Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2008, Article ID 140548, 10 pages
doi:10.1155/2008/140548

Research Article

On Chung-Teicher Type Strong Law of Large Numbers for ρ^*-Mixing Random Variables

Anna Kuczmaszewska

Department of Applied Mathematics, Lublin University of Technology, Nadbystrzycka 38 D, 20-618 Lublin, Poland

Correspondence should be addressed to Anna Kuczmaszewska, a.kuczmaszewska@pollub.pl

Received 23 December 2007; Revised 05 March 2008; Accepted 20 March 2008

Recommended by Stevo Stevic

In this paper the classical strong laws of large number of Kolmogorov, Chung, and Teicher for independent random variables were generalized on the case of ρ^*-mixing sequence. The main result was applied to obtain a Marcinkiewicz SLLN.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with value in a real space $\mathbb{R}$ and let $S_n = \sum_{i=1}^{n} X_i$. We say that the sequence $\{X_n, n \geq 1\}$ satisfies the strong law of large numbers (SLLN) if there exists some increasing sequence $\{b_n, n \geq 1\}$ and some sequence $\{a_n, n \geq 1\}$ such that

$$\frac{\sum_{i=1}^{n}(X_i - a_i)}{b_n} \longrightarrow 0 \quad \text{a.s. as } n \longrightarrow \infty. \tag{1.1}$$

In this paper, we consider the strong law of large numbers for sequences of dependent random variables which are said to be ρ^*-mixing. To introduce the concept of ρ^*-mixing sequence, we need the maximal correlation coefficient defined as follows:

$$\rho^*(k) = \sup_{S,T}\left(\sup_{X\in L^2(\mathcal{F}_S),\, Y\in L^2(\mathcal{F}_T)} \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{Var} X \cdot \operatorname{Var} Y}}\right), \tag{1.2}$$

where S, T are the finite subsets of positive integers such that dist $(S,T) = \inf_{x\in S, y\in T}|x-y| \geq k$ and $\mathcal{F}_W$ is the σ-field generated by the random variables $\{X_i, i \in W \subset \mathbb{N}\}$.

Definition 1.1. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be a ρ^*-mixing sequence if

$$\lim_{n\to\infty} \rho^*(n) < 1. \tag{1.3}$$

ρ^*-mixing random variables were investigated by many authors. Various moment inequalities for sums and maximum of partial sums can be found in papers by Bradley [1], Bryc and Smoleński [2], Peligrad [3], Peligrad and Gut [4], and Utev and Peligrad [5]. These inequalities are used in many papers concerning the problems of invariance principle (Utev and Peligrad [5]), CLT (Peligrad [3]), or complete convergence for some stochastically dominated sequence of ρ^*-mixing random variables (Cai [6]), and for an array of rowwise ρ^*-mixing random variables (Zhu [7]). They will be also important in our further consideration.

The aim of this paper is to give some sufficient conditions for SLLN for a sequence of ρ^*-mixing random variables without assumptions of identical distribution and stochastical domination. The result presented in this paper is obtained by using the maximal type inequality and the following strong law of large numbers proved by Fazekas and Klesov [8].

Theorem 1.2 (Fazekas and Klesov [8]). *Let $\{b_n, n \geq 1\}$ be a nondecreasing, unbounded sequence of positive numbers. Let $\{\alpha_n, n \geq 1\}$ be nonnegative numbers. Let r be a fixed positive number. Assume that for each $n \geq 1$:*

$$E\left[\max_{1\leq i\leq n} |S_i|\right]^r \leq \sum_{i=1}^{n} \alpha_i. \tag{1.4}$$

If

$$\sum_{i=1}^{\infty} \frac{\alpha_i}{b_i^r} < \infty, \tag{1.5}$$

then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \quad a.s. \tag{1.6}$$

Using this theorem, we are going to show that classical Kolmogorov, Chung, and Teicher's strong law of large numbers for independent random variables $\{X_n, n \geq 1\}$ (Chung [9] and Teicher [10]) can be generalized to the case of ρ^*-mixing sequences.

In our further consideration, we need the following result.

Lemma 1.3 (Utev and Peligrad [5]). *Let $\{X_n, n \geq 1\}$ be a ρ^*-mixing sequence with $EX_n = 0$, $E|X_n|^q < \infty$, $n \geq 1$, $q \geq 2$. Then, there exists a positive constant c such that*

$$E \max_{1\leq k\leq n} \left|\sum_{i=1}^{k} X_i\right|^q \leq c\left[\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2\right)^{q/2}\right], \quad \forall n \geq 1. \tag{1.7}$$

Let C denote a constant which is not necessary the same in its each appearance.

2. The main result

Let $\{\varphi_n,\ n \geq 1\}$ be a sequence of nonnegative, even, continuous and nondecreasing on $(0,\infty)$ functions $\varphi_n : \mathbb{R} \to \mathbb{R}^+$ with $\lim_{x\to\infty}\varphi_n(x) = \infty$ and such that for all $n \geq 1$ and $1 < p \leq 2$:

$$\text{(a)} \quad \varphi_n(x)/x \searrow \quad \text{or} \quad \text{(b)} \quad \varphi_n(x)/x \nearrow, \quad \varphi_n(x)/x^p \searrow \quad \text{as } x \longrightarrow \infty. \tag{2.1}$$

Theorem 2.1. *Let $\{X_n,\ n \geq 1\}$ be a sequence of ρ^*-mixing random variables and let $\{b_n,\ n \geq 1\}$ be an increasing sequence of positive real numbers. Let $1 < p \leq 2$.*

Assume that $\{\varphi_n,\ n \geq 1\}$ satisfies (a) in (2.1) with

$$\text{(A)} \quad \sum_{i=2}^{\infty} b_i^{-p} E\left(\frac{\varphi_i^p(|X_i|)}{\varphi_i^p(b_i) + \varphi_i^p(|X_i|)}\right) \sum_{j=1}^{i-1} b_j^p E\left(\frac{\varphi_j^p(|X_j|)}{\varphi_j^p(b_j) + \varphi_j^p(|X_j|)}\right) < \infty, \tag{2.2}$$

$$\text{(B)} \quad \sum_{n=1}^{\infty} P\left[|X_n| \geq a_n\right] < \infty, \tag{2.3}$$

for some sequence $\{a_n,\ n \geq 1\}$ of positive numbers such that

$$\text{(C)} \quad \sum_{n=1}^{\infty} E\left(\frac{\varphi_n^{2p}\left[\min\left(|X_n|, a_n\right)\right]}{\varphi_n^{2p}(b_n) + \varphi_n^{2p}(|X_n|)}\right) < \infty, \tag{2.4}$$

or $\{\varphi_n,\ n \geq 1\}$ satisfies (b) in (2.1) with

$$\text{(A}_1\text{)} \quad \sum_{i=2}^{\infty} b_i^{-p} E\left(\frac{\varphi_i(|X_i|)}{\varphi_i(b_i) + \varphi_i(|X_i|)}\right) \sum_{j=1}^{i-1} b_j^p E\left(\frac{\varphi_j(|X_j|)}{\varphi_j(b_j) + \varphi_j(|X_j|)}\right) < \infty, \tag{2.5}$$

and (B) for some sequence $\{a_n,\ n \geq 1\}$ of positive numbers such that

$$\text{(C}_1\text{)} \quad \sum_{n=1}^{\infty} E\left(\frac{\varphi_n^{2}\left[\min\left(|X_n|, a_n\right)\right]}{\varphi_n^{2}(b_n) + \varphi_n^{2}(|X_n|)}\right) < \infty. \tag{2.6}$$

Then,

$$b_n^{-1} \sum_{i=1}^{n} \left\{X_i - E\left(X_i I\left[|X_i| < b_i\right]\right)\right\} \longrightarrow 0 \quad a.s.\ n \longrightarrow \infty. \tag{2.7}$$

Proof. Let $X_i' = X_i I[|X_i| < b_i]$, $X_i^* = X_i' - EX_i'$, $S_n' = \sum_{i=1}^n X_i^*$, and $S_n = \sum_{i=1}^n X_i$.

Then,

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left[X_n' \neq X_n\right] \\
&\quad = \sum_{n=1}^{\infty} P\left[|X_n| \geq b_n\right] \leq \sum_{n=1}^{\infty} P\left[2\varphi_n^{2r}(|X_n|) \geq \varphi_n^{2r}(b_n) + \varphi_n^{2r}(|X_n|)\right] \\
&\quad \leq C \sum_{n=1}^{\infty} \left\{E\left(\frac{\varphi_n^{2r}(|X_n|)}{\varphi_n^{2r}(b_n) + \varphi_n^{2r}(|X_n|)} \cdot I\left[|X_n| \geq a_n\right]\right) + E\left(\frac{\varphi_n^{2r}(|X_n|)}{\varphi_n^{2r}(b_n) + \varphi_n^{2r}(|X_n|)} \cdot I\left[|X_n| < a_n\right]\right)\right\} \\
&\quad \leq C\left\{\sum_{n=1}^{\infty} P\left[|X_n| \geq a_n\right] + \sum_{n=1}^{\infty} E\left(\frac{\varphi_n^{2r}\left[\min\left(|X_n|, a_n\right)\right]}{\varphi_n^{2r}(b_n) + \varphi_n^{2r}(|X_n|)}\right)\right\} < \infty
\end{aligned} \tag{2.8}$$

for $r = p$ in the case (a) or $r = 1$ in the case (b). Hence, the sequences $\{X_n, n \geq 1\}$ and $\{X'_n, n \geq 1\}$ are equivalent in Khinchin's sense.

Thus, we need only to show that

$$\frac{S'_n}{b_n} \longrightarrow 0 \quad \text{a.s. } n \longrightarrow \infty. \tag{2.9}$$

By Lemma 1.3, for $q = 4$, we have

$$\begin{aligned} E\left[\max_{1\leq i\leq n}\left|S'_i\right|\right]^4 &\leq C\left[\sum_{i=1}^{n} EX_i^{*4} + \left(\sum_{i=1}^{n} EX_i^{*2}\right)^2\right] \leq C\left(2\sum_{i=1}^{n} EX_i^{*4} + 2\sum_{i=2}^{n} EX_i^{*2}\sum_{j=1}^{i-1} EX_j^{*2}\right) \\ &\leq C\left[EX_1'^4 + \sum_{i=2}^{n}\left(EX_i'^4 + EX_i'^2\sum_{j=1}^{i-1} EX_j'^2\right)\right]. \end{aligned} \tag{2.10}$$

By Theorem 1.2 applied with

$$\alpha_1 = EX_1'^4, \qquad \alpha_i = EX_i'^4 + EX_i'^2\sum_{j=1}^{i-1} EX_j'^2 \quad \text{for } i = 2, 3, \ldots, n, \tag{2.11}$$

we see that in order to show (2.9) it is enough to prove that

$$\sum_{i=1}^{\infty} \frac{\alpha_i}{b_i^4} < \infty, \tag{2.12}$$

which holds if

$$\sum_{i=1}^{\infty} \frac{EX_i'^4}{b_i^4} < \infty, \qquad \sum_{i=2}^{\infty} \frac{EX_i'^2}{b_i^4}\sum_{j=1}^{i-1} EX_j'^2 < \infty. \tag{2.13}$$

Put $I := \sum_{i=1}^{\infty} P[|X_i| \geq a_i]$. Then, we have

$$\sum_{i=1}^{\infty} \frac{EX_i'^4}{b_i^4} \leq \sum_{i=1}^{\infty} \frac{E\left(X_i I\left[|X_i| < \min(a_i, b_i)\right]\right)^4}{b_i^4} + I \leq \sum_{i=1}^{\infty} \frac{E\left(|X_i|^{2p} I\left[|X_i| < \min(a_i, b_i)\right]\right)}{b_i^{2p}} + I. \tag{2.14}$$

Moreover, we note that

$$\frac{a}{b} \cdot I[a \leq b] \leq 2 \cdot \frac{a}{a+b} \quad \forall\, a, b > 0. \tag{2.15}$$

Hence, in case (a) we get, by (B) and (C),

$$\sum_{i=1}^{\infty} \frac{EX_i'^4}{b_i^4} \leq \sum_{i=1}^{\infty} E\left(\frac{\varphi_i^{2p}(|X_i|)}{\varphi_i^{2p}(b_i)} I\left[|X_i| < \min(a_i, b_i)\right]\right) + I \leq 2\sum_{i=1}^{\infty} E\left(\frac{\varphi_i^{2p}\left[\min(|X_i|, a_i)\right]}{\varphi_i^{2p}(b_i) + \varphi_i^{2p}(|X_i|)}\right) + I < \infty, \tag{2.16}$$

while in case (b) we get, by (B) and (C_1),

$$\sum_{i=1}^{\infty} \frac{EX_i'^4}{b_i^4} \le 2\sum_{i=1}^{\infty} E\left(\frac{\varphi_i^2[\min(|X_i|, a_i)]}{\varphi_i^2(b_i) + \varphi_i^2(|X_i|)}\right) + I < \infty. \tag{2.17}$$

Using the fact that

$$b_i^{-4} E(X_i'^2) E(X_j'^2) \le b_i^{-2p} E|X_i'|^p E|X_j'|^p \quad \forall j < i \tag{2.18}$$

both in either case (a)

$$\sum_{i=2}^{\infty} \frac{EX_i'^2}{b_i^4} \sum_{j=1}^{i-1} EX_j'^2$$

$$\le \sum_{i=2}^{\infty} \frac{E|X_i'|^p}{b_i^{2p}} \sum_{j=1}^{i-1} E|X_j'|^p \le 4\sum_{i=2}^{\infty} b_i^{-p} E\left(\frac{\varphi_i^p(|X_i|)}{\varphi_i^p(b_i) + \varphi_i^p(|X_i|)}\right) \sum_{j=1}^{i-1} b_j^p E\left(\frac{\varphi_j^p(|X_j|)}{\varphi_j^p(b_j) + \varphi_j^p(|X_j|)}\right) < \infty, \tag{2.19}$$

or case (b)

$$\sum_{i=2}^{\infty} \frac{EX_i'^2}{b_i^4} \sum_{j=1}^{i-1} EX_j'^2$$

$$\le \sum_{i=2}^{\infty} \frac{E|X_i'|^p}{b_i^{2p}} \sum_{j=1}^{i-1} E|X_j'|^p \le 4\sum_{i=2}^{\infty} b_i^{-p} E\left(\frac{\varphi_i(|X_i|)}{\varphi_i(b_i) + \varphi_i(|X_i|)}\right) \sum_{j=1}^{i-1} b_j^p E\left(\frac{\varphi_j(|X_j|)}{\varphi_j(b_j) + \varphi_j(|X_j|)}\right) \infty. \tag{2.20}$$

Thus, we have established (2.9) and consequently (2.7). □

Corollary 2.2. *Let $\{X_n, n \ge 1\}$ be a sequence of ρ^*-mixing random variables satisfying the condition*

$$\sum_{i=1}^{\infty} E\left(\frac{|X_i|^{rt}}{i^r + |X_i|^{rt}}\right) < \infty \tag{2.21}$$

with $r = p$ for $0 < t < 1$ and all $1 < p \le 2$, or $r = 2/t$ for $1 \le t < 2$.

Then,

$$n^{-1/t} \sum_{i=1}^{n} \{X_i - E(X_i I[|X_i| < i^{1/t}])\} \longrightarrow 0 \quad \text{a.s. as } n \longrightarrow \infty, \tag{2.22}$$

Proof. Let $a_n = b_n = n^{1/t}$ for any $0 < t < 2$. Then, the assumption (B) of Theorem 2.1 is fulfilled. Indeed we see

$$\text{(B)}: \quad \sum_{n=1}^{\infty} P[|X_n| \ge a_n]$$

$$= \sum_{n=1}^{\infty} P[|X_n| \ge n^{1/t}] \le C\sum_{n=1}^{\infty} P[2|X_n|^{rt} \ge n^r + |X_n|^{rt}] < C \cdot \sum_{n=1}^{\infty} E\left(\frac{|X_n|^{rt}}{n^r + |X_n|^{rt}}\right) < \infty \tag{2.23}$$

by (2.21) with $r = p$ for $0 < t < 1$ or $r = 2/t$ for $1 \le t < 2$.

Let now $0 < t < 1$. Then, for $\varphi_n(x) = x^t$, $x > 0$, $n \geq 1$, the conditions (A) and (C) with $1 < p \leq 2$ are fulfilled:

$$(\mathrm{A}): \quad \sum_{i=2}^{\infty} b_i^{-p} E\left(\frac{\varphi_i^p(|X_i|)}{\varphi_i^p(b_i)+\varphi_i^p(|X_i|)}\right)\sum_{j=1}^{i-1} b_j^p E\left(\frac{\varphi_j^p(|X_j|)}{\varphi_j^p(b_j)+\varphi_j^p(|X_j|)}\right) \leq \left(\sum_{i=1}^{\infty} E\left(\frac{|X_i|^{pt}}{i^p+|X_i|^{pt}}\right)\right)^2 < \infty, \tag{2.24}$$

by (2.21) with $r = p$.

$$\begin{aligned}(\mathrm{C}): \quad \sum_{n=1}^{\infty} E\left(\frac{\varphi_n^{2p}[\min(|X_n|, a_n)]}{\varphi_n^{2p}(b_n)+\varphi_n^{2p}(|X_n|)}\right) &= \sum_{n=1}^{\infty} E\left(\frac{[\min(|X_n|, a_n)]^{2pt}}{n^{2p}+|X_n|^{2pt}}\right) \\ &\leq \sum_{n=1}^{\infty} E\left(\frac{|X_n|^{2pt}}{n^{2p}+|X_n|^{2pt}} I[|X_n| \geq n^{1/t}]\right) \\ &\quad + \sum_{n=1}^{\infty} E\left(\frac{|X_n|^{2pt}}{|X_n|^{pt}((n^p/|X_n|^{pt})\cdot n^p+|X_n|^{pt})} I[|X_n| < n^{1/t}]\right) \\ &\leq C\sum_{n=1}^{\infty} P[|X_n| \geq n^{1/t}] + \sum_{n=1}^{\infty} E\left(\frac{|X_n|^{pt}}{n^p+|X_n|^{pt}}\right) < \infty\end{aligned} \tag{2.25}$$

by (2.23) and (2.21) with $r = p$, $1 < p \leq 2$.

Thus, by Theorem 2.1, we have

$$n^{-1/t}\sum_{i=1}^{n}\{X_i - E(X_i I[|X_i| < i^{1/t}])\} \longrightarrow 0 \quad \text{a.s. as } n \longrightarrow \infty \tag{2.26}$$

for $0 < t < 1$.

Now, we need to show that (2.26) also holds for $1 \leq t < 2$.

Let $\varphi_n(x) = x^2$. Then, for $p = 2$, by the similar calculations as in case $0 < t < 1$, we get

$$(\mathrm{A}_1): \quad \sum_{i=2}^{\infty} b_i^{-p} E\left(\frac{\varphi_i(|X_i|)}{\varphi_i(b_i)+\varphi_i(|X_i|)}\right)\sum_{j=1}^{i-1} b_j^p E\left(\frac{\varphi_j(|X_j|)}{\varphi_j(b_j)+\varphi_j(|X_j|)}\right) \leq \left(\sum_{i=1}^{\infty} E\left(\frac{|X_i|^2}{i^{2/t}+|X_i|^2}\right)\right)^2 < \infty \tag{2.27}$$

by (2.21) with $r = 2/t$ and

$$\begin{aligned}(\mathrm{C}_1): \quad &\sum_{n=1}^{\infty} E\left(\frac{\varphi_n^2[\min(|X_n|, a_n)]}{\varphi_n^2(b_n)+\varphi_n^2(|X_n|)}\right) \\ &= \sum_{n=1}^{\infty} E\left(\frac{[\min(|X_n|, a_n)]^4}{n^{4/t}+|X_n|^4}\right) \leq C\sum_{n=1}^{\infty} P[|X_n| \geq n^{1/t}] + \sum_{n=1}^{\infty} E\left(\frac{|X_n|^2}{n^{2/t}+|X_n|^2}\right) < \infty\end{aligned} \tag{2.28}$$

by (2.23) and (2.21) with $r = 2/t$.

Therefore, by Theorem 2.1, we get (2.26) for $1 \leq t < 2$.

This completes the proof of Corollary 2.2. □

Corollary 2.3. *Let $\{X, X_n, n \geq 1\}$ be a sequence of ρ^*-mixing random variables with $E|X|^t < \infty$, $0 < t < 2$. Let*

$$P\left[\left|X_i\right| > x\right] \leq CP\left[|X| > x\right] \tag{2.29}$$

for all $x > 0$, $i \geq 1$ and some positive constant C.

Moreover, when $1 \leq t < 2$, let $EX = 0$. Then,

$$n^{-1/t}\sum_{i=1}^{n} X_i \longrightarrow 0 \quad a.s.\ as\ n \longrightarrow \infty. \tag{2.30}$$

Proof. We first note that (2.29) implies

$$E\left(\left|X_i\right|^s I\left[\left|X_i\right| < a\right]\right) \leq C\left\{E\left(|X|^s I\left[|X| < a\right]\right) + a^s P\left[|X| \geq a\right]\right\} \tag{2.31}$$

for any $a > 0$ and $s > 0$.

Put now $a_n = b_n = n^{1/t}$ for any $0 < t < 2$.

It is easy to see that $E|X|^t < \infty$, (2.29) and (2.31) with $s = rt$ imply convergence of the series:

$$\sum_{n=1}^{\infty} E\left(\frac{\left|X_n\right|^{rt}}{n^r + \left|X_n\right|^{rt}}\right) \tag{2.32}$$

for $r = p$, $(1 < p \leq 2)$ in the case $0 < t < 1$ and $r = 2/t$ in the case $1 \leq t < 2$.

We have

$$\begin{aligned}
\sum_{n=1}^{\infty} E\left(\frac{\left|X_n\right|^{rt}}{n^r + \left|X_n\right|^{rt}}\right) &= \sum_{n=1}^{\infty} E\left(\frac{\left|X_n\right|^{rt}}{n^r + \left|X_n\right|^{rt}} I\left[\left|X_n\right| < n^{1/t}\right]\right) + \sum_{n=1}^{\infty} E\left(\frac{\left|X_n\right|^{rt}}{n^r + \left|X_n\right|^{rt}} I\left[\left|X_n\right| \geq n^{1/t}\right]\right) \\
&\leq \sum_{n=1}^{\infty} E\left(\frac{\left|X_n\right|^{rt}}{n^r} I\left[\left|X_n\right| < n^{1/t}\right]\right) + \sum_{n=1}^{\infty} P\left[\left|X_n\right| \geq n^{1/t}\right] \\
&\leq C\left\{\sum_{n=1}^{\infty} E\left(\frac{|X|^{rt}}{n^r} I\left[|X| < n^{1/t}\right]\right) + \sum_{n=1}^{\infty} P\left[|X| \geq n^{1/t}\right]\right\} \\
&\leq C\left\{\sum_{n=1}^{\infty} \frac{E\left(|X|^{t+t(r-1)} I\left[|X| < n^{1/t}\right]\right)}{n^{1+1/t\cdot t(r-1)}} + E|X|^t\right\} < \infty.
\end{aligned} \tag{2.33}$$

Because of Corollary 2.2, this proves that (2.26) holds for any $0 < t < 2$.

To complete the proof we should show that

$$n^{-1/t}\left|\sum_{i=1}^{n} E\left(X_i I\left[\left|X_i\right| < i^{1/t}\right]\right)\right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{2.34}$$

for any $0 < t < 2$.

For $0 < t < 1$ and $I_1 := n^{-1/t}\sum_{i=1}^{n} i^{1/t}P[|X| \geq i^{1/t}]$, we have

$$\begin{aligned} n^{-1/t}\left|\sum_{i=1}^{n} E\left(X_i I\left[\left|X_i\right| < i^{1/t}\right]\right)\right| &\leq n^{-1/t}\sum_{i=1}^{n} E\left(\left|X_i\right| I\left[\left|X_i\right| < i^{1/t}\right]\right) \\ &\leq C\left\{n^{-1/t}\sum_{i=1}^{n} E\left(|X| I\left[|X| < i^{1/t}\right]\right) + I_1\right\} \\ &\leq C\left\{n^{-1/t}\sum_{i=1}^{n} E\left(|X| I\left[|X| < n^{1/t}\right]\right) + I_1\right\} \\ &\leq C\left\{n^{1-1/t}E\left(|X| I\left[|X| < n^{1/t}\right]\right) + I_1\right\} \\ &= C\left\{n^{1-1/t}\sum_{j=1}^{n} E\left(|X| I\left[j-1 \leq |X|^t < j\right]\right) + I_1\right\}. \end{aligned} \tag{2.35}$$

But

$$\sum_{n=1}^{\infty} P\left[|X| \geq n^{1/t}\right] \leq CE|X|^t < \infty, \tag{2.36}$$

so by Kronecker's lemma we get

$$I_1 := n^{-1/t}\sum_{i=1}^{n} i^{1/t}P\left[|X| \geq i^{1/t}\right] \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.37}$$

Moreover, we note

$$\sum_{j=1}^{\infty} j^{1-1/t}E\left(|X| I\left[j-1 \leq |X|^t < j\right]\right) \leq \sum_{j=1}^{\infty} E\left(|X| I\left[j-1 \leq |X|^t < j\right]\right) = E|X|^t < \infty, \tag{2.38}$$

and by Kronecker's lemma

$$n^{1-1/t}\sum_{j=1}^{n} E\left(|X| I\left[j-1 \leq |X|^t < j\right]\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{2.39}$$

which together with (2.35) and (2.37) gives (2.34) for $0 < t < 1$.

Let $1 \leq t < 2$. First, we will show that

$$\lim_{n\to\infty} P\left[\left|\sum_{i=1}^{n} X_i\right| > \varepsilon n^{1/t}\right] = 0. \tag{2.40}$$

To achieve this, we put $Y_i = X_i I[|X_i| < n^{1/t}]$ for $1 \leq i \leq n$.

Because of $EX_i = 0$ and $E|X|^t < \infty$, we have

$$n^{-1/t}\left|\sum_{i=1}^{n} EY_i\right| \leq n^{-1/t}\sum_{i=1}^{n} E\left(\left|X_i\right| I\left[\left|X_i\right| \geq n^{1/t}\right]\right) \leq CE\left(|X|^t I\left[|X| \geq n^{1/t}\right]\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.41}$$

Therefore, for large enough n, we obtain

$$P\left[\left|\sum_{i=1}^{n} X_i\right| > \varepsilon n^{1/t}\right] \le \sum_{i=1}^{n} P\left[\left|X_i\right| \ge n^{1/t}\right] + P\left[\left|\sum_{i=1}^{n} (Y_i - EY_i)\right| > \varepsilon n^{1/t}\right]. \tag{2.42}$$

Hence, we need only to show that

$$\sum_{i=1}^{n} P\left[\left|X_i\right| \ge n^{1/t}\right] \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{2.43}$$

$$P\left[\left|\sum_{i=1}^{n} (Y_i - EY_i)\right| > \varepsilon n^{1/t}\right] \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.44}$$

By $E|X|^t < \infty$, we get

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=1}^{n} P\left[\left|X_i\right| \ge n^{1/t}\right] \le C \sum_{n=1}^{\infty} P\left[|X| \ge n^{1/t}\right] \le CE|X|^t < \infty, \tag{2.45}$$

which implies (2.43).

By Lemma 1.3 and (2.23) with $s = 2$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n} P\left[\left|\sum_{i=1}^{n} (Y_i - EY_i)\right| > \varepsilon n^{1/t}\right] \\
&\quad \le C \sum_{n=1}^{\infty} n^{-(1+2/t)} \sum_{i=1}^{n} EY_i^2 \\
&\quad \le C \left\{ \sum_{n=1}^{\infty} n^{-(1+2/t)} \sum_{i=1}^{n} E\left(X^2 I\left[|X| < n^{1/t}\right]\right) + \sum_{n=1}^{\infty} n^{-(1+2/t)} \sum_{i=1}^{n} n^{2/t} P\left[|X| \ge n^{1/t}\right] \right\} \\
&\quad \le C \left\{ \sum_{n=1}^{\infty} n^{-2/t} \sum_{j=1}^{n} E\left(X^2 I\left[j-1 \le |X|^t < j\right]\right) + \sum_{n=1}^{\infty} P\left[|X| \ge n^{1/t}\right] \right\} \\
&\quad \le C \left\{ \sum_{k=1}^{\infty} E\left(X^2 I\left[k-1 \le |X|^t < k\right]\right) \sum_{n=k}^{\infty} n^{-2/t} + E|X|^t \right\} \\
&\quad \le C \left\{ \sum_{k=1}^{\infty} k^{-2/t+1} E\left(X^2 I\left[k-1 \le |X|^t < k\right]\right) + E|X|^t \right\} \\
&\quad \le C \left\{ \sum_{k=1}^{\infty} kP\left[k-1 \le |X|^t < k\right] + E|X|^t \right\} < CE|X|^t < \infty
\end{aligned} \tag{2.46}$$

which gives (2.44).

Thus, we have established that (2.40) holds true. Equations (2.40) and (2.26) imply (2.34) which completes the proof of Corollary 2.3. □

Corollary 2.3 gives the Marcinkiewicz SLLN for ρ^*-mixing random variables $\{X_n, n \ge 1\}$ stochastically dominated by a random variable X for $0 < t < 2$. The identical result was obtained by Cai [6] as a consequence of complete convergence theorem. Both of them, for $0 < t < 1$, are a special case of more general result of Fazekas and Tómács [11] obtained for stochastically dominated random variables without assuming any kind of dependence.

Acknowledgment

The author thanks the referees for their comments and suggestions which allowed to improve this paper.

References

[1] R. C. Bradley, "On the spectral density and asymptotic normality of weakly dependent random fields," *Journal of Theoretical Probability*, vol. 5, no. 2, pp. 355–373, 1992.
[2] W. Bryc and W. Smoleński, "Moment conditions for almost sure convergence of weakly correlated random variables," *Proceedings of the American Mathematical Society*, vol. 119, no. 2, pp. 629–635, 1993.
[3] M. Peligrad, "Maximum of partial sums and an invariance principle for a class of weak dependent random variables," *Proceedings of the American Mathematical Society*, vol. 126, no. 4, pp. 1181–1189, 1998.
[4] M. Peligrad and A. Gut, "Almost-sure results for a class of dependent random variables," *Journal of Theoretical Probability*, vol. 12, no. 1, pp. 87–104, 1999.
[5] S. Utev and M. Peligrad, "Maximal inequalities and an invariance principle for a class of weakly dependent random variables," *Journal of Theoretical Probability*, vol. 16, no. 1, pp. 101–115, 2003.
[6] G.-H. Cai, "Strong law of large numbers for ρ^*-mixing sequences with different distributions," *Discrete Dynamics in Nature and Society*, vol. 2006, Article ID 27648, 7 pages, 2006.
[7] M.-H. Zhu, "Strong laws of large numbers for arrays of rowwise ρ^*-mixing random variables," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 74296, 6 pages, 2007.
[8] I. Fazekas and O. Klesov, "A general approach to the strong laws of large numbers," *Theory of Probability and Its Applications*, vol. 45, no. 3, pp. 436–449, 2000.
[9] K.-L. Chung, "Note on some strong laws of large numbers," *American Journal of Mathematics*, vol. 69, no. 1, pp. 189–192, 1947.
[10] H. Teicher, "Some new conditions for the strong law," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 59, no. 3, pp. 705–707, 1968.
[11] I. Fazekas and T. Tómács, "Strong laws of large numbers for pairwise independent random variables with multidimensional indices," *Publicationes Mathematicae Debrecen*, vol. 53, no. 1-2, pp. 149–161, 1998.