Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2007, Article ID 89107, 11 pages
doi:10.1155/2007/89107

Research Article

Precise Rates in Log Laws for NA Sequences

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Received 27 September 2006; Revised 23 December 2006; Accepted 30 January 2007

Let $X_1, X_2, \ldots$ be a strictly stationary sequence of negatively associated (NA) random variables with $EX_1 = 0$, set $S_n = X_1 + \cdots + X_n$, suppose that $\sigma^2 = EX_1^2 + 2\sum_{n=2}^{\infty} EX_1X_n > 0$ and $EX_1^2 < \infty$, if $-1 < \alpha \le 1$; $EX_1^2(\log|X_1|)^{\alpha} < \infty$, if $\alpha > 1$. We prove $\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} ((\log n)^{\alpha}/n) P(|S_n| \ge \sigma(\epsilon + \kappa_n)\sqrt{2n\log n}) = 2^{-(\alpha+1)}(\alpha+1)^{-1} E|N|^{2\alpha+2}$, where $\kappa_n = O(1/\log n)$ and N is the standard normal random variable.

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1. Introduction

A finite family of random variables, $X_1, X_2, \ldots, X_n$, is said to be NA if, for every pair of disjoint subsets T_1 and T_2 of $\{1, 2, \ldots, n\}$,

$$\operatorname{Cov}\left(f_1(X_i, i \in T_1),\ f_2(X_j, j \in T_2)\right) \le 0, \tag{1.1}$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family is NA if every finite subfamily is NA. This definition was introduced by Alam and Saxena [1] and Joag-Dev and Proschan [2], and has found many applications in percolation theory, multivariate statistical analysis, and reliability theory (see, e.g., Barlow and Proschan [3]).

Let $\{X_n : n \ge 1\}$ be a sequence of NA random variables on some probability space $(\Omega, \mathcal{F}, P)$ with mean zero and finite variance. As usual, set $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, $n \ge 1$, and write $\sigma_n^2 = ES_n^2$. Under appropriate covariance conditions, many limit theorems have been obtained. For example, the central limit theorem was proved by Newman [4].

Theorem 1.1. *Let $\{X_n : n \geq 1\}$ be strictly stationary NA sequences with mean zero and*

$$0 < \sigma^2 = EX_1^2 + 2\sum_{n=2}^{\infty} EX_1X_n < \infty, \tag{1.2}$$

then

$$\frac{S_n}{(\sigma\sqrt{n})} \xrightarrow{\mathcal{D}} N(0,1), \quad \text{as } n \longrightarrow \infty. \tag{1.3}$$

Further results are three series theorems (see, e.g., Matula [5]), probability inequalities (cf. Roussas [6], Shao [7]), weak convergence (see, e.g., Zhang [8]), the complete convergence (cf. Liang and Su [9], Liang [10]), and the law of the iterated logarithm (see, e.g., Shao and Su [11], Zhang [12]), and so forth.

Note that in the above-mentioned limit theorems, the convergence rates of logarithm are little known, the purpose of the present paper is to investigate the precise asymptotics in the law of the logarithm for NA sequences. It is well known that NA sequences can contain independent random variables as special case, many authors have given lots of beautiful results for independent variables. Let us first recall parts of those results, it is very convenient to adopt the following notations: let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) nondegenerate random variables with $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$, set $S_n = X_1 + \cdots + X_n$, $\log x = \log_e(x \vee e)$. Chow and Lai [13] studied the following results.

Theorem 1.2. *Suppose that* $\operatorname{Var} X_1 = \sigma^2$ *and* $\alpha \geq 1$. *Then the following are equivalent:*

$$\begin{gathered}
\sum_{n=1}^{\infty} n^{\alpha-2} P\left(|S_n| \geq \epsilon\sqrt{2n\log n}\right) < \infty, \quad \forall \epsilon > \sigma\sqrt{\alpha-1}; \\
\sum_{n=1}^{\infty} n^{\alpha-2} P\left(\max_{1\leq k\leq n} |S_k| \geq \epsilon\sqrt{2n\log n}\right) < \infty, \quad \forall \epsilon > \sigma\sqrt{\alpha-1}; \\
\sum_{n=1}^{\infty} n^{\alpha-2} P\left(|S_n| \geq \epsilon\sqrt{2n\log n}\right) < \infty, \quad \text{for some } \epsilon > 0; \\
EX_1 = 0, \qquad \frac{E|X_1|^{2\alpha}}{(\log|X_1|)^{\alpha}} < \infty.
\end{gathered} \tag{1.4}$$

Heyde [14] presented an interesting and beautiful result.

Theorem 1.3. *If* $EX_1 = 0$ *and* $EX_1^2 < \infty$, *then*

$$\lim_{\epsilon\downarrow 0} \epsilon^2 \sum_{n=1}^{\infty} P(|S_n| \geq \epsilon n) = EX_1^2. \tag{1.5}$$

This is a precise estimate for the convergence rate of probability series as $\epsilon \downarrow 0$, which has been generalized and extended in several directions. For $\alpha = 1$ in Theorem 1.2, Gut and Spătaru [15] obtained the results as follows.

Theorem 1.4. *Suppose that $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$. Then, for $0 \le \delta \le 1$,*

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\delta+2} \sum_{n=1}^{\infty} \frac{(\log n)^{\delta}}{n} P\left(|S_n| \ge \epsilon \sqrt{n \log n} \right) = \frac{\sigma^{2\delta+2} E|N|^{2\delta+2}}{\delta+1}, \tag{1.6}$$

where N is a standard normal random variable.

Our starting point is Theorem 1.4, the present work will give the analogue of (1.6) for NA sequences. From now on, we adopt the following notations: let $X_1, X_2, \ldots$ be strictly stationary NA sequences with $EX_1 = 0$ and $EX_1^2 < \infty$, $\sigma^2 = EX_1^2 + 2\sum_{n=2}^{\infty} EX_1X_n > 0$, and set $S_n = X_1 + \cdots + X_n$, $M_n = \max_{1 \le k \le n} |S_k|$, write log for the natural logarithm, $\log x = \log_e(x \vee e)$, $[z]$ denotes the largest integer which is not larger than z, C denotes positive constant, independent of ϵ, it may take different values in each appearance. The paper is organized as follows: we first introduce our main results, after which the proofs of Theorems 2.1 and 2.4 are exposed in Sections 3 and 4, respectively. We now state the main results.

2. Main results

Theorem 2.1. *Let $\kappa_n = O(1/\log n)$, $EX_1^2 < \infty$, if $-1 < \alpha \le 1$; $EX_1^2(\log|X_1|)^{\alpha} < \infty$, if $\alpha > 1$. Then*

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^{\alpha}}{n} P\left(|S_n| \ge \sigma(\epsilon + \kappa_n) \sqrt{2n \log n} \right) \\ = 2^{-(\alpha+1)} (\alpha+1)^{-1} E|N|^{2\alpha+2}, \end{aligned} \tag{2.1}$$

where N is a standard normal random variable.

Corollary 2.2. *Under the conditions in Theorem 2.1,*

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^{\alpha}}{n} P\left(|S_n| \ge \sigma\epsilon \sqrt{2n \log n} \right) = \frac{E|N|^{2\alpha+2}}{2^{(\alpha+1)}(\alpha+1)}. \tag{2.2}$$

Corollary 2.3. *Suppose that $EX_1^2 < \infty$. Then*

$$\lim_{\epsilon \downarrow 0} \epsilon^{2} \sum_{n=1}^{\infty} n^{-1} P\left(|S_n| \ge \sigma(\epsilon + \kappa_n) \sqrt{2n \log n} \right) = \frac{EN^2}{2}. \tag{2.3}$$

Theorem 2.4. *Let $\kappa_n = O(1/\log n)$, $EX_1^2 < \infty$, if $-1 < \alpha \le 1/2$; $EX_1^2(\log|X_1|)^{\alpha} < \infty$, if $\alpha > 1/2$. Then*

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^{\alpha}}{n} P\left(M_n \ge \sigma(\epsilon + \kappa_n) \sqrt{2n \log n} \right) \\ = 2^{-\alpha} (\alpha+1)^{-1} E|N|^{2\alpha+2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2\alpha+2}}. \end{aligned} \tag{2.4}$$

Without loss of generality, throughout the paper, we will suppose that $\sigma^2 = 1$. Let $\Phi(x)$ denote the standard normal distribution function, and put $\Psi(x) = 1 - \Phi(x) + \Phi(-x)$, $x \geq 0$.

3. Proof of Theorem 2.1

In order to prove this result easily, we separate the proof into two propositions, the first one can be formulated as follows.

Proposition 3.1. *Suppose that N be a nondegenerate Gaussian random variable. Then*

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^{\alpha}}{n} P\left(|N| \geq (\epsilon + \kappa_n)\sqrt{2\log n}\right) = 2^{-(\alpha+1)}(\alpha+1)^{-1} E|N|^{2(\alpha+1)}. \tag{3.1}$$

Proof. Noting the definition of κ_n, we first show that

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^{\alpha}}{n} P\left(|N| \geq \epsilon\sqrt{2\log n}\right) = 2^{-(\alpha+1)}(\alpha+1)^{-1} E|N|^{2\alpha+2}. \tag{3.2}$$

By integral formula and transformation, it is enough to show that for any $\alpha > -1$,

$$\begin{aligned} &\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^{\alpha}}{n} P\left(|N| \geq \epsilon\sqrt{2\log n}\right) \\ &= \lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \int_{n}^{n+1} \frac{(\log x)^{\alpha}}{x} P\left(|N| \geq \epsilon\sqrt{2\log x}\right) dx \\ &= 2^{-\alpha} \int_{0}^{\infty} y^{2\alpha+1} P(|N| \geq y)\, dy \\ &= 2^{-(\alpha+1)}(\alpha+1)^{-1} E|N|^{2\alpha+2}. \end{aligned} \tag{3.3}$$

Write

$$A_n(\epsilon) = \left| P\left(|N| \geq \epsilon\sqrt{2\log n}\right) - P\left(|N| \geq (\epsilon + \kappa_n)\sqrt{2\log n}\right) \right|. \tag{3.4}$$

The proof of (3.1) should be completed, if one could show that

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^{\alpha}}{n} A_n(\epsilon) = 0, \tag{3.5}$$

the proof of (3.5) is similar to that of Proposition 2.2 in Huang and Zhang [16].

Before giving the second proposition, the following lemma is necessary. □

Lemma 3.2 [17]. *Suppose that $\{X_k : k \geq 1\}$ be NA sequences with $EX_k = 0$, $E|X_k|^p < \infty$, for $p \geq 2$. Then, for any $t > p/2$, $x > 0$,*

$$P(|S_n| \geq x) \leq \sum_{k=1}^{n} P\left(|X_k| \geq \frac{x}{t}\right) + 2e^t\left(1 + \frac{x^2}{t\sum_{k=1}^{n} EX_k^2}\right)^{-t}. \tag{3.6}$$

Proposition 3.3. *Suppose that $EX_1^2 < \infty$, if $-1 < \alpha \leq 1$; $EX_1^2(\log|X_1|)^\alpha < \infty$, if $\alpha > 1$. Then*

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^\alpha}{n} \left| P\left(|S_n| \geq (\epsilon + \kappa_n)\sqrt{2n\log n}\right) - P\left(|N| \geq (\epsilon + \kappa_n)\sqrt{2\log n}\right) \right| = 0. \tag{3.7}$$

Proof. Set $H(\epsilon) = [\exp(M/\epsilon^2)]$, where $M > 4$, $0 < \epsilon < 1/4$. It is easy to get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(\log n)^\alpha}{n} \left| P\left(|S_n| \geq (\epsilon + \kappa_n)\sqrt{2n\log n}\right) - P\left(|N| \geq (\epsilon + \kappa_n)\sqrt{2\log n}\right) \right| \\
&\quad = \sum_{n \leq H(\epsilon)} \frac{(\log n)^\alpha}{n} \left| P\left(|S_n| \geq (\epsilon + \kappa_n)\sqrt{2n\log n}\right) - P\left(|N| \geq (\epsilon + \kappa_n)\sqrt{2\log n}\right) \right| \\
&\qquad + \sum_{n > H(\epsilon)} \frac{(\log n)^\alpha}{n} \left| P\left(|S_n| \geq (\epsilon + \kappa_n)\sqrt{2n\log n}\right) - P\left(|N| \geq (\epsilon + \kappa_n)\sqrt{2\log n}\right) \right| \triangleq I_1 + I_2.
\end{aligned} \tag{3.8}$$

We first consider I_1. Let $\Delta_n = \sup_x |P(|S_n| \geq x\sqrt{n}) - P(|N| \geq x)|$, noting Theorem 1.1, since $\Psi(x)$ is a continuous function, then, for any $x \geq 0$, we have $\lim_{n\to\infty} \Delta_n = 0$. It follows that

$$\begin{aligned}
\epsilon^{2\alpha+2} I_1 &\leq \epsilon^{2\alpha+2} \sum_{n \leq H(\epsilon)} \frac{(\log n)^\alpha}{n} \Delta_n = \epsilon^{2\alpha+2} \sum_{n \leq H(\epsilon)} \frac{(\log n)^\alpha}{n} \Delta_n \\
&\leq C\epsilon^{2\alpha+2} (\log n)^{\alpha+1} \frac{1}{(\log n)^{\alpha+1}} \sum_{n \leq H(\epsilon)} \frac{(\log n)^\alpha}{n} \Delta_n \\
&\leq CM^{\alpha+1} \frac{1}{(\log n)^{\alpha+1}} \sum_{n \leq H(\epsilon)} \frac{(\log n)^\alpha}{n} \Delta_n \longrightarrow 0, \quad \text{as } \epsilon \downarrow 0.
\end{aligned} \tag{3.9}$$

Note that $(1/(\log n)^{\alpha+1}) \sum_{n \leq H(\epsilon)} ((\log n)^\alpha/n)\Delta_n \to 0$, $\epsilon \downarrow 0$, then (3.9) holds. Turn to I_2, one can get

$$\begin{aligned}
I_2 &\leq \sum_{n > H(\epsilon)} \frac{(\log n)^\alpha}{n} P\left(|N| \geq (\epsilon + \kappa_n)\sqrt{2\log n}\right) \\
&\quad + \sum_{n > H(\epsilon)} \frac{(\log n)^\alpha}{n} P\left(|S_n| \geq (\epsilon + \kappa_n)\sqrt{2n\log n}\right) \triangleq I_3 + I_4.
\end{aligned} \tag{3.10}$$

Without loss of generality, we can assume that $0 < \epsilon < 1/4$, $M > 4$, then one can get $H(\epsilon) - 1 \geq \sqrt{H(\epsilon)}$. Note, in particular, that the definition of κ_n, for n large enough, we have $|\kappa_n| < \epsilon/4$. Then, for I_3, it follows that

$$\begin{aligned}
\epsilon^{2\alpha+2} I_3 &\leq \epsilon^{2\alpha+2} \sum_{n>H(\epsilon)} \frac{(\log n)^\alpha}{n} P\left(|N| \geq \frac{\epsilon}{2}\sqrt{2\log n}\right) \\
&\leq \epsilon^{2\alpha+2} \int_{H(\epsilon)-1}^{\infty} \frac{(\log x)^\alpha}{x} P\left(|N| \geq \frac{\epsilon}{2}\sqrt{2logx}\right) dx \\
&\leq \epsilon^{2\alpha+2} \int_{\sqrt{H(\epsilon)}}^{\infty} \frac{(\log x)^\alpha}{x} P\left(|N| \geq \frac{\epsilon}{2}\sqrt{2logx}\right) dx \\
&\leq C \int_{\sqrt{M/4}}^{\infty} y^{2\alpha+1} P(|N| > y)\, dy \longrightarrow 0, \quad \text{as } M \longrightarrow \infty,
\end{aligned} \tag{3.11}$$

uniformly with respect to $0 < \epsilon < 1/4$. We finally estimate I_4, by Lemma 3.2, which yields, for n large enough,

$$\begin{aligned}
P\left(|S_n| \geq (\epsilon + \kappa_n)\sqrt{2n\log n}\right) &\leq P\left(|S_n| \geq \frac{\epsilon}{2}\sqrt{2n\log n}\right) \\
&\leq nP\left(|X_1| \geq \frac{\epsilon}{2m}\sqrt{2n\log n}\right) + 2e^m \left(1 + \frac{\epsilon^2 \log n}{2mEX_1^2}\right)^{-m} \\
&\triangleq I_5 + I_6,
\end{aligned} \tag{3.12}$$

where m is a positive integer to be specified later. Then, observe that $n > H(\epsilon)$ implies $\log n > M/\epsilon^2$, for I_5, if $-1 < \alpha \leq 1$, the proof is similar to that of Lemma 3.2 [15]; if $\alpha > 1$, applying Fubini's theorem, it turns out that

$$\begin{aligned}
\sum_{n>H(\epsilon)} \frac{(\log n)^\alpha}{n} I_5 &= \sum_{n>H(\epsilon)} (\log n)^\alpha P\left(|X_1| \geq \frac{\epsilon\sqrt{2n\log n}}{2m}\right) \\
&\leq C \sum_{n>H(\epsilon)} (\log n)^\alpha \sum_{j=n}^{\infty} P\left(\frac{\sqrt{2Mj}}{2m} \leq |X_1| < \frac{\sqrt{2M(j+1)}}{2m}\right) \\
&\leq C \sum_{j>H(\epsilon)} P\left(\frac{\sqrt{2Mj}}{2m} \leq |X_1| < \frac{\sqrt{2M(j+1)}}{2m}\right) \sum_{n=H(\epsilon)}^{j} (\log n)^\alpha \\
&\leq C \sum_{j>H(\epsilon)} j(\log j)^\alpha P\left(j \leq \frac{2m^2 X_1^2}{M} < j+1\right) \leq \frac{CEX_1^2(\log|X_1|)^\alpha}{M}.
\end{aligned} \tag{3.13}$$

Furthermore, one can easily get $\limsup_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n>H(\epsilon)} (\log n)^\alpha I_5/n = 0$. We finally estimate I_6, by the arbitrarity of $m(>1)$, one can obviously choose an appropriate positive

integer m, such that $m > \alpha + 1$. Then we have

$$\begin{aligned}
\sum_{n>H(\epsilon)} \frac{(\log n)^{\alpha}}{n} I_6 &\le C \sum_{n>H(\epsilon)} \frac{(\log n)^{\alpha}}{n} \left(\frac{\epsilon^2 \log n}{2mEX_1^2} \right)^{-m} \\
&\le C \sum_{n>H(\epsilon)} \frac{(\log n)^{\alpha}}{n} (\epsilon^2 \log n)^{-m} \\
&\le C\epsilon^{-2m} \int_{H(\epsilon)-1}^{\infty} \left(\frac{(\log x)^{\alpha-m}}{x} \right) dx \\
&\le C\epsilon^{-2m} (\log(H(\epsilon)))^{\alpha+1-m} \\
&\le C\epsilon^{-2\alpha-2} M^{\alpha+1-m},
\end{aligned} \tag{3.14}$$

it is easy to get $\lim_{M\to\infty} \epsilon^{2\alpha+2} \sum_{n>H(\epsilon)} (\log n)^{\alpha} I_6/n = 0$, uniformly with respect to $0 < \epsilon < 1/4$. Thus the proof of Proposition 3.3 is completed. □

Proof of Theorem 2.1. Combining Propositions 3.1 and 3.3, one can complete the proof of this theorem immediately. □

4. Proof of Theorem 2.4

The following propositions will simplify the proof of Theorem 2.4, which are stated as follows.

Proposition 4.1. *Suppose that $\{W(t) : t \ge 0\}$ be a standard Wiener process (Brownian motion). Then*

$$\begin{aligned}
\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^{\alpha}}{n} P\left(\sup_{0\le s\le 1} |W(s)| \ge (\epsilon + \kappa_n)\sqrt{2\log n} \right) \\
= 2^{-\alpha}(\alpha+1)^{-1} E|N|^{2\alpha+2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2\alpha+2}}.
\end{aligned} \tag{4.1}$$

Proof. Noting the result of Billingsley [18],

$$\begin{aligned}
P\left(\sup_{0\le s\le 1} |W(s)| \ge x \right) &= 1 - \sum_{k=-\infty}^{\infty} (-1)^k P((2k-1)x \le N \le (2k+1)x) \\
&= 4\sum_{k=0}^{\infty} (-1)^k P(N \ge (2k+1)x) = 2\sum_{k=0}^{\infty} (-1)^k P(|N| \ge (2k+1)x),
\end{aligned} \tag{4.2}$$

where N is the standard normal random variable. Then, according to Proposition 3.1, one can complete the proof easily. □

Lemma 4.2 [7]. *Suppose that $\{X_n : n \geq 1\}$ be strictly stationary NA sequences, $EX_1 = \mu$, $0 < \operatorname{Var} X_1 = \sigma^2 < \infty$, and $B^2 = EX_1^2 + 2\sum_{n=2}^{\infty} EX_1X_n > 0$, set $S_m = \sum_{k=1}^{m} X_k$, write*

$$W_n(t) = \frac{1}{B\sqrt{n}}\left(S_m + (nt - m)X_{m+1} - nt\mu\right), \quad m \leq n < m+1,\ 0 \leq t \leq T. \tag{4.3}$$

Then

$$W_n(t) \xrightarrow{\mathscr{D}} W(t) \quad \text{in } C[0,T], \tag{4.4}$$

where $W(t)$ is the standard Wiener process and $C[0,T]$ is the usual C space on $[0,T]$.

Lemma 4.3 [11]. *Let $\{X_n : n \geq 1\}$ be a sequence of NA random variable with mean zero and finite second moments. Set $S_n = X_1 + \cdots + X_n$ and $B_n^2 = \sum_{k=1}^{n} EX_k^2$. Then for all $x > 0$, $a > 0$, and $0 < \beta < 1$,*

$$P\left(\max_{1\leq k\leq n} |S_k| \geq x\right) \leq 2P\left(\max_{1\leq k\leq n} |X_k| \geq a\right) + \frac{2}{1-\beta}\exp\left(-\frac{\beta x^2}{2(ax + B_n^2)}\right). \tag{4.5}$$

Proposition 4.4. *Suppose that $EX_1^2 < \infty$, if $-1 < \alpha \leq 1/2$; $EX_1^2(\log|X_1|)^\alpha < \infty$, if $\alpha > 1/2$. Then*

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\epsilon^{2\alpha+2}\sum_{n\geq 1}\frac{(\log n)^\alpha}{n}\Big|P\Big(M_n \geq (\epsilon+\kappa_n)\sqrt{2n\log n}\Big) \\ - P\Big(\sup_{0\leq s\leq 1}|W(s)| \geq (\epsilon+\kappa_n)\sqrt{2\log n}\Big)\Big| = 0.\end{aligned} \tag{4.6}$$

Proof. Let $H(\epsilon)$ be as above, it follows that

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(\log n)^\alpha}{n}\left|P\left(M_n \geq (\epsilon+\kappa_n)\sqrt{2n\log n}\right) - P\left(\sup_{0\leq s\leq 1}|W(s)| \geq (\epsilon+\kappa_n)\sqrt{2\log n}\right)\right| \\ &\quad = \sum_{n\leq H(\epsilon)}\frac{(\log n)^\alpha}{n}\left|P\left(M_n \geq (\epsilon+\kappa_n)\sqrt{2n\log n}\right) - P\left(\sup_{0\leq s\leq 1}|W(s)| \geq (\epsilon+\kappa_n)\sqrt{2\log n}\right)\right| \\ &\qquad + \sum_{n>H(\epsilon)}\frac{(\log n)^\alpha}{n}\left|P\left(M_n \geq (\epsilon+\kappa_n)\sqrt{2n\log n}\right) - P\left(\sup_{0\leq s\leq 1}|W(s)| \geq (\epsilon+\kappa_n)\sqrt{2\log n}\right)\right| \\ &\quad \triangleq I_1' + I_2'.\end{aligned} \tag{4.7}$$

Noting Lemma 4.2, we have $M_n/\sqrt{n} \xrightarrow{\mathscr{D}} \sup_{0\le t\le 1}|W(t)|$, as $n\to\infty$. Similar to Theorem 2.1, one can get $\lim_{\epsilon\downarrow 0}\epsilon^{2\alpha+2}I_1'=0$. We now estimate I_2', it turns out that

$$\begin{aligned} I_2' &\le \sum_{n>H(\epsilon)} \frac{(\log n)^\alpha}{n} P\Big(\sup_{0\le s\le 1}|W(s)| \ge (\epsilon+\kappa_n)\sqrt{2\log n}\Big) \\ &\quad + \sum_{n>H(\epsilon)} \frac{(\log n)^\alpha}{n} P\Big(\max_{1\le k\le n}|S_k| \ge (\epsilon+\kappa_n)\sqrt{2n\log n}\Big) \triangleq I_3'+I_4'. \end{aligned} \tag{4.8}$$

Observe that $P(\sup_{0\le s\le 1}|W(s)|\ge x)\le 2P(|N|\ge x)$, see [18]. Similar to Theorem 2.1, we have $\lim_{\epsilon\downarrow 0}\epsilon^{2\alpha+2}I_3'=0$. We then consider I_4', as a matter of fact, by Lemma 4.3, take $x=\epsilon\sqrt{2n\log n}/2$, $a=(2n\epsilon\sqrt{\log n})^{1/2}$. For n large enough, one could get

$$\begin{aligned} P\Big(\max_{1\le k\le n}|S_k| \ge (\epsilon+\kappa_n)\sqrt{2n\log n}\Big) &\le P\Big(\max_{1\le k\le n}|S_k| \ge \frac{\epsilon}{2}\sqrt{2n\log n}\Big) \\ &\le 2nP\Big(|X_1| \ge \Big(2n\epsilon\sqrt{\log n}\Big)^{1/2}\Big) + \frac{2}{1-\beta}\exp\left(-\frac{\beta\epsilon^2\log n}{8\Big(\Big(\epsilon\sqrt{\log n}\Big)^{3/2}+1\Big)}\right) \triangleq I_5'+I_6'. \end{aligned} \tag{4.9}$$

Without loss of generality, we can assume $0<\epsilon<1/4$, $M>16$, Notice that $n>H(\epsilon)$ if and only if $\log n > M/\epsilon^2$, then for I_6', we have

$$\begin{aligned} \epsilon^{2\alpha+2}\sum_{n>H(\epsilon)}\frac{(\log n)^\alpha I_6'}{n} &\le C\epsilon^{2\alpha+2}\sum_{n>H(\epsilon)}(\log n)^\alpha \exp\frac{\Big(-\beta\Big(\epsilon\sqrt{\log n}\Big)^{1/2}/9\Big)}{n} \\ &\le C\epsilon^{2\alpha+2}\int_{H(\epsilon)-1}^{\infty}(\log x)^\alpha\exp\left(\frac{-\beta\Big(\epsilon\sqrt{\log x}\Big)^{1/2}}{9}\right)\frac{dx}{x} \\ &\le C\epsilon^{2\alpha+2}\int_{\sqrt{H(\epsilon)}}^{\infty}(\log x)^\alpha\exp\left(\frac{-\beta\Big(\epsilon\sqrt{\log x}\Big)^{1/2}}{9}\right)\frac{dx}{x} \\ &\le C\epsilon^{2\alpha+2}\int_{\beta\sqrt[4]{M}/9}^{\infty}\epsilon^{-2\alpha-2}y^{4\alpha+3}\exp(-y)dy \\ &\le C\int_{\beta\sqrt[4]{M}/9}^{\infty}y^{4\alpha+3}\exp(-y)dy \longrightarrow 0, \quad \text{as } M\longrightarrow\infty. \end{aligned} \tag{4.10}$$

We then estimate I_5'. If $-1 < \alpha \le 1/2$, the following result is obvious according to $EX_1^2 < \infty$; if $\alpha > 1/2$, by Fubini's theorem, it follows that

$$\begin{aligned}
\sum_{n>H(\epsilon)} \frac{(\log n)^\alpha I_5'}{n} &= \sum_{n>H(\epsilon)} (\log n)^\alpha P\left(|X_1| \ge \left(2n\epsilon\sqrt{\log n}\right)^{1/2}\right) \\
&\le C \sum_{n>H(\epsilon)} (\log n)^\alpha \sum_{k=n}^{\infty} P\left(\sqrt{k} \le \frac{|X_1|}{\sqrt[4]{4M}} < \sqrt{k+1}\right) \\
&\le C \sum_{k>H(\epsilon)} P\left(\sqrt{k} \le \frac{|X_1|}{\sqrt[4]{4M}} < \sqrt{k+1}\right) \sum_{n=H(\epsilon)}^{k} (\log n)^\alpha \\
&\le C \sum_{k>H(\epsilon)} k(\log k)^\alpha P\left(\sqrt{k} \le \frac{|X_1|}{\sqrt[4]{4M}} < \sqrt{k+1}\right) \\
&\le CEX_1^2 \frac{(\log|X_1|)^\alpha}{\sqrt{4M}} < \infty.
\end{aligned} \tag{4.11}$$

□

Proof of Theorem 2.4. The proof follows from Propositions 4.1 and 4.4. □

Acknowledgments

The author would like to express his deep thanks to the referee for careful reading and valuable suggestions. This work is supported by the Natural Science Foundation of Department of Education of Zhejiang Province (Grant no. 20060237).

References

[1] K. Alam and K. M. L. Saxena, "Positive dependence in multivariate distributions," *Communications in Statistics*, vol. 10, no. 12, pp. 1183–1196, 1981.

[2] K. Joag-Dev and F. Proschan, "Negative association of random variables, with applications," *The Annals of Statistics*, vol. 11, no. 1, pp. 286–295, 1983.

[3] R. E. Barlow and F. Proschan, *Statistical Theory of Reliability and Life Testing: Probability Models*, To Begin With, Silver Spring, Md, USA, 1981.

[4] C. M. Newman, "Asymptotic independence and limit theorems for positively and negatively dependent random variables," in *Inequalities in Statistics and Probability (Lincoln, Neb., 1982)*, Y. L. Tong, Ed., vol. 5 of *IMS Lecture Notes Monogr. Ser.*, pp. 127–140, Inst. Math. Statist., Hayward, Calif, USA, 1984.

[5] P. Matuła, "A note on the almost sure convergence of sums of negatively dependent random variables," *Statistics & Probability Letters*, vol. 15, no. 3, pp. 209–213, 1992.

[6] G. G. Roussas, "Exponential probability inequalities with some applications," in *Statistics, Probability and Game Theory*, T. S. Ferguson, L. S. Shapley, and J. B. MacQueen, Eds., vol. 30 of *IMS Lecture Notes Monogr. Ser.*, pp. 303–319, Inst. Math. Statist., Hayward, Calif, USA, 1996.

[7] Q.-M. Shao, "A comparison theorem on moment inequalities between negatively associated and independent random variables," *Journal of Theoretical Probability*, vol. 13, no. 2, pp. 343–356, 2000.

[8] L.-X. Zhang, "The weak convergence for functions of negatively associated random variables," *Journal of Multivariate Analysis*, vol. 78, no. 2, pp. 272–298, 2001.

[9] H.-Y. Liang and C. Su, "Complete convergence for weighted sums of NA sequences," *Statistics & Probability Letters*, vol. 45, no. 1, pp. 85–95, 1999.

[10] H.-Y. Liang, "Complete convergence for weighted sums of negatively associated random variables," *Statistics & Probability Letters*, vol. 48, no. 4, pp. 317–325, 2000.

[11] Q.-M. Shao and C. Su, "The law of the iterated logarithm for negatively associated random variables," *Stochastic Processes and Their Applications*, vol. 83, no. 1, pp. 139–148, 1999.

[12] L. X. Zhang, "Some limit results on the law of the iterated logarithm for NA sequences," *Acta Mathematica Sinica. Chinese Series*, vol. 47, no. 3, pp. 541–552, 2004 (Chinese).

[13] Y. S. Chow and T. L. Lai, "Some one-sided theorems on the tail distribution of sample sums with applications to the last time and largest excess of boundary crossings," *Transactions of the American Mathematical Society*, vol. 208, pp. 51–72, 1975.

[14] C. C. Heyde, "A supplement to the strong law of large numbers," *Journal of Applied Probability*, vol. 12, no. 1, pp. 173–175, 1975.

[15] A. Gut and A. Spătaru, "Precise asymptotics in the Baum-Katz and Davis laws of large numbers," *Journal of Mathematical Analysis and Applications*, vol. 248, no. 1, pp. 233–246, 2000.

[16] W. Huang and L. Zhang, "Precise rates in the law of the logarithm in the Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 304, no. 2, pp. 734–758, 2005.

[17] C. Su, L. C. Zhao, and Y. B. Wang, "The moment inequalities and weak convergence for negatively associated sequences," *Science in China, Series A*, vol. 26, no. 12, pp. 1091–1099, 1996 (Chinese).

[18] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, New York, NY, USA, 1968.

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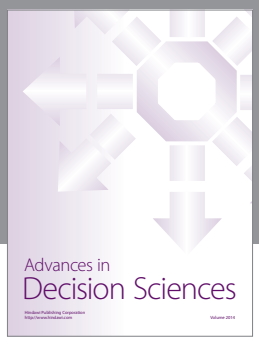

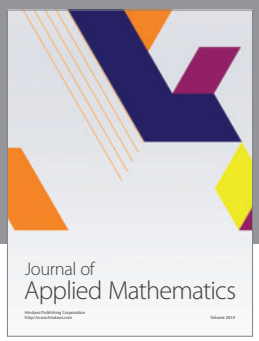

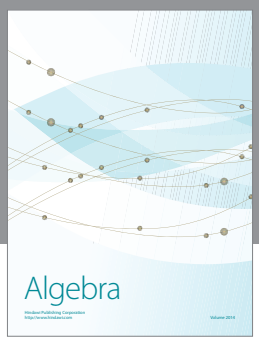

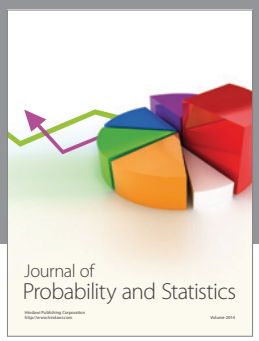

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