corollary 4.2. *If Theorem 4.1 holds for some sequence of samples* $\mathrm{SI}(\mu)$ *then it also holds for any arbitrary sequence of samples* $\mathrm{SI}(\rho) \supset \mathrm{SI}(\mu)$ *for any* $\rho \geq \mu$ *including* $\rho = n$ *so that the system is reachable for sequences of sampling instants of arbitrary finite cardinal exceeding* μ.

Proof. For any natural number $\rho \geq \mu$, note that if the continuous-time controllability matrix $C(A,B)$ is full rank then $M(A,B,\rho)$ is also full rank. Note also that by replacing $T_{\mathrm{SI}(\mu)} \to T_{\mathrm{SI}(\rho)}$, sequences of samples might be found defined as

$$T_{\mathrm{SI}(\rho)} := \begin{bmatrix} \gamma_{00}(0,T_0,t_\rho)I_m & \cdots & \gamma_{0,\rho-1}(t_{\rho-1},T_{\rho-1},t_\rho)I_m \\ \vdots & \vdots & \vdots \\ \gamma_{\rho-1,0}(0,T_0,t_\rho)I_m & \cdots & \gamma_{\rho-1,\rho-1}(t_{\rho-1},T_{\rho-1},t_\rho)I_m \end{bmatrix} \tag{4.6}$$

guaranteeing $\operatorname{Det} T_{\mathrm{SI}(\rho)} \neq 0$ by calculating a sufficiently large number and unique (for each given ρ) of linearly independent real $\gamma_{\rho(\cdot,\cdot)}$-functions according to Lemma 3.5((ii)-(iii)). Furthermore, since $\widehat{C}_S(\Psi,\Gamma,\mathrm{SI}(\mu))$ is monomial then there is a monomial square submatrix of any $\widehat{M}(\Psi,\Gamma,\mathrm{SI}(\rho))$ provided that $\mathrm{SI}(\rho) \supset \mathrm{SI}(\mu)$. □

Corollary 4.2 is useful for potential applications since the number of sampling instants might be increased while maintaining the reachability of the discrete-time system provided that the continuous one is reachable. This allows the choice of the time interval used to drive the system to the desired final state and to generically choose the distribution of the sampling instants under rather weak constraints. The increase in the number of samples also allows the improving of the noise influence in the numerical results since more data are processed. On the other hand, particular distributions of sampling instants might be chosen, for instance, to optimize the condition number of the coefficient matrix of the algebraic problem associated with the reachability one. This results in improving the relative errors in the solution generated by those of the measured data and the associated with the entries of the coefficient matrix. Such issues have been previously addressed in the context of general linear dynamic time-invariant systems [1, 2]. Corollary 4.2 addresses the way of arbitrarily increasing the number of sampling instants while keeping the stability in order to take advantages such as to improve the measuring errors influence

from the measurements to the results. From an algebraic point of view it is, however, interesting to solve the problem with the smallest possible number of calculations by using a square coefficient matrix of nth-order. This only requires the injection of a number of nonzero input components being equal to n for the whole number of sampling instants at hand, with the remaining input components being zeroed, in order to algebraically solve the reachability problem. The subsequent result addresses that issue.

Corollary 4.3. *Assume that Theorem 4.1 holds. Then, there is a (in general, nonunique) set of m nonnegative integer numbers m_i ($i \in \overline{m}$) such that $\sum_{i=1}^{m} m_i = n$ and $1 \le n_c := \mathrm{Max}_{1\le i\le m}(m_i) \le \mu$ being the controllability index of (2.1) such that*

(1) *it exists a (in general nonunique) nonsingular n-square real submatrix of $C(A,B)$*

$$C_{n_c}(A,B) := [C_{n_1}(A,b_1), C_{n_2}(A,b_2), \ldots, C_{n_m}(A,b_m)], \tag{4.7}$$

where $C_{n_i}(A,b_i) := [b_i, Ab_i, \ldots, Ab_i^{n_i-1}]$ and b_i is the ith column of B, $i \in \overline{m}$,

(2) *the reachability of the discrete-time system (2.3) is guaranteed by a minimum number of nonunique and nonuniquely distributed n_c controls*

$$\left(u^T(t_0), u^T(t_1), \ldots, u^T(t_{n_c-1})\right)^T \in \mathbb{R}_+^{mn_{n_c}} \tag{4.8}$$

fulfilling $u_i(t_j) = 0$, for all $j(\ge m_i) \in \overline{n_c - 1} \cup \{0\}$, for all $i \in \overline{m}$ being injected at sequences of sampling instants $\mathrm{SI}(n_c) := \{t_0 \equiv 0, t_1, t_2, \ldots, t_{n_c-1}\}$ of cardinal equalizing the controllability index.

Outline of Proof. Introducing the constraints $u_i(t_j) = 0$, for all $j(\ge m_i) \in \overline{n_c - 1} \cup \{0\}$, for all $i \in \overline{m}$ in (4.3), it is trivial to deduce

$$\begin{aligned} x(t_{n_c}) &= \widehat{M}(\Psi,\Gamma,\mathrm{SI}(n_c))\left[u^T(t_0), u^T(t_1), \ldots, u^T(t_{n_c-1})\right]^T \\ &= \widehat{\widehat{M}}(\Psi,\Gamma,\mathrm{SI}(n_c))\widehat{\widehat{u}}(\mathrm{SI}(n_c)) = C_{n_c}(A,B)QQ^T\widehat{T}_{\mathrm{SI}(n_c)}\widehat{u}(\mathrm{SI}(n_c)) \\ &= C_{n_c}(A,B)\widehat{T}_{\mathrm{SI}(n_c)}\widehat{u}(\mathrm{SI}(n_c)), \end{aligned} \tag{4.9}$$

where the two vectors below are identical after appropriately reordering the components in any of them:

$$\begin{aligned} \widehat{\widehat{u}}(\mathrm{SI}(n_c)) &:= \left[\widehat{\widehat{u}}_1^T(\mathrm{SI}(n_c)), \widehat{\widehat{u}}_2^T(\mathrm{SI}(n_c)), \ldots, \widehat{\widehat{u}}_m^T(\mathrm{SI}(n_c))\right]^T, \\ \widehat{u}(\mathrm{SI}(n_c)) &:= \left[\widehat{u}_1^T(\mathrm{SI}(m_1)), \widehat{u}_2^T(\mathrm{SI}(m_2)), \ldots, \widehat{u}_m^T(\mathrm{SI}(m_m))\right]^T \end{aligned} \tag{4.10}$$

with

$$\begin{aligned} \widehat{\widehat{u}}_i^T(\mathrm{SI}(n_c)) &:= \left[\widehat{u}_i^T(\mathrm{SI}(m_i)), 0, \ldots, 0\right] \in \mathbb{R}_+^{n_c}, \quad \forall i \in \overline{m}, \\ \widehat{u}_i^T(\mathrm{SI}(m_i)) &:= \left[u_i(t_0), u_i(t_1), \ldots, u_i(t_{m_i-1})\right], \quad \forall i \in \overline{m} \end{aligned} \tag{4.11}$$

and $u_i(t_j) = 0$, for all $j(\geq m_i) \in \overline{n_c - 1} \cup \{0\}$, for all $i \in \overline{m}$, so that

$$\widehat{\widehat{M}}(\Psi, \Gamma, \mathrm{SI}(n_c)) = C_{n_c}(A,B)QQ^T\widehat{T}_{\mathrm{SI}(n_c)} = Z_1\widehat{M}(\Psi, \Gamma, \mathrm{SI}(n_c))Z_2 \tag{4.12}$$

is a similar matrix to $\widehat{M}(\Psi, \Gamma, \mathrm{SI}(n_c))$ via some equivalence transformation defined by the $n \times \mu m$ and $\mu m \times n$ real matrices Z_1 and Z_2, respectively, Q being a real full row rank $n \times n_{\mu m}$ matrix, so that QQ^T is a square nonsingular n-matrix for $\mu \geq n/m$, which reorders the columns of $C(A,B)$, and potentially reduces its number to n. By construction, the distribution of sampling instants may be chosen such that $C_{n_c}(A,B)\widehat{T}_{\mathrm{SI}(n_c)}$ be monomial since a submatrix of the controllability matrix of the discrete-time system (2.3) is monomial from Theorem 4.1. Thus, the discrete-time system (2.3) is positive and reachable in n_c samples at some sampling instant t_{n_c} through some sequence of n_c sampling instants $\mathrm{SI}(n_c) := \{t_0 = 0, t_1, \ldots, t_{n_c-1}\}$. The proof is complete. □

Now, the close property of controllability is investigated. Controllability refers to drive any nonzero arbitrary initial condition in the first orthant to some arbitrary prescribed point or proper or improper region in the first orthant (see Definitions 2.2–2.4). First, note that the state transition matrix is never nilpotent at any time. Discrete or digital systems not being related to discretization of continuous-time systems, are globally controllable in finite time if they are reachable and its state-transition matrix Ψ_d is nilpotent. In such a case, there is a natural number ν such that $\Psi_d^\nu = 0$ so that $x^* - \Psi_d^q x_0 = x^* \in \mathbb{R}_+^n$ for any $q \geq \nu$ and any given pair $(x_0^T, x^*)^T \in \mathbb{R}_+^{2n}$ and the system is globally controllable in any finite number of step non less than ν. A sequence of the nonzero components of the control input driving x_0 to x^* is calculated from the formula $\widehat{C}_S^{-1}(\Psi_d, \Gamma, \mathrm{SI}(\mu))x^*$ (in a similar way as that used for reachability in the proof of Theorem 4.1 for discrete-time systems) for some appropriate distribution of the sampling instants including potentially the case of constant sampling periods for appropriate values. Global controllability is then guaranteed for any initial and final conditions in the open first orthant for any $1 \leq q \leq \nu$ and pairs fulfilling $(x_0^T, x^*)^T \in \mathbb{R}_+^{2n}$ such that $(x^* - \Psi_d x_0) \in \mathbb{R}_+^n$, alternative nonzero control components valid to drive the state from the initial to the final position might instead be calculated as $\widehat{C}_S^{-1}(\Psi_d, \Gamma, \mathrm{SI}(\mu))(x^* - \Psi_d x_0)$. Since it is unfeasible a nilpotent state transition matrix of a discrete-time system when arising from the discretization of a continuous-time one, global controllability in finite time is unfeasible. Then, global controllability to a specific region and global asymptotic controllability are now investigated. Define $\mathbf{R}_\varepsilon^n := \{z \in \mathbb{R}_+^n : z_i \geq \varepsilon$, for all $i \in \overline{n}\}$, for all $\varepsilon \in \mathbb{R}$. Note that $\mathbb{R}_0^n \equiv \mathbb{R}_+^n$. The following result follows directly from Theorem 4.1 and Corollaries 4.2-4.3.

Theorem 4.4. *The following properties hold*

(i) *The discrete system (2.3) is positive and globally controllable from $x_0 \in \mathbf{B_R} \subset \mathbb{R}_+^n$ ($\mathbf{B_R}$ being a bounded domain of the first orthant) to any region $\mathbb{R}_\varepsilon^n := \{z \in \mathbb{R}_+^n : z_i \geq \varepsilon$, for all $i \in \overline{n}\}$ being a proper subset of the closed first orthant, for any $\varepsilon \geq \varepsilon_0$ and some $\varepsilon_0 > 0$, for a finite sequence of sampling instants $\mathrm{SI}(\rho)$, ρ being a finite natural number dependent on ε, if and only if*

(1) *the continuous-time system (2.1) is positive, reachable, and globally asymptotically Lyapunov's stable, that is,* $A \in M_E^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$, $\operatorname{rank} C(A,B) = n$, *which requires the necessary condition* $\mu \geq n/m$, *and* A *is a stability matrix;*

(2) *A* n*-square real submatrix* $\widehat{C}_S(\Psi,\Gamma,\mathrm{SI}(n))$ *of the controllability matrix of (2.3) is nonsingular and the sequence of sampling instants* $\mathrm{SI}(\rho)$ *satisfies that the matrix*

$$T_{\mathrm{SI}(\rho)} := \begin{bmatrix} \gamma_{00}(0,T_0,t_\rho)I_m & \cdots & \gamma_{0,\rho-1}(t_{\rho-1},T_{\rho-1},t_\rho)I_m \\ \vdots & \vdots & \vdots \\ \gamma_{\rho-1,0}(0,T_0,t_\rho)I_m & \cdots & \gamma_{\rho-1,\rho-1}(t_{\rho-1},T_{\rho-1},t_\rho)I_m \end{bmatrix} \tag{4.13}$$

is nonsingular. Then, the system is also reachable for the finite sequence of sampling instants $\mathrm{SI}(\rho)$. *The property is always guaranteed to hold for some finite sequence of sampling instants of cardinal* $\rho \geq \mu$ *provided that it holds for some finite* ρ, *and also for any finite or infinite sequence of sampling instants of cardinal* $\rho' \geq \rho$ *provided that it holds for some* ρ.

(ii) *If Property (i) holds then the discrete system (2.3) is also globally asymptotically controllable to the origin with a nonnegative control.*

Proof. (i) Property (i) follows directly from Theorem 3.4 and from Corollary 4.2, which allows to extend the sequence of sampling instants, since for sufficiently large, but finite t, $(x^* - \Psi(t)x_0 - \boldsymbol{\varepsilon}) \in \mathbb{R}_+^n$ where $\boldsymbol{\varepsilon} \in \mathbb{R}_+^n$ has all its components identical to ε, since A is a stability matrix. Then for any finite time $t_\rho \geq t$ a sequence of sampling instants $\mathrm{SI}(\rho) := \{0,t_1,\ldots,t_{\rho-1},t_\rho\}$ satisfying the nonsingularity of $T_{\mathrm{SI}(\rho)}$ exists and a submatrix of the discrete controllability matrix is monomial. The remaining of the proof follows as in Theorem 4.1. A sequence of nonnegative controls driving the state from the initial to the final position is calculated as in Theorem 4.1 with the replacement $x^* \to (x^* - \Psi(t_\rho)x_0)$.

(ii) Define $x^*(t_{k+1},x_0) := \Psi(t_{k-k_1+1})x_0 (\in \mathbb{R}_+^n) \to x^* \equiv 0$ as $t_k \to \infty$ for all finite $k_1 \in \mathbb{N}$ and any finite $x_0 \in \mathbb{R}_+^n$. From the properties of the state-transition matrix:

$$\Psi(t_{k+1}) = \Psi(t_{k_1})\Psi(t_{k-k_1+1}) = \left(\prod_{i=1}^{k_1}[\Psi(T_{k-k_1+i})]\right)\Psi(t_{k-k_1+1}) \tag{4.14}$$

so that, for each given real constant $\delta \in (0,1)$, there exists $k^* = k^*(\delta)$ such that $\| \prod_{i=1}^{k_1} \|_2 < \delta$ where $\|(\cdot)\|_2$ is the ℓ_2 (spectral) matrix or (induced) vector norm for any $k_1 \geq k^*$. Then

$$\begin{aligned} x^*(t_{k+1},x_0) - x(t_{k+1}) &= (\Psi(t_{k-k_1+1}) - \Psi(t_{k+1}))x_0 = \left(I_n - \prod_{i=1}^{k_1}[\Psi(T_{k-k_1+i})]\right)\Psi(t_{k-k_1+1})x_0 \\ &= \sum_{j=0}^{k}\prod_{\ell=j+1}^{k}[\Psi(T_\ell)]Bu(t_j) \end{aligned} \tag{4.15}$$

and $\|x^*(t_{k+1},x_0)-x(t_{k+1})\|_2 \geq (1-\delta)\|\Psi(t_{k-k_1+1})x_0\|_2$. If now, $k_1 \to \infty$ and $(k-k_1) \to \infty$ then $\delta \to 0$ so that

$$\begin{aligned}&\left(I_n - \prod_{i=1}^{k_1}[\Psi(T_{k-k_1+i})]\right)\Psi(t_{k-k_1+1})\\&\quad = (I_n - o_{n\times n}(\delta))o_{n\times n}(\delta) \geq o_{n\times n}(\delta) - \left|o^2_{n\times n}(\delta)\right| = o_{n\times n}(\delta) \in \mathbb{R}^{n\times n}_+,\end{aligned} \tag{4.16}$$

where $\prod_{i=1}^{k_1}[\Psi(T_{k-k_1+i})] = o_{n\times n}(\delta) \in \mathbb{R}^{n\times n}_+$ and $\Psi(t_{k-k_1+1}) = o_{n\times n}(\delta) \in \mathbb{R}^{n\times n}_+$. Furthermore, both matrices are also $O_{n\times n}(\delta)$, with the extended "Big-O," "Small-o" Landau's notations as follows.

(i) A real n-matrix F is $O_{n\times n}(\delta)$ if $(\delta I_n - |F|) \succcurlyeq 0$, where $\succcurlyeq 0$ stands for positive semidefinite and $|F| = (|f_{ij}|)$ is the matrix of entries of the absolute values of the matrix $F = (f_{ij})$.

(ii) A real n-matrix F is $o_{n\times n}(\delta)$ or, respectively, $o^2_{n\times n}(\delta)$ if it is $O_{n\times n}(\delta)$ and, furthermore, $\lim_{\delta\to 0}(f_{ij}/\delta) = 0$, respectively, $\lim_{\delta\to 0}(f_{ij}/\delta^2) = 0$.

From (4.15)-(4.16), it follows that

$$x^*(t_{k+1},x_0) - x(t_{k+1}) = \sum_{j=0}^{k}\prod_{\ell=j+1}^{k}[\Psi(T_\ell)]Bu(t_j) = o_n(\delta) \in \mathbb{R}^n_+ \tag{4.17}$$

for all $x_0 \in \mathbb{R}^n_+$ being bounded as $\delta \to 0$ for $k_1 \to \infty$ and $(k-k_1) \to \infty$, that is, $\lim_{k\to\infty}(\sum_{j=0}^{k}\prod_{\ell=j+1}^{k}[\Psi(T_\ell)]Bu(t_j)) = 0$ and $\lim_{\substack{k_1\to\infty\\(k-k_1)\to\infty}}(x^*(t_{k+1},x_0)) = 0$ so that a nonnegative infinite sequence of controls generated at appropriately distributed infinite sequences of sampling instants is able to asymptotically drive any bounded initial state $x_0 \in \mathbb{R}^n_+$ to zero from property (i) according to the vanishing real vector sequence $x^*(t_{k+1},x_0) := \Psi(t_{k-k_1+1})x_0(\in \mathbb{R}^n_+) \to x^* \equiv 0$. The proof is complete. □

Note related to the proof of Theorem 4.4(ii) that although $(-\prod_{i=1}^{k_1}[\Psi(T_{k-k_1+i})])\Psi(t_{k-k_1+1}) = o^2_{n\times n}(\delta)$ is not necessarily nonnegative, $(I_n - \prod_{i=1}^{k_1}[\Psi(T_{k-k_1+i})])\Psi(t_{k-k_1+1}) = o_{n\times n}(\delta) \in \mathbb{R}^{n\times n}_+$ from (4.16).

A crucial constraint for reachability and controllability of linear positive systems is that the controllability matrix be monomial. Thus, it is interesting to derive conditions for the controllability matrix to be monomial under alternative sets of sampling instants or state transformations. This idea is addressed in the subsequent result as follows.

Theorem 4.5. *Consider the state transformation $x' = Qx$, where Q is an n-real square matrix, so that the discrete state transition and control matrices are related as $\Psi' = Q^{-1}\Psi Q$ and $\Gamma' = Q^{-1}\Gamma$, respectively; and also that two different sets of μ sampling instants $\mathrm{SI}(\mu)$ and $\mathrm{SI}'(\mu)$. Then, the following properties hold.*

(i) *The controllability matrices are related as*

$$\widehat{C}(\Psi',\Gamma',\mathrm{SI}'(\mu)) = Q^{-1}\widehat{C}(\Psi,\Gamma,\mathrm{SI}'(\mu))T^{-1}_{\mathrm{SI}(\mu)}T_{\mathrm{SI}'(\mu)} \tag{4.18}$$

provided that $T_{\mathrm{SI}(\mu)}$ is nonsingular.

(ii) *If* $\widehat{C}(\Psi,\Gamma,\mathrm{SI}(\mu))$ *is a monomial matrix M, so that the positive system (2.3) is reachable for the set of sampling instants* $\mathrm{SI}(\mu)$*, then* $\widehat{C}(\Psi',\Gamma',\mathrm{SI}'(\mu))$ *is monomial if and only if* $(T^{-1}_{\mathrm{SI}'(\mu)}T_{\mathrm{SI}(\mu)}M^{-1}Q)^{-1}$ *is monomial and then* $T^{-1}_{\mathrm{SI}'(\mu)}T_{\mathrm{SI}(\mu)}M^{-1}Q$ *is also monomial. As a result the system (2.3) is positive and reachable for the new state variables and sampling instants.If* $Q = I_n$ *(i.e., the state vector is not transformed) then* $\widehat{C}(\Psi,\Gamma,\mathrm{SI}'(\mu))$ *if and only if* $\widehat{C}(\Psi,\Gamma,\mathrm{SI}(\mu))$ *provided that* $T^{-1}_{\mathrm{SI}'(\mu)}T_{\mathrm{SI}(\mu)}$ *is monomial.*

Proof. (i) Direct calculations yield

$$\widehat{C}(\Psi',\Gamma',\mathrm{SI}'(\mu)) = Q^{-1}C(A,B)T_{\mathrm{SI}'(\mu)} = Q^{-1}\widehat{C}(\Psi,\Gamma,\mathrm{SI}'(\mu))T^{-1}_{\mathrm{SI}(\mu)}T_{\mathrm{SI}'(\mu)} \tag{4.19}$$

provided that $T_{\mathrm{SI}(\mu)}$ is nonsingular. Property (ii) follows directly from the above expression since the inverse of a monomial matrix is nonsingular and monomial [38]. □

Theorem 4.5 may be directly extended to controllability and also to observability/constructibility by considering pairs (C,Ψ) and (C',Ψ'). Parallel results for output reachability and controllability might be obtained directly by extending Theorems 4.1–4.5 and Corollaries 4.2-4.3 by using from (2.3):

$$\begin{aligned} y(t_{i+1}) &= C(\Psi(T_i)x(t_i)+\Gamma(T_i)u(t_i)) = C\Psi(t_{i+1})x_0 + \int_0^{t_{i+1}} C\Psi(t_{i+1}-\tau)Bu(\tau)d\tau \\ &= \prod_{j=0}^{i}[C\Psi(T_j)]x_0 + \sum_{j=0}^{i}\prod_{\ell=j+1}^{i} C[\Psi(T_\ell)]Bu(t_j); \quad \forall i \in \overline{q-1}\cup\{0\} \end{aligned} \tag{4.20}$$

by noting that the output controllability matrix of (2.1) and (2.3) are, respectively,

$$\widehat{C}_0(C,A,B) := C\cdot\widehat{C}(A,B); \qquad \widehat{C}_0(C,\Psi,\Gamma,\mathrm{SI}(n)) := C\cdot\widehat{C}(C,\Psi,\Gamma,\mathrm{SI}(n)) \tag{4.21}$$

so that necessary conditions for $\operatorname{rank}\widehat{C}_0(C,\Psi,\Gamma,\mathrm{SI}(n)) = p$ are $\operatorname{rank}C = \operatorname{rank}\widehat{C}(A,B) \geq p$, $\operatorname{Det}T_{\mathrm{SI}(\mu)} \neq 0$ for some sequence of μ sampling instants $\mathrm{SI}(\mu)$. The discrete output controllability matrix has also to possess a monomial square real p-matrix. Those conditions guarantee directly output reachability and some extra ones accordingly modifying those supplied in Theorem 4.4 guarantee directly controllability and asymptotic controllability of the discretized system (2.3).

5. Main results concerning observability and constructibility

Now, the properties of observability and constructibility are formulated. The observability and constructibility of positive systems under arbitrary sampling are dual properties to reachability and controllability, respectively (see, for instance, [41]). From Lemma 3.5(i), the unforced output trajectory may be expressed as follows:

$$y(t) = \sum_{k=0}^{\mu-1}\alpha_k(t)CA^kx_o = [\alpha_0(t)I_p,\alpha_1(t)I_p,\ldots,\alpha_{\mu-1}(t)I_p]M(C^T,A^T,\mu)^Tx_0. \tag{5.1}$$

Since the properties of observability and constructibility hold or not independent of the control, it suffices the above unforced trajectory to characterize them. Consider a sequence of μ sampling instants is $\mathrm{SI}(\mu) := (t_1, t_2, \ldots, t_\mu)$ and the associate output vector sequence

$$\hat{y}_{\mathrm{SI}(\mu)} := \left[y^T(t_1), y^T(t_2), \ldots, y^T(t_\mu)\right]^T = \Pi_{\mathrm{SI}(\mu)} M(C^T, A^T, \mu)^T x_0, \tag{5.2}$$

where

$$\Pi_{\mathrm{SI}(\mu)} := \begin{bmatrix} \alpha_0(t_1) I_p \alpha_1(t_1) I_p & \cdots & \alpha_{\mu-1}(t_1) I_p \\ \vdots & \cdots & \vdots \\ \alpha_0(t_\mu) I_p \alpha_1(t_\mu) I_p & \cdots & \alpha_{\mu-1}(t_\mu) I_p \end{bmatrix}. \tag{5.3}$$

The algebraic system of linear equations (5.2), if solvable, allows solving the observability problem consisting of the calculation of the initial state from future output measurements. Since the state transition matrix is a fundamental matrix of the differential system, it is nonsingular for all finite time. Thus, for any finite time $t_{\mu+1} > t_\mu$,

$$x_0 = \Psi^{-1}(t_{\mu+1}) x(t_{\mu+1}) = \Psi(-t_{\mu+1}) x(t_{\mu+1}) \tag{5.4}$$

which replaced into (5.2) yields

$$\hat{y}_{\mathrm{SI}(\mu)} := \left[y^T(t_1), y^T(t_2), \ldots, y^T(t_\mu)\right]^T = \Pi_{\mathrm{SI}(\mu)} M(C^T, A^T, \mu)^T \Psi^{-1}(t_{\mu+1}) x(t_{\mu+1}). \tag{5.5}$$

The algebraic system of linear equations (5.5) is solvable if and only if (5.2) is solvable which allows solving the global constructibility problem consisting of the calculation of a future state from previous output measurements. However, note that each of the coefficient matrices is not guaranteed to possess a monomial submatrix if the other coefficient matrix possesses that property. Thus, constructibility and observability are equivalent properties for nonpositive discretized systems for the same sequence of sampling instants but the equivalence does not hold in the general case for positive systems. Using those features, Theorems 4.1–4.4 and Corollaries 4.2-4.3 lead to close results for observability/global constructibility as follows.

Theorem 5.1. *The following properties hold.*

(i) *The discrete system (2.3) is positive and observable if and only if the continuous-time system (2.1) is positive and observable; that is,* $A \in M_E^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{p\times n}$, *and* $\operatorname{rank} O(C, A) = n$ *which requires the necessary condition* $\mu \geq n/m$ *and, furthermore, an* n*-square real submatrix* $\hat{O}_S(C, \Psi, \mathrm{SI}(n))$ *of the observability matrix* $\hat{O}(\Psi, \Gamma, \mathrm{SI}(n))$ *of the discrete-time system (2.3) is monomial. A necessary condition for the existence of a monomial submatrix* $\hat{O}_S(C, \Psi, \mathrm{SI}(n))$, *and then the observability matrix to be full rank, is that a sequence of sampling instants* $\mathrm{SI}(\mu) := \{t_1, t_2, \ldots, t_\mu\} \subset \mathrm{SI}(n)$ *real square* μp*-matrix* $\Pi_{\mathrm{SI}(\mu)}$ *defined in (5.3) be nonsingular.*

If the system (2.3) is observable for a sequence $\mathrm{SI}(\mu)$, *then it is also observable for sequences of sampling instants* $\mathrm{SI}(\rho)$ *of a larger number of samples.*

If the system (2.3) is observable for a sequence $\mathrm{SI}(\mu)$*, then there exists a set of nonnegative integer numbers* p_i $(i \in \overline{p})$ *such that* $\sum_{i=1}^{m} p_i = n$ *and* $1 \le n_o := \mathrm{Max}_{1\le i\le p}(p_i) \le \mu$*,* n_o *being the observability index of both (2.1) and (2.3) such that t he system is still observable for sequences of sampling instants* $\mathrm{SI}(\rho)$ *of any finite cardinal* $\rho \ge n_0$*.*

(ii) *Property (i) may be reformulated for global constructibility of a positive discretized system (2.3) by replacing the existence of a monomial square n-matrix* $\widehat{O}_S(C,\Psi,\mathrm{SI}(n))$ *of the observability one by the existence of a monomial matrix of the constructibility matrix* $[\widehat{O}(C,\Psi,\mathrm{SI}(n))\Psi^{-1}(t_\mu)]$*. Global constructibility is guaranteed by observability for a sequence of sampling instants* $\mathrm{SI}(\mu)$ *if the state-transition matrix is monomial at some finite* $t_{\mu+1} > t_\mu$*.*

The proof of Theorem 5.1 is omitted since the reasoning is very close to those used in the proofs of Theorems 4.1–4.4 and Corollaries 4.2-4.3. Note that the square matrix $\Pi_{\mathrm{SI}(\mu)}$ is guaranteed to be nonsingular by almost any arbitrary distribution of the samples $\mathrm{SI}(\mu)$ since the linearly independent functions $\alpha_{(\cdot)}$ are, furthermore, a Tchebyshev system on each real interval $[\zeta,\zeta+\pi/\omega)$ where $\zeta \in \mathbb{R}_+$ is arbitrary and ω is an upper bound of the maximum eigenfrequency of (2.1) [1, 2]. It suffices then to take the sampling intervals distinct and belonging to such intervals. Note also that to prove the last part of Theorem 5.1(ii), the property that the product of monomial matrices is monomial and the inverse of a monomial matrix is monomial is used [38, 39]. It seems promising to extend in the future the above formulation to neural networks, which are very useful in computation and for describing certain dynamical systems which often have hybrid disposals and possess constant or time-varying delays [42] and to polytopic parameterizations of dynamic systems [43–45].

6. Examples

Example 6.1. Consider the second-order positive continuous-time system with one single input of state equation parameterized by $A = \mathrm{Diag}(\lambda_1,\lambda_2)$, $b = (b_1,b_2) \in \mathbb{R}_+^2$, $\lambda_{1,2} \in \mathbb{R}$. The controllability matrix of the obtained discretized system for a constant sampling period T is

$$C(\Psi(T),\Gamma(T)) = \begin{bmatrix} \dfrac{e^{\lambda_1 T}-1}{\lambda_1}b_1 & \dfrac{e^{\lambda_1 T}(e^{\lambda_1 T}-1)}{\lambda_1}b_1 \\ \dfrac{e^{\lambda_2 T}-1}{\lambda_2}b_2 & \dfrac{e^{\lambda_2 T}(e^{\lambda_2 T}-1)}{\lambda_2}b_2 \end{bmatrix} \tag{6.1}$$

which cannot be monomial for any parameterization of the form $b = (b_1,b_2) \in \mathbb{R}_+^2$, $\lambda_{1,2} \in \mathbb{R}$ and no bounded positive sampling period. Then, the discretized system cannot be positive and reachable/controllable for any set of sampling instants with constant or aperiodic associate sampling periods. If $A = \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix}$ is the real canonical matrix associated with a pair of complex conjugate eigenvalues $\lambda_{1,2} = \mu \pm i\nu$ then the associate system cannot be positive since $A \notin M_E^{2\times 2}$.

Example 6.2. Now, consider a second-order positive continuous-time system with two input components with its matrix A in diagonal Jordan form with either a Jordan block

or two Jordan blocks defined by $A_g = \begin{bmatrix} \lambda & g \\ 0 & \lambda \end{bmatrix}$, $B = (b_{ij}) \in \mathbb{R}_+^{2\times 2}$, $\lambda \in \mathbb{R}$ with $g = 1$, respectively, $g = 0$ for the case of one, respectively two, Jordan blocks. For a sampling period T, the discrete state-transition and control-transition matrices are given by

$$\Psi_g(T) = \begin{bmatrix} e^{\lambda T} & gTe^{\lambda T} \\ 0 & e^{\lambda T} \end{bmatrix},$$

$$\Gamma_g(T) = \frac{1}{\lambda} \begin{bmatrix} b_{11}(e^{\lambda T}-1)+b_{21}g\left(e^{\lambda T}\left(T-\frac{1}{\lambda}\right)+\frac{1}{\lambda}\right) & b_{12}(e^{\lambda T}-1)+b_{22}g\left(e^{\lambda T}\left(T-\frac{1}{\lambda}\right)+\frac{1}{\lambda}\right) \\ e^{\lambda T}b_{21}(e^{\lambda T}-1) & e^{\lambda T}b_{22}(e^{\lambda T}-1) \end{bmatrix}. \tag{6.2}$$

In the case of two Jordan blocks, that is, $g = 0$, the discrete system is positive and reachable in two steps $t_0 = 0$ and $t_1 = T$ for any $\lambda \in (-\infty, -\varepsilon) \cup (\varepsilon, \infty)$, $T \in (\varepsilon, \infty)$ provided that $b_{ii} = 0$ for $i \in \overline{2}$ and $b_{ij} > 0$, $i, j(\neq i) \in \overline{2}$ and in the case that $b_{ii} > 0$ for $i \in \overline{2}$ and $b_{ij} = 0$, $i, j(\neq i) \in \overline{2}$. For the case $g = 1$, the system is reachable if $b_{ii} > 0$ for $i \in \overline{2}$ and $b_{ij} = 0$ for $i, j(\neq i) \in \overline{2}$ since then the $(1,1)$ and $(2,2)$ entries of $\Gamma_1(T)$ are positive, its $(2,1)$ entry is zero while its $(1,2)$ entry is also zero provided that $\lambda = f(\lambda, T) := (e^{\lambda T} - 1)/(Te^{\lambda T}) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ since $g = 1$. If $\lambda > 0$, this always holds for some sufficiently large $T > 0$ since $0 < (e^{\lambda T} - 1)/(Te^{\lambda T}) \to 0$ as $T \to \infty$ so that there is a real solution to the constraint $\lambda = (e^{\lambda T} - 1)/(Te^{\lambda T})$. For $\lambda < 0$ it also exists such a sampling period implying that the above constraint holds since $f(\lambda, 0) = 0$, $f(\lambda, \infty) := \lim_{T\to\infty} f(\lambda, \infty) = -\infty$ and $f(\lambda, T)$ is continuous on $\mathbb{R} \times \mathbb{R}_\varepsilon$, for all $\varepsilon \in \mathbb{R}_+$. In this last case, the discrete system is also globally asymptotically controllable to any region $\mathbb{R}_\varepsilon^n := \{z \in \mathbb{R}_+^n : z_i \geq \varepsilon, \text{ for all } i \in \overline{n}\}$, for all $\varepsilon \in \mathbb{R}_+$.

If, in addition to the above conditions, $\lambda < 0$ then the discretized system is globally asymptotically controllable to any region $\mathbb{R}_\varepsilon^n := \{z \in \mathbb{R}_+^n : z_i \geq \varepsilon, \text{ for all } i \in \overline{n}\}$, for all $\varepsilon \in \mathbb{R}_+$. If the parameterization changes to $A = \mathrm{Diag}(\lambda_1, \lambda_2)$, $\lambda_{1,2} \in \mathbb{R}$ (i.e., the two eigenvalues are real and distinct or they are equal with two Jordan blocks discussed above) then $\Psi_0(T) = \mathrm{Diag}(e^{\lambda_1 T}, e^{\lambda_2 T})$ and $\Gamma(T) = \Gamma_0(T)$, defined above with the replacements $\lambda \to \lambda_1$ and $\lambda \to \lambda_2$ in the first and second row vectors, respectively. The same conclusions about reachability and global asymptotic controllability to $\mathbb{R}_\varepsilon^n := \{z \in \mathbb{R}_+^n : z_i \geq \varepsilon, \text{ for all } i \in \overline{n}\}$, for all $\varepsilon \in \mathbb{R}_+$, provided that $\lambda_i < 0$, $i \in \overline{2}$, as in the case $g = 0$. The number of samples might be increased as stated in Corollary 4.2 under weak conditions.

Example 6.3. Consider again the single-input Example 6.1. The system is reachable for all bounded positive sampling periods if and only if the eigenvalues are distinct but its positivity is lost. However, the discretized system is positive reachable for any given sampling instants $t_0 = 0$, $t_1 = T > 0$ and a state-space transformation in the continuous-time system $x(t) = Qz(t)$, $Q \in \mathbb{R}^{2\times 2}$ being nonsingular such that $\widehat{C}(\Psi_z(T), \Gamma_z(T)) = M$ (monomial) so that $A_z = Q^{-1}AQ$ and $B_z = Q^{-1}B$ are the new dynamics and control matrices in the transformed state variables. Similar considerations as those in the above examples can be derived for observability and constructibility with the manipulation of the observability matrix.

7. Conclusions

This paper has been devoted to investigate basic properties of linear time-invariant systems under discretization with arbitrary, in general, aperiodic sampling. The properties investigated have been reachability and controllability and their dual properties of observability and constructibility. The main issue is that the properties hold if the corresponding ones of the continuous-time system hold and some extra ones concerning the distribution of the sampling instants hold as well. The conditions on the sampling instants hold generically based on the properties of the linearly independent functions used to expand the fundamental matrix of the differential system from its infinitesimal generator and their associate Tchebyshev system which possesses a nonzero determinant [1, 2]. It is pointed out that a possible practical usefulness is the choice of the samples so that the coefficient matrix of the linear algebraic system associated with each of the investigated properties has a condition number as small as possible in order to improve the transmission of the measuring, parameterization, and rounding relative errors from the data and parameters to the solution.

Acknowledgments

The author is very grateful to the Spanish Ministry of Education and to Basque government by their support through Grants DPI 2006-00714 and SAIOTEK 2006 Ref.S-PE06UN10, respectively.

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M. De la Sen: Department of Electricity and Electronics, Institute for Research and Development of Processes, Faculty of Science and Technology, Campus of Leioa (Bizkaia). Aptdo. 644,
48080 Bilbao, Spain
Email address: manuel.delasen@ehu.es